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Calculating Fuzzy Inverse Matrix Using Linear Programming Problem: An Improved Approach

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Abstract

Calculating the matrix inverse is a key point in solving linear equation system, which involves complex calculations, particularly when the matrix elements are LR (Left and Right) fuzzy numbers. In this paper, first, the method of Kaur and Kumar for calculating the matrix inverse is reviewed, and its disadvantages are discussed. Then, a new method is proposed to determine the inverse of LR fuzzy matrix based on linear programming problem. It is demonstrated that the proposed method is capable of overcoming the shortcomings of the previous matrix inverse. Numerical examples are utilized to verify the performance and applicability of the proposed method.

Key Words: LR fuzzy matrix, fuzzy identity matrix, fuzzy inverse matrix.

Mathematical Subject Classification: 15A09

1. Introduction

Matrix is an important tool in mathematics that has many applications in engineering and economics such as fuzzy linear regression (M. A. Basaran & Simonetti, 2021), soliton theory (Zhang, Gao, & Xu, 2020), estimating marine survival (Pardo & Hutchings, 2020), and fuzzy singular differential equations (Najariyan & Zhao, 2020). If the elements of a matrix are ambiguous, the matrix is fuzzy. In recent decade, fuzzy matrix has gained the attention of many researchers (Khalili, Naseri, & Taghi-Nezhad, 2020; Nazari, Fathali, & Taghi-Nezhad, 2020; N. Taghi-nezhad, Naseri, Khalili Goodarzi, & Taleshian Jelodar, 2015; N.A. Taghi-Nezhad, 2019; N. A. Taghi-Nezhad, Moradi, & Karamali, 2021). Since, matrix division has not been defined, the concept of inverse matrix has been introduced, and various methods have been proposed to solve fuzzy linear equations (Abbasi & Allahviranloo, 2021; Allahviranloo & Babakordi, 2017; Fatemeh. Babakordi, 2020; Fatemeh Babakordi & Allahviranloo, 2021; F. Babakordi, Allahviranloo, & Adabitabarfirozja, 2016; F. Babakordi & Firozja, 2020; F Babakordi & Taghi-Nezhad, 2019; Golbabai & Panjeh Ali Beik, 2015). One of these methods is the matrix inverse method. Therefore, study of matrix inversion is of special importance.
When the matrix is singular and rectangle, one encounters generalized inverse matrix. Babakordi reviewed different types of generalized inverse matrices, and investigated their structure in (fatemeh babakordi, 2021). Generalized inverse matrix has also been studied in (Jianmiao Cen, 1999; J Cen, 2005). If a matrix is non-singular and square, the matrix is said to be inversible. If a matrix is fuzzy, then its inverse matrix is also fuzzy.
Hashimoto (Hashimoto, 1984) used the Gödel implication operator, and showed some properties of sub-inverses of fuzzy matrices of the first class. In (Dehghan, Ghatee, & Hashemi, 2009), Dehghan et. al. pursued two ideas to obtain the inverse of a fuzzy matrix: The first idea is called “scenario-based”, and the second one is named as “arithmetic-based”. Recently, in (Farahani, Ebadi, & Jafari, 2021), the problem of calculating a fuzzy inverse matrix was converted to a problem of solving a system of fuzzy polynomial equations, where a fuzzy system was transformed to an equivalent system of crisp polynomial equations. The solution of the system of crisp polynomial equations was calculated using Wu’s method, and a criterion was introduced for inevitability of a fuzzy matrix

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(FM). In addition, an algorithm was proposed to calculate the fuzzy inverse matrix. Petchimuthu & Kamacı (Petchimuthu & Kamacı, 2020) investigated some of the basic operations and properties of the inverse fuzzy soft matrices. Meenakshi (Meenakshi, 2019) studied fuzzy matrices, and assigned a chapter of her book to inverting fuzzy matrices.
In this paper, In Section 2, basic definitions are presented, and the method of Kaur and Kumar is described. In Section 3, the shortcomings and disadvantages of the Kaur and Kumar method are described. In Section 4, a new method is proposed for calculating the matrix inverse using linear programming problem. The effectiveness and the applicability of the proposed method are shown using numerical examples in Section 5. Finally, the conclusions are given in Section 6.

1.1. Literature review and problem statement

The paper (M. Basaran, 2012) presented the results of a research on calculation of LR fuzzy inverse matrix by solving the system equation $\tilde{A} \otimes \tilde{X} = \tilde{I}$, where $\tilde{X} = \tilde{A}^{-1}$. But, there was an unresolved solution related to the sign of the fuzzy matrix, since only positive LR fuzzy matrix was considered by Basaran in (M. Basaran, 2012). A way to overcome this insufficiency could be the definition of LR fuzzy number, which stated that a fuzzy number is positive, if its core is positive, and is negative, if its core is negative. This definition was used by Mosleh and Otadi (Mosleh & Otadi, 2015) to obtain LR fuzzy inverse matrix for both positive and negative cases. Finally, Kaur and Kumar (Kaur & Kumar, 2017) obtained the exact product inverse of fuzzy matrix by improving and modifying the work of Mosleh and Otadi. However, there are still shortcomings in the previous methods. Two main conditions which must be considered in the calculation of a fuzzy inverse matrix are: First, $\tilde{A} \otimes \tilde{A}^{-1} = \tilde{I}$ must hold, and, second, the inverse matrix must be fuzzy as well. These conditions did not hold in the previous inversion methods. All this suggests that it is advisable to conduct a study on calculation of LR fuzzy inverse matrix.

2. Basic Definitions

Definition 2.1 (Bose, 1995) A fuzzy number $\tilde{u}$ is said to be a LR fuzzy number, if:

$$\tilde{u}(x) = \begin{cases} L\left(\frac{a-x}{\alpha}\right) & x \le a, \alpha > 0 \\ R\left(\frac{x-a}{\beta}\right) & x \ge a, \beta > 0 \end{cases}$$

Where a is the mean value of $\tilde{u}$, α and β are left and right spreads, respectively, and the function $L(.)$, which is called left shape function, satisfies :
(1) $L(x) = L(-x)$,
(2) $L(0) = 1$, and $L(1) = 0$,
(3) $L(x)$ is non-increasing on $[0, \infty)$.
The definition of a right shape function $R(.)$ is usually similar to that of left shape function $L(.)$.
The mean value, left and right spreads, and shape functions of a LR fuzzy number $\tilde{u}$ are symbolically shown as $\tilde{u} = (a, \alpha, \beta)_{LR}$, and for two LR fuzzy numbers $\tilde{u} = (a, \alpha, \beta)_{LR}$ and $\tilde{v} = (b, \gamma, \delta)_{LR}$, there is:

$$\tilde{u} \oplus \tilde{v} = (a + b, \alpha + \gamma, \beta + \delta)_{LR}$$

$$\tilde{u} \otimes \tilde{v} = \begin{cases} (ab, a\gamma + b\alpha, a\delta + \beta b)\tilde{u}, \tilde{v} \ge 0, \\ (ab, b\alpha - a\delta, \beta b - a\gamma)\tilde{u} \le 0, \tilde{v} \ge 0, \\ (ab, a\gamma + b\alpha, a\delta + \beta b) \qquad 0 \epsilon \tilde{u}, \tilde{v} \ge 0, \end{cases} \tag{1}$$

Definition 2.2 (Dehghan et al., 2009) A $n \times n$ matrix systemdefined as below is called a fully fuzzy matrix system, and denoted as $\tilde{A} \otimes \tilde{X} = \tilde{B}$ in compact form:

$$\begin{bmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1n} \\ \vdots & \vdots & \vdots \\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn} \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} & \cdots & \tilde{x}_{1n} \\ \vdots & \vdots & \vdots \\ \tilde{x}_{n1} & \cdots & \tilde{x}_{nn} \end{bmatrix} = \begin{bmatrix} \tilde{b}_{11} & \cdots & \tilde{b}_{1n} \\ \vdots & \vdots & \vdots \\ \tilde{b}_{n1} & \cdots & \tilde{b}_{nn} \end{bmatrix}$$

where$\tilde{A} = [a_{ij}]$ and $\tilde{B} = [b_{ij}]$ are $n \times n$known fuzzy matrices, and $\tilde{X} = [x_{ij}]$is a $n \times n$ unknown fuzzy matrix.

Definition 2.3 (Jianmiao Cen, 1999) If $\tilde{a}_{\mathrm{ij}} = (a_{ij}, \gamma_{ij}, \delta_{ij})$, $\tilde{x}_{ij} = (x_{jk}, \alpha_{jk}, \beta_{jk})$, and $\tilde{a}_{\mathrm{ij}} \otimes \tilde{x}_{ij} = (m_{ik}, \rho_{ik}, \phi_{ik})$, then we have:

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$$(m_{ik},\rho_{ik},\phi_{ik}) \approx \begin{cases}(m_{ik1},\rho_{ik1},\phi_{ik1}) & a_{ij}\geq 0, a_{ij}-\gamma_{ij}<0,\\ (m_{ik2},\rho_{ik2},\phi_{ik2}) & a_{ij}<0, \quad a_{ij}+\delta_{ij}\geq 0,\\ (m_{ik3},\rho_{ik3},\phi_{ik3}) & (a_{ij},\gamma_{ij},\delta_{ij})\leq 0,\\ (m_{ik4},\rho_{ik4},\phi_{ik4}) & (a_{ij},\gamma_{ij},\delta_{ij})\geq 0,\end{cases}$$

$$(m_{ik1},\rho_{ik1},\varphi_{ik1}) = (a_{ij}x_{jk},$$
$$a_{ij}x_{jk}-\min\{a_{ij}x_{jk}+a_{ij}\beta_{jk}-\gamma_{ij}x_{jk}-\gamma_{ij}\beta_{jk}, a_{ij}x_{jk}-a_{ij}\alpha_{jk}+\delta_{ij}x_{jk}-\delta_{ij}\alpha_{jk}\},$$
$$\max\{a_{ij}x_{jk}-a_{ij}\alpha_{jk}-\gamma_{ij}x_{jk}+\gamma_{ij}\alpha_{jk}, a_{ij}x_{jk}+a_{ij}\beta_{jk}+\delta_{ij}x_{jk}+\delta_{ij}\beta_{jk}\}-a_{ij}x_{jk}\}),$$

$$(m_{ik2},\rho_{ik2},\varphi_{ik2}) = (a_{ij}x_{jk},$$
$$a_{ij}x_{jk}-min\{a_{ij}x_{jk}+a_{ij}\beta_{jk}-\gamma_{ij}x_{jk}-\gamma_{ij}\beta_{jk}, a_{ij}x_{jk}-a_{ij}\alpha_{jk}+\delta_{ij}x_{jk}-\delta_{ij}\alpha_{jk}\},$$
$$max\{a_{ij}x_{jk}-a_{ij}\alpha_{jk}-\gamma_{ij}x_{jk}+\gamma_{ij}\alpha_{jk}, a_{ij}x_{jk}+a_{ij}\beta_{jk}+\delta_{ij}x_{jk}+\delta_{ij}\beta_{jk}\}-a_{ij}x_{jk}\}), \tag{2}$$

$$(m_{ik3},\rho_{ik3},\varphi_{ik3}) = (a_{ij}x_{jk},$$
$$a_{ij}x_{jk}-\min\{a_{ij}x_{jk}+a_{ij}\beta_{jk}-\gamma_{ij}x_{jk}-\gamma_{ij}\beta_{jk}, a_{ij}x_{jk}+a_{ij}\beta_{jk}+\delta_{ij}x_{jk}+\delta_{ij}\beta_{jk}\},$$
$$\max\{a_{ij}x_{jk}-a_{ij}\alpha_{jk}+\delta_{ij}x_{jk}-\delta_{ij}\alpha_{jk}, a_{ij}x_{jk}-a_{ij}\alpha_{jk}-\gamma_{ij}x_{jk}+\gamma_{ij}\alpha_{jk}\}-a_{ij}x_{jk}\}),$$

$$(m_{ik4},\rho_{ik4},\varphi_{ik4}) = (a_{ij}x_{jk},$$
$$a_{ij}x_{jk}-min\{a_{ij}x_{jk}-a_{ij}\alpha_{jk}-\gamma_{ij}x_{jk}+\gamma_{ij}\alpha_{jk}, a_{ij}x_{jk}-a_{ij}\alpha_{jk}+\delta_{ij}x_{jk}-\delta_{ij}\alpha_{jk}\},$$
$$max\{a_{ij}x_{jk}+a_{ij}\beta_{jk}-\gamma_{ij}x_{jk}-\gamma_{ij}\beta_{jk}, a_{ij}x_{jk}+a_{ij}\beta_{jk}+\delta_{ij}x_{jk}+\delta_{ij}\beta_{jk}\}-a_{ij}x_{jk}\}).$$

2.1. Kaur and Kumar Method

In the Kaur and Kumar method, first, each LR fuzzy number, which its core was one and its spreads were between zero and one,was considered as LR fuzzy one number, and each LR fuzzy number, which its core was zero and its spreads were between zero and one, was considered as LR fuzzy zero number. Then, by defining fuzzy identity matrix and solving the linear equation system $\tilde{A}\otimes\tilde{X}=\tilde{I}$, the fuzzy inverse matrix was calculated. As a numerical example, the fuzzy inverse matrix of $\tilde{A}=\begin{bmatrix}(10,4,4) & (8,3,3)\\ (6,2,2) & (4,3,3)\end{bmatrix}$ was obtained by solving the following matrix system:

$$\begin{bmatrix}(10,4,4) & (8,3,3)\\ (6,2,2) & (4,3,3)\end{bmatrix}\otimes\begin{bmatrix}\tilde{x}_{11} & \tilde{x}_{12}\\ \tilde{x}_{21} & \tilde{x}_{22}\end{bmatrix}=\begin{bmatrix}(1,\alpha,\beta) & (0,\gamma,\delta)\\ (0,\gamma,\delta) & (1,\alpha,\beta)\end{bmatrix}$$

where $\alpha=1,\ \beta=1.07,\ \gamma=0,\ \delta=1.25$.
The solution was obtained as:

$$\tilde{A}^{-1}=\begin{bmatrix}(-0.5,0.5,0.11) & (1,1,0.65)\\ (0.75,0.75,0.35) & (-1.25,1.25,0..52)\end{bmatrix}.$$

In the following, the shortcomings and drawbacks of the Kaur and Kumar method are explained.

3. Kaur and Kumar Method Limitations

It should be noted that as Kaur and Kumar used the definition of 5-7 in (Kaur & Kumar, 2017) for the identity matrix, they had to consider both left and right spreads between zero and one for fuzzy zero and one numbers. However, they solved an example, where $\alpha=1,\ \beta=1.07$, and $\delta=1.25$. That is, the spreads were equal to or greater than one, which was contrary to the definition they had utilized for the identity matrix. If another definition was applied for the identity matrix, they had to mention it clearly in their paper.

Theorem 3.1 Consider a fuzzy number $\tilde{a}=(a,\gamma,\delta)$ and its inverse $\tilde{a}^{-1}=(x,\alpha,\beta)$. According to the Kaur and Kumar method proposed in (Kaur & Kumar, 2017), there must be $\tilde{a}\otimes\tilde{a}^{-1}=(1,\alpha_1,\beta_1)$ such that $0<\alpha_1,\ \beta_1<1$, whilst:

a)If $a\geq 0$, then $a-\gamma<0$, and the fuzzy inverse matrix does not exist.
b)If $a<0$, then $a+\delta\geq 0$, and the fuzzy inverse matrix does not exist.
c)If $(a,\gamma,\delta)\leq 0$, then only when $x+\beta<0$ and $\gamma+a<0$ satisfying $0<\alpha<\frac{a+\gamma}{a(\gamma-a)}$, the matrix inverse exists.

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d) If $(a,\gamma,\delta) \geq 0$, then only when $x-\alpha>0$ and $a-\delta>0$, satisfying $0<\beta<\frac{a-\delta}{a(a+\delta)}$, the matrix inverse exists.

Proof: $\tilde{a}^{-1}$ must satisfy $\tilde{a} \otimes \tilde{a}^{-1} = (1,\alpha_1,\beta_1)$, therefore, based on multiplication (2), four cases can occur:

- **a)** $a \geq 0, a-\gamma<0$,
- **b)** $a<0, a+\delta \geq 0$,
- **c)** $(a,\gamma,\delta) \leq 0$,
- **d)** $(a,\gamma,\delta) \geq 0$.

In the following, the theorem is proved based on the above cases.

- **a)** $a \geq 0,\ a-\gamma<0$: Based on multiplication (2), the condition $\tilde{a} \otimes \tilde{a}^{-1} = (1,\alpha_1,\beta_1)$ can be considered as follows:

$$\tilde{a} \otimes \tilde{a}^{-1} = (ax, ax - \min\{ax+a\beta-\gamma x-\gamma\beta, ax-a\alpha+\delta x-\delta\alpha\}, \max\{ax-a\alpha-\gamma x+\gamma\alpha, ax+a\beta+\delta x+\delta\beta\}-ax\}) = (1,\alpha_1,\beta_1)$$

Therefore:

$$ax = 1, \quad (1)$$

$$ax - \min\{ax+a\beta-\gamma x-\gamma\beta, ax-a\alpha+\delta x-\delta\alpha\} = \alpha_1, \quad (2)$$

$$\max\{ax-a\alpha-\gamma x+\gamma\alpha, ax+a\beta+\delta x+\delta\beta\}-ax\} = \beta_1 \quad (3)$$

It must be noticed that always $a \neq 0$, as a result, it can be achieved from (1) that:

$$x = \frac{1}{a}$$

On the other hand, using (2), the following two cases can occur:

$$\min\{ax+a\beta-\gamma x-\gamma\beta, ax-a\alpha+\delta x-\delta\alpha\} = ax+a\beta-\gamma x-\gamma\beta$$

or

$$\min\{ax+a\beta-\gamma x-\gamma\beta, ax-a\alpha+\delta x-\delta\alpha\} = ax-a\alpha+\delta x-\delta\alpha$$

If $\min\{ax+a\beta-\gamma x-\gamma\beta, ax-a\alpha+\delta x-\delta\alpha\} = ax+a\beta-\gamma x-\gamma\beta$ occurs, by substituting the minimum value in (2), it can be obtained that:

$$-a\beta+\frac{\gamma}{a}+\gamma\beta = \alpha_1$$

Now, in order to satisfy $\alpha_1<1$, the following inequality must hold:

$$-a\beta+\frac{\gamma}{a}+\gamma\beta<1 \quad \Rightarrow \quad \left(\frac{\gamma-a}{a}\right) < -\beta(\gamma-a) \Rightarrow \beta < -\frac{1}{a}$$

Since $\alpha>0$, no β can be found, where $\tilde{a} \otimes \tilde{a}^{-1} = (1,\alpha_1,\beta_1)$ and $\alpha_1<1$. Hence, when $a \geq 0$ and $a-\gamma<0$ hold, $\min\{ax+a\beta-\gamma x-\gamma\beta, ax-a\alpha+\delta x-\delta\alpha\} = ax+a\beta-\gamma x-\gamma\beta$ is not inversible.

Now, if the case $\min\{ax+a\beta-\gamma x-\gamma\beta, ax-a\alpha+\delta x-\delta\alpha\} = ax-a\alpha+\delta x-\delta\alpha$ occurs, by substituting the minimum value in (2), it can be achieved that:

$$a\alpha-\frac{\delta}{a}+\delta\alpha = \alpha_1$$

However, for $\alpha_1<1$, the following inequality must hold:

$$a\alpha-\frac{\delta}{a}+\delta\alpha<1 \quad \Rightarrow \quad (a+\delta)\alpha < \frac{a+\delta}{a} \Rightarrow \alpha < \frac{1}{a}$$

Since $\alpha<x$, then $ax-a\alpha+\delta x-\delta\alpha$ cannot be determined as the minimum value. Thus, in this case, the fuzzy inverse matrix does not exist.

It can be seen that since there is no α that satisfy (2), the fuzzy inverse matrix does not exist, and there is no need to investigate (3).

b) $a<0$, $a+\delta \geq 0$: Based on multiplication (2), the condition $\tilde{a} \otimes \tilde{a}^{-1} = (1,\alpha_1,\beta_1)$ is considered as follows:

$$\tilde{a} \otimes \tilde{a}^{-1} = (ax, ax - min\{ax+a\beta-\gamma x-\gamma\beta, ax-a\alpha+\delta x-\delta\alpha\}, max\{ax-a\alpha-\gamma x+\gamma\alpha, ax+a\beta+\delta x+\delta\beta\}-ax\}) = (1,\alpha_1,\beta_1)$$

Consequently, we have:

$$ax - min\{ax+a\beta-\gamma x-\gamma\beta, ax-a\alpha+\delta x-\delta\alpha\} = \alpha_1, \quad (4)$$

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$$max\{ax - a\alpha - \gamma x + \gamma\alpha, ax + a\beta + \delta x + \delta\beta\} - ax\} = \beta_1. \quad (5)$$

On the other hand, using (4), the following two cases can occur:

$$min\{ax + a\beta - \gamma x - \gamma\beta, ax - a\alpha + \delta x - \delta\alpha\} = ax + a\beta - \gamma x - \gamma\beta$$

Or

$$min\{ax + a\beta - \gamma x - \gamma\beta, ax - a\alpha + \delta x - \delta\alpha\} = ax - a\alpha + \delta x - \delta\alpha$$

If the case $min\{ax + a\beta - \gamma x - \gamma\beta, ax - a\alpha + \delta x - \delta\alpha\} = ax + a\beta - \gamma x - \gamma\beta$ occurs, by substituting the minimum value in (4),the following can be obtained:

$$-a\beta + \gamma x + \gamma\beta = \alpha_1$$

However, in order to satisfy $\alpha_1 < 1$, there must be:

$$-a\beta + \frac{\gamma}{a} + \gamma\beta < 1 \quad \Longrightarrow \quad (\gamma - a)\beta < \frac{a-\gamma}{a} \Longrightarrow \beta < \frac{-1}{a}$$

Thus, in this case, the fuzzy inverse matrix does not exist, since if $\beta + x < 0$, then $min\{ax + a\beta - \gamma x - \gamma\beta, ax - a\alpha + \delta x - \delta\alpha\} = ax - a\alpha + \delta x - \delta\alpha$.

If the case $min\{ax + a\beta - \gamma x - \gamma\beta, ax - a\alpha + \delta x - \delta\alpha\} = ax - a\alpha + \delta x - \delta\alpha$ occurs, by substituting the minimum value in (4), there is:

$$a\alpha - \frac{\delta}{a} + \delta\alpha = \alpha_1$$

However, for $\alpha_1 < 1$, the following must hold:

$$a\alpha - \frac{\delta}{a} + \delta\alpha < 1 \Longrightarrow (a+\delta)\alpha < \frac{a+\delta}{a} \Longrightarrow \alpha < \frac{1}{a}$$

Since $0 < x - \alpha$, the fuzzy inverse matrix does not exist in this case.

It can be seen that since there is no α that satisfy (4), the fuzzy inverse matrix does not exist, and it is not required to investigate (5).

c)$(a, \gamma, \delta) \leq 0$: According to multiplication (2), the condition $\tilde{a} \otimes \tilde{a}^{-1} = (1, \alpha_1, \beta_1)$ can be considered as following

$$\begin{aligned}&\tilde{a} \otimes \tilde{a}^{-1} = (ax,\\ &ax - min\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\},\\ &max\{ax - a\alpha + \delta x - \delta\alpha, ax - a\alpha - \gamma x + \gamma\alpha\} - ax\}) = (1, \alpha_1, \beta_1),\end{aligned}$$

Therefore:

$$ax - min\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} = \alpha_1 \quad (6)$$

$$max\{ax - a\alpha + \delta x - \delta\alpha, ax - a\alpha - \gamma x + \gamma\alpha\} - ax\} = \beta_1 \quad (7)$$

On the other hand, using (6), the following two cases can occur:

$$min\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} = ax + a\beta - \gamma x - \gamma\beta$$

Or

$$min\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} = ax + a\beta + \delta x + \delta\beta$$

If the case $min\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} = ax + a\beta - \gamma x - \gamma\beta$ occurs, by substituting the minimum value in equation (6), it can be achieved that:

$$-a\beta + \frac{\gamma}{a} + \gamma\beta = \alpha_1$$

In order to have $\alpha_1 < 1$, the following must hold:

$$-a\beta + \frac{\gamma}{a} + \gamma\beta < 1 \quad \Longrightarrow \quad (\gamma - a)\beta < \frac{a-\gamma}{a} \quad \Longrightarrow \quad \beta < \frac{-1}{a}$$

In this case, the fuzzy solution does not exist, since if $\beta + x < 0$, then $min\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} = ax + a\beta + \delta x + \delta\beta$.

If the case $min\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} = ax + a\beta + \delta x + \delta\beta$ occurs, by substituting the minimum value in (6), one can achieve:

$$-a\beta - \delta x - \delta\beta = \alpha_1$$

Since $\alpha_1 < 1$, then:

$$-a\beta - \frac{\delta}{a} - \delta\beta < 1 \Longrightarrow (\gamma - a)\beta < \frac{a-\gamma}{a}$$

On the other hand, as:

$$max\{ax - a\alpha + \delta x - \delta\alpha, ax - a\alpha - \gamma x + \gamma\alpha\} = max\{(a+\delta)(x-\alpha), (x-\alpha)(a-\gamma)\}$$

and

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$$(a-\gamma) < (a+\delta) \xrightarrow{x-\alpha<0} (a-\gamma)(x-\alpha) > (a+\delta)(x-\alpha)$$

hence, in this case, there is always $max\{ax - a\alpha + \delta x - \delta\alpha, ax - a\alpha - \gamma x + \gamma\alpha\} = ax - a\alpha - \gamma x + \gamma\alpha$. By substituting the maximum value in (7), one can see that:

$$-a\alpha - \gamma x + \gamma\alpha = \beta_1$$

For $\beta_1 < 1$, the following must hold:

$$-a\alpha - \frac{\gamma}{a} + \gamma\alpha < 1 \Rightarrow \alpha(\gamma - a) < \frac{a+\gamma}{a} \Longrightarrow \alpha < \frac{a+\gamma}{a(\gamma - a)}$$

For $a + \gamma < 0$ satisfying the condition $0 < \alpha < \frac{a+\gamma}{a(\gamma-a)}$, the fuzzy inverse matrix exists.

d)$(a, \gamma, \delta) \geq 0$: The condition $\tilde{a} \otimes \tilde{a}^{-1} = (1, \alpha_1, \beta_1)$ is considered as follows using multiplication (2):

$$\begin{aligned}&\tilde{a} \otimes \tilde{a}^{-1} = (ax,\\ &ax - min\{ax - a\alpha - \gamma x + \gamma\alpha, ax - a\alpha + \delta x - \delta\alpha\},\\ &max\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} - ax\}) = (1, \alpha_1, \beta_1).\end{aligned}$$

Therefore:

$$ax - min\{ax - a\alpha - \gamma x + \gamma\alpha, ax - a\alpha + \delta x - \delta\alpha\} = \alpha_1 \quad (8)$$

$$max\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} - ax = \beta_1 \quad (9)$$

Moreover, the following two cases can occur in this situation:

$$min\{ax - a\alpha - \gamma x + \gamma\alpha, ax - a\alpha + \delta x - \delta\alpha\} = ax - a\alpha - \gamma x + \gamma\alpha$$

or

$$min\{ax - a\alpha - \gamma x + \gamma\alpha, ax - a\alpha + \delta x - \delta\alpha\} = ax - a\alpha + \delta x - \delta\alpha$$

If the case $min\{ax - a\alpha - \gamma x + \gamma\alpha, ax - a\alpha + \delta x - \delta\alpha\} = ax - a\alpha - \gamma x + \gamma\alpha$ occurs, by substituting the minimum value in (8), there is:

$$a\alpha + \gamma x - \gamma\alpha = \alpha_1$$

In order to have $\alpha_1 < 1$, the following inequality must hold:

$$a\alpha + \frac{\gamma}{a} - \gamma\alpha < 1 \Longrightarrow (a - \gamma)\alpha < \frac{a-\gamma}{a} \Longrightarrow \alpha < \frac{1}{a}$$

Therefore, for $0 < \alpha < \frac{1}{a}$, the fuzzy inverse matrix exists.

If the case $min\{ax - a\alpha - \gamma x + \gamma\alpha, ax - a\alpha + \delta x - \delta\alpha\} = ax - a\alpha + \delta x - \delta\alpha$ occurs, by substituting the minimum value in (8), there is:

$$a\alpha - \delta x + \delta\alpha = \alpha_1$$

To have $\alpha_1 < 1$, there mut be:

$$a\alpha - \frac{\delta}{a} + \delta\alpha < 1 \Longrightarrow (a + \delta)\alpha < \frac{a+\delta}{a} \Longrightarrow \alpha < \frac{1}{a}$$

In this case, the problem does not have a fuzzy solution.

On the other hand, the following two cases can occur for the maximum conditions:

$$max\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} = ax + a\beta - \gamma x - \gamma\beta$$

or

$$max\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} = ax + a\beta + \delta x + \delta\beta$$

However, since:

$$max\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} = max\{(a-\gamma)(x+\beta), (a+\delta)(x+\beta)\}$$

and

$$a - \gamma < a + \delta \xrightarrow{x+\beta>0} (a-\gamma)(x+\beta) < (a+\delta)(x+\beta)$$

as a result, we have:

$$max\{ax + a\beta - \gamma x - \gamma\beta, ax + a\beta + \delta x + \delta\beta\} = ax + a\beta + \delta x + \delta\beta$$

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By substituting the maximum value in (9), it is obtained that:

$$a\beta + \delta x + \delta\beta = \beta_1$$

Since $\beta_1 < 1$ holds, thus:

$$a\beta + \frac{\delta}{a} + \delta\beta < 1 \Rightarrow (a+\delta)\beta < \frac{a-\delta}{a} \Rightarrow \beta < \frac{a-\delta}{a(a+\delta)}$$

Therefore, only for $a > \delta$ satisfying the condition $0 < \beta < \frac{a-\delta}{a(a+\delta)}$, the fuzzy inverse matrix exists.

4. Proposed Method

In this section, using linear programming problem, a method is proposed to calculate the inverse of a LR fuzzy matrix that overcomes the aforementioned shortcomings.

Definition 4.1 A fuzzy zero number is defined as a number that its core is zero, and its spreads are non-negative real numbers, and is denoted by $\tilde{0} = (0, \alpha, \beta)$.

Definition 4.2 A fuzzy one number is defined as a number that its core is one, and its spreads are non-negative real numbers, and is denoted by $\tilde{1} = (1, \alpha, \beta)$.

Definition 4.3 A $n \times n$ fuzzy matrix is called fuzzy identity matrix, and is denoted by $\tilde{I}$, when its main diagonal is $\tilde{1}_{ii} = (1, \lambda_{ii}, \eta_{ii})$, and its non-diagonal elements are $\tilde{0}_{ij} = (0, \theta_{ij}, \vartheta_{ij})$, $0 \le i, j \le n$.

For calculating the fuzzy inverse matrix, by assuming $\tilde{a}_{\mathrm{ij}} = [(a_{ij}, \gamma_{ij}, \delta_{ij})]$ and $\tilde{x}_{ij} = [(x_{jk}, \alpha_{jk}, \beta_{jk})]$, the fuzzy system $\tilde{A} \otimes \tilde{X} = \tilde{I}$ is considered as follows:

$$\begin{bmatrix} (a_{11}, \gamma_{11}, \delta_{11}) & \cdots & (a_{1n}, \gamma_{1n}, \delta_{1n}) \\ \vdots & \ddots & \vdots \\ (a_{n1}, \gamma_{n1}, \delta_{n1}) & \cdots & (a_{nn}, \gamma_{nn}, \delta_{nn}) \end{bmatrix} \begin{bmatrix} (x_{11}, \alpha_{11}, \beta_{11}) & \cdots & (x_{1n}, \alpha_{1n}, \beta_{1n}) \\ \vdots & \ddots & \vdots \\ (x_{n1}, \alpha_{n1}, \beta_{n1}) & \cdots & (x_{nn}\alpha_{nn}, \beta_{nn}) \end{bmatrix} = \begin{bmatrix} (1, \lambda_{11}, \eta_{11}) & \cdots & (0, \theta_{1n}, \vartheta_{1n}) \\ \vdots & \ddots & \vdots \\ (0, \theta_{n1}, \vartheta_{n1}) & \cdots & (1, \lambda_{nn}, \eta_{nn}) \end{bmatrix} \tag{4}$$

Using matrix multiplication definition leads to:

$$\begin{bmatrix} (a_{11}, \gamma_{11}, \delta_{11})(x_{11}, \alpha_{11}, \beta_{11}) + \cdots + (a_{1n}, \gamma_{1n}, \delta_{1n})(x_{n1}, \alpha_{n1}, \beta_{n1}) & \cdots & (a_{11}, \gamma_{11}, \delta_{11})(x_{1n}, \alpha_{1n}, \beta_{1n}) + \cdots + (a_{1n}, \gamma_{1n}, \delta_{1n})(x_{nn}\alpha_{nn}, \beta_{nn}) \\ \vdots & \ddots & \vdots \\ (a_{n1}, \gamma_{n1}, \delta_{n1})(x_{11}, \alpha_{11}, \beta_{11}) + \cdots + (a_{nn}, \gamma_{nn}, \delta_{nn})(x_{n1}, \alpha_{n1}, \beta_{n1}) & \cdots & (a_{n1}, \gamma_{n1}, \delta_{n1})(x_{1n}, \alpha_{1n}, \beta_{1n}) + \cdots + (a_{nn}, \gamma_{nn}, \delta_{nn})(x_{nn}\alpha_{nn}, \beta_{nn}) \end{bmatrix} = \begin{bmatrix} (1, \lambda_{11}, \eta_{11}) & \cdots & (0, \theta_{1n}, \vartheta_{1n}) \\ \vdots & \ddots & \vdots \\ (0, \theta_{n1}, \vartheta_{n1}) & \cdots & (1, \lambda_{nn}, \eta_{nn}) \end{bmatrix} \tag{5}$$

Different cases can happen according to the sign of $\tilde{a}_{\mathrm{ij}}$:

1. If for each i and j, $(a_{ij}, \gamma_{ij}, \delta_{ij}) \ge 0$:

Using the summation (1) and multiplication (2) definitions, and assuming:

$$z_{\tilde{a}_{ij\otimes\tilde{x}_{ij}}} = -min\{a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}, a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}\}$$

$$z'\tilde{a}_{ij\otimes\tilde{x}_{ij}} = \max\{a_{ij}x_{jk} + a_{ij}\beta_{jk} - \gamma_{ij}x_{jk} - \gamma_{ij}\beta_{jk}, a_{ij}x_{jk} + a_{ij}\beta_{jk} + \delta_{ij}x_{jk} + \delta_{ij}\beta_{jk}\}$$

and by balancing both sides of (5), it is suggested that the following linear programming problem to be solved in order to determine the matrix inverse:

$$min \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} z_{\tilde{a}_{ik\otimes\tilde{x}_{kj}}} + z'_{\tilde{a}_{ik\otimes\tilde{x}_{kj}}} \tag{6}$$

$$s.t.$$

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$$\sum_{k=1}^{n} a_{ik}\ x_{kj} = 1 \qquad i = j$$
$$\sum_{k=1}^{n} a_{ik}\ x_{kj} = 0 \qquad i \neq j$$
$$\sum_{k=1}^{n} a_{ik}\ x_{kj} + \sum_{k=1}^{n} z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} = \lambda_{ij}$$
$$\sum_{k=1}^{n} z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} - \sum_{k=1}^{n} a_{ik}\ x_{kj} = \eta_{ij}$$
$$z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq -a_{ij}x_{jk} + a_{ij}\alpha_{jk} + \gamma_{ij}x_{jk} - \gamma_{ij}\alpha_{jk}$$
$$z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq -a_{ij}x_{jk} + a_{ij}\alpha_{jk} - \delta_{ij}x_{jk} + \delta_{ij}\alpha_{jk}$$
$$z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq a_{ij}x_{jk} + a_{ij}\beta_{jk} - \gamma_{ij}x_{jk} - \gamma_{ij}\beta_{jk}$$
$$z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq a_{ij}x_{jk} + a_{ij}\beta_{jk} + \delta_{ij}x_{jk} + \delta_{ij}\beta_{jk}$$
$$\alpha_{jk}, \beta_{jk}, \lambda_{ij}, \eta_{ij} \geq 0$$

2. If for each i and j, $(a_{ij}, \gamma_{ij}, \delta_{ij}) \leq 0$:
Assuming that

$$z_{\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = -\min\{a_{ij}x_{jk} + a_{ij}\beta_{jk} - \gamma_{ij}x_{jk} - \gamma_{ij}\beta_{jk}, a_{ij}x_{jk} + a_{ij}\beta_{jk} + \delta_{ij}x_{jk} + \delta_{ij}\beta_{jk}\}$$

$$z'_{\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = \max\{a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}, a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}\}$$

the following linear programming problem must be solved to achieve the matrix inverse:

$$min \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} + z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}}$$
$$s.t.$$
$$\sum_{k=1}^{n} a_{ik}\ x_{kj} = 1 \qquad i = j$$
$$\sum_{k=1}^{n} a_{ik}\ x_{kj} = 0 \qquad i \neq j$$
$$\sum_{k=1}^{n} a_{ik}\ x_{kj} + \sum_{k=1}^{n} z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} = \lambda_{ij} \tag{7}$$
$$\sum_{k=1}^{n} z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} - \sum_{k=1}^{n} a_{ik}\ x_{kj} = \eta_{ij}$$
$$z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq -a_{ij}x_{jk} - a_{ij}\beta_{jk} + \gamma_{ij}x_{jk} + \gamma_{ij}\beta_{jk}$$
$$z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq -a_{ij}x_{jk} - a_{ij}\beta_{jk} - \delta_{ij}x_{jk} - \delta_{ij}\beta$$
$$z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}$$
$$z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}$$
$$\alpha_{jk}, \beta_{jk}, \lambda_{ij}, \eta_{ij} \geq 0$$

3. If for each i and j, $a_{ij} < 0,\ a_{ij} + \delta_{ij} \geq 0$, then:

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$$z_{\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = -\min\{a_{ij}x_{jk} + a_{ij}\beta_{jk} - \gamma_{ij}x_{jk} - \gamma_{ij}\beta_{jk}, a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}\}$$
$$z'_{\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = \max\{a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}, a_{ij}x_{jk} + a_{ij}\beta_{jk} + \delta_{ij}x_{jk} + \delta_{ij}\beta_{jk}\}$$

In order to obtain the matrix inverse, the following linear programming problem must be solved:

$$\begin{gathered}
min\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} + z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \\
s.t. \\
\sum_{k=1}^{n} a_{ik}\ x_{kj} = 1 \qquad i = j \\
\sum_{k=1}^{n} a_{ik}\ x_{kj} = 0 \qquad i \neq j \\
\sum_{k=1}^{n} a_{ik}\ x_{kj} + \sum_{k=1}^{n} z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} = \lambda_{ij} \\
\sum_{k=1}^{n} z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} - \sum_{k=1}^{n} a_{ik}\ x_{kj} = \eta_{ij} \\
z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq -a_{ij}x_{jk} - a_{ij}\beta_{jk} + \gamma_{ij}x_{jk} + \gamma_{ij}\beta_{jk} \\
z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq -a_{ij}x_{jk} + a_{ij}\alpha_{jk} - \delta_{ij}x_{jk} + \delta_{ij}\alpha_{jk} \\
z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk} \\
z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq a_{ij}x_{jk} + a_{ij}\beta_{jk} + \delta_{ij}x_{jk} + \delta_{ij}\beta_{jk} \\
\alpha_{jk}, \beta_{jk}, \lambda_{ij}, \eta_{ij} \geq 0
\end{gathered} \tag{8}$$

4. If for each i and j,$a_{ij} \geq 0$, $a_{ij} - \gamma_{ij} < 0$:
Considering

$$z_{\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = -\min\{a_{ij}x_{jk} + a_{ij}\beta_{jk} - \gamma_{ij}x_{jk} - \gamma_{ij}\beta_{jk}, a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}\}$$
$$z'_{\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = \max\{a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}, a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}\}$$

the following linear programming problem must be solved to find the inverse matrix:

$$\begin{gathered}
min\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} + z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \\
s.t. \\
\sum_{k=1}^{n} a_{ik}\ x_{kj} = 1 \qquad i = j \\
\sum_{k=1}^{n} a_{ik}\ x_{kj} = 0 \qquad i \neq j \\
\sum_{k=1}^{n} a_{ik}\ x_{kj} + \sum_{k=1}^{n} z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} = \lambda_{ij} \\
\sum_{k=1}^{n} z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} - \sum_{k=1}^{n} a_{ik}\ x_{kj} = \eta_{ij} \\
z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq -a_{ij}x_{jk} - a_{ij}\beta_{jk} + \gamma_{ij}x_{jk} + \gamma_{ij}\beta_{jk}
\end{gathered} \tag{9}$$

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$$z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq -a_{ij}x_{jk} + a_{ij}\alpha_{jk} - \delta_{ij}x_{jk} + \delta_{ij}\alpha_{jk}$$
$$z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}$$
$$z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} \geq a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}$$
$$\alpha_{jk}, \beta_{jk}, \lambda_{ij}, \eta_{ij} \geq 0$$

5.If for some i and j, $(a_{ij}, \gamma_{ij}, \delta_{ij}) \geq 0$, and for some other i and j,$(a_{ij}, \gamma_{ij}, \delta_{ij}) \leq 0$, if for some i and j, $a_{ij} < 0, a_{ij} + \delta_{ij} \geq 0$ and for some other i and j, $a_{ij} \geq 0, a_{ij} - \gamma_{ij} < 0$, assuming that:

$$z_{1\,\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = -min\{a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}, a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}\}$$
$$z_{2\,\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = -\min\{a_{ij}x_{jk} + a_{ij}\beta_{jk} - \gamma_{ij}x_{jk} - \gamma_{ij}\beta_{jk}, a_{ij}x_{jk} + a_{ij}\beta_{jk} + \delta_{ij}x_{jk} + \delta_{ij}\beta_{jk}\}$$

$$z_{3\,\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = -\min\{a_{ij}x_{jk} + a_{ij}\beta_{jk} - \gamma_{ij}x_{jk} - \gamma_{ij}\beta_{jk}, a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}\}$$
$$z_{4\,\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = -\min\{a_{ij}x_{jk} + a_{ij}\beta_{jk} - \gamma_{ij}x_{jk} - \gamma_{ij}\beta_{jk}, a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}\}$$

$$z'_{1\,\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = \max\{a_{ij}x_{jk} + a_{ij}\beta_{jk} - \gamma_{ij}x_{jk} - \gamma_{ij}\beta_{jk}, a_{ij}x_{jk} + a_{ij}\beta_{jk} + \delta_{ij}x_{jk} + \delta_{ij}\beta_{jk}\}$$
$$z'_{2\,\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = \max\{a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}, a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}\}$$

$$z'_{3\,\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = \max\{a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}, a_{ij}x_{jk} + a_{ij}\beta_{jk} + \delta_{ij}x_{jk} + \delta_{ij}\beta_{jk}\}$$
$$z'_{4\,\tilde{a}_{ij}\otimes\tilde{x}_{ij}} = \max\{a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk}, a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk}\}$$

the following linear programming problem is just required to be solved in order to find the matrix inverse:

$$min \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} z_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}} + z'_{\tilde{a}_{ik}\otimes\tilde{x}_{kj}}$$
$$s.t.$$
$$\sum_{k=1}^{n} a_{ik}\ x_{kj} = 1 \qquad i = j$$
$$\sum_{k=1}^{n} a_{ik}\ x_{kj} = 0 \qquad i \neq j$$
$$\sum_{k=1}^{n} a_{ik}\ x_{kj} + \sum_{\tilde{a}_{ik}\geq 0} z_{1\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} + \sum_{\tilde{a}_{ik}\leq 0} z_{2\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} + \sum_{a_{ik}<0, a_{ik}+\delta_{ik}\geq 0} z_{3\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} + \sum_{a_{ik}\geq 0, a_{ik}+\delta_{ik}<0} z_{3\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} = \lambda_{ij}$$

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$$\sum_{\tilde{a}_{ik}\geq 0} z'_{1\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} + \sum_{\tilde{a}_{ik}\leq 0} z'_{2\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} + \sum_{a_{ik}<0, a_{ik}+\delta_{ik}\geq 0} z'_{3\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} + \sum_{a_{ik}\geq 0, a_{ik}+\delta_{ik}<0} z'_{4\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} - \sum_{k=1}^{n} a_{ik}\ x_{kj} = \eta_{ij}$$

$$\begin{aligned}
z_{1\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq -a_{ij}x_{jk} + a_{ij}\alpha_{jk} + \gamma_{ij}x_{jk} - \gamma_{ij}\alpha_{jk} \\
z_{1\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq -a_{ij}x_{jk} + a_{ij}\alpha_{jk} - \delta_{ij}x_{jk} + \delta_{ij}\alpha_{jk} \\
z_{2\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq -a_{ij}x_{jk} - a_{ij}\beta_{jk} + \gamma_{ij}x_{jk} + \gamma_{ij}\beta_{jk} \\
z_{2\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq -a_{ij}x_{jk} - a_{ij}\beta_{jk} - \delta_{ij}x_{jk} - \delta_{ij}\beta \\
z_{3\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq -a_{ij}x_{jk} - a_{ij}\beta_{jk} + \gamma_{ij}x_{jk} + \gamma_{ij}\beta_{jk} \\
z_{3\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq -a_{ij}x_{jk} + a_{ij}\alpha_{jk} - \delta_{ij}x_{jk} + \delta_{ij}\alpha_{jk} \\
z_{4\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq -a_{ij}x_{jk} - a_{ij}\beta_{jk} + \gamma_{ij}x_{jk} + \gamma_{ij}\beta_{jk} \\
z_{4\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq -a_{ij}x_{jk} + a_{ij}\alpha_{jk} - \delta_{ij}x_{jk} + \delta_{ij}\alpha_{jk} \\
z'_{1\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq a_{ij}x_{jk} + a_{ij}\beta_{jk} - \gamma_{ij}x_{jk} - \gamma_{ij}\beta_{jk} \\
z'_{1\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq a_{ij}x_{jk} + a_{ij}\beta_{jk} + \delta_{ij}x_{jk} + \delta_{ij}\beta_{jk} \\
z'_{2\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk} \\
z'_{2\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk} \\
z'_{3\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk} \\
z'_{3\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq a_{ij}x_{jk} + a_{ij}\beta_{jk} + \delta_{ij}x_{jk} + \delta_{ij}\beta_{jk} \\
z'_{4\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq a_{ij}x_{jk} - a_{ij}\alpha_{jk} + \delta_{ij}x_{jk} - \delta_{ij}\alpha_{jk} \\
z'_{4\,\tilde{a}_{ik}\otimes\tilde{x}_{kj}} &\geq a_{ij}x_{jk} - a_{ij}\alpha_{jk} - \gamma_{ij}x_{jk} + \gamma_{ij}\alpha_{jk} \\
&\alpha_{jk}, \beta_{jk}, \lambda_{ij}, \eta_{ij} \geq 0
\end{aligned} \tag{10}$$

Theorem 4.1 Let $\tilde{A} = [(a_{ij}, \gamma_{ij}, \delta_{ij})]$, and $\tilde{X} = [(x_{ij}, \alpha_{ij}, \beta_{ij})]$. The fuzzy matrix $\tilde{A}$ is invertible, and its inverse is as $\tilde{A}^{-1} = \tilde{X}$, when for $A = [a_{ij}]$, we have $|A| \neq 0$, and the linear programming problem$\tilde{A} \otimes \tilde{X} = \tilde{I}$ has a feasible solution.

Proof. If $|A| \neq 0$, then A is invertible, and when the linear programming problem $\tilde{A} \otimes \tilde{X} = \tilde{I}$ has a feasible solution, then the positive α_{ij} and β_{ij} are obtained by solving the linear programming problem.

5. Numerical Examples

In this section, two numerical examples are presented through which the performance and effectiveness of the proposed method are verified.

Example 5.1 To calculate the inverse of the fuzzy number (5,10,20), the following equation is considered:

$$(5,10,20) \otimes (x_1, \alpha_1, \beta_1) = (1, \alpha, \beta).$$

By solving the following linear programming problem:

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$$
\begin{aligned}
&Min\, W = z_1 + z_2\\
&s.t.\\
&5x_1 = 1,\\
&5x_1 + z_1 = \alpha,\\
&z_2 - 5x_1 = \beta,\\
&z_1 \geq 5x_1 - 5\beta_1,\\
&z_1 \geq -25x_1 + 25\alpha_1,\\
&z_2 \geq -5x_1 + 5\alpha_1,\\
&z_2 \geq 25x_1 + 25\beta_1,\\
&\alpha, \beta, \alpha_1, \beta_1 \geq 0
\end{aligned}
$$

it can be achieved that:

$$x_1 = 0.2, \qquad \alpha_1 = 0, \qquad \beta_1 = 0, \qquad \alpha = 2, \qquad \beta = 4$$

That is:

$$\tilde{a}^{-1} = (0.2,0,0)$$

and

$$\tilde{a} \otimes \tilde{a}^{-1} = (5,10,20) \otimes (0.2,0,0) = (1,2,4).$$

The above example shows that the proposed method doesn't have the shortcomings of the Kaur and Kumar method.

Example 5.2 To determine the inverse of the matrix $\tilde{A} = \begin{bmatrix} (5,1,1) & (6,2,2) \\ (4,2,2) & (7,1,1) \end{bmatrix}$, the following system is formulated:

$$\begin{bmatrix} (5,1,1) & (6,2,2) \\ (4,2,2) & (7,1,1) \end{bmatrix} \begin{bmatrix} (x_{11}, \alpha_{11}, \beta_{11}) & (x_{12}, \alpha_{12}, \beta_{12}) \\ (x_{21}, \alpha_{21}, \beta_{21}) & (x_{22}, \alpha_{22}, \beta_{22}) \end{bmatrix} = \begin{bmatrix} (1, \alpha_1, \beta_1) & (0, \gamma_1, \delta_1) \\ (0, \gamma_2, \delta_2) & (1, \alpha_2, \beta_2) \end{bmatrix}$$

Since $\forall\, i, j \;\; a_{ij} > 0$, using (6), we have:

$$x_{11} = 0.636,\ x_{12} = -0.545, x_{21} = -0.364, x_{22} = 0.455,$$
$$\alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = \beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0,$$
$$\alpha_{11} = \alpha_{12} = \alpha_{21} = \alpha_{22} = \beta_{11} = \beta_{12} = \beta_{21} = \beta_{22} = 0,$$

Therefore:

$$\tilde{A} \otimes \tilde{A}^{-1} = \begin{bmatrix} (5,1,1) & (6,2,2) \\ (4,2,2) & (7,1,1) \end{bmatrix} \begin{bmatrix} (0.636,0,0) & (-0.545,0,0) \\ (-0..364,0,0) & (0.455,0,0) \end{bmatrix} = \begin{bmatrix} (1,1.364,1.364) & (0,1.455,1.455) \\ (0,1.636,1.636) & (1,1.545,1.545) \end{bmatrix}$$

Namely, $\tilde{A} \otimes \tilde{A}^{-1} = \tilde{I}$.

6. Conclusion

In this paper, first, the Kaur and Kumar method in the field of fuzzy inverse matrix is shortly reviewed, and its disadvantages are studied. Then, a new method is proposed for calculating the fuzzy inverse matrix using the linear programming problem.

One of the advantages of the proposed method is that the inverse of a fuzzy matrix can be obtained by solving a minimization problem. The other advantage is that the fuzzy inverse matrix, if it exists, is also fuzzy, and for each fuzzy matrix $\tilde{A}$, it is always $\tilde{A} \otimes \tilde{A}^{-1} = \tilde{I}$.

6.1. Discussion of Experimental Results

In this paper, fuzzy inverse matrix calculation is discussed. There are two important principles in the definition of the fuzzy inverse matrix;

1) The multiplication of a fuzzy matrix by its inverse matrix should always be a fuzzy identical matrix,
and
2) The inverse of a fuzzy matrix should always be a fuzzy matrix as well.

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In this paper, it is shown that these two basic conditions are not satisfied in the matrix inversion methods proposed so far. Therefore, in the current study, an effective method is presented for fuzzy inverse matrix calculation considering these two conditions.

One of the limitations of the proposed method is that the suggested fuzzy zero and one numbers are defined in such a way so that they include a large set of fuzzy numbers. In the future studies, we will try to modify these definitions so that fuzzy zero and one spreads be more limited, and thus, we can have an identical matrix with less ambiguity.

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