Pakistan Journal of Statistics and Operation Research

Three Parameters Quasi Gamma Distribution and with Properties and Applications

Qaisar Rashid[1*], Dr. Hafiz Muhammad Yaseen[2] Muhammad Uzair[3] and Muhammad Tariq Jamshaid[4]

* Corresponding Author

[1] Pakistan Bureau of Statistics, Sargodha, Pakistan. rajaqaisar89@gmail.com
[2]Department of Statistics, GC University Faisalabad, Pakistan. hafizmy_313@yahoo.com
[3]Department of Statistics, NUML, Islamabad, Pakistan. muzair553@gmail.com
[4]Primary & Secondary Healthcare Department Lahore, Pakistan. jamshaidsamtia77@gmail.com

Abstract

This paper introduced a new life time data analysis distribution name three parameters quasi gamma distribution discussed about its some properties including moment generating function, rth moment about origin and mean, mean deviations, reliability measurements, Bonferroni and Lorenz curve, Order statistics, Renyi entropy, also discussed about maximum likelihood method and real-life data applications.

Key words: Quasi Gamma, Moments, Reliability measures, Order statistics

Mathematical Subject Classification: 60E05, 62E15

1. Introduction

The gamma distribution playing important role in life time data applications, It used in field of sciences like engineering, meteorology, demography, business etc. The gamma function, a generalization of the factorial function to non-integral values, Milton and Stegun (1972) discussed Euler gamma function which in the 18th century was introduced by Swiss mathematician Leonhard Euler . For values of $y > 0$, the it is known as gamma function is defined using an integral formula as $\Gamma n = \int_0^\infty y^{n-1} e^{-y}\, dy$ if at least α events occurred in a fix time then it follow gamma distribution its pdf and cdf defined as following:

$$f_G(y;\ \alpha,\theta) = \frac{y^{\alpha-1} e^{-\theta y}}{\Gamma(\alpha)} \qquad \alpha > 0;\ \theta > 0; y > 0 \tag{1.1}$$

$$F_G(y;\ \alpha,\theta) = \frac{\gamma(\alpha,y)}{\Gamma(\alpha)} \qquad \alpha > 0;\ \theta > 0; y > 0 \tag{1.2}$$

$\gamma(\alpha, y)$ *is incomplete gamma function*

If $\alpha \le 1$ (1.1) is a J-shaped function, if $\alpha > 1$ it is uni-model maximum at $y = \alpha - 1$; for $\alpha = 1$ is an exponential distribution. Gupta and Kundu (2006) studied the closeness between the gamma distribution and the generalized exponential distribution and observed that if the shape parameter of the gamma distribution is not very high then a gamma distribution approximately very well by a generalized exponential distribution. This closeness between the two distributions studied by Kolmogorov discrepancy measure and by Kullback-Leibler discrepancy measure, it is observed that for shape parameter close to one the two distributions are almost indistinguishable. Shanker et al. (2018) introduced a new quasi exponential distribution and discussed its properties, maximum likelihood method and applications. The probability density function pdf and cumulative density function cdf of quasi exponential distribution defined below;

$$f_{QE}(y;\ \theta) = \frac{2\theta^{\frac{1}{2}} e^{-\theta y^2}}{\Gamma(\frac{1}{2})} \qquad \theta > 0;\ y > 0 \tag{1.3}$$

$$F_{QE}(y;\ \theta) = 1 - \frac{\Gamma(\frac{1}{2}, \theta y^2)}{\Gamma(\frac{1}{2})} \qquad \theta > 0; y > 0 \tag{1.4}$$

Survival function of Quasi exponential defined as

$$S_{QE}(y;\ \theta) = P(Y > y) = \frac{2\theta^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} \int_y^{\infty} e^{-\theta t^2} dt = \frac{\Gamma(\frac{1}{2}, \theta y^2)}{\Gamma(\frac{1}{2})} \qquad \theta > 0; y > 0 \tag{1.5}$$

The quasi exponential is better fitted than exponential distribution and least standard error, showed using biomedical data application. There are no of distribution have been introduced for life time data as a Quasi type distributions some of them are Quasi Lindley distribution by Shanker and Mishra (2013), New Quasi Lindley distribution by Shanker and Amanuel (2013 b), Quasi Sujatha distribution with properties and application by Shaker (2016), Quasi Shanker distribution introduced by Shanker & Shukla, (2017), Quasi Ardhana distribution with properties and application disscsed by Shanker & Shukla, (2018), Quasi Lindley Pareto distribution by Asad et al. (2018), Quasi Amrendra introduced by Rashid et al. (2019). A two parameters quasi gamma distribution (QGD) as a weighted version of quasi exponential distribution (QED) with weight function $y^{2\alpha-1}$; $\alpha > 0$ introduced by Shanker (2018). It reliability function defined as;

$$S_{QG}(y;\ \theta, \alpha) = P(Y > y) = \frac{2\theta^{\alpha}}{\Gamma\alpha} \int_y^{\infty} t^{2\alpha-1} e^{-\theta t^2} dt \qquad \text{put } \theta t^2 = u,\ t = \left(\frac{u}{\theta}\right)^{\frac{1}{2}}\ dt = \frac{du}{(\theta u)^{\frac{1}{2}}}$$

$$= \frac{1}{\Gamma\alpha} \int_{\theta y^2}^{\infty} u^{\alpha-1} e^{-u} du = 1 - \frac{\boldsymbol{\Gamma(\alpha, \theta y^2)}}{\boldsymbol{\Gamma\alpha}} \tag{1.6}$$

The pdf of Quasi gamma distribution defined as;

$$f_{QG}(y;\ \theta, \alpha) = \frac{2\theta^{\alpha} y^{2\alpha-1} e^{-\theta y^2}}{\Gamma\alpha} \qquad \alpha > 0;\ \theta > 0; y > 0 \tag{1.7}$$

In chapter 17 the detailed discussions are available on gamma distributions with their properties and applications done by different researchers that described by Johnson et al (1994). Another generalization of gamma distribution and discussed its application to drought data by Nadarajah and Gupta (2007). The exponentiated generalized gamma distribution proposed by Cordeiro et al (2013) and discussed its application to lifetime data. The Weibull distribution is the power transformation of exponential distribution developed by Weibull (1951) and proved to be better lifetime distribution than exponential distribution due to an additional parameter. The statistical modeling and analysis of lifetime data are crucial in almost all branches of physical, technical, engineering and biomedical sciences. For modeling lifetime data analyzing the one parameter exponential distribution, the two-parameter Weibull and gamma distributions are common in statistics literature. Due to theoretical or applied point of view it has been noted that these lifetime distributions are not always a suitable model either. In this respect an attempt has been made to determined three parameters lifetime distribution which strives well with exponential, quasi exponential, gamma and quasi gamma distributions.

In this paper a three parameters quasi gamma distribution (TPQGD) of which one parameter quasi exponential as well as quasi gamma distribution is a particular case has been proposed. Its statistical properties including shapes of pdf for varying values of parameters, moment generating function, rth moment about origin and mean, mean deviations, reliability measurements, Bonferroni and Lorenz curve, Order statistics, Renyi entropy, also discussed about maximum likelihood method Finally, applications related to a real lifetime data from physical sciences, engineering and biomedical has been presented to test its goodness of fit over one parameter exponential and quasi exponential distributions and two-parameter gamma and quasi gamma distributions, the new three parameters Quasi Gamma distribution have discussed in following.

2. New Three Parameters Quasi Gamma Distribution (TPQGD)

The quasi gamma distribution (TPQGD) probability density function (pdf) and cumulative density function (cdf) as a weighted version of quasi exponential distribution (QED) with weight function $y^{2k\alpha-1}$; $\alpha > 0, k > 0$; scale parameter θ and shape parameters α and k is defined following as;

$$f_{TPQGD}(y;\ \alpha, \theta, k) = \frac{2k\theta^{\alpha} y^{2k\alpha-1} e^{-\theta y^{2k}}}{\Gamma\alpha} \qquad k > 0;\ \alpha > 0;\ \theta > 0; y > 0 \tag{2.1}$$

By definition of survival function

$S_{TPQGD}(y;\ \alpha,\theta,k) = P(Y > y) = \frac{2k\theta^{\alpha}}{\Gamma\alpha}\int_y^{\infty} t^{2k\alpha-1}e^{-\theta t^{2k}}dt$

put $\theta t^{2k} = u,\ t = \left(\frac{u}{\theta}\right)^{\frac{1}{2k}}\ ;\ dt = \frac{du\left(\frac{u}{\theta}\right)^{\frac{1}{2}}}{2uk}$ or make transformation $y = x^{\frac{1}{k}}$ in quasi gamma distribution in eq. (1.7) we determined pdf of three parameters quasi gamma distribution (TPQGD).

$S_{TPQGD}(y;\ \alpha,\theta,k) = 1 - \frac{\boldsymbol{\Gamma(\alpha,\theta y^{2k})}}{\boldsymbol{\Gamma\alpha}} \qquad k > 0;\ \alpha > 0;\ \theta > 0; y > 0$

$$F_{TPQGD}(y;\ \alpha,\theta,k) = 1 - S_{TPQGD}(y;\ \alpha,\theta,k) \tag{2.2}$$

$$F_{TPQGD}(y;\ \alpha,\theta,k) = 1 - \frac{\boldsymbol{\Gamma(\alpha,\theta y^{2k})}}{\boldsymbol{\Gamma\alpha}} \qquad k > 0;\ \alpha > 0;\ \theta > 0; y > 0 \tag{2.3}$$

$\Gamma(\alpha,\theta y^{2k})$ *is upper gamma incomplete fucntion.*

If put in eq. (2.1); $k = \frac{1}{2}$ it reduced to gamma; for $k = 1$ in Quasi gamma distribution; reduced to exponential distribution and if $k = 1, \alpha = \frac{1}{2}$ it reduced to Quasi exponential distribution, for $\alpha = 1, k = \frac{1}{2}$ it reduced to exponential distribution, $k = \frac{1}{2}, \alpha = 2$ it reduced to Rayleigh distribution.

3. Properties of TPQGD

Moment generating function: The moment generating function of TPQGD defined following as;

$$M_y(t) = \sum_{m=0}^{\infty} \frac{\Gamma(\frac{m+1}{\alpha})}{\theta^{\frac{m}{\alpha}}\Gamma(\frac{1}{\alpha})}\left(\frac{t}{\theta}\right)^m \tag{3.1}$$

Rth moments: The rth moment function of TPQGD defined following such that;

$$\mu_r{}' = \frac{\boldsymbol{\Gamma\frac{(2k\alpha+r)}{2k}}}{\boldsymbol{\theta^{\frac{r}{2k}}\Gamma\alpha}} \qquad r = 1,2,3,\ldots \tag{3.2}$$

From (1.10) 1st four moment about origin are:

$\mu_1{}' = \frac{\Gamma\left(\frac{2k\alpha+1}{2k}\right)}{\theta^{\frac{1}{2k}}\Gamma\alpha};\ \mu_2{}' = \frac{\Gamma\left(\frac{k\alpha+1}{k}\right)}{\theta^{\frac{1}{k}}\Gamma\alpha};\ \mu_3{}' = \frac{\Gamma\left(\frac{2k\alpha+3}{2k}\right)}{\theta^{\frac{3}{2k}}\Gamma\alpha};\ \mu_4{}' = \frac{\Gamma\left(\frac{k\alpha+2}{k}\right)}{\theta^{\frac{4}{2k}}\Gamma\alpha}$

By using above moments about origin; the moments about mean are obtain as:

$$\mu_2 = \mu_2{}' - (\mu_1{}')^2 \tag{3.3}$$

$$\mu_3 = \mu_3{}' + 3\mu_2{}'\mu_1{}' - 2(\mu_1{}')^3 \tag{3.4}$$

$$\mu_4 = \mu_4{}' - 4\mu_1{}'\mu_3{}' + 6(\mu_1{}')^2\mu_2{}' - 3(\mu_1{}')^4 \tag{3.5}$$

Using moments about origin in (3.3), (3.4) and (3.5) obtain moments about mean following;

$$\mu_2 = \frac{\theta^{\frac{1}{4k}}\Gamma\alpha\Gamma\left(\frac{k\alpha+1}{k}\right) - \left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^2}{\left\{\theta^{\frac{1}{2k}}\Gamma\alpha\right\}^2} \tag{3.6}$$

$$\mu_3 = \frac{(\Gamma\alpha)^2\Gamma\left(\frac{2k\alpha+3}{2k}\right) + 3\Gamma\alpha\Gamma\left(\frac{k\alpha+1}{k}\right)\Gamma\left(\frac{2k\alpha+1}{2k}\right) - 2\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^3}{\theta^{\frac{3}{2k}}(\Gamma\alpha)^3} \tag{3.7}$$

$$\mu_4 = \frac{(\Gamma\alpha)^3\Gamma\left(\frac{k\alpha+2}{k}\right) - 4(\Gamma\alpha)^2\Gamma\left(\frac{2k\alpha+1}{2k}\right)\Gamma\left(\frac{2k\alpha+3}{2k}\right) + 6\Gamma\alpha\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^2\Gamma\left(\frac{k\alpha+1}{k}\right) - 3\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^4}{\theta^{\frac{4}{2k}}(\Gamma\alpha)^4} \tag{3.8}$$

The mean, variance, coefficient of variation, coefficient of skewness, coefficient of kurtosis, index of dispersion, mean deviation about mean and median of TPQGD are obtained as:

Mean: $\frac{\Gamma\left(\frac{2k\alpha+1}{2k}\right)}{\theta^{\frac{1}{2k}}\Gamma\alpha}$; **Variance:** $\frac{\theta^{\frac{1}{4k}}\Gamma\alpha\Gamma\left(\frac{k\alpha+1}{k}\right)-\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^2}{\left\{\theta^{\frac{1}{2k}}\Gamma\alpha\right\}^2}$; `

Coefficient of Variation: $C.V=\frac{\sigma}{\mu_1{}'}\times 100=\frac{\sqrt{\left[\theta^{\frac{1}{4k}}\Gamma\alpha\Gamma\left(\frac{k\alpha+1}{k}\right)-\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^2\right]}}{\Gamma\left(\frac{2k\alpha+1}{2k}\right)}\times 100$

Index of Dispersion: $\gamma=\frac{\sigma^2}{\mu_1{}'}=\frac{\left[\theta^{\frac{1}{4k}}\Gamma\alpha\Gamma\left(\frac{k\alpha+1}{k}\right)-\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^2\right]}{\theta^{\frac{1}{2k}}\Gamma\alpha.\Gamma\left(\frac{2k\alpha+1}{2k}\right)}$

Coefficient of Skewness: $\sqrt{\beta_1}=\frac{\mu_3}{\mu_2{}^{3/2}}$ (3.9)

By using 2nd and 3rd moments about mean in (3.9) have;

$$\sqrt{\beta_1}=\frac{\left[(\Gamma\alpha)^2\Gamma\left(\frac{2k\alpha+3}{2k}\right)+3\Gamma\alpha\Gamma\left(\frac{k\alpha+1}{k}\right)\Gamma\left(\frac{2k\alpha+1}{2k}\right)-2\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^3\right]}{\left[\theta^{\frac{1}{4k}}\Gamma\alpha\Gamma\left(\frac{k\alpha+1}{k}\right)-\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^2\right]^{3/2}} \quad (3.10)$$

Coefficient of Kurtosis:

$$\beta_2=\frac{\mu_4}{\mu_2{}^2} \quad (3.11)$$

By using 2nd and 4th moments about mean in (3.11);

$$\beta_2=\frac{\left[(\Gamma\alpha)^3\Gamma\left(\frac{k\alpha+2}{k}\right)-4(\Gamma\alpha)^2\Gamma\left(\frac{2k\alpha+1}{2k}\right)\Gamma\left(\frac{2k\alpha+3}{2k}\right)+6\Gamma\alpha\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^2\Gamma\left(\frac{k\alpha+1}{k}\right)-3\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^4\right]}{\left[\theta^{\frac{1}{4k}}\Gamma\alpha\Gamma\left(\frac{k\alpha+1}{k}\right)-\left\{\Gamma\left(\frac{2k\alpha+1}{2k}\right)\right\}^2\right]^2} \quad (3.12)$$

Mean Deviations:

The amount of variation in a population is generally measured to some extent by the totality of deviations usually either from the mean or the median. These are known as the mean deviation about the mean and the mean deviation about the median and are defined;

$$\varphi_1(y)=\int_0^\infty|Y-\mu|f_{TPQGD}(\mathrm{y})dy \quad (3.13)$$

$$\mu=E(y)$$

$$\varphi_2(y)=\int_0^\infty|y-M|f_{TPQGD}(\mathrm{y})dy \quad (3.14)$$

$M=Median(y)$ The measure of $\varphi_1(y)$ and $\varphi_2(y)$ can be calculated as:

$$\varphi_1(y)=\int_0^\mu(\mu-y)f_{TPQGD}(\mathrm{y})dy+\int_\mu^\infty(y-\mu)f_{TPQGD}(\mathrm{y})dy$$

$$=\mu F(\mu)-\int_0^\mu yf_{TPQGD}(\mathrm{y})dy-\mu[1-F(\mu)]+\int_\mu^\infty yf_{TPQGD}(\mathrm{y})dy$$

$$=2\mu F(\mu)-2\mu+2\int_\mu^\infty yf_{TPQGD}(\mathrm{y})dy$$

$$=2\mu F(\mu)+2\int_0^\mu yf_{TPQGD}(\mathrm{y})dy \quad (3.15)$$

$\varphi_2(y) = \int_0^{\infty}|y - M| f_{TPQGD}(\mathrm{y})dy$

$\varphi_2(y) = \int_0^{M}(M - y) f_{TPQGD}(\mathrm{y})dy + \int_M^{\infty}(y - M) f_{TPQGD}(\mathrm{y})dy$

$= MF(M) - \int_0^{M} y f_{TPQGD}(\mathrm{y})dy - M[1 - F(M)] + \int_M^{\infty} y f_{TPQGD}(\mathrm{y})dy$

$= 2MF(M) - \int_0^{M} y f_{TPQGD}(\mathrm{y})dy - M + \int_M^{0} y f_{TPQGD}(\mathrm{y})dy + \int_0^{\infty} y f_{TPQGD}(\mathrm{y})dy$

$= -\int_0^{M} y f_{TPQGD}(\mathrm{y})dy + \int_M^{0} y f_{TPQGD}(\mathrm{y})dy + \mu$

$$= \mu - 2\int_0^{M} y f_{TPQGD}(\mathrm{y})dy \tag{3.16}$$

So by using pdf (2.1) found that:

$$\int_0^{\mu} y f_{TPQGD}(\mathrm{y})dy = \frac{\gamma(\frac{2k\alpha+1}{2k}, \theta\mu^{2k})}{\theta^{\frac{1}{2k}}\Gamma\alpha} \tag{3.17}$$

$\gamma(\frac{2k\alpha+1}{2k}, \theta\mu^{2k})$ TPQGD lower incomplete gamma function

$$\int_0^{M} y f_{TPQGD}(\mathrm{y})dy = \frac{\gamma(\frac{2k\alpha+1}{2k}, \theta M^{2k})}{\theta^{\frac{1}{2k}}\Gamma\alpha} \tag{3.18}$$

$\gamma(\frac{2k\alpha+1}{2k}, \theta M^{2k})$ TPQGD lower incomplete gamma function

Put (3.17) into (3.15) and (3.18) into (3.16) get following mean deviation about mean and median respectively:

$$\varphi_1(y) = \frac{2[\theta^{\frac{1}{2k}}\{\mu\Gamma\alpha - \mu\Gamma(\alpha+1, \theta\mu^{2k})\} + \gamma(\frac{2k\alpha+1}{2k}, \theta\mu^{2k})}{\theta^{\frac{1}{2k}}\Gamma\alpha} \tag{3.19}$$

$$\varphi_2(y) = \frac{\mu\theta^{\frac{1}{2k}}\Gamma\alpha - 2\gamma(\frac{2k\alpha+1}{2k}, \theta M^{2k})}{\theta^{\frac{1}{2k}}\Gamma\alpha} \tag{3.20}$$

4. Reliability Measures

There are different reliability measures namely Survival Function, Hazard Rate Function, Cumulative Hazard Function and Reversed Cumulative Hazard Function.
Let Y be a continuous random variable with pdf $f_{TPQGD}(y;\ \alpha, \theta, k)$ and cdf $F_{TPQGD}(y;\ \alpha, \theta, k)$ of TPQGD. Then the Survival $S_{TPQGD}(y;\ \alpha, \theta, k)$, Hazard rate $h_{TPQGD}(y;\ \alpha, \theta, k)$, Cumulative hazard function $CH_{TPQGD}(y;\ \alpha, \theta, k)$ and $H_{TPQGD}(y;\ \alpha, \theta, k)$ Reversed cumulative hazard function given below:

Survival function:

Let Y be a continuous random variable with pdf $f_{TPQGD}(y;\ \alpha, \theta, k)$ (2.1) and cdf $F_{TPQGD}(y;\ \alpha, \theta, k)$ (2.3) of TPQGD the Survival function obtain as:

$$S_{TPQGD}(y;\ \alpha, \theta, k) = P(Y > y) = \frac{\boldsymbol{\Gamma(\alpha, \theta y^{2k})}}{\boldsymbol{\Gamma\alpha}} \tag{4.1}$$

Hazard function:

Let Y be a continuous random variable with pdf $f_{TPQGD}(y;\ \alpha, \theta, k)$ (2.1) and cdf $F_{TPQGD}(y;\ \alpha, \theta, k)$ (2.3) of TPQGD. The hazard rate function known as the failure rate function defined as;

$$h_{TPQGD}(\mathrm{y}) = \lim_{\Delta y \to 0} \frac{P(Y < y + \Delta y | Y > y)}{\Delta y} = \frac{f(y)}{1 - F(y)} \tag{4.2}$$

By using (2.1)and (2.3) in (4.2) find as:

$$h_{TPQGD}(y;\ \alpha,\theta,k) = \frac{2k\theta^{\alpha}y^{2k\alpha-1}e^{-\theta y^{2k}}}{\Gamma(\boldsymbol{\alpha},\boldsymbol{\theta y^{2k}})} \quad (4.3)$$

Here note that:

$$h_{TPQGD}(0;\alpha,\theta) = f_{TPQGD}(0;\alpha,\theta)$$

Cumulative hazard function:

Let Y be a continuous random variable with pdf $f_{TPQGD}(y;\ \alpha,\theta,k)$ and c.d.f. $F_{TPQGD}(y;\ \alpha,\theta,k)$ of TPQGD then the Cumulative hazard function defined as:

$$CH_{TPQGD}(y;\ \alpha,\theta,k) = -\ln|F(y;\ \alpha,\theta,k)| \quad (4.4)$$

By putting (2.3) into (4.4) have,

$$CH_{TPQGD}(y;\ \alpha,\theta,k) = -\ln\left|1 - \frac{\Gamma(\boldsymbol{\alpha},\boldsymbol{\theta y^{2k}})}{\boldsymbol{\Gamma\alpha}}\right| \quad (4.5)$$

Reversed hazard function:

Let Y be a continuous random variable with pdf $f_{TPQGD}(y;\ \alpha,\theta,k)$ and c.d.f. $F_{TPQGD}(y;\ \alpha,\theta,k)$ of TPQGD then the reversed hazard function defined as:

$$H_{TPQGD}(\mathrm{y}) = \frac{f(y;\alpha,\theta,k)}{F(y;\alpha,\theta,k)} \quad (4.6)$$

By putting (2.1) and (2.3) into (4.6) have following;

$$H_{TPQGD}(y;\ \alpha,\theta,k) = \frac{2k\theta^{\alpha}y^{2k\alpha-1}e^{-\theta y^{2k}}}{\Gamma\alpha-\Gamma(\boldsymbol{\alpha},\boldsymbol{\theta y^{2k}})} \quad (4.7)$$

Mean residual function:

$$m_{TPQGD}(\mathrm{y}) = E(Y-y|Y>y) = \frac{1}{1-F_{TPQGD}(y;\alpha,\theta,k)}\int_y^{\infty}\{1-F_{TPQGD}(t;\alpha,\theta,k)\}dt$$

$$= \frac{1}{1-F_{TPQGD}(y;\alpha,\theta,k)}\int_x^{\infty} tf_{TPQGD}(t;\alpha,\theta,k)dt - y \quad (4.8)$$

By using (2.1) and (2.3) in (4.8) have as:

$$m_{TPQGD}(\mathrm{y}) = \frac{\Gamma\left(\frac{2k\alpha+1}{2k},\theta y^{2k}\right) - y\theta^{\frac{1}{2k}}\Gamma(\boldsymbol{\alpha},\boldsymbol{\theta y^{2k}})}{\theta^{\frac{1}{2k}}\Gamma(\boldsymbol{\alpha},\boldsymbol{\theta y^{2k}})} \quad (4.9)$$

Also note that put y=0 in (4.9) and get 1st moment about origin such as:

$$m_{TPQGD}(0;\alpha,\theta,k) = \mu_1{}' = \frac{\Gamma\left(\frac{2k\alpha+1}{2k}\right)}{\theta^{\frac{1}{2k}}\Gamma\alpha}$$

5. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves Bonferroni, (1930), and Bonferroni and Gini indices have applications not only in economics to study the variation in income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni and Lorenz curves are defined as;

$$B(p) = \frac{1}{p\mu}\int_0^q yf_{TPQGD}(\mathrm{y})dy \quad (5.1)$$

$$= \frac{1}{p\mu}\int_0^{\infty} yf_{TPQGD}(\mathrm{y})dy - \frac{1}{p\mu}\int_q^{\infty} yf_{TPQGD}(\mathrm{y})dy$$

$$= \frac{1}{p} - \frac{1}{p\mu}\int_q^{\infty} yf_{TPQGD}(\mathrm{y})dy$$

By using (2.1) pdf of TPQGD determined as;

$$\int_0^q y f_{TPQGD}(y)dy = \frac{\gamma(\frac{2k\alpha+1}{2k},\theta q^{2k})}{\theta^{\frac{1}{2k}}\Gamma\alpha} \tag{5.2}$$

Put (5.2) into (5.1).

$$B(p) = \frac{1}{p\mu}\frac{\gamma(\frac{2k\alpha+1}{2k},\theta q^{2k})}{\theta^{\frac{1}{2k}}\Gamma\alpha} \tag{5.3}$$

$$L(p) = \frac{1}{\mu}\int_0^q y f_{TPQGD}(y)dy \tag{5.4}$$

$$= \frac{1}{\mu}\int_0^\infty y f_{TPQGD}(y)dy - \frac{1}{\mu}\int_q^\infty y f_{TPQGD}(y)dy$$

$$= 1 - \frac{1}{\mu}\int_q^\infty y f_{TPQGD}(y)dy$$

Put (5.2) into (5.4) have;

$$L(p) = \frac{\gamma(\frac{2k\alpha+1}{2k},\theta q^{2k})}{\mu\theta^{\frac{1}{2k}}\Gamma\alpha} \tag{5.5}$$

Or both equivalent to $B(p) = \frac{1}{p\mu}\int_0^p F(y)^{-1}dy$ and $L(p) = \frac{1}{\mu}\int_0^p F(y)^{-1}dy$ where

$q = F(p)^{-1}$

The Bonferroni indice is defined as;

$$B = 1 - \int_0^1 B(p)dp \tag{5.6}$$

Put (5.3) into (5.6) have,

$$B = 1 - \frac{1}{p\mu}\frac{\gamma(\frac{2k\alpha+1}{2k},\theta q^{2k})}{\theta^{\frac{1}{2k}}\Gamma\alpha} \tag{5.7}$$

The Gini indice defined as following;

$$G = 1 - 2\int_0^1 L(p)dp \tag{5.8}$$

Put (5.5) into (5.8) determined as;

$$G = 1 - \frac{2\gamma(\frac{2k\alpha+1}{2k},\theta q^{2k})}{\mu\theta^{\frac{1}{2k}}\Gamma\alpha} \tag{5.9}$$

6. Oder Statistics (OS)

The density function $f_{(i,j)}(y)$ of "ith" order statistics (i=1, 2, …, n) from independent and identically distributed (i.i.d), random variable $y_1, y_2, \dots y_n$. The order statistics say as $y_{(1)}, y_{(2)}, \dots y_{(n)}$ the function of order statistics defined as:

$$f_{(i)}(y;\ \alpha,\theta,k) = \frac{f(y)}{B(i,n-i+1)}\sum_{j=0}^{n-i}(-1)^j\binom{n-j}{j}[F(y)]^{i+j-1} \tag{6.1}$$

Where $B(i, n-i+1)$ is the beta function. Here, present an expansion for the density function of TPQGD (2.1), $f_{TPQGD}(y;\ \alpha,\theta,k)$ *and* cdf (2.3)$F_{TPQGD}(y;\ \alpha,\theta,k)$ into (6.1) have pdf of order statistics is;

$$f_{(i)}(y;\ \alpha,\theta,k) = \frac{2k\theta^\alpha y^{2k\alpha-1}e^{-\theta y^{2k}}}{B(i,n-i+1)\Gamma\alpha}\sum_{j=0}^{n-i}(-1)^j\binom{n-j}{j}\sum_{m=0}^{\infty}\binom{i+j-1}{m}\left[\frac{\Gamma(\alpha,\theta y^{2k})}{\Gamma\alpha}\right]^m \tag{6.2}$$

Also we know that expansion series; $(1-y)^n = \sum_{i=1}^{\infty} \binom{n}{i} (-1)^i (y)^i$

The ith order statistics cdf of TPQGD is:

$$F_{(i)}(y;\ \alpha,\theta,k) = \sum_{j=0}^{n} \sum_{i=0}^{n-j} (-1)^i \binom{n}{j} \binom{n-j}{i} \left\{1 - \frac{\boldsymbol{\Gamma}(\boldsymbol{\alpha},\boldsymbol{\theta y^{2k}})}{\boldsymbol{\Gamma\alpha}}\right\}^{j+i} \tag{6.3}$$

For maximum order statistics put i=n, for minimum order statistics put i=1 in equation.

7. Renyi Entropy Measure

A popular entropy measure is Renyi entropy (1961), an entropy of a random variable Y is a measure of the variation of uncertainty. Let Y is a continuous random variable having probability density function (TPQGD) $f_{TPQGD}(y;\ \alpha,\theta,k)$, then Renyi entropy is defined as

$$T_{RE}(y) = \frac{1}{1-\delta} \log\left\{\int_0^{\infty} f(y)^{\delta}\, dy\right\} \tag{7.1}$$

Let $f_{TPQGD}(y;\ \alpha,\theta,k)$ pdf of (TPQGD) the Renyi entropy such that:

$$T_{RE}(y;\ \alpha,\theta,k) = \frac{1}{1-\delta} \log\left\{\int_0^{\infty} f_{TPQGD}(y;\ \alpha,\theta,k)^{\delta}\, dy\right\}$$

$$= \frac{1}{1-\delta} \log\left\{\int_0^{\infty} \left[\frac{2k\theta^{\alpha} y^{2k\alpha-1} e^{-\theta y^{2k}}}{\Gamma\alpha}\right]^{\delta} dy\right\}$$

$$= \frac{1}{1-\delta} \log\left\{\frac{(2k\theta^{\alpha})^{\delta}}{(\Gamma\alpha)^{\delta}} \int_0^{\infty} y^{\delta(2k\alpha-1)} e^{-\delta\theta y^{2k}}\, dy\right\}$$

$$= \frac{1}{1-\delta} \log\left\{\frac{(2k)^{\delta-1}\Gamma\left(\frac{\delta(2k\alpha-1)+1}{2k}\right)}{(\Gamma\alpha)^{\delta}\theta^{\frac{\delta(2k\alpha-\alpha-1)+1}{2k}}\delta^{\frac{\delta(2k\alpha-1)+1}{2k}}}\right\} \tag{7.2}$$

8. Maximum Likelihood Method

Let be a random sample $y_1, y_2, \dots y_n$ from $f_{TPQGD}(y_i;\alpha,\theta,k)$ TPQGD, then the Maximum Likelihood (ML) function such as;

$$L(y;\ \theta,\alpha,k) = \prod_{i=0}^{n} \frac{2k\theta^{\alpha} {y_i}^{2k\alpha-1} e^{-\theta {y_i}^{2k}}}{\Gamma\alpha} \tag{8.1}$$

$$lnL(y;\ \theta,\alpha,k) = nln2 + nlnk + n\alpha ln\theta + (2k\alpha-1)\sum_{i=0}^{n} \ln(y_i) - \theta \sum_{i=0}^{n} {y_i}^{2k} - nln(\Gamma\alpha)$$

$$\frac{\partial lnL(y;\theta,\alpha,k)}{\partial\theta} = \frac{n\alpha}{\theta} - \sum_{i=0}^{n} {y_i}^{2k} \tag{8.2}$$

$$\theta = \frac{n\alpha}{\sum_{i=0}^{n} {y_i}^{2k}}$$

$$\frac{\partial lnL(y;\theta,\alpha,k)}{\partial k} = \frac{n}{k} + 2\alpha \sum_{i=0}^{n} \ln(y_i) - \theta \sum_{i=0}^{n} {y_i}^{2k} \ln({y_i}^2) \tag{8.3}$$

$$\frac{\partial lnL(y;\theta,\alpha,k)}{\partial\alpha} = nln\theta + 2k\sum_{i=0}^{n} \ln(y_i) - \frac{\frac{dlog\{\Gamma\alpha\}}{d\alpha}}{\Gamma\alpha} \tag{8.4}$$

$\Psi(\alpha) = \frac{dlog\{\Gamma\alpha\}}{d\alpha} = -\gamma + \sum_{k=1}^{n} \frac{(-1)^{k+1}(\alpha)!}{(\alpha)k.k!(\alpha-k-1)!}$ Digmma function or Psi function;

Where $\gamma = 0.5772156649$ Euler-Mascheroni constant;

By putting (8.2), (8.3) and (8.4) as $\frac{\partial lnL(y;\theta,\beta,k)}{\partial\theta} = 0; \frac{\partial lnL(y;\theta,\beta,k)}{\partial k} = 0;\ \frac{\partial lnL(y;\theta,\beta,k)}{\partial\beta} = 0$

The (8.2), (8.3) and (8.4) natural log likelihood equations do not seem to be solved directly because they are not in closed form, so the Fisher's scoring method can be applied to solve these equations. We have;

$$\frac{\partial^2 lnL(y;\theta,\alpha,k)}{\partial\theta^2} = -\frac{n\alpha}{\theta^2} \tag{8.5}$$

$$\frac{\partial^2 lnL(y;\theta,\alpha,k)}{\partial k^2} = -\frac{n}{k^2} - \theta \sum_{i=0}^{n} y_i^{2k} \{\ln(y_i^2)\}^2 \quad (8.6)$$

$$\frac{\partial^2 lnL(y;\theta,\alpha,k)}{\partial \alpha^2} = \frac{\frac{d^2 log\{\Gamma\alpha\}}{d\alpha^2}}{(\Gamma\alpha)^2} \quad (8.7)$$

$$\frac{\partial^2 lnL(y;\theta,\alpha,k)}{\partial \theta \partial \alpha} = \frac{n}{\theta} \quad (8.8)$$

$$\frac{\partial^2 lnL(y;\theta,\alpha,k)}{\partial \theta \partial k} = -\sum_{i=0}^{n} y_i^{2k} \ln(y_i^2) \quad (8.9)$$

$$\frac{\partial lnL(y;\theta,\alpha,k)}{\partial k \partial \alpha} = 2\sum_{i=0}^{n} \ln(y_i) \quad (8.10)$$

The iterative solution of the equations (8.5) to (8.10) using matrix given following will be the MLEs $\hat{\theta},\ \hat{\alpha}\ and\ \hat{k}$of parameters θ , α and k of TPQGD.

$$\begin{bmatrix} \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial \theta^2} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial \alpha \partial \theta} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial k \partial \theta} \\ \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial \theta \partial \alpha} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial \alpha^2} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial k \partial \alpha} \\ \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial \theta \partial k} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial \beta \partial k} & \frac{\partial lnL^2(y;\theta,\alpha,k)}{\partial k^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}-\alpha_0 \\ \hat{k}=k_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \\ \hat{k}-k_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial lnL(y;\theta,\alpha,k)}{\partial \theta} \\ \frac{\partial lnL(y;\theta,\alpha,k)}{\partial \alpha} \\ \frac{\partial lnL(y;\theta,\alpha,k)}{\partial k} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}-\alpha_0 \\ \hat{k}=k_0}}$$

Here θ_0, α_0 and k_0 initial values of parameters of θ, α and k of TPQGD.

1. Applications

Data Set 1. The following data represent the tensile strength, measured in GPa, of 69 carbon fibres tested under tension at gauge lengths of 20 mm Bader and Priest, (1982) {1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585 and 3.585};

Data set 2. The second data corresponds to 46 observations reported on active repair times (hours) for an airborne communication transceiver discussed by Alven (1964). are {0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0 and 24.5};

Data set 3. The third data set is taken from Nichols and Padgett (2006) consisting of 100 observations on breaking stress of carbon fibers (in Gba) given as; {3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05 and 3.65};

Data set 4. The data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross et al. (1975) are; {1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6 and 2};

Data set 5. The data set is the strength data of glass of the aircraft window reported by Fuller et al. (1994) given as;{18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29 and 45.381};

Anderson Darling statistics:

$$A^2 = -n - S$$

Where $S = \sum_{i=1}^{n}(2i-1)/n[lnF(y_i) + \ln\{1 - F(y_{n+1-i})\}]$

F is the cumulative distribution function of the specified distribution and y_i are ordered data.

Cramer Von Mises Statistics:

Let $y_1, y_2, \dots y_n$ be the (independent and identically-distributed) random variable have continuous distribution with cdf $F(y)$, the Cramér–von Mises test statistic based on following statistic

$$\omega_n^2[\psi(F(y))] = \int_{-\infty}^{+\infty}\left[\sqrt{n}\{F_n(y) - F(y)\}\right]^2 \psi(F(y))dF(y)$$

Here $F_n(y)$is the empirical distribution function constructed from the sample size n and $\psi(F(y))$ is a certain nonnegative function defined and integrated on the interval [0, 1]. Simplified as;

$$\omega_n^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(y_{(i)}) - \frac{2i-1}{2n}\right]^2$$

Here $y_{(1)} \le y_{(2)} \le \cdots \le y_{(n)}$ is the varied series based on the sample $y_1, y_2, \dots y_n$.

Conclusion

The new proposed Quasi Gamma lifetime data distribution have three parameters θ *and* k preforming as shape parameters and α is scale parameter, if in eq. (2.1) put $k = \frac{1}{2}$ it reduced to gamma; for $k = 1$ in quasi gamma distribution; reduced to exponential distribution and if $k = 1, \alpha = \frac{1}{2}$ it reduced to quasi exponential distribution, for $\alpha = 1, k = \frac{1}{2}$ it reduced to exponential distribution, $k = \frac{1}{2}, \alpha = 2$ it reduced to Rayleigh distribution. Further obtained properties including moment generating function, rth moment about origin and mean, mean deviations, reliability measurements including survival function, hazard function, reversed hazard function, cumulative hazard function and mean residual error; Bonferroni and Lorenz curve, Order statistics, Renyi entropy is measure variation of uncertainty have been discussed. The estimation of parameters using ML method determined MLEs given in table 9.1 to 9.5, on basis of real life data sets from 1 to 5 respectively. The graphical presentation of PDF and CDF of TPQGD from Fig. 2.1 to Fig. 2.6, with different values of parameters showing flexibility of distribution. The goodness of fit of distributions with data distribution determined by using Anderson−Darling and Cramér–von Mises test statistics results are given in table 9.1 to 9.5 have least values of TPQGD, also the from Fig. 9.1 to Fig. 9.5 showing goodness of fit of TPQGD better than other distributions. The new proposed TPQGD distribution may performed better in other real life applications related to stress and strength of items, effectiveness or relief of medicine, repairing and depreciation of items base data with respect to time, for lifetime data analysis.

Table. 9.1 The MLE's, Anderson-Darling and Cramér–von Mises statistics values using data set 1.

Data	Distributions	MLEs			Anderson-Darling	Cramér–von Mises
		$\hat{\theta}$	$\hat{\alpha}$	$\hat{k}$		
Set 1	TPQGD	0.10145	2.4108	1.68356	0.14756	0.01590
	QGD	0.99557	6.2229		0.20675	0.02564
	QED	0.07999			20.8592	4.50131
	Gamma	23.3819	0.1048		0.33390	0.04492
	Exponential	0.40794			20.4128	4.30654

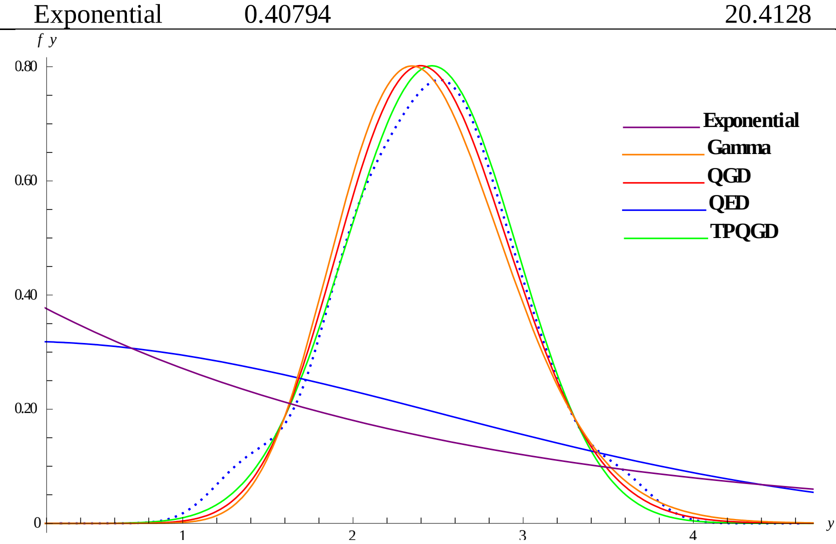

Fig. 9.1 Goodness of fit of distributions with distribution of data set 1.

Table. 9.2 The MLE's , Anderson-Darling and Cramér–von Mises statistics values using data set 2.

Data	**Distributions**	**MLEs**			**Anderson-Darling**	**Cramér–von Mises**
		$\hat{\theta}$	$\hat{\alpha}$	$\hat{k}$		
Set 2	TPQGD	234.418	244.022	0.02893	0.35478	0.05644
	QGD	0.00814	0.30069		2.63331	0.46991
	QED	0.01353			8.55164	1.60026
	Gamma	0.93229	3.86844		1.10392	0.17525
	Exponential	0.27727			1.26289	0.21348

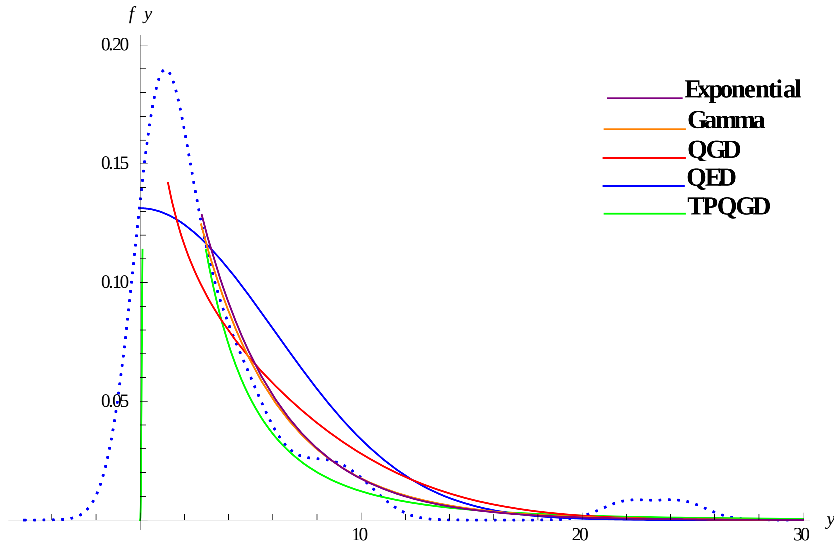

Fig. 9.2 Goodness of fit of distributions with distribution of data set 2.

Table. 9.3 The MLE's , Anderson-Darling and Cramér–von Mises statistics values using data set 3.

Data	**Distributions**	**MLEs**			**Anderson-Darling**	**Cramér–von Mises**
		$\hat{\theta}$	$\hat{\alpha}$	$\hat{k}$		
Set 3	TPQGD	0.11901	1.37172	1.16125	0.40879	0.06934
	QGD	0.22269	1.75696		0.44316	0.08124
	QED	0.06337			14.3263	2.90103
	Gamma	5.95261	0.44037		0.75892	0.15016
	Exponential	0.38147			17.3021	3.43402

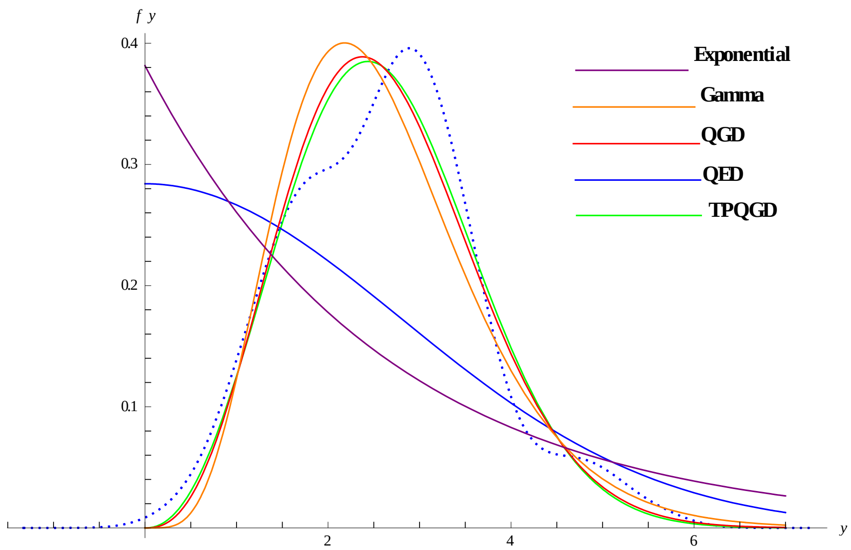

Fig. 9.3 Goodness of fit of distributions with distribution of data set 3.

Table. 9.4 The MLE's , Anderson-Darling and Cramér–von Mises statistics values using data set 4.

Data	Distributions	MLEs			Anderson-Darling	Cramér–von Mises
		$\hat{\theta}$	$\hat{\alpha}$	$\hat{k}$		
Set 4	TPQGD	490.673	532.825	0.06913	0.42634	0.07242
	QGD	0.57526	2.34767		0.86904	0.15339
	QED	0.12251			4.04417	0.86039
	Gamma	9.66948	0.19649		0.59902	0.10251
	Exponential	0.52631			4.60349	0.96296

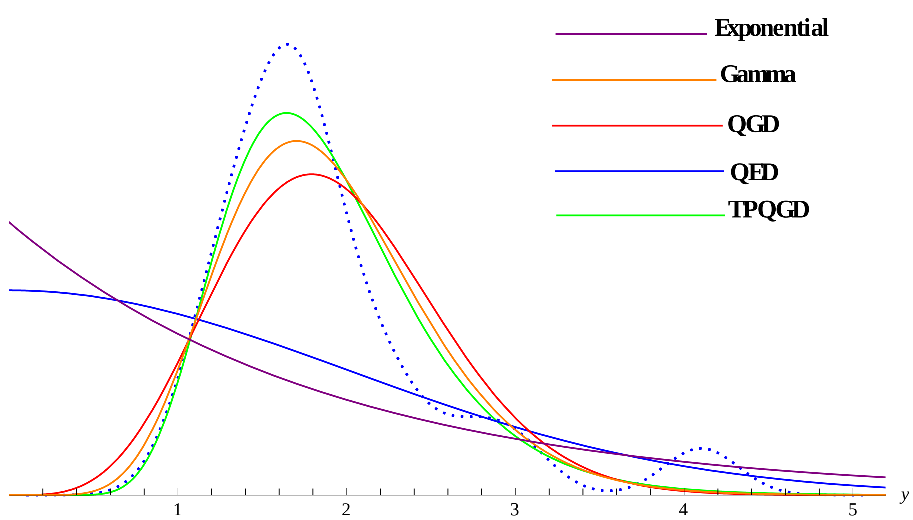

Fig. 9.4 Goodness of fit of distributions with distribution of data set 4.

Pak.j.stat.oper.res. Vol.18 No. 3 2022 pp 759-773 DOI: http://dx.doi.org/10.18187/pjsor.v18i3.3759

Table. 9.5 The MLE's , Anderson-Darling and Cramér–von Mises statistics values using data set 5.

Data	**Distributions**	**MLEs**			**Anderson-Darling**	**Cramér–von Mises**
		$\hat{\theta}$	$\hat{\alpha}$	$\hat{k}$		
Set 5	TPQGD	113.693	275.880	0.13004	0.41523	0.07895
	QGD	0.00486	4.87065		0.47678	0.08479
	QED	0.00049			8.38258	1.79875
	Gamma	18.9315	1.62751		0.43868	0.08155
	Exponential	0.03245			8.52724	1.78829

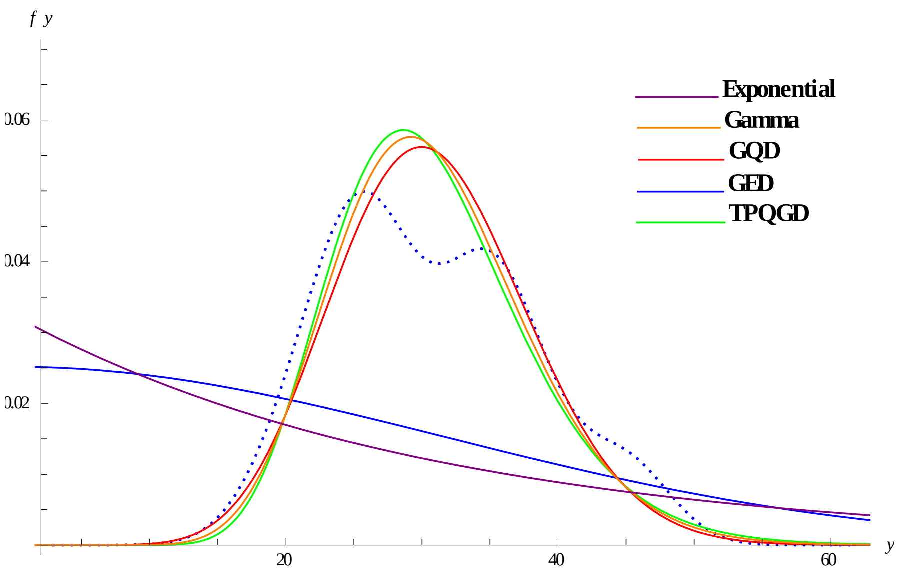

Fig. 9.5 Goodness of fit of distributions with distribution of data set 5.

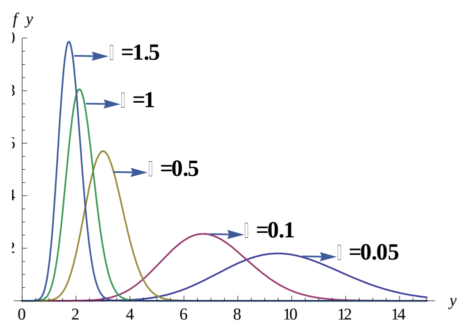

Fig. 2.1 PDF of TPQGD with different value of θ with fix values of k and α

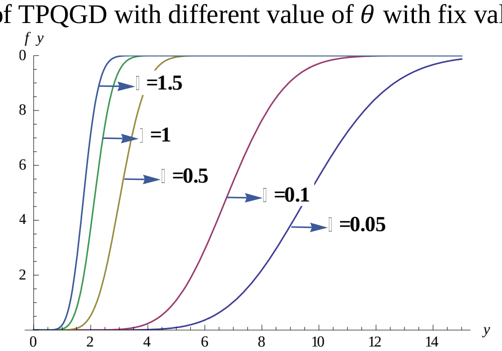

Fig. 2.2 CDF of TPQGD with different value of θ with fix values of k and α

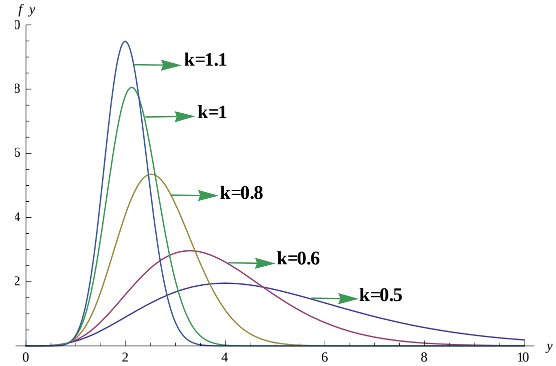

Fig. 2.3 PDF of TPQGD with different value of k with fix values of θ *and* α

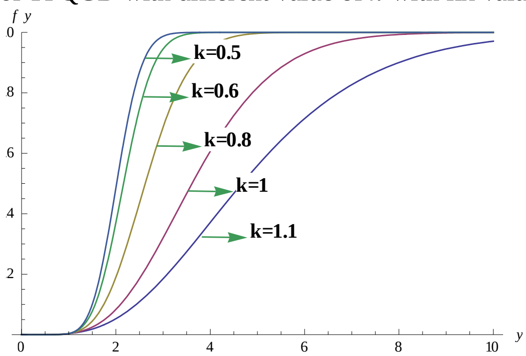

Fig. 2.4 CDF of TPQGD with different value of k with fix values of θ *and* α

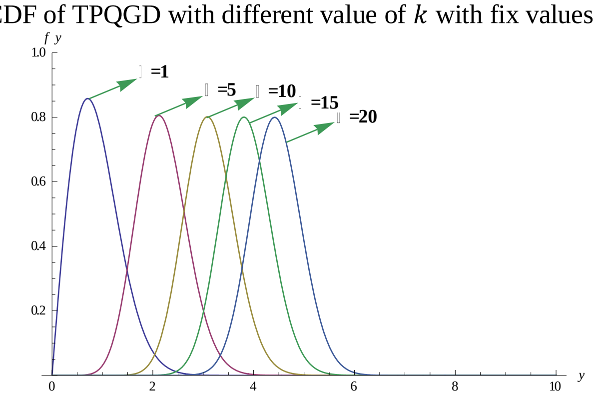

Fig. 2.5 PDF of TPQGD with different value of α with fix values of θ *and* k

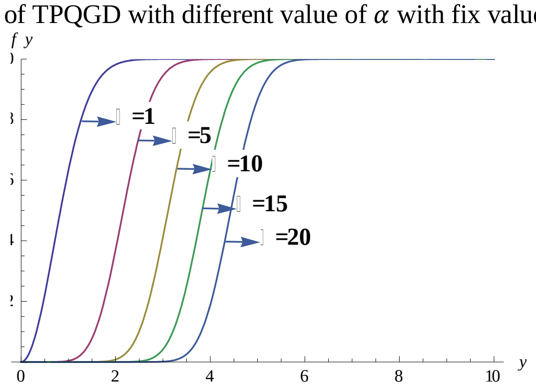

Fig. 2.6 CDF of TPQGD with different value of α with fix values of θ *and* k

References

1. Ali, A. Rashid, Q. Zubair, M. and Jamshaid, M. T. (2018). A Quasi Lindley Pareto Distribution. Proceedings of the Pakistan Academy of Sciences, 55 (2): 32–40.
2. Alven, W. (1964). Reliability Engineering by ARINC. Prentice-hall,New Jersey.
3. Bader, M. G. and Priest, A. M. (1982). Statistical aspects of fiber and bundle strength in hybrid composites . In T. K. Hayashi, Progressin Science in Engineering Composites, ICCM-IV (pp. 1129 -1136). Tokyo.
4. Bonferroni, C. E. (1930). Elementi di Statistca generale . Seeber, Firenze.
5. Cordeiro, G. M., Ortega, E. M. M., and Silva, G. O. (2013). The exponentiated generalized gamma distribution with application to lifetime data. Journal of Statistical Computation and Simulation, 81:827-842.
6. Fuller, E. J., Frieman, S., Quinn, J., Quinn, G. and Carter. W. (1994). Fracture mechanics approach to the design of glass. International Society for Optical Engineering, 2286, pp. 419-430.
7. Gross, A. J. and Clark, V. A. (1975). Survival Distributions: Reliability Applications in the Biometrical Sciences . New York: John Wiley.
8. Gupta, R. D. and Kundu, D. (2006). Closeness of Gamma and Generalized Exponential Distribution. Communications in Statistics - Theory and Methods, Pages 705-721 .
9. Johnson, N. L., Kotz, S., and Balakrishanan, N. (1994). Continuous Univariate Distributions, Second edition,. New York: John Wiley and Sons.
10. Milton, A., and Stegun, I. A. (1972). "Handbook of Mathematical Functions with Formulas", Graphs, and Mathematical Tables. New York: Dover: eds.
11. Nadarajah, S., Gupta, A. K. (2007). A generalized gamma distribution with application to drought data. Mathematics and Computers in Simulation, 74:1-7.
12. Nichols, M.D. and Padgett, W.J. (2006). A Bootstrap control chart for Weibull percentiles . Quality and Reliability Engineering International, 22, 141-151. .
13. Rashid, Q. Yaseen, D. H. M. and Jamshaid, M. T. (2019). Quasi Amarendra Distribution and Its Properties. Global Scientific Journals, (7) 490-502.
14. Renyi, A. (1961). On measures of entropy and information. in proceedings of the 4th Berkeley symposium on Mathematical Statistics and Probability,Berkeley, (pp. 1: 547-561).
15. Shanker, R. and Mishra, A. (2013a). A quasi Lindley distribution. African Journal of Mathematics and Computer Science and Research, Vol. 6(4), pp. 64-71.
16. Shanker, R. and Shukla, K. K. (2018). A Quasi Ardhana distribution and Its Application. International Journal of Statistics and Systems, ISSN 0973-2675 Volume 13, Number 1, pp. 61-80.
17. Shanker, R. Shukla, K. K. Sharma, and S. Shanker, R. (2018). A quasi exponential distribution. Biometrics & Biostatistics International Journal, 7(2):90–94.
18. Shanker, R., and Amanuel, A. G. (2013 b). A new quasi Lindley distribution. International Journal of Statistics and Systems, 8 (2), pp. 143–156.
19. Shanker, R., and Shukla, K. K. (2017). A Quasi Shanker Distribution and Its Application. Biometrics & Biostatistics International Journal, 6(1), 1 – 10.
20. Shanker, R., and Shukla, K. K. Sharma, S. Shanker, R. (2018). A quasi exponential distribution. Biometrics & Biostatistics International Journal, 7(2):90–94.
21. Shanker, R., Shukla, K. K., Sharma, S. and Shanker, R. (2018). A Quasi Gamma Distribution. International Journal of Statistics and Applied Mathematics, 3(4): 208-217.
22. Weibull, W. (1951). A statistical distribution of wide applicability. Journal of Applied Mathematics, 18:293-297.