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Asymptotic Normality of the Conditional Hazard Rate Function Estimator for Right Censored Data under Association

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Abstract
In this paper, we study a smooth estimator of the conditional hazard rate function in the censorship model when the data exhibit some dependence structure. We show, under some regularity conditions, that, when suitably normalized, the kernel estimator of the conditional hazard rate function is asymptotically normally distributed.

Key Words: Association, Asymptotic normality, Censored data, Conditional hazard rate function, Kaplan-Meier estimator, Kernel estimator.

Mathematical Subject Classification: 62G20, 62E20, 62G05.

1. Introduction

It is well known that, in many areas of life testing studies such as medical follow-ups and engineering, it is not always possible to observe the variable of interest called the lifetime.

Formally, let $T_1, \dots, T_n$ be a sequence of survival times of $n \geq 1$ individuals in a life table. These random variables (r.v's) are strictly stationary with common unknown absolutely continuous distribution function (d.f) F. That is, assuming that $\{C_i;\ i \geq 1\}$ is a sequence of independent and identically distributed (i.i.d) censoring r.v's with common unknown d.f. G, we only observe the n pairs $\{(Y_i, \delta_i),\ i = 1,2,\dots,n\}$, with $Y_i = \min(T_i, C_i)$ and $\delta_i = \mathbb{I}_{(T_i \leq C_i)}$, where $\mathbb{I}_A$ denotes the indicator function of a Borel-set A.

Let $X_1, \dots, X_n$ be a stationary sequence of real-valued r.v's with probability density function (p.d.f) l and let $F(.|.)$ be the conditional d.f of T given $X = x$. We assume that C and (X, T) are independent and we observe $\{(Y_i, \delta_i, X_i), i = 1,2,\dots,n\}$.

In classical statistical inference, the observed r.v's of interest are generally assumed to be i.i.d. However, in some real-life situations, these r.v's are not independent. For example, in reliability and survival analysis, the lifetimes of components are not independent but associated. For a list of relevant examples and ample bibliographical references, we refer the reader to the seminal book of Bulinski and Shashkin (2007).

The concept of association was introduced by Esary et al. (1967), in the context of reliability studies. The reader may refer to Prakasa Rao (2012) for a long list of examples of associated r.v's. Recall that a set of r.v's $Z = (Z_1, Z_2, \dots, Z_n)$ is said to be associated if for every pair of non-decreasing (component wise) functions $g_1(.)$ and $g_2(.)$ from $\mathbb{R}^N$ to $\mathbb{R}$, we have

$$cov(g_1(Z), g_2(Z)) \geq 0,$$

whenever the covariance is defined. An infinite sequence $\{Z_N, N \geq 1\}$ of r.v's is said to be associated if every finite sub-family of r.v's is associated.

In this random censorship model, the true survival times $\{T_i, i \geq 1\}$ are assumed to be positively associated, which implies that the random sequences $\{Y_i, i \geq 1\}$ and $\{\delta_i, i \geq 1\}$ are positively associated.(see Lemma 2.2 in Cai and Roussas (1998)). The stationary sequence $\{X_i, i \geq 1\}$ is also assumed to be positively associated. Positive association seems to be a natural assumption in certain practical trials like those described in Ying and Wei (1994) and Cai and Roussas (1998). In this paper, we are concerned with the nonparametric estimation of the unknown conditional hazard rate function, based on the observations $\{(Y_i, \delta_i, X_i),\ i \geq 1\}$. We use the form of the ratio of two estimators by estimating the density of probability function and the survival function separately as done in many papers as, for instance, in Lemdani and Ould Saïd (2007), Ferraty, Rabhi and Vieu (2008) and Dialo and Louani (2013) in the i.i.d case. Note that many other estimation methods can be used. A local linear estimator of the conditional hazard rate for censored data has been studied by Spierdijk (2008). Gamis, Martinez-Miranda and Neilsen (2013) developed indirect cross-validation for the local linear estimator starting from the multivariate local linear estimator of Nielsen (1998).

In the context of right response data, with associated r.v's, no much researches are done for this kind of model. Cai and Roussas (1998) established uniform strong consistency along with the asymptotic normality to Kaplan-Meier estimator. Ferrani, Ould Saïd, and Tatachak (2016) established the strong uniform consistency of the kernel estimator of the underlying p.d.f and the almost sure convergence of a smooth kernel mode estimator under a right censored model.

The goal of this paper is to establish the asymptotic normality of a kernel estimator of the conditional hazard rate function when the data are strictly stationary sequence of associated r.v's under right censoring.

This paper is organized as follows. In section 2, we recall the notations and the definitions of our estimators. The main results and the assumptions are listed in section 3 while the proofs are relegated to section 4.

2. Notations and definitions

The conditional hazard rate function, also known as the force of mortality or the failure rate, of T given $X = x$ is defined by $h(.|x) = \frac{f(.|x)}{1-F(.|x)}$, for x such that $F(.|x) < 1$.

In medical trials, $h(t|x)dt$ can be interpreted as the instantaneous risk of death at time t, conditioned by the fact that the subject is still alive at time x. In the literature, there exist several methods of estimating the hazard rate function. In nonparametric estimation, the method using a kernel smoothing has received considerable attention. For a review of kernel smoothing approaches, we refer the reader to Silverman (1986) and Izenman (1991) for uncensored data, and to Singpurwalla and Wong (1983), Tanner and Wong (1983), Padgett and McNichols (1984), Lo Mack and Wang (1989), Lecoute and Ould Saïd (1995), Gonzàlez-Manteiga, Cao and Marron (1996), Nassiri, Delacroix and Bonneu (2000), and Van Keilegom and Veraverbeke (2001) in the case of right censored data.

Our estimator of the conditional hazard function is obtained by estimating the conditional density $f(.|.)$ and the conditional survival function $1 - F(.|.)$. We consider the following smooth estimator of $F(.|.)$

$$\tilde{F}_n(t|x) = \frac{\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}(Y_i)} K\left(\frac{x-X_i}{h}\right) H\left(\frac{t-Y_i}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)} \tag{2.1}$$

where $\bar{G} = 1 - G$, K is a p.d.f (called kernel function), $h := h_n$ is a sequence of positive real numbers (called bandwidth) tending to zero as n goes to infinity and H is a d.f.

Recall that (2.1) can be rewritten as

$$\tilde{F}_n((t|x) = \frac{\tilde{F}_{1,n}(t,x)}{\mathrm{l_n}(\mathrm{x})} \tag{2.2}$$

where

$$\tilde{F}_{1,n}(t,x) = \frac{1}{nh}\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}(Y_i)} K(\frac{x - X_i}{h}) H\left(\frac{t - Y_i}{h}\right)$$

and $\mathrm{l_n}(\mathrm{x}) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)$, the well known kernel estimator of the marginal density of X.

In practice G is usually unknown, hence it impossible to use the estimator (2.1). Then, to construct a practically feasible estimator, we replace the unknown df G by the Kaplan-Meier estimate G_n (see Kaplan and Meier (1958)) given by

$$1 - G_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1 - \delta_{(i)}}{n - i + 1}\right)^{\mathbb{I}_{\{Y_i \le t\}}} & \text{if } t < Y_n \\ 0 & \text{if } t \ge Y_n \end{cases}$$

Therefore, the feasible estimator of the conditional d.f $F(.|.)$ is given by

$$F_n(t|x) = \frac{\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}(Y_i)} K\left(\frac{x-X_i}{h}\right) H\left(\frac{t-Y_i}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)} =: \frac{F_{1,n}(t,x)}{l_n(x)} \tag{2.3}$$

where

$$F_{1,n}(t,x) = \frac{1}{nh}\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}(Y_i)} K\left(\frac{x - X_i}{h}\right) H\left(\frac{t - Y_i}{h}\right). \tag{2.4}$$

We respectively define the first partial derivative with respect to the second component of $F_{1,n}(t,x)$ and $\widetilde{F}_{1,n}(t,x)$ by

$$\frac{\partial F_{1,n}(t,x)}{\partial t} = F'_{1,n}(t,x) = \frac{1}{nh^2}\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}_n(Y_i)} K\left(\frac{x - X_i}{h}\right) H'\left(\frac{t - Y_i}{h}\right),$$

and

$$\frac{\partial \tilde{F}_{1,n}(t,x)}{\partial t} = \tilde{F}'_{1,n}(t,x) = \frac{1}{nh^2}\sum_{i=1}^{n} \frac{\delta_i}{\bar{G}(Y_i)} K\left(\frac{x - X_i}{h}\right) H'\left(\frac{t - Y_i}{h}\right),$$

where H' is the derivative of H.

The conditional density estimator is given by

$$f_n(t|x) = \frac{F'_{1,n}(t,x)}{\mathrm{l_n}(\mathrm{x})}.$$

Then the natural estimator of the hazard rate function, $h(.|.)$ is given by

$$h_n(t|x) = \frac{f_n(t|x)}{1 - F_n(t|x)}.$$

The infeasible estimator $\tilde{h}_n$ is the same but it involves infeasible estimators $\tilde{f}_n$ and $1-\tilde{F}_n$ of the conditional density and survival functions.

3. Assumptions and main result

Our assumptions are gathered for easy references. Let, for any d.f L, $\tau_L = sup\{y, L(y) < 1\}$, be its right endpoint. Throughout this paper, we assume that $\tau_F < \tau_G$ and that C and (X,T) are independent.

We employ an appropriate form of Bernstein big-block and small-block procedure to prove that the estimator $h_n(.|.)$ is asymptotically normal. Our approach consists in splitting the set $\{1, \dots, n\}$ into k large p −blocks and small q −blocks where $p = p_n$, $q = q_n$ are positive integers tending to ∞, as $n \to \infty$, and $k = k_n \coloneqq [n|(p+q)]$ f with $[x]$ standing for the integer part of x (see for example Masry (2005) and Roussas (2000)).

Let Ω be a compact set such that $\Omega \subset \Omega_0 = \{x \in \mathbb{R}/l(x) > 0\}$, $\mathcal{C}$ a compact set included in $]-\infty, \tau_F[$ and τ a positive real number such that $\tau < \tau_F$.

A1. The kernel K satisfies

- (i) K is strictly positive and bounded with compact support. Moreover, there exist constant M^* and m^* such that $0 < M^* < \infty$, $0 < m^* < \infty$, $\sup K(u) = M^*$, and $inf K(u) = m^*$,
- (ii) $K$is Lipschizian,
- (iii) $\int_{\mathbb{R}} K(u)du = 0$, $\int_{\mathbb{R}} |u|K(u)du < +\infty$ and $\int_{\mathbb{R}} u^2 \ \mathrm{K(u)du} < +\infty$,
- (iv) $\int_{\mathbb{R}} K^2(u)du = \kappa < +\infty$.

A2. The bandwidth$h$ satisfies $\sqrt{\frac{\log\log n}{n}} = \mathcal{O}(h^3)$.

A3. The joint density $F_1^{'}(.,.)$ is bounded and differentiable up to order 2 and $sup_{x,t}\left|D_{(i,j)}F_1^{'}(x,t)\right| < \infty$ for $i + j \leq 2$, where $D_{(i,j)}F_1^{'}(x,t) = \frac{\partial^{i+j}}{\partial x^i \partial t^j}F_1^{'}(x,t)$.

A4. The d.f H has a first derivative $H^{'}$ which is positive and bounded such that

- (i) There exist two constants $0 < M < \infty$, and $0 < m < \infty$, $sup_{\mathbb{R}}H^{'}(t) = M$ and $inf_{\mathbb{R}}H^{'}(t) = m$,
- (ii) $\int_{\mathbb{R}} H^{'}(t)dt = 1$ and $\int_{\mathbb{R}} |t|\, H^{'}(t)dt < +\infty$.
- (iii) $\int_{\mathbb{R}} (H^{'}(s))^2 ds = \sigma < +\infty$.

A5. The marginal density $l(.)$ satisfies the Lipshitz condition and there exists $\gamma_0 < 0$ such that $l(x) < \gamma_0$ for all $x \in \Omega$.

A6. For positively associated r.v's $\{X_i, 1 \leq i \leq n\}$, we suppose that

$$u(n) = sup_{n\geq 1}\sum_{i=1}^{n}\sum_{j=1_{|i-j|>0}}^{n}\theta_{|i-j|}^{\gamma} < \infty,$$

where $0 < \gamma < 1$, and $\theta_{|i-j|} \coloneqq \left|cov(X_i, X_j)\right|$.

A7. If $f_{(X_i,Y_i,X_j,Y_j)}$ is the joint p.d.f of (X_i, Y_i, X_j, Y_j) and $f_{(X_i,Y_i)}$the joint p.d.f of (X_i, Y_i), then $sup_{|i-j|>0}\|g_{i,j}\|_\infty < \infty$, where $g_{i,j} = f_{(X_i,Y_i,X_j,Y_j)} - f_{(X_i,Y_i)} \times f_{(X_j,Y_j)}$.

A8. For the large p −blocks and small q −blocks, we suppose that

(i) $\frac{pk}{n} \to 1$, (ii) $\frac{qk}{n} \to 0$, (iii) $\frac{p^2}{nh^2} \to 0$, as $n \to \infty$, (iv) $\frac{1}{h^4}\sum_{j=q}^{\infty}|cov(X_1, X_{j+1}| \to 0$.

Remark. 3.1. Assumptions **A1**(i)-(ii), **A2**, **A4**(i)-(ii), and **A5** are common in kernel estimation. A similar assumption as **A6** is also employed in many papers dealing with the association, see Cai and Roussas(1998), Roussas (1991), Guessoum, Ould-Said, Sadki, and Tatachak.(2012). Assumption **A3** is needed to deal with the bias term of the joint density $F_1'(.,.)$. Assumption **A7** is used when dealing with the covariance term to make it negligible, as done in many papers, for examples in Adjoudj and Tatachak (2019) and Djelladj and Tatachak (2019).

The Lipshitz conditions on K and l are often present in this case of dependent data in order to use exponential type inequalities as in Gheliem and Guessoum (2022) .

Assumptions **A1**(ii) and **A4**(iii) are added to get the asymptotic variance term. Assumption **A8** is inspired from Roussas (2000) and Masry (2005) and used in the proof following the familiar pattern for dependent situations using large and small blocks.

Theorem 3.1. Under assumptions **A1-A8**, and for any $x \in \Omega_0$ such that $l(x) > 0$, and for n large enough we have

$$(nh^2)^{\frac{1}{2}}\left(h_n(t|x) - h(t|x)\right) \xrightarrow{D} N(0, \sigma^2(t,x)),$$

where $\xrightarrow{D}$ denotes the convergence in distribution, and

$$\sigma^2(t,x) = \frac{1}{l^2(x)[1-F(t|x)]^2}\left[\frac{\kappa\sigma F_1'(t,x)}{\bar{G}(t)}\right].$$

4. Auxiliary results and proofs

The proof of our main result is split up into several lemmas. The first lemma deals with the convergence of the conditional probability density estimator $\tilde{f}_n(t|x)$ to $f(t|x)$.

Lemma 4.1. Under assumptions **A1**(iv), **A2**, **A4**(i), **A6** and **A7** and for n large enough

$$Var\left(\tilde{F}_{1,n}'(t,x)\right) \to 0.$$

Proof. Recall that

$$\tilde{F}_{1,n}'(t,x) = \frac{1}{nh^2}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right),$$

then

$$\begin{aligned} var\left(\tilde{F}'_{1,n}(t,x)\right) &= var\left(\frac{1}{nh^2}\sum_{i=1}^{n}\frac{\delta_i}{\bar{G}(Y_i)}K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\right) \\ &= \frac{1}{nh^4}\mathbb{E}\left[K^2\left(\frac{x-X_1}{h}\right)\left(H'\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\bar{G}(Y_1)}\right)^2\right] \\ &- \frac{1}{nh^4}\left[\mathbb{E}\left[K\left(\frac{x-X_1}{h}\right)H'\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\bar{G}(Y_1)}\right]\right]^2 \\ &+ \frac{1}{n^2h^4}\sum_{i=1}^{n}\sum_{j=1_{|i-j|>0}}^{n} cov(K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)},K\left(\frac{x-X_j}{h}\right)H'\left(\frac{t-Y_j}{h}\right)) \\ &=: \psi_1 - \psi_2 + \psi_3. \end{aligned}$$

By assumptions **A1**(iv) and **A4**(i), we have

$$\begin{aligned} \psi_1 &\leq \frac{1}{nh^4}\frac{M^2}{\bar{G}^2(\tau)}\mathbb{E}\left[K^2\left(\frac{x-X_1}{h}\right)\right] \\ &= \frac{1}{nh^3}\frac{M^2}{\bar{G}^2(\tau)}\kappa l(x) + \mathcal{O}\left(\frac{1}{nh^3}\right). \end{aligned}$$

Then by **A2**, $\psi_1 \to 0$ as $n \to \infty$.

Using the same arguments as before, we have

$$\begin{aligned} \psi_2 &\leq \frac{1}{nh^4}\frac{M^2}{\bar{G}^2(\tau)}\left[\mathbb{E}\left[K\left(\frac{x-X_1}{h}\right)\right]\right]^2 \\ &= \frac{1}{nh^2}\frac{M^2}{\bar{G}^2(\tau)}l^2(x) + \mathcal{O}\left(\frac{1}{nh^2}\right) \end{aligned}$$

and ψ_2 tends again to zero as n goes to infinity.
For ψ_3, by the association of the variables (X_i), we have under **A4**(i)

$$\begin{aligned} \left|cov\left(K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)},K\left(\frac{x-X_j}{h}\right)H'\left(\frac{t-Y_j}{h}\right)\frac{\delta_j}{\bar{G}(Y_j)}\right)\right| &\leq \frac{M^2(LipK)^2}{\bar{G}^2(\tau)h^2}\left|cov\left(X_i,X_j\right)\right| \qquad (4.1) \\ &\leq \frac{C}{h^2}\theta_{|i-j|}. \end{aligned}$$

On the other hand, under assumption **A7**, we have

$$\begin{aligned} \left|cov\left(K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)},K\left(\frac{x-X_j}{h}\right)H'\left(\frac{t-Y_j}{h}\right)\frac{\delta_j}{\bar{G}(Y_j)}\right)\right| &\leq \frac{1}{\bar{G}^2(\tau)}h^4\left\|g_{i,j}\right\|_\infty\|K\|_1^2\|H'\|_1^2 \qquad (4.2) \\ &\leq Ch^4 \leq Ch^2. \end{aligned}$$

By elevating (4.1) to γ and (4.2) to $(1-\gamma)$, where $0<\gamma<1$, we have

$$\begin{aligned} \left|cov\left(K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)},K\left(\frac{x-X_j}{h}\right)H'\left(\frac{t-Y_j}{h}\right)\frac{\delta_j}{\bar{G}(Y_j)}\right)\right| &\leq \frac{C^\gamma}{(h^{2\gamma})}\theta_{|i-j|}^\gamma C^{1-\gamma}(h^2)^{1-\gamma} \\ &< C\theta_{|i-j|}^\gamma h^{2-4\gamma}. \end{aligned}$$

Then we get

$$\frac{1}{n^2h^4}\sum_{i=1}^{n}\sum_{j=1,|i-j|>0}^{n} cov(K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)},K\left(\frac{x-X_j}{h}\right)H'\left(\frac{t-Y_j}{h}\right)\frac{\delta_j}{\bar{G}(Y_j)}) < \frac{1}{n^2h^4}h^{2-4\gamma}\sum_{i=1}^{n}\sum_{j=1_{|i-j|>}}^{n} < \frac{1}{(nh^3)^2}Cu(n).$$

Therefore by **A2** and **A6**, $\psi_3 \to 0$, as $n \to \infty$. This completes the proof of Lemma 4.1. ■

Remark 4.1. Under the assumptions of Lemma 4.1, and making use of Tchebychev's inequality we have for n large enough $\tilde{F}'_{1,n}(t,x) \to F_1^{'}(t,x)$, in probability and we add assumption **A5** then

$$\tilde{f}_n(t|x) = \frac{\tilde{F}'_{1,n}(t,x)}{\mathrm{l_n(x)}} \to f(t|x)$$

in probability as $n \to \infty$.
The following Lemma deals with the convergence of the conditional probability density estimator $f_n(t|x)$ to $f(t|x)$.

Lemma 4.2. Under assumptions **A1**(ii), **A2**, and **A4**(i), we have

$$f_n(t|x) - f(t|x) \to 0 \text{ in probability, as } n \to \infty.$$

Proof. We have

$$f_n(t|x) - f(t|x) = \left(f_n(t|x) - \tilde{f}_n(t|x)\right) + \left(\tilde{f}_n(t|x) - f(t|x)\right).$$

$$\begin{aligned}
\left|f_n(t|x) - \tilde{f}_n(t|x)\right| &= \frac{1}{\mathrm{l_n(x)}}\left|F'_{1,n}(t,x) - \tilde{F}'_{1,n}(t,x)\right| \\
&= \frac{1}{\mathrm{l_n(x)}}\frac{1}{nh^2}\left|\sum_{i=1}^{n}\delta_i K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\left(\frac{1}{\bar{G}_n(Y_i)} - \frac{1}{\bar{G}(Y_i)}\right)\right| \\
&\leq \frac{1}{\mathrm{l_n(x)}}\frac{1}{h^2}\frac{MM^*}{\bar{G}_n(\tau)\bar{G}(\tau)} sup_{0\leq t\leq\tau}|G_n(t) - G(t)|\frac{1}{n}\sum_{i=1}^{n}\delta_i.
\end{aligned}$$

Since $\mathrm{l_n(x)} \geq inf_x\left(\mathrm{l_n(x)}\right) \geq \gamma > 0$ and $\bar{G}(\tau) > 0$, in conjunction with the SLLN in the associated case (see Bagai and Rao, (1995)) applied to the sequence $\{\delta_i\}_{i\geq 1}$ and the LIL on the independent censoring times $\{C_i\}$, we have

$$\left|f_n(t|x) - \tilde{f}_n(t|x)\right| \leq \frac{cMM^*}{\bar{G}^2(\tau)}\frac{1}{h^2}\left(\sqrt{\frac{loglogn}{n}}\right) a.s.$$

where c is a positive constant.
Assumption **A2** gives us that $\left|f_n(t|x) - \tilde{f}_n(t|x)\right| \to 0 \; a.s.$ as $n \to \infty$.■

Remark 4.2. Using the same idea as the last one, we can prove that

$$\left|F_n(t|x) - \tilde{F}_n(t|x)\right| \to 0 \text{ in probability, as } n \to \infty.$$

In order to prove Theorem 3.1, we will use the following decomposition and the lemmas bellow.

$$\begin{aligned}
\tilde{h}_n(t|x) - h(t|x) &= \frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)} - \mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right) + \mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right) - \frac{f(t|x)}{1-F(t|x)} \\
&= \frac{l(x)}{\mathrm{l_n(x)}}\left\{\left[\frac{\tilde{F}'_{1,n}(t,x) - \mathbb{E}\left(\tilde{F}'_{1,n}(t,x)\right)}{l(x)}\right] \times \frac{1}{1-\tilde{F}_n(t|x)}\right\} \\
&+ \left\{\frac{\mathbb{E}\left(\tilde{f}_n(t|x)\right)}{1-\tilde{F}_n(t|x)} - \mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right) + \mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right) - \frac{f(t|x)}{1-F(t|x)}\right\}.
\end{aligned}$$

Note that

$$(nh^2)^{\frac{1}{2}}[h_n(t|x) - h(t|x)] = \frac{l(x)}{\mathrm{l_n(x)}}\left[(nh^2)^{\frac{1}{2}} A_n(t,x)\right] + (nh^2)^{\frac{1}{2}} C_n(t,x)$$

where

$$A_n(t,x) = \frac{1}{1-\tilde{F}_n(t|x)} \times \left[\frac{\tilde{F}'_{1,n}(t,x) - \mathbb{E}\left(\tilde{F}'_{1,n}(t,x)\right)}{l(x)}\right] =: \frac{J_n(t,x)}{1-\tilde{F}_n(t|x)}$$

$$\begin{aligned}
B_n(t,x) &= \left[\frac{\mathbb{E}\left(\tilde{f}_n(t|x)\right)}{1-\tilde{F}_n(t|x)} - \mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right)\right] + \left[\mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right) - \frac{f(t|x)}{1-F(t|x)}\right] \\
&=: \upsilon_1(t,x) + \upsilon_2(t,x)
\end{aligned}$$

and

$$\begin{aligned}
C_n(t,x) &= \left[h_n(t|x) - \tilde{h}_n(t|x)\right] + B_n(t,x) \\
&=: J_1(t,x) + B_n(t,x).
\end{aligned}$$

The next Lemmas show the asymptotic normality of $(nh^2)^{\frac{1}{2}} A_n(t,x)$ and the convergence in probability of $(nh^2)^{\frac{1}{2}} C_n(t,x)$ to zero. We begin by showing the second part.

Lemma 4.3. Under the assumptions of Lemma 4.2, we have

$J_1(t,x) \to 0$ in probability, as $n \to \infty$.

Proof.

$$\begin{aligned}
J_1(t,x) &= \frac{f_n(t|x)}{1-F_n(t|x)} - \frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)} \\
&= \frac{f_n(t|x)[F_n(t|x) - \tilde{F}_n(t|x)] + \left[f_n(t|x) - \tilde{f}_n(t|x)\right][1-F_n(t|x)]}{[1-F_n(t|x)][1-\tilde{F}_n(t|x)]} \\
&= \frac{f_n(t|x)[F_n(t|x) - \tilde{F}_n(t|x)]}{[1-F_n(t|x)][1-\tilde{F}_n(t|x)]} + \frac{\left[f_n(t|x) - \tilde{f}_n(t|x)\right]}{[1-\tilde{F}_n(t|x)]} \\
&\leq A\left[f_n(t|x)[F_n(t|x) - \tilde{F}_n(t|x)]\right] + B\left[f_n(t|x) - \tilde{f}_n(t|x)\right]
\end{aligned}$$

where $B = sup_{x\in\Omega} sup_{t\in C} \frac{1}{1-F(t|x)}$, and $A = B^2$. ■

Making use of the Lemma 4.2 we get that $J_1(t,x)$ goes to zero in probability, as n goes to infinity.

Lemma 4.4. Under the assumptions of Lemma 4.1, we have

$B_n(t,x) \to 0$ in probability, as $n \to \infty$.

Proof. For $\upsilon_1(t,x)$, it can be written as

$$\begin{aligned}
\upsilon_1(t,x) &= \frac{\mathbb{E}\left(\tilde{f}_n(t|x)\right)}{1-\tilde{F}_n(t|x)} - \frac{1-\tilde{F}_n(t|x)}{1-\tilde{F}_n(t|x)}\mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right) \\
&= \frac{1}{1-\tilde{F}_n(t|x)}\left[\mathbb{E}\left(\tilde{f}_n(t|x)\right) - \left(1-\tilde{F}_n(t|x)\right)\mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right)\right] \\
&= \mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right)\left[\mathbb{E}\left(\tilde{h}_n(t|x)\left(1-\tilde{F}_n(t|x)\right)\right) - \left(1-\tilde{F}_n(t|x)\right)\mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right)\right] \\
&\leq B\left[\mathbb{E}\left(\tilde{h}_n(t|x)\right) - \mathbb{E}\left(\tilde{h}_n(t|x)\right)\right].
\end{aligned}$$

The latter quantity goes to zero, which implies that $\upsilon_1(t,x) \to 0$ as $n \to \infty$.
Let us now examine the term $\upsilon_2(t,x)$. We have

$$\begin{aligned}
\upsilon_2(t,x) &= \mathbb{E}\left(\frac{\tilde{f}_n(t|x)}{1-\tilde{F}_n(t|x)}\right) - \frac{f(t|x)}{1-F(t|x)} \\
&\leq A\mathbb{E}\left[sup_{x\in\Omega}sup_{t\in C}\left|\tilde{f}_n(t|x) - f(t|x)\right| + sup_{x\in\Omega}sup_{t\in C}\left|\tilde{F}_n(t|x)\left(f(t|x) - \tilde{f}_n(t|x)\right)\right|\right] \\
&+ A\mathbb{E}\left[sup_{x\in\Omega}sup_{t\in C}\left|f_n(t|x)\left(\tilde{F}_n(t|x) - F(t|x)\right)\right|\right].
\end{aligned}$$

Then we can conclude, by Lemma 4.3, that the $\upsilon_2(t,x) \to 0$ in robability as n goes to infinity. ■

Lemma 4.5. Under assumptions **A1, A3,** and **A4,** and for n large enough we have

$$nh^2 Var\left(A_n(t,x)\right) = nh^2 Var\left(\frac{J_n(t,x)}{1-\tilde{F}_n(t|x)}\right) \to \left(\sigma(t,x)\right)^2$$

Proof.

$$\begin{aligned}
J_n(t,x) &= \frac{\tilde{F}'_{1,n}(t,x) - \mathbb{E}\left(\tilde{F}'_{1,n}(t,x)\right)}{l(x)} \\
&= \frac{1}{nh^2}\left[\frac{1}{l(x)}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)}\right] \\
&\quad -\frac{1}{nh^2}\mathbb{E}\left[\frac{1}{l(x)}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)}\right] \\
&=: \frac{1}{nh^2 l(x)}\sum_{i=1}^{n} N_i(t,x),
\end{aligned}$$

where

$$\begin{aligned}
N_i(t,x) &= \left[K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)}\right] \\
&\quad -\mathbb{E}\left[K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)}\right].
\end{aligned}$$

Then

$$
\begin{aligned}
nh^2 Var J_n(t,x) &= \frac{1}{h^2 l^2(x)} Var\big(N_1(t,x)\big) + \frac{1}{nh^2 l^2(x)} \sum_{i=1}^{n} \sum_{j=1_{|i-j|>0}}^{n} cov\big(N_i, N_j\big) \qquad (4.3)\\
&= \frac{1}{h^2 l^2(x)} \mathbb{E}\left({N_1}^2(t,x)\right) + \frac{1}{nh^2 l^2(x)} \sum_{i=1}^{n} \sum_{j=1_{|i-j|>0}}^{n} cov\big(N_i, N_j\big)\\
&= \frac{1}{h^2 l^2(x)} \mathbb{E}\left[K^2\left(\frac{x-X_1}{h}\right)\left(H'\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\bar{G}(Y_1)}\right)^2\right]\\
&\quad - \frac{1}{h^2 l^2(x)}\left[\mathbb{E}\left[K\left(\frac{x-X_1}{h}\right)H'\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\bar{G}(Y_1)}\right]\right]^2\\
&\quad + \frac{1}{nh^2 l^2(x)} *\\
&\quad \sum_{i=1}^{n} \sum_{j=1_{|i-j|>0}}^{n} cov\left(K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)}, K\left(\frac{x-X_j}{h}\right)H'\left(\frac{t-Y_j}{h}\right)\frac{\delta_j}{\bar{G}(Y_j)}\right)\\
&=: \beta_{1,n} - \beta_{2,n} + \beta_{3,n}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{1,n} &= \frac{1}{h^2 l^2(x)} \mathbb{E}\left[K^2\left(\frac{x-X_1}{h}\right)\left(H'\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\bar{G}(Y_1)}\right)^2\right]\\
&= \frac{1}{h^2 l^2(x)} \mathbb{E}\left[\bar{G}^{-2}(T_1) K^2\left(\frac{x-X_1}{h}\right)(H')^2\left(\frac{t-T_1}{h}\right)\mathbb{E}\left[\mathbb{I}_{\{T_1 \le C_1\}} | X_1, T_1\right]\right].
\end{aligned}
$$

Using a change of variables, we can write

$$
\beta_{1,n} = \frac{1}{l^2(x)} \iint_{-\infty}^{+\infty} \frac{K^2(r)(H')^2(s)}{\bar{G}(t-sh)} F_1'(x-rh, t-sh)\, dr ds.
$$

Then, since $G(.)$ is continuous, we have under **A1, A3,** and **A4**

$$
\begin{aligned}
\beta_{1,n} &= \frac{F_1'(x,t)}{l^2(x)\bar{G}(t)} \iint_{-\infty}^{+\infty} K^2(r)(H')^2\,(s)\, dr ds + o(1)\\
&= \frac{\kappa\sigma}{l^2(x)} \frac{F_1'(x,t)}{\bar{G}(t)} + o(1).
\end{aligned}
$$

Now let us turn to the second term of (4.3),

$$
\begin{aligned}
\beta_{2,n} &= \frac{1}{h^2 l^2(x)}\left[\mathbb{E}\left[K\left(\frac{x-X_1}{h}\right)\mathbb{E}\left(H'\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\bar{G}(Y_1)} | X_1\right)\right]\right]^2\\
&= \frac{1}{h^2 l^2(x)}\left[\mathbb{E}\left[\bar{G}^{-1}(T_1) K\left(\frac{x-X_1}{h}\right)H'\left(\frac{t-T_1}{h}\right)\mathbb{E}\left[\mathbb{I}_{\{T_1 \le C_1\}} | X_1, T_1\right]\right]\right]^2.
\end{aligned}
$$

A Taylor expansion, and assumptions **A1**(iii), **A3,** and **A4**(ii) permit us to write

$$
\begin{aligned}
\beta_{2,n} &= \frac{h^2}{l^2(x)}\left(\iint_{-\infty}^{+\infty} K(s)H'(r)F_1'(x-rh, t-sh)\, dr ds\right)^2\\
&= o(h^6).
\end{aligned}
$$

By the association of the variables (X_i) and under assumption **A4** (i)and **A7,** we have

$$\beta_{3,n} < \frac{1}{nl^2(x)} \sum_{i=1}^{n} \sum_{j=1_{|i-j|>0}}^{n} C\theta_{|i-j|}^{\gamma}.$$

This implies that $\beta_{3,n} \to 0$ as $n \to \infty$. Then

$$\begin{aligned} \lim_{n\to\infty} nh^2 Var\big(A_n(t,x)\big) &= \frac{1}{l^2(x)[1-F(t|x)]^2}\left[\frac{\kappa\sigma F_1'(x,t)}{\bar{G}(t)}\right] \\ &= \big(\sigma(t,x)\big)^2. \end{aligned}$$

■

The asymptotic normality will be established by splitting the sum $\frac{1}{nh^2 l(x)}\sum_{i=1}^{n} N_i(t,x)$, into large p −blocks and small q −blocks. To this end, for $m = 1, \dots, k,$ split the set $\{1, \dots, n\}$ into k large p −blocks and small q −blocks, to be denoted by I_m and J_m, respectively as follows

$$\begin{aligned} I_m &= \{i; i = (m-1)(p+q)+1, \dots, (m-1)(p+q)+p\} \\ J_m &= \{j; j = (m-1)(p+q)+p+1, \dots, m(p+q)\} \end{aligned} \tag{4.4}$$

where $p, q,$ and k are given by assumption **A8.** Set

$$K_{ni}(x) = K\left(\frac{x-X_i}{h}\right), \qquad Z_{ni} = \frac{N_i(t,x)}{(nh^2 l^2(x))^{\frac{1}{2}}},$$

and

$$S_n = (nh^2)^{\frac{1}{2}} J_n(t,x) = \sum_{i=1}^{n} Z_{ni}(t,x).$$

For $m = 1, \dots, k$, set $y_{nm}, y'_{nm}, y''_{nm}$ as follows

$$y_{nm} = \sum_{i=(m-1)(p+q)+1}^{(m-1)(p+q)+p} Z_{ni}, \quad y'_{nm} = \sum_{j=(m-1)(p+q)+p+1}^{m(p+q)} Z_{nj}, \quad y''_{nk} = \sum_{l=k(p+q)+1}^{n} Z_{nl}. \tag{4.5}$$

Also set

$$T_n = \sum_{m=1}^{k} y_{nm}, \quad T'_n = \sum_{m=1}^{k} y'_{nm}, \; T''_n = y''_{nk} \tag{4.6}$$

Remark that $S_n = T_n + T'_n + T''_n$.

To prove the asymptotic normality of $(nh^2)^{\frac{1}{2}} J_n(t,x)$, it suffices to establish that

$$T_n \to N(0, \sigma^2(t,x)) \text{as } n \to \infty, \tag{4.7}$$

and

$$\mathbb{E}(T'_n)^2 + \mathbb{E}(T''_n)^2 \to 0 \text{ as } n \to \infty. \tag{4.8}$$

Now we have to establish (4.7), this is done in two steps. First, it will be shown that the characteristic function of T_n minus the product of the characteristic function of $Z_{nj}, j = 1, \dots, k$ converges to 0. Therefore, its suffices to prove that

$$\left|\mathbb{E}\left(e^{it\sum_{m=1}^{k} y_{nm}}\right) - \prod_{m=1}^{k} \mathbb{E}(e^{it y_{nm}})\right| \to 0. \tag{4.9}$$

This proves that the rv's Z_{nj} are asymptotical independent.

Secondly,it is proved that the distribution determined by the product of the characteristic function of $\{Z_{nj} \; , j = 1, \dots, k\}$ is asymptotically, the distribution $N(0, \sigma^2(t,x))$.

Therefore we have to verify the standard Lindeberg-Feller condition

$$kVar(y_{n1}) \to \sigma^2(t,x), \quad k\mathbb{E}\big(y_{n1}^2 \mathbb{I}_{\{|y_{n1}|>\varepsilon\sigma(t,x)\}}\big) \to 0 \tag{4.10}$$

Firstly, in the two next Lemmas we establish that $kVar(y_{n1}) \to \sigma^2(t,x)$, and $Var(T_n) \to \sigma^2(t,x)$.

Lemma 4.6. Under assumptions of Lemma 4.5 and **A8**, the following hold

i) $kVar(y'_{n1}) \to 0,$

ii) $\left|cov(y'_{n1}, y'_{n,l+1})\right| \leq \frac{B'^2 qM^2}{nl^2(x)\bar{G}^2(\tau)} \frac{1}{h^4} \sum_{r=l(p+q)-(q-1)}^{l(p+q)+(q-1)} |cov(X_1, X_{r+1})|,$

iii) $\sum_{1 \leq i < j \leq k} \left|cov(y'_{ni}, y'_{nj})\right| \to 0.$

Proof .i) We have

$$\begin{aligned}
kVar(y'_{n1}) &= kVar\left(\sum_{i=p+1}^{p+q} Z_{ni}(t,x)\right) \\
&= kqVar\big(Z_{n1}(t,x)\big) + 2k \sum_{1 \leq i<j \leq q} \left|cov\left(Z_{ni}(t,x), Z_{nj}(t,x)\right)\right| \\
&= kqVar\left(\frac{1}{(nh^2 l^2(x))^{\frac{1}{2}}}\left[\begin{array}{c} \left[K\left(\frac{x-X_1}{h}\right)H'\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\bar{G}(Y_1)}\right] \\ -\mathbb{E}\left[K\left(\frac{x-X_1}{h}\right)H'\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\bar{G}(Y_1)}\right]\end{array}\right]\right) \\
&\quad +2k \sum_{1 \leq i<j \leq q} \left|cov\left(Z_{ni}(t,x), Z_{nj}(t,x)\right)\right| \\
&= \frac{kq}{nh^2 l^2(x)} Var\left[K\left(\frac{x-X_1}{h}\right)H'\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\bar{G}(Y_1)}\right] \\
&\quad + \frac{2k}{nh^2 l^2(x)} \\
&\quad * \sum_{1 \leq i<j \leq q} \left|cov\left(K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)}, K\left(\frac{x-X_j}{h}\right)H'\left(\frac{t-Y_j}{h}\right)\frac{\delta_j}{\bar{G}(Y_j)}\right)\right|.
\end{aligned}$$

The result follows from Lemma 4.5 and assumption **A8(ii).**

ii) We have

$$\begin{aligned}
\left|cov(y'_{n1}, y'_{n,l+1})\right| &= \left|\sum_{i=p+1}^{p+q} \sum_{j=l(p+q)+p+1}^{(l+1)(p+q)} cov(Z_{ni}, Z_{nj})\right| \\
&= \left|\sum_{r=1}^{q}(q-r+1)cov(Z_{n1}, Z_{n,l(p+q)+r}) + \sum_{r=1}^{q}(q-r)(Z_{n,r+1}, Z_{n,l(p+q)+1})\right| \\
&\leq q \sum_{r=l(p+q)-(q-1)}^{l(p+q)+(q-1)} \left|cov(Z_{n1}, Z_{n,r+1})\right| \\
&\leq \frac{qM^2}{nh^2 l^2(x)\bar{G}^2(\tau)} \sum_{r=l(p+q)-(q-1)}^{l(p+q)+(q-1)} \left|cov(K_{n1}, K_{n,r+1})\right|.
\end{aligned} \tag{4.11}$$

Lemma 1 in Bulinski (1996) gives

$$\left|cov(K_{n1}, K_{n,r+1})\right| \leq \frac{B'^2}{h^2} |cov(X_1, X_{r+1})|.$$

Therefore, (4.11) provides the desired result.

iii) By stationarity and **A8(iv)**, we have

$$
\begin{aligned}
\sum_{1\le i<j\le k}\left|cov(y'_{ni,}y'_{nj})\right| &= \sum_{l=1}^{k-1}(k-l)\left|cov(y'_{n1,}y'_{n,l+1})\right| \\
&\le k\sum_{l=1}^{k-1}\left|cov(y'_{n1,}y'_{n,l+1})\right| \\
&\le \frac{B'^2M^2}{\bar{G}^2(\tau)}\frac{qk}{nh^4l^2(x)}\sum_{l=1}^{k-1}\sum_{r=l(p+q)-(q-1)}^{l(p+q)+(q-1)}|cov(X_1,X_{r+1})| \\
&\le \frac{B'^2M^2}{l^2(x)\bar{G}^2(\tau)}\frac{qk}{n}\frac{1}{h^4}\sum_{r=p}^{\infty}|cov(X_1,X_{r+1})| \to 0.
\end{aligned}
$$

■

Lemma 4.7. Under the assumptions of the previous Lemma, we have

i) $kVar(y_{n1}) \to \sigma^2(t,x)$

ii) $\sum_{1\le i<j\le k}\left|cov(y_{ni},y_{nj})\right| \to 0$

iii) $Var(T_n) \to \sigma^2(t,x)$

Proof.

$$
\begin{aligned}
kVar(y_{n1}) \;=\; & k\frac{p}{nh^2l^2(x)}Var\left(K\left(\frac{x-X_1}{h}\right)H'\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\bar{G}(Y_1)}\right) \\
&+\frac{2k}{nh^2l^2(x)} \\
&*\sum_{1\le i<j\le p}\left|cov\left(K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)},K\left(\frac{x-X_j}{h}\right)H'\left(\frac{t-Y_j}{h}\right)\frac{\delta_j}{\bar{G}(Y_j)}\right)\right|.
\end{aligned}
$$

Thus we get i) by lemma 4.5 and assumption **A8(**i).
For ii) working as In Lemma 4.6(ii), we have

$$
\begin{aligned}
\left|cov(y_{n1},y_{n,l+1})\right| &= \left|\sum_{i=1}^{p}\sum_{j=l(p+q)-p}^{l(p+q)+p}cov\left(Z_{ni}(t,x),Z_{nj}(t,x)\right)\right| \\
&\le \frac{pM^2}{nh^2l^2(x)}\sum_{r=l(p+q)-p}^{l(p+q)+p}\left|cov\left(K_{n1},K_{n,r+1}\right)\right| \\
&\le \frac{B'^2pM^2}{nh^4l^2(x)\bar{G}^2(\tau)}\sum_{r=l(p+q)-p}^{l(p+q)+p}|cov(X_1,X_{r+1})|.
\end{aligned}
$$

Thus, by assumptions **A8** (i) and (iv**)**, one has

$$
\sum_{1\le i<j\le k}\left|cov(y_{ni}y_{nj})\right| \le \frac{B'^2M^2}{l^2(x)\bar{G}^2(\tau)}\frac{pk}{n}\frac{1}{h^4}\sum_{r=q}^{\infty}|cov(X_1,X_{r+1})| \to 0.
$$

$$
Var(T_n)=\sum_{m=1}^{k}y_{nm}=kVar(y_{n1})+2\sum_{1\le i<j\le k}\left|cov(y_{ni}y_{nj})\right|
$$

and this converges to $\sigma^2(t,x)$ by parts (i) and (ii).
Now, we prove (4.8). We have

$$
Var(T'_n)=kVar(y'_{n1})+2\sum_{1\le i<j\le k}\left|cov(y'_{ni,}y'_{nj})\right|,
$$

$$\begin{aligned} Var(T_n'') &= (n-k(p+q))VarZ_{n1}(t,x) + 2\sum_{k(p+q)+1\le i<j\le n}\left|cov\left(Z_{ni}(t,x),Z_{nj}(t,x)\right)\right| \\ &\le pVarZ_{n1}(t,x) + 2\sum_{1\le i<j\le p}\left|cov\left(Z_{ni}(t,x),Z_{nj}(t,x)\right)\right|. \end{aligned}$$

Hence (4.8) holds by Lemma 4.6.

■

On the other hand, let us show (4.9). We have

$$\begin{aligned} \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k}y_{nm}}\right)-\prod_{m=1}^{k}\mathbb{E}(e^{ity_{nm}})\right| &\le \left|cov(e^{it\sum_{m=1}^{k-1}y_{nm}},e^{ity_{nk}})\right| \\ &+ \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k-1}y_{nm}}\right)-\prod_{m=1}^{k-1}\mathbb{E}(e^{ity_{nm}})\right|. \end{aligned}$$

Therefore, by Lemma 1 in Bulinski (1996), we have

$$\begin{aligned} \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k}y_{nm}}\right)-\prod_{m=1}^{k}\mathbb{E}(e^{ity_{nm}})\right| &\le cB'^2M^2\frac{t^2}{nh^4l^2(x)}\left[\sum_{i\in I_1}\sum_{j\in I_2}|cov(X_i,X_j)|\right. \\ &+\sum_{i\in(I_1\cup I_2)}\sum_{j\in I_3}|cov(X_i,X_j)|+\cdots \\ &\left.+\sum_{i\in(I_1\cup\ldots\cup I_{k-1})}\sum_{j\in I_k}|cov(X_i,X_j)|\right]. \end{aligned} \tag{4.13}$$

By stationarity, the right-hand side of (4.13) is equal to

$$(k-1)\sum_{i\in I_1}\sum_{j\in I_2}|cov(X_i,X_j)|+(k-2)\sum_{i\in I_1}\sum_{j\in I_3}\left|cov\left(X_i,X_j\right)\right|+\cdots+\sum_{i\in I_1}\sum_{j\in I_k}\left|cov\left(X_i,X_j\right)\right|,$$

So inequality (4.13) becomes

$$\begin{aligned} \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k}y_{nm}}\right)-\prod_{m=1}^{k}\mathbb{E}(e^{ity_{nm}})\right| &\le cB'^2M^2\frac{t^2}{nh^4l^2(x)}\left[(k-1)\sum_{i\in I_1}\sum_{j\in I_2}|cov(X_i,X_j)|\right. \\ &+(k-2)\sum_{i\in I_1}\sum_{j\in I_3}\left|cov\left(X_i,X_j\right)\right| \\ &\left.+\cdots+\sum_{i\in I_1}\sum_{j\in I_k}\left|cov\left(X_i,X_j\right)\right|\right]. \end{aligned} \tag{4.14}$$

Once again, by stationarity, and by taking the means of these expressions, inequality (4.14) becomes

$$\begin{aligned} \left|\mathbb{E}\left(e^{it\sum_{m=1}^{k}y_{nm}}\right)-\prod_{m=1}^{k}\mathbb{E}(e^{ity_{nm}})\right| &\le cB'^2M^2\frac{t^2}{nh^4l^2(x)}pk\sum_{j=(p+q)+1}^{(k-1)(p+q)+p}\left|cov\left(X_1,X_j\right)\right| \\ &\le c\frac{B'^2M^2t^2}{l^2(x)}\frac{pk}{n}\frac{1}{h^4}\sum_{j=p}^{\infty}\left|cov\left(X_1,X_j\right)\right| \end{aligned}$$

which goes to zero by assumptions **A8**(i) and (iv).
Last, let establish (4.10). We have

$$
\begin{aligned}
|y_{n1}| &\leq \frac{1}{(nh^2l^2(x))^{\frac{1}{2}}}\sum_{i=1}^{p}\left|\left[\left[K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)}\right]-\mathbb{E}\left[K\left(\frac{x-X_i}{h}\right)H'\left(\frac{t-Y_i}{h}\right)\frac{\delta_i}{\bar{G}(Y_i)}\right]\right]\right| \\
&\leq \frac{MM^*}{(nh^2l^2(x))^{\frac{1}{2}}\bar{G}(\tau)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
k\mathbb{E}\left(y_{n1}^2\mathbb{I}_{\{|y_{n1}|>\varepsilon\sigma(t,x)\}}\right) &\leq \frac{k(MM^*p)^2}{\bar{G}^2(\tau)nh^2l^2(x)}P(|y_{n1}|>\varepsilon\sigma(t,x)) \\
&\leq \frac{M^2(M^*)^2}{\bar{G}^2(\tau)l^2(x)}\frac{kVar(y_{n1})}{\varepsilon^2\sigma^2(t,x)}\frac{p^2}{nh^2}.
\end{aligned}
$$

This concludes the proof of Theorem 3.1.

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