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The Balakrishnan-Alpha-Beta-Skew-Normal Distribution: Properties and Applications

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Abstract

In this paper, a new form of alpha-beta-skew distribution is proposed under Balakrishnan (2002) mechanism and investigated some of its related distributions. The most important feature of this new distribution is that it is versatile enough to support both unimodal and bimodal as well as multimodal behaviors of the distribution. The moments, distributional properties and some extensions of the proposed distribution have also been studied. Finally, the suitability of the proposed distribution has been tested by conducting data fitting experiment and comparing the values of Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) with the values of some other related distributions. Likelihood Ratio testis used for discriminating between normal and the proposed distributions.

Keywords: Skew-Normal Distribution, Alpha-Beta-Skew-Normal Distribution, Bimodal Distribution, Likelihood Ratio Test, AIC, BIC.

Mathematical Subject Classification: 60E05, 62E10

1. Introduction:

The application of skewed distributions arises in every area of the sciences, engineering and medicine because the data are likely to come from asymmetrical populations. One of the simple and common approaches for the construction of skewed distributions is to introduce the skewness into some known symmetrical distributions. Azzalini (1985) first introduced the skew normal distribution using a natural extension of symmetrical normal distribution and adding an additional parameter to introduce asymmetry. Its probability density function (pdf) is given by

$$f_Z(z;\lambda) = 2\varphi(z)\Phi(\lambda z);\quad -\infty < z < \infty \tag{1}$$

where $\varphi(.)$ and $\Phi(.)$ are, respectively, the pdf and cumulative distribution function (cdf) of standard normal variable Z and $\lambda \in R$, the skewness parameter. The family of distributions given by Equation (1) and the skew-normal class have been studied and extended by many authors (for detail see Chakraborty and Hazarika, 2011). Balakrishnan (2002) proposed the generalization of skew normal distribution (as a discussant in Arnold and Beaver, 2002) and studied its properties and the pdf of the same is given by

$$f_Z(z;\lambda,n) = \varphi(z)[\Phi(\lambda z)]^n \big/ C_n(\lambda);\ -\infty < z < \infty, \lambda \in R \tag{2}$$

Where n is a positive integer and $C_n(\lambda) = \mathrm{E}(\Phi^n(\lambda U))$, $U \sim N(0,1)$.

Huang and Chen (2007) proposed a general methodology for the construction of skew-symmetric distributions by implementing the concept of skew function $G(.)$ instead of cdf in Equation (1) where, $G(.)$ is a Lebesgue measurable function satisfying, $0 \le G(z) \le 1$ and $G(z)+G(-z)=1, \quad z \in R$, almost everywhere. The pdf of the same is given by

$$f(z) = 2\varphi(z)G(z);\; z \in R. \tag{3}$$

Obviously, by selecting different skew functions in Equation (3), one can develop a wide number of skewed distributions. Elal-Olivero (2010) introduced, using the approach of the Equation (3), a new form of skew distribution which has both unimodal as well as bimodal behavior and is known as alpha-skew-normal distribution, denoted by $ASN(\alpha)$ and its pdf is given by

$$f(z;\alpha) = \left(\frac{(1-\alpha z)^2+1}{2+\alpha^2}\right)\varphi(z);\; z \in R. \tag{4}$$

Shafiei et al. (2016) further extended the above family of skew distributions to provide more flexibility than the Azzalini (1985) and the Elal-Olivero (2010) distributions and called it the alpha-beta-skew-normal distribution, denoted by $ABSN(\alpha,\beta)$ with pdf given by

$$f(z;\alpha,\beta) = \left(\frac{(1-\alpha z-\beta z^3)^2+1}{2+\alpha^2+15\beta^2+6\alpha\beta}\right)\varphi(z);\; z,\alpha,\beta \in R. \tag{5}$$

Hazarika et al. (2020) introduced a new form of skew normal distribution known as Balakrishnan-alpha-skew-normal $BASN_2(\alpha)$ distribution and the pdf is given by

$$f(z;\alpha) = \frac{[(1-\alpha z)^2+1]^2}{C_2(\alpha)}\varphi(z),\; z \in R \tag{6}$$

where $C_2(\alpha) = 4+8\alpha^2+3\alpha^4$.

There are more skewed distributions related to Huang and Chen (2007) by considering different skew functions such as Harandi and Alamatsaz (2013), Hazarika and Chakraborty (2014), Chakraborty et al. (2012, 2014, and 2015), Shah et al. (2020a, 2020b, 2020c), Shah and Hazarika (2019) etc.

In the present article, the Balakrishnan (2002) and the Shafiei et al. (2016) methodology have been implemented to propose a new alpha-beta-skew-normal distribution, known as Balakrishnan-alpha-beta-skew-normal distribution which is versatile enough to support both unimodal and bimodal behavior and investigate some of its distributional properties. To exhibit the applicability of the proposed distribution, three real life datasets are considered which give better fit when compared to some other known distributions. The article is organized as follows. In Section 2, the Balakrishnan-alpha-beta-skew-normal distribution is defined and some of its important distributional properties are discussed. The random number generation of the proposed distribution is defined in Section 3. The location-scale extension and maximum likelihood estimation are given in Section 4. In Section 5, some numerical examples based on real life data and Likelihood ratio test are provided. Finally, the article ends with conclusions in Section 6.

2. A New Alpha-Beta-Skew-Normal Distribution

In this section, we define a new family of distribution known as Balakrishnan-alpha-beta-skew-normal (BABSN) distribution and discuss some of its distributional properties.

Definition 1: If a random variable Z has a pdf given by

$$f_Z(z;\alpha,\beta) = \frac{[(1-\alpha z-\beta z^3)^2+1]^2}{C_2(\alpha,\beta)}\varphi(z);\; z \in R \tag{7}$$

where $C_2(\alpha,\beta) = C_2(\alpha)+60\alpha^3\beta+12\alpha\beta(4+315\beta^2)+630\alpha^2\beta^2+15\beta^2(8+693\beta^2)$, $C_2(\alpha)$ is defined before and $\varphi(.)$ is the pdf of standard normal distribution, then it is said to be a Balakrishnan-alpha-beta-skew-normal distribution with skewness parameters $\alpha \in R$ and $\beta \in R$. In the rest of this article we shall refer the distribution in Equation (7) as $BABSN_2(\alpha,\beta)$.

Remark 1: The pdf of the proposed distribution is constructed using the formulae in Equation (2) and Equation (3), by taking $\Phi(.) = G(.) = \dfrac{(1-\alpha z-\beta z^3)^2+1}{2+\alpha^2+15\beta^2+6\alpha\beta}$ and $n=2$.

If $Z \sim BABSN_2(\alpha,\beta)$, the following properties are deduced immediately from the definition:

- If $\beta = 0$, then we get the $BASN_2(\alpha)$ distribution of Hazarika et al. (2020) and is given by $f(z;\alpha) = [(1-\alpha z)^2+1]^2\varphi(z)/C_2(\alpha)$.

- If $\alpha = 0$, then we get $f(z;\beta) = [(1-\beta z^3)^2 + 1]^2 \varphi(z)/[4 + 15\beta^2(8 + 693\beta^2)]$.
  This is a new distribution referred to as Balakrishnan-beta-skew-normal $BBSN_2(\beta)$ distribution.
- If $\alpha = \beta = 0$, then we get the standard normal $N(0,1)$ distribution and is given by $f(z) = \varphi(z)$.
- If $\alpha \to \pm\infty$, then we get the bimodal-normal $BN(4)$ distribution (see Hazarika et al. 2020) given by $f(z) = (z^4/3)\varphi(z)$.
- If $\beta \to \pm\infty$, then we get the bimodal-normal $BN(12)$ distribution (see Hazarika et al. 2020) given by $f(z) = (z^{12}/10395)\varphi(z)$.
- If $Z \sim BABSN_2(\alpha,\beta)$, then $-Z \sim BABSN_2(-\alpha,-\beta)$.

The pdf of $BABSN_2(\alpha,\beta)$ for different choices of the parameters α and β are plotted in Figure 1.It is observed from Figure 1 that the proposed distribution is very flexible to support unimodal, bimodal and multimodal behaviors and has at most four modes and is also proved in Proposition 2.1.The parameters of the proposed distribution have significant effects on the skewness and the probable number of modes.

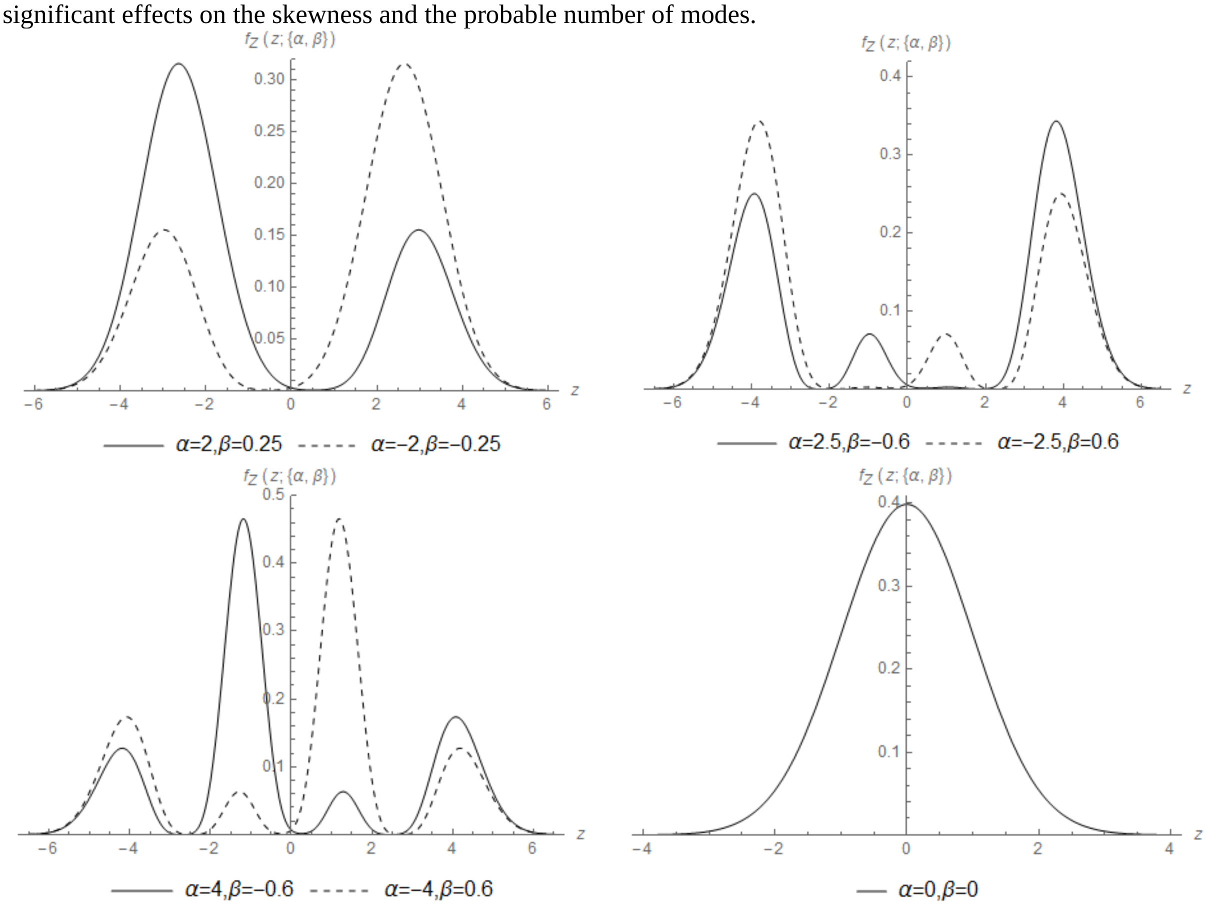

Figure 1: Plots of the pdf of $BABSN_2(\alpha,\beta)$.

Proposition2.1: The pdf of $BABSN_2(\alpha,\beta)$ distribution has at most four modes.

Proof2.1: See Appendix A.

Proposition2.2: The cdf of $BABSN_2(\alpha,\beta)$ distribution can be written as

$$F_Z(z;\alpha,\beta) = \Phi(z) + \frac{\varphi(z)}{2C_2(\alpha,\beta)}\begin{pmatrix} 2\alpha(8+\alpha(8\alpha - z(8-4\alpha z+(3+z^2)\alpha^2))) - 8(-2(2+z^2)+ \\ 4\alpha z(3+z^2) - 3\alpha^2 b_1 + \alpha^3 z b_2)\beta - 4(4zb_2 - 6\alpha b_3 + 3\alpha^2 z b_4)\beta^2 - \\ 8(-384 + z(-z(192+48z^2+8z^4+z^6)+\alpha b_5))\beta^3 - 2z\beta^4 b_6 \end{pmatrix} \tag{8}$$

where $b_1=(8+4z^2+z^4)$, $b_2=(15+5z^2+z^4)$, $b_3=(48+24z^2+6z^4+z^6)$, $b_4=(105+35z^2+7z^4+z^6)$, $b_5=(945+315z^2+63z^4+9z^6+z^8)$, and $b_6=(10395+3465z^2+693z^4+99z^6+11z^8+z^{10})$, $\varphi(.)$ and $\Phi(.)$ are as defined before.

Proof2.2: See Appendix B.

The cdf is plotted in Figure 2 for studying variation in its shape with respect to the parameters α and β.

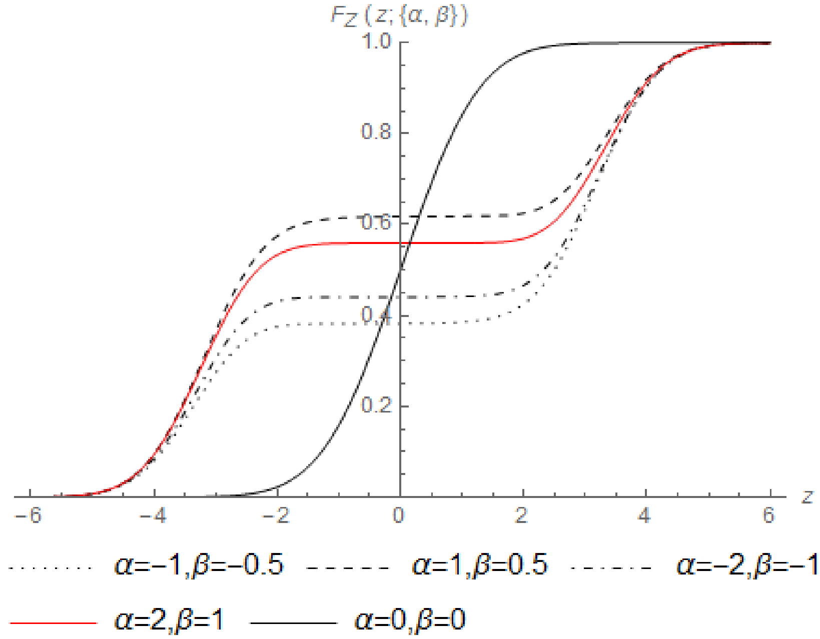

Figure 2: Plots of the cdf of $BABSN_2(\alpha,\beta)$.

Corollary 1: In particular, by taking the limit $\alpha\to\pm\infty$ of $F_Z(z;\alpha,\beta)$ in Equation (8), we get the cdf of $BN(4)$ distribution as $F_Z(z)=\Phi(z)-\frac{z(3+z^2)}{3}\varphi(z)$. Again, in particular, by taking the limit $\beta\to\pm\infty$ of $F_Z(z;\alpha,\beta)$ in Equation (8), we get the cdf of $BN(12)$ distribution as $F_Z(z)=\Phi(z)-\frac{z\,b_6}{10395}\varphi(z)$ where b_6 is defined above.

Proposition2.3: The moment generating function (mgf) of $BABSN_2(\alpha,\beta)$ distribution can be written as

$$M_Z(t)=\frac{M_X(t)}{C_2(\alpha,\beta)}\begin{pmatrix}4+\alpha(-8t+8\alpha(1+t^2)-4t\alpha^2(3+t^2)+\alpha^3c_1)+4(-2t(3+t^2)+4\alpha\,c_1-\\ 3t\alpha^2c_2+\alpha^3c_3)\beta+2(60+4t^2c_4-6\alpha\,t\,c_5+3\alpha^2c_6)\beta^2+4(-t\,c_7+\\ \alpha\,c_8)\beta^3+c_8\,\beta^4\end{pmatrix} \tag{9}$$

where $M_X(t)$ is the mgf of $X\sim N(0,1)$, $c_1=(3+6t^2+t^4)$, $c_2=(15+10t^2+t^4)$, $c_3=(15+45t^2+15t^4+t^6)$, $c_4=(45+15t^2+t^4)$, $c_5=(105+105t^2+21t^4+t^6)$, $c_6=(105+420t^2+210t^4+28t^6+t^8)$, $c_7=(945+1260t^2+378t^4+36t^6+t^8)$, $c_8=(945+4725t^2+3150t^4+630t^6+45t^8+t^{10})$ and $c_9=(10395+62370\,t^2+51975\,t^4+13860\,t^6+1485\,t^8+66t^{10}+t^{12})$.

Proof 2.3: See Appendix C.

Corollary 2: In particular, by taking the limit $\alpha\to\pm\infty$ of $M_Z(t)$ in Equation (9), we get the mgf of $BN(4)$ distribution as $M_Z(t)=\frac{c_1}{3}M_X(t)$ where c_1 is defined above. Again, in particular, by taking the limit $\beta\to\pm\infty$ of $M_Z(t)$ in Equation (9), we get the mgf of $BN(12)$ distribution as $M_Z(t)=\frac{c_9}{10395}M_X(t)$ where c_9 is defined above.

Proposition2.4: The n^{th} order moment of $BABSN_2(\alpha,\beta)$ distribution can be written as

$$E(Z^{2n}) = \frac{\begin{bmatrix} 4E_N(Z^{2n}) + 8\alpha^2 E_N(Z^{2n+2}) + \alpha(\alpha^3+16\beta)E_N(Z^{2n+4}) + 4\beta(\alpha^3+2\beta) \\ E_N(Z^{2n+6}) + 6\alpha^2\beta^2 E_N(Z^{2n+8}) + 4\alpha\beta^3 E_N(Z^{2n+10}) + \beta^4 E_N(Z^{2n+12}) \end{bmatrix}}{C_2(\alpha,\beta)}$$

and

$$E(Z^{2n-1}) = \frac{\begin{bmatrix} -8\alpha E_N(Z^{2n}) - 4(\alpha^3+2\beta)E_N(Z^{2n+2}) - 12\alpha^2\beta E_N(Z^{2n+4}) \\ -12\alpha\beta^2 E_N(Z^{2n+6}) - 4\beta^3 E_N(Z^{2n+8}) \end{bmatrix}}{C_2(\alpha,\beta)}$$

where $E_N(Z^n) = \frac{2^{(-1+\frac{n}{2})}[1+(-1)^n]\Gamma(\frac{n+1}{2})}{\sqrt{\pi}}, n > 0$ is the moment of standard normal distribution.

Proof2.4: See Appendix D.

From the above equation of n^{th} order moment, we get

$$E(Z) = \frac{-4[3\alpha^3 + 6\beta + 45\alpha^2\beta + 945\beta^3 + \alpha(2+315\beta^2)]}{C_2(\alpha,\beta)}$$

$$E(Z^2) = \frac{4 + 15\alpha^4 + 420\alpha^3\beta + 840\beta^2 + 135135\beta^4 + 60\alpha\beta(4+693\beta^2) + 6\alpha^2(4+945\beta^2)}{C_2(\alpha,\beta)}$$

$$E(Z^3) = \frac{-12[5\alpha^3 + 105\alpha^2\beta + 5\beta(2+693\beta^2) + \alpha(2+945\beta^2)]}{C_2(\alpha,\beta)}$$

$$E(Z^4) = \frac{3[4 + 40\alpha^2 + 35\alpha^4 + 140\alpha\beta(4+9\alpha^2) + 630\beta^2(4+33\alpha^2) + 180180\alpha\beta^3 + 675675\beta^4]}{C_2(\alpha,\beta)}$$

$$Var(Z) = \frac{\begin{bmatrix} -16[3\alpha^3 + 6\beta + 45\alpha^2\beta + 945\beta^3 + \alpha(2+315\beta^2)]^2 + [4 + 3\alpha^4 + 60\alpha^3\beta + \\ 12\alpha\beta(4+315\beta^2) + \alpha^2(8+630\beta^2) + 15\beta^2(8+693\beta^2)][4 + 15\alpha^4 + 420\alpha^3\beta + \\ 840\beta^2 + 135135\beta^4 + 60\alpha\beta(4+693\beta^2) + 6\alpha^2(4+945\beta^2)] \end{bmatrix}}{[C_2(\alpha,\beta)]^2}$$

By numerically optimizing $E(Z)$ and $Var(Z)$ with respect to α and β, we get the following bounds for mean and variance as $-1.52269 \le E(Z) \le 1.52269$ and $0.605494 \le Var(Z) \le 16.1254$. The plots of the mean and the variance are given respectively, in Figure 3 and Figure 4 to study their behaviors. These plots also verify these bounds.

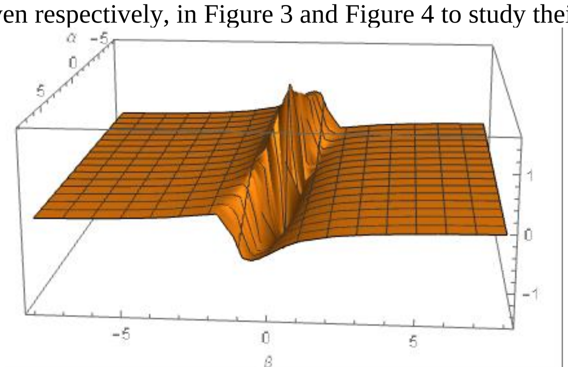

Figure 3: Plots of mean

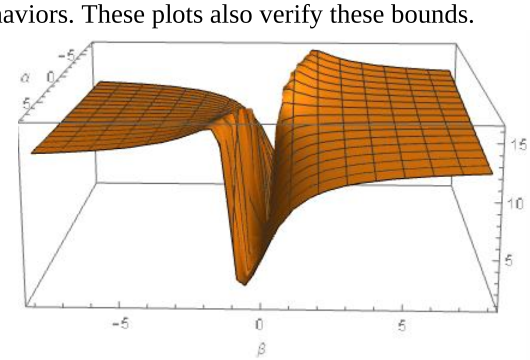

Figure 4: Plots of variance

Remark 2: By taking the limit $\alpha \to \pm\infty$ in the moments of $BABSN_2(\alpha,\beta)$ distribution, we can derive the moments of $BN(4)$ distribution as $E(Z) = 0,\ Var(Z) = 5$ .Again, by taking the limit $\beta \to \pm\infty$ in the moments of $BABSN_2(\alpha,\beta)$ distribution, we can derive the moments of $BN(12)$ distribution as $E(Z) = 0,\ Var(Z) = 13$.

2.1. Skewness and Kurtosis

The skewness (β_1) and kurtosis (β_2) of $BABSN_2(\alpha,\beta)$ distribution are respectively, given by

$$\beta_1 = \frac{16(32d_1^3 + 3C_2(\alpha,\beta)^2 d_2 - 3C_2(\alpha,\beta)d_1 d_3)^2}{(-16d_1^2 + C_2(\alpha,\beta)d_3)^3}$$

where $d_1 = 3\alpha^3 + 6\beta + 45\alpha^2\beta + 945\beta^3 + \alpha(2 + 315\beta^2)$, $d_2 = 5\alpha^3 + 105\alpha^2\beta + 5\beta(2 + 693\beta^2) + \alpha(2 + 945\beta^2)$, and $d_3 = 4 + 15\alpha^4 + 420\alpha^3\beta + 840\beta^2 + 135135\beta^4 + 60\alpha\beta(4 + 693\beta^2) + 6\alpha^2(4 + 945\beta^2)$.

In particular, when $\alpha = \beta = 1$, the value of $\beta_1 = 0.0470217$ and when $\alpha = \beta = 0$, then $\beta_1 = 0$ which is the skewness of standard normal distribution and the distribution has symmetric normal curve.

Again,

$$\beta_2 = \frac{3(C_2(\alpha,\beta)^3 d_4 - 256d_1^4 - 64C_2(\alpha,\beta)^2 d_1 d_2 + 32C_2(\alpha,\beta)d_1^2 d_3)}{(-16d_1^2 + C_2(\alpha,\beta)d_3)^2}$$

where $d_4 = 4 + 40\alpha^2 + 35\alpha^4 + 140\alpha\beta(4 + 9\alpha^2) + 630\beta^2(4 + 33\alpha^2) + 180180\alpha\beta^3 + 675675\beta^4$.

In particular, when $\alpha = \beta = 1$, the value of $\beta_2 = 1.22609$ and when $\alpha = \beta = 0$, then $\beta_2 = 3$ which is the kurtosis of standard normal distribution and the distribution has mesokurtic curve. The numerical optimization of β_1 and β_2 with respect to α and β, gives the following bounds for skewness and kurtosis as $0 \le \beta_1 \le 6.27667$ and $1.10797 \le \beta_2 \le 16.237$. The plots of the skewness and kurtosis are respectively given in Figure 5 and Figure 6 to study their behaviors. These plots also verify these bounds.

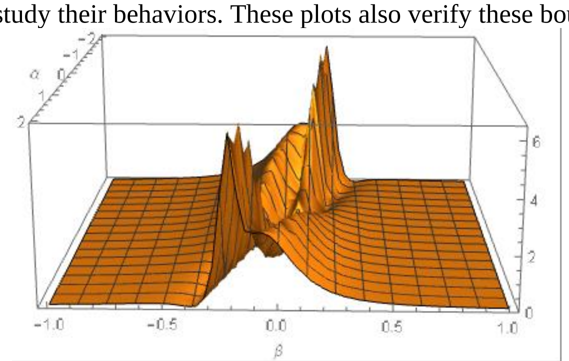

Figure 5: Plots of skewness

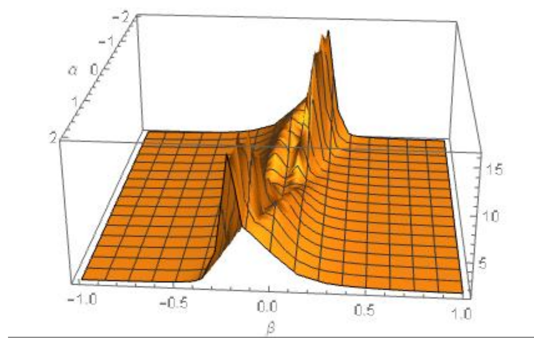

Figure 6: Plots of kurtosis

Remark 3: By taking the limit $\alpha \to \pm\infty$ in the results of $BABSN_2(\alpha,\beta)$ distribution, we can derive the skewness and kurtosis of $BN(4)$ distribution as $\beta_1 = 0,\ \beta_2 = 1.4$. Again, by taking the limit $\beta \to \pm\infty$ in the results of $BABSN_2(\alpha,\beta)$ distribution, we can derive the skewness and kurtosis of $BN(12)$ distribution as $\beta_1 = 0,\ \beta_2 = 1.15385$.

3. Random Number Generation

In this section, first we represent $BABSN_2(\alpha,\beta)$ distribution into symmetric and asymmetric form and then generate a random number from $BABSN_2(\alpha,\beta)$ distribution.

Lemma1: The density function in Equation (7) of model $BABSN_2(\alpha,\beta)$ can be represented as sum of two functions

$$f_Z(z;\alpha,\beta) = \frac{4 + z^2(\alpha+\beta z^2)^2(8 + z^2(\alpha+\beta z^2)^2)}{C_2(\alpha,\beta)}\varphi(z) + \frac{-4z(\alpha+\beta z^2)(2 + z^2(\alpha+\beta z^2)^2)}{C_2(\alpha,\beta)}\varphi(z) \quad (10)$$

In Equation (10), the first term is symmetric and the second term is asymmetric and the symmetric part, which is defined below, is symbolically denoted by $SCBABSN_2(\alpha,\beta)$. For $\alpha = \beta = 0$, $Z \sim N(0,1) = \varphi(z)$.

Remark 4: To generate data from $BABSN_2(\alpha,\beta)$ distribution, we use the acceptance-rejection algorithm which was introduced by Von Neumann (1951) as follows:

Let $f(z)$ be the density function of $Z \sim BABSN_2(\alpha,\beta)$ and $f_1(z)$ the density function of $S \sim SCBABSN_2(\alpha,\beta)$, with

$$M = Sup\left[\frac{f(x)}{f_1(x)}\right] = \frac{1}{3}(3 + 2\sqrt{2}).$$

To generate a random variable $Z \sim BABSN_2(\alpha,\beta)$, we shall carry out the following steps:

a) *Generate a random variable* $S \sim SCBABSN_2(\alpha,\beta)$.

b) *Generate* $U \sim Uniform(0,1)$ *independently from* S.

c) *If* $U < \frac{1}{M}\frac{f(S)}{f_1(S)} = \frac{3[(1-\alpha S-\beta S^3)^2+1]^2}{(3+2\sqrt{2})[4+S^2(\alpha+S^2\beta)^2(8+S^2(\alpha+S^2\beta)^2)]}$, *set* $Z = S$ *accept; otherwise, go back to step one (reject).*

By the acceptance-rejection method, any choice of this random variable will be accepted with probability $\frac{1}{M}$, i.e., $P\left(U < \frac{1}{M}\frac{f(S)}{f_1(S)}\right) = \frac{1}{M} = \frac{3}{(3+2\sqrt{2})}$. Thus, since the number of trials is geometric with $p = \frac{1}{M}$, the expected value for this number is $M = \frac{(3+2\sqrt{2})}{3} = 1.9428$.

4. Parameter Estimation of $BABSN_2(\alpha,\beta)$ Distribution

Here, we present the problem of parameter estimation of a location and scale extension of $BABSN_2(\alpha,\beta)$ distribution. If $Z \sim BABSN_2(\alpha,\beta)$ then $Y = \mu+\sigma Z$ is said to be the location $(\mu \in R)$ and scale $(\sigma > 0)$ extension of Z and has the pdf given by

$$f_Y(y;\mu,\sigma,\alpha,\beta) = \frac{1}{C_2(\alpha,\beta)}\left[\left(1-\alpha\left(\frac{y-\mu}{\sigma}\right)-\beta\left(\frac{y-\mu}{\sigma}\right)^3\right)^2+1\right]^2\varphi\left(\frac{y-\mu}{\sigma}\right) \qquad (11)$$

where $y \in R, \alpha \in R$, $\beta \in R$ and $C_2(\alpha,\beta)$ is as defined before and the distribution of Y is denoted by $Y \sim BABSN_2(\mu,\sigma,\alpha,\beta)$.

4.1. Maximum Likelihood Estimation

The log-likelihood function of the random sample $y_1, y_2, ..., y_n$ from $Y \sim BABSN_2(\mu,\sigma,\alpha,\beta)$ distribution for the parameters $\theta = (\mu,\sigma,\alpha,\beta)$ is given by

$$\begin{aligned} l(\theta) = {} & 2\sum_{i=1}^{n}\log\left[\left\{1-\alpha\left(\frac{y_i-\mu}{\sigma}\right)-\beta\left(\frac{y_i-\mu}{\sigma}\right)^3\right\}^2+1\right]-n\log C_2(\alpha,\beta)-n\log(\sigma) \\ & -\frac{n}{2}\log(2\pi)-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{y_i-\mu}{\sigma}\right)^2. \end{aligned} \qquad (12)$$

By differentiating Equation (12) partially with respect to the parameters $\theta = (\mu,\sigma,\alpha,\beta)$, we get the following likelihood equations as follows:

$$\frac{\partial l(\theta)}{\partial\mu} = -\sum_{i=1}^{n} -\frac{(y_i-\mu)}{\sigma^2} + 2\sum_{i=1}^{n}\frac{2b_i}{(1+b_i{}^2)}\left(\frac{\alpha}{\sigma}+\frac{3\beta(y_i-\mu)^2}{\sigma^3}\right)$$

$$\frac{\partial l(\theta)}{\partial\sigma} = -\frac{n}{\sigma} - \sum_{i=1}^{n} -\frac{(y_i-\mu)^2}{\sigma^3} + 2\sum_{i=1}^{n}\frac{2b_i}{(1+b_i{}^2)}\left(\frac{\alpha(y_i-\mu)}{\sigma^2}+\frac{3\beta(y_i-\mu)^3}{\sigma^4}\right)$$

$$\frac{\partial l(\theta)}{\partial\alpha} = -\frac{n[4(4\alpha+3\alpha^3+12\beta+45\alpha^2\beta+315\alpha\beta^2+945\beta^3)]}{C_2(\alpha,\beta)} + 2\sum_{i=1}^{n} -\frac{2(y_i-\mu)b_i}{\sigma(1+b_i{}^2)}$$

$$\frac{\partial l(\theta)}{\partial\beta} = -\frac{n[12(5\alpha^3+105\alpha^2\beta+5\beta(4+693\beta^2)+\alpha(4+945\beta^2))]}{C_2(\alpha,\beta)} + 2\sum_{i=1}^{n} -\frac{2(y_i-\mu)^3 b_i}{\sigma^3(1+b_i{}^2)}$$

where $b_i = \left(1-\frac{\alpha(y_i-\mu)}{\sigma}-\frac{\beta(y_i-\mu)^3}{\sigma^3}\right)$.

Now, the solutions of the above system of likelihood equations by numerical maximization of the Equation (12) with respect to the parameters $\theta = (\mu,\sigma,\alpha,\beta)$ gives the maximum likelihood estimates of the parameters $\theta = (\mu,\sigma,\alpha,\beta)$.

5. Real life applications: comparative data fitting

Here we have considered three datasets, first (Dataset 1) is related to N latitude degrees in 69 samples from world lakes, which appear in Column 5 of the Diversity data set in website: http://users.stat.umn.edu/sandy/courses/8061/datasets/lakes.lsp, second (Dataset 2) is the exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003. The data obtained from the website http://www.globalfindata.com and third (Dataset 3) consists of the velocities of 82 distant galaxies, diverging from our own galaxy. The data set is available at http://www.stats.bris.ac.uk/~peter/mixdata.The summary statistics of the datasets are given in Table 1 below.

Table1: Summary Statistic for the Datasets.

Datasets	Min.	Median	Mean	Max.	SD	Skewness	Kurtosis
1	28	43	45.165	74.7	9.619	1.662	5.598
2	1.158	4.753	4.117	11.091	1.384	0.026	5.282
3	9.172	20.834	20.831	34.279	4.568	-0.431	5.259

We then compared the proposed $BABSN_2(\mu,\sigma,\alpha,\beta)$ distribution with the normal distribution $N(\mu,\sigma^2)$, the logistic distribution $LG(\mu,\sigma)$, the Laplace distribution $La(\mu,\sigma)$, the skew-normal distribution $SN(\mu,\sigma,\lambda)$ of Azzalini (1985), the skew-logistic distribution $SLG(\mu,\sigma,\lambda)$ of Wahed and Ali (2001), the skew-Laplace distribution $SLa(\mu,\sigma,\lambda)$ of Aryal and Nadarajah (2005), the alpha-skew-normal distribution $ASN(\mu,\sigma,\alpha)$ of Elal-Olivero (2010), the alpha-skew-Laplace distribution $ASLa(\mu,\sigma,\alpha)$ of Harandi and Alamatsaz (2013), the alpha-skew-logistic distribution $ASLG(\mu,\sigma,\alpha)$ of Hazarika and Chakraborty (2014), the alpha-beta-skew-normal distribution $ABSN(\mu,\sigma,\alpha,\beta)$ and the beta-skew-normal distribution $BSN(\mu,\sigma,\beta)$ of Shafiei et al. (2016), and the Balakrishnan-beta-skew-normal distribution $BBSN(\mu,\sigma,\beta)$.

Using R software package (See GenSA package version-1.0.3, Xiang et al. 2013), the MLE of the parameters are obtained by using numerical optimization routine. AIC and BIC are used for comparison of the models. Tables 2, 3 and 4 give the MLE's, log-likelihood, AIC and BIC of the above mentioned distributions. The graphical representations of the results taking only the top three competitors for the proposed model are given in Figures7, 8 and 9.

Table 2: MLE's, log-likelihood, AIC and BIC for N latitude degrees in 69 samples from world lakes.

Parameters → / Distributions ↓	μ	σ	λ	α	β	$\log L$	AIC	BIC
$N(\mu,\sigma^2)$	45.165	9.549	--	--	--	-253.599	511.198	515.666
$LG(\mu,\sigma)$	43.639	4.493	--	--	--	-246.645	497.290	501.758
$SN(\mu,\sigma,\lambda)$	35.344	13.69	3.687	--	--	-243.036	492.072	498.774
$BSN(\mu,\sigma,\beta)$	54.47	5.52	--	--	0.74	-242.528	491.06	497.76
$SLG(\mu,\sigma,\lambda)$	36.787	6.417	2.828	--	--	-239.053	490.808	490.808
$La(\mu,\sigma)$	43.000	5.895	--	--	--	-239.248	482.496	486.964
$ASLG(\mu,\sigma,\alpha)$	49.087	3.449	--	0.861	--	-237.351	480.702	487.404
$SLa(\mu,\sigma,\lambda)$	42.300	5.943	0.255	--	--	-236.900	479.799	486.501
$ASLa(\mu,\sigma,\alpha)$	42.300	5.439	--	-0.220	--	-236.079	478.159	484.861
$ASN(\mu,\sigma,\alpha)$	52.147	7.714	--	2.042	--	-235.370	476.739	483.441
$BBSN_2(\mu,\sigma,\beta)$	55.586	4.461	--	--	0.202	-235.124	476.247	482.949
$ABSN(\mu,\sigma,\alpha,\beta)$	47.690	7.15	--	1.72	-0.37	-230.767	469.53	478.48
$BABSN_2(\mu,\sigma,\alpha,\beta)$	54.991	8.275	--	2.823	-0.157	**-225.938**	**459.877**	**468.813**

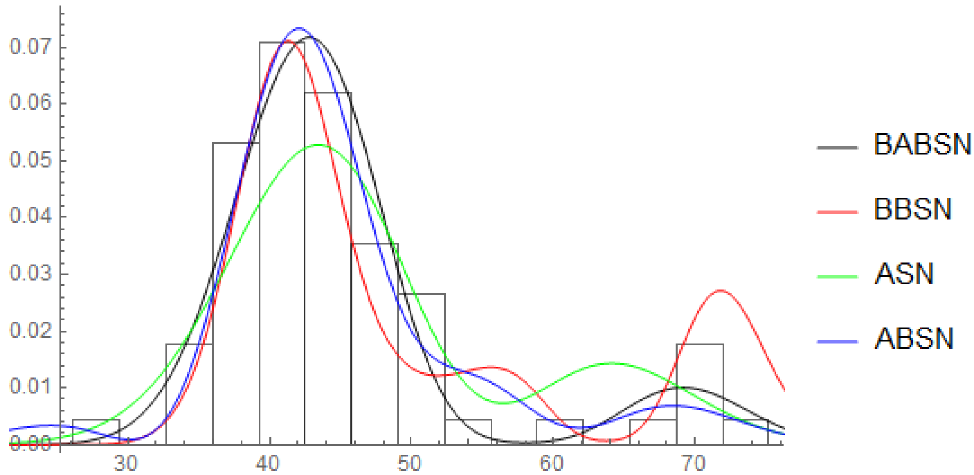

Figure 7: Plots of observed and expected densities for N latitude degrees in 69 samples from world lakes.

From Table 2, it is seen that the proposed Balakrishnan-alpha-beta-skew-normal $BABSN_2(\mu,\sigma,\alpha,\beta)$ distribution provides better fit to the data set under consideration in terms of all criteria, namely the log-likelihood, the AIC as well as the BIC. The plots of observed (in histogram) and expected (lines) densities presented in Figure 7, also confirm our findings.

Table 3: MLE's, log-likelihood, AIC and BIC for the exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003.

Parameters → / Distributions ↓	μ	σ	λ	α	β	$\log L$	AIC	BIC
$SN(\mu,\sigma,\lambda)$	3.589	1.478	0.501	--	--	-355.217	716.434	726.388
$N(\mu,\sigma^2)$	4.117	1.381	--	--	--	-355.265	714.529	721.165
$LG(\mu,\sigma)$	4.251	0.753	--	--	--	-351.192	706.385	713.021
$SLG(\mu,\sigma,\lambda)$	5.360	1.018	-2.371	--	--	-341.391	688.782	698.736
$La(\mu,\sigma)$	4.754	0.971	--	--	--	-339.315	682.630	689.265
$BSN(\mu,\sigma,\beta)$	4.526	0.974	--	--	0.181	-334.4876	674.975	684.929
$BBSN_2(\mu,\sigma,\beta)$	4.689	0.844	--	--	0.085	-325.8177	657.635	667.589
$ASN(\mu,\sigma,\alpha)$	3.656	0.883	--	-3.504	--	-317.946	641.892	651.847
$SLa(\mu,\sigma,\lambda)$	4.855	1.000	1.506	--	--	-311.318	628.636	638.590
$ABSN(\mu,\sigma,\alpha,\beta)$	3.497	1.075	--	-7.583	1.196	-301.3475	610.695	623.968
$ASLG(\mu,\sigma,\alpha)$	3.764	0.403	--	-2.025	--	-301.963	609.927	619.881
$ASLa(\mu,\sigma,\alpha)$	4.861	0.677	--	0.539	--	-301.443	608.885	618.840
$BABSN_2(\mu,\sigma,\alpha,\beta)$	4.352	0.639	--	-0.631	0.157	**-294.9735**	**597.947**	**611.219**

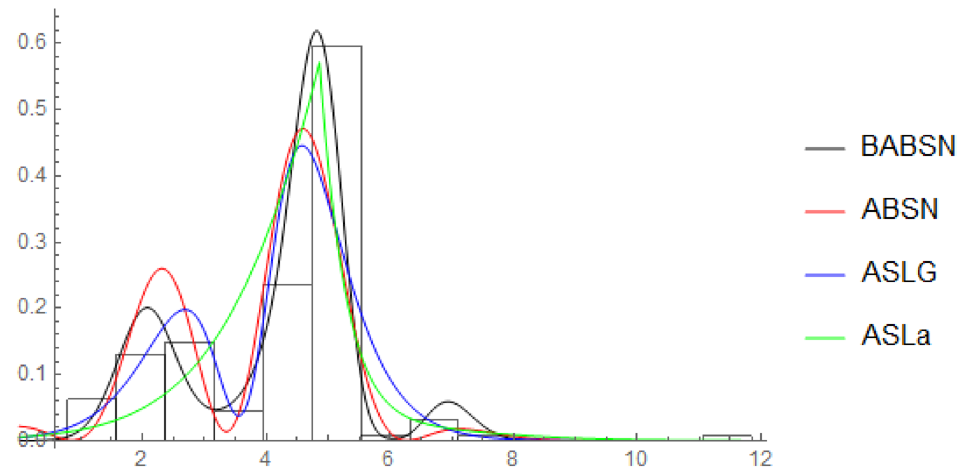

Figure 8: Plots of observed and expected densities for exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003.

From Table 3, it is observe that the proposed Balakrishnan-alpha-beta-skew-normal $BABSN_2(\mu,\sigma,\alpha,\beta)$ distribution provides better fit to the data set under consideration in terms of all criteria, namely the log-likelihood, the AIC as well as the BIC. The plots of observed (in histogram) and expected (lines) densities presented in Figure 8, also confirm our findings.

Table 4: MLE's, log-likelihood, AIC and BIC for the velocities of 82 distant galaxies, diverging from our own galaxy.

Parameters → / Distributions ↓	μ	σ	λ	α	β	$\log L$	AIC	BIC
$N(\mu,\sigma^2)$	20.832	4.540	--	--	--	-240.417	484.833	489.646
$SN(\mu,\sigma,\lambda)$	24.610	5.907	-1.395	--	--	-239.21	484.420	491.640
$SLG(\mu,\sigma,\lambda)$	21.532	2.219	-0.154	--	--	-233.314	472.628	479.849
$LG(\mu,\sigma)$	21.075	2.204	--	--	--	-233.649	471.299	476.113
$BSN(\mu,\sigma,\beta)$	20.596	3.260	--	--	-0.158	-232.220	470.440	477.660
$BBSN_2(\mu,\sigma,\beta)$	21.602	3.017	--	--	0.070	-230.666	467.332	474.552
$ASN(\mu,\sigma,\alpha)$	17.417	3.869	--	-1.656	--	-230.088	466.175	473.395
$SLa(\mu,\sigma,\lambda)$	20.846	2.997	1.002	--	--	-228.829	463.658	470.878
$La(\mu,\sigma)$	20.838	2.997	--	--	--	-228.830	461.660	466.474
$ASLG(\mu,\sigma,\alpha)$	18.482	1.646	--	-0.833	--	-224.877	455.754	462.974
$ABSN(\mu,\sigma,\alpha,\beta)$	19.448	3.462	--	-1.392	0.323	-220.055	448.109	457.736
$ASLa(\mu,\sigma,\alpha)$	19.473	1.805	--	-0.842	--	-220.793	447.586	454.806
$BABSN_2(\mu,\sigma,\alpha,\beta)$	15.933	4.137	--	-3.220	0.342	**-216.228**	**440.456**	**450.083**

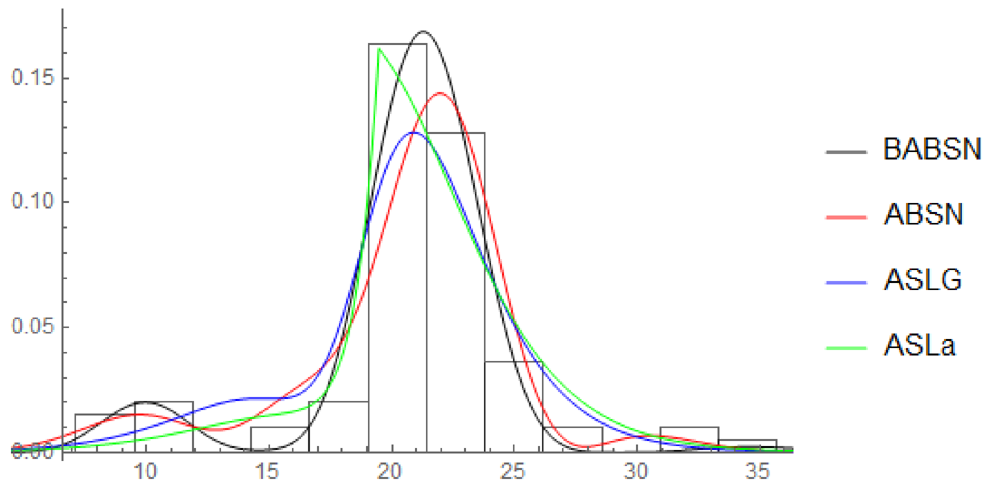

Figure 9: Plots of observed and expected densities for velocities of 82 distant galaxies, diverging from our own galaxy.

From Table 4, it is seen that the proposed Balakrishnan-alpha-beta-skew-normal $BABSN_2(\mu,\sigma,\alpha,\beta)$ distribution provides better fit to the data set under consideration in terms of all criteria, namely the log-likelihood, the AIC as well as the BIC. The plots of observed (in histogram) and expected (lines) densities presented in Figure 9, also confirm our findings.

5.1. Likelihood Ratio Test

Since $N(\mu,\sigma^2)$ and $BABSN_2(\mu,\sigma,\alpha,\beta)$ are nested models, the likelihood ratio (LR) test is used to discriminate between them. The LR test is carried out to test the following hypothesis: $H_0:\alpha=\beta=0$, that is the sample is drawn from $N(\mu,\sigma^2)$; against the alternative $H_1:\alpha\neq 0,\beta\neq 0$, that is the sample is drawn from $BABSN_2(\mu,\sigma,\alpha,\beta)$. The values of the LR test statistic for the above three datasets are given in Table 5.

Table 5: The values of the LR test statistic for different datasets

Datasets	Dataset 1	Dataset 2	Dataset 3	Degrees of Freedom	Critical value
LR test statistic	55.322	120.583	48.378	2	9.210

The values of LR test statistic for the Datasets 1, 2 and 3 are respectively, 55.322, 120.583 and 48.378 which exceed the critical value at 1% level of significance for two (2) degrees of freedom, i.e., 9.210. Thus there is evidence in favor of the alternative hypothesis that the sampled data comes from $BABSN_2(\mu,\sigma,\alpha,\beta)$, and not from $N(\mu,\sigma^2)$.

6. Conclusions and Future Scope

In this study a new alpha-beta-skew-normal distribution is constructed which includes unimodal, bimodal as well as multimodal shapes and some of its properties are studied. Our findings adequately supported the proposed $BABSN_2(\mu,\sigma,\alpha,\beta)$ distribution as the better fitted one to the datasets under consideration in terms of model selection criteria, namely AIC and BIC. The plots of observed and expected densities presented above also confirm our findings. Furthermore, there is scope of extending the present work by considering the Logistic and the Laplace distributions. Moreover, logarithmic forms and bivariate generalizations can also be considered as future work.

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Appendix

A: Derivation of Mode (Proof 2.1.)

Differentiating the Equation (7) with respect to z, we have

$$f_Z'(z;\alpha,\beta)=\frac{\partial f_Z(z;\alpha,\beta)}{\partial z}=\frac{\partial}{\partial z}\frac{[(1-\alpha z-\beta z^3)^2+1]^2}{C_2(\alpha,\beta)}\varphi(z)$$

$$=\frac{1}{C_2(\alpha,\beta)}\frac{\partial}{\partial z}[(1-\alpha z-\beta z^3)^2+1]^2\varphi(z)$$

$$=\frac{1}{C_2(\alpha,\beta)}\frac{\partial}{\partial z}\begin{bmatrix}4-8\alpha z+8\alpha^2z^2-4(\alpha^3-2\beta)z^3+\alpha(\alpha^3+16\beta)z^4-12\alpha^2\beta z^5+\\4\beta(\alpha^3+2\beta)z^6-12\alpha\beta^2z^7+6\alpha^2\beta^2z^8-4\beta^3z^9+4\alpha\beta^3z^{10}+\beta^4z^{12}\end{bmatrix}\varphi(z)$$

$$=\frac{1}{C_2(\alpha,\beta)}\Bigg[4\frac{\partial}{\partial z}\{\varphi(z)\}-8\alpha\frac{\partial}{\partial z}\{z\varphi(z)\}+8\alpha^2\frac{\partial}{\partial z}\{z^2\varphi(z)\}-4(\alpha^3-2\beta)\frac{\partial}{\partial z}\{z^3\varphi(z)\}+\alpha(\alpha^3+16\beta)$$
$$\frac{\partial}{\partial z}\{z^4\varphi(z)\}-12\alpha^2\beta\frac{\partial}{\partial z}\{z^5\varphi(y)\}+4\beta(\alpha^3+2\beta)\frac{\partial}{\partial z}\{z^6\varphi(z)\}-12\alpha\beta^2\frac{\partial}{\partial z}\{z^7\varphi(z)\}+6\alpha^2\beta^2$$
$$\frac{\partial}{\partial z}\{z^8\varphi(z)\}-4\beta^3\frac{\partial}{\partial z}\{z^9\varphi(z)\}+4\alpha\beta^3\frac{\partial}{\partial z}\{z^{10}\varphi(z)\}+\beta^4\frac{\partial}{\partial z}\{z^{12}\varphi(z)\}\Bigg]$$

(A1)

Now, we have

$\frac{\partial}{\partial z}\left\{\varphi(z)\right\}=-z\varphi(z)$, $\frac{\partial}{\partial z}\left\{z\varphi(z)\right\}=-(z^2-1)\varphi(z)$, $\frac{\partial}{\partial z}\left\{z^2\varphi(z)\right\}=-z(z^2-2)\varphi(z)$, $\frac{\partial}{\partial z}\left\{z^3\varphi(z)\right\}=-z^2(z^2-3)\varphi(z)$,
$\frac{\partial}{\partial z}\left\{z^4\varphi(z)\right\}=-z^3(z^2-4)\varphi(z)$, $\frac{\partial}{\partial z}\left\{z^5\varphi(z)\right\}=-z^4(z^2-5)\varphi(z)$, $\frac{\partial}{\partial z}\left\{z^6\varphi(z)\right\}=-z^5(z^2-6)\varphi(z)$,
$\frac{\partial}{\partial z}\left\{z^7\varphi(z)\right\}=-z^6(z^2-7)\varphi(z)$, $\frac{\partial}{\partial z}\left\{z^8\varphi(z)\right\}=-z^7(z^2-8)\varphi(z)$, $\frac{\partial}{\partial z}\left\{z^9\varphi(z)\right\}=-z^8(z^2-9)\varphi(z)$,
$\frac{\partial}{\partial z}\left\{z^{10}\varphi(z)\right\}=-z^9(z^2-10)\varphi(z)$, $\frac{\partial}{\partial z}\left\{z^{12}\varphi(z)\right\}=-z^{11}(z^2-12)\varphi(z)$.

Putting these values in the Equation (A1), we get

$$f_Z{}'(z;\alpha,\beta)=-\frac{\varphi(z)}{C_2(\alpha,\beta)}[(2-2\alpha z+\alpha^2z^2-2\beta z^3+2\alpha\beta z^4+\beta^2z^6)\{4\alpha+(2-4\alpha^2)z$$
$$-2(\alpha-6\beta)z^2+\alpha(\alpha-16\beta)z^3-2\beta z^4+2\beta(\alpha-6\beta)z^5+\beta^2z^7\}] \quad \text{(A2)}$$

Since the Equation (A2) has at most seven zeros, the function $f_Z(z;\alpha,\beta)$ can have at most four modes.

B: Derivation of cdf (Proof 2.2.)

$$F_Z(z)=P(Z\le z)=\int_{-\infty}^{z}\frac{[(1-\alpha z-\beta z^3)^2+1]^2}{C_2(\alpha,\beta)}\varphi(z)\,dz$$

$$=\frac{1}{C_2(\alpha,\beta)}\int_{-\infty}^{z}\left[\begin{array}{l}4-8\alpha z+8\alpha^2z^2-4(\alpha^3-2\beta)z^3+\alpha(\alpha^3+16\beta)z^4-12\alpha^2\beta z^5+\\4\beta(\alpha^3+2\beta)z^6-12\alpha\beta^2z^7+6\alpha^2\beta^2z^8-4\beta^3z^9+4\alpha\beta^3z^{10}+\beta^4z^{12}\end{array}\right]\varphi(z)\,dz$$

$$=\frac{1}{C_2(\alpha,\beta)}\Bigg[4\int_{-\infty}^{z}\varphi(z)\,dz-8\alpha\int_{-\infty}^{z}z\,\varphi(z)\,dz+8\alpha^2\int_{-\infty}^{z}z^2\,\varphi(z)\,dz-4\,(\alpha^3+2\beta)\int_{-\infty}^{z}z^3\,\varphi(z)\,dz+$$
$$\alpha(\alpha^3+16\beta)\int_{-\infty}^{z}z^4\,\varphi(z)\,dz-12\alpha^2\beta\int_{-\infty}^{z}z^5\,\varphi(z)\,dz+4\beta(\alpha^3+2\beta)\int_{-\infty}^{z}z^6\,\varphi(z)\,dz-12\alpha\beta^2\int_{-\infty}^{z}z^7\,\varphi(z)\,dz+ \quad \text{(B1)}$$
$$6\alpha^2\beta^2\int_{-\infty}^{z}z^8\,\varphi(z)\,dz-4\beta^3\int_{-\infty}^{z}z^9\,\varphi(z)\,dz+4\alpha\beta^3\int_{-\infty}^{z}z^{10}\,\varphi(z)\,dz+\beta^4\int_{-\infty}^{z}z^{12}\,\varphi(z)\,dz\Bigg]$$

Now, we have

$\int_{-\infty}^{z}\varphi(z)\,dz=\Phi(z)$, $\int_{-\infty}^{z}z\,\varphi(z)\,dz=-\varphi(z)$, $\int_{-\infty}^{z}z^2\varphi(z)\,dz=-z\,\varphi(z)+\Phi(z)$, $\int_{-\infty}^{z}z^3\varphi(z)\,dz=-(2+z^2)\varphi(z)$,

$\int_{-\infty}^{z}z^4\varphi(z)\,dz=-z(3+z^2)\varphi(z)+3\Phi(z)$, $\int_{-\infty}^{z}z^5\varphi(z)\,dz=-(8+4z^2+z^4)\varphi(z)$, $\int_{-\infty}^{z}z^6\varphi(z)\,dz=-z(15+5z^2+z^4)\varphi(z)+15\Phi(z)$

, $\int_{-\infty}^{z}z^7\varphi(z)\,dz=-(48+24z^2+6z^4+z^6)\varphi(z)$, $\int_{-\infty}^{z}z^8\varphi(z)\,dz=-z(105+35z^2+7z^4+z^6)\varphi(z)+105\Phi(z)$,

$\int_{-\infty}^{z}z^9\varphi(z)\,dz=-(384+192z^2+48z^4+8z^6+z^8)\varphi(z)$, $\int_{-\infty}^{z}z^{10}\varphi(z)\,dz=-z(945+315z^2+63z^4+9z^6+z^8)\varphi(z)+945\Phi(z)$,

$\int_{-\infty}^{z}z^{12}\varphi(z)\,dz=-z(10395+3465z^2+693z^4+99z^6+11z^8+z^{10})\varphi(z)+10395\Phi(z)$.

Putting these values in the Equation (B1), we get the desired result in the Equation (8).

C: Derivation of mgf (Proof 2.3.)

$$M_Z(t)=E(e^{tz})=\int_{-\infty}^{\infty}e^{tz}\,\frac{[(1-\alpha z-\beta z^3)^2+1]^2}{C_2(\alpha,\beta)}\varphi(z)\,dz$$

$$=\frac{1}{C_2(\alpha,\beta)}\int_{-\infty}^{\infty}e^{tz}\left[\begin{array}{l}4-8\alpha z+8\alpha^2z^2-4(\alpha^3-2\beta)z^3+\alpha(\alpha^3+16\beta)z^4-12\alpha^2\beta z^5+\\4\beta(\alpha^3+2\beta)z^6-12\alpha\beta^2z^7+6\alpha^2\beta^2z^8-4\beta^3z^9+4\alpha\beta^3z^{10}+\beta^4z^{12}\end{array}\right]\varphi(z)\,dz$$

$$= \frac{1}{C_2(\alpha,\beta)}\Bigg[4\int_{-\infty}^{\infty} e^{tz}\varphi(z)\,dz - 8\alpha\int_{-\infty}^{\infty} z\,e^{tz}\varphi(z)\,dz + 8\alpha^2\int_{-\infty}^{\infty} z^2 e^{tz}\varphi(z)\,dz - 4(\alpha^3+2\beta)\int_{-\infty}^{\infty} z^3 e^{tz}\varphi(z)\,dz +$$

$$\alpha(\alpha^3+16\beta)\int_{-\infty}^{\infty} z^4 e^{tz}\varphi(z)\,dz - 12\alpha^2\beta\int_{-\infty}^{\infty} z^5 e^{tz}\varphi(z)\,dz + 4\beta(\alpha^3+2\beta)\int_{-\infty}^{\infty} z^6 e^{tz}\varphi(z)\,dz - 12\alpha\beta^2$$

$$\int_{-\infty}^{\infty} z^7 e^{tz}\varphi(z)\,dz + 6\alpha^2\beta^2\int_{-\infty}^{\infty} z^8 e^{tz}\varphi(z)\,dz - 4\beta^3\int_{-\infty}^{\infty} z^9 e^{tz}\varphi(z)\,dz + 4\alpha\beta^3\int_{-\infty}^{\infty} z^{10} e^{tz}\varphi(z)\,dz +$$

$$\beta^4\int_{-\infty}^{\infty} z^{12} e^{tz}\varphi(z)\,dz\Bigg] \tag{C1}$$

Now, we have

$$\int_{-\infty}^{\infty} e^{tz}\varphi(z)\,dz = e^{\frac{t^2}{2}} = M_X(t),\ \int_{-\infty}^{\infty} z\,e^{tz}\varphi(z)\,dz = t\,M_X(t),\ \int_{-\infty}^{\infty} z^2 e^{tz}\varphi(z)\,dz = (1+t^2)M_X(t),\ \int_{-\infty}^{\infty} z^3 e^{tz}\varphi(z)\,dz = -(2+z^2)\varphi(z),$$

$$\int_{-\infty}^{\infty} z^4 e^{tz}\varphi(z)\,dz = (3+6t^2+t^4)M_X(t),\ \int_{-\infty}^{\infty} z^5 e^{tz}\varphi(z)\,dz = t(15+10t^2+t^4)M_X(t),$$

$$\int_{-\infty}^{\infty} z^6 e^{tz}\varphi(z)\,dz = (15+45t^2+15t^4+t^6)M_X(t),\ \int_{-\infty}^{\infty} z^7 e^{tz}\varphi(z)\,dz = t(105+105t^2+21t^4+t^6)M_X(t),$$

$$\int_{-\infty}^{\infty} z^8 e^{tz}\varphi(z)\,dz = (105+420t^2+210t^4+28t^6+t^8)M_X(t),\ \int_{-\infty}^{\infty} z^9 e^{tz}\varphi(z)\,dz = t(945+1260t^2+378t^4+36t^6+t^8)M_X(t),$$

$$\int_{-\infty}^{\infty} z^{10} e^{tz}\varphi(z)\,dz = (945+4725t^2+3150t^4+630t^6+45t^8+t^{10})M_X(t),$$

$$\int_{-\infty}^{\infty} z^{12} e^{tz}\varphi(z)\,dz = (10395+62370t^2+51975t^4+13860t^6+1485t^8+66t^{10}+t^{12})M_X(t).$$

Putting these values in the Equation (C1), we get the desired result in the Equation (9).

D: Derivation of Nth Moments (Proof 2.4.)

$$E(Z^{2n}) = \int_{-\infty}^{\infty} z^{2n}\frac{[(1-\alpha z-\beta z^3)^2+1]^2}{C_2(\alpha,\beta)}\varphi(z)\,dz$$

$$= \frac{1}{C_2(\alpha,\beta)}\int_{-\infty}^{\infty}\begin{bmatrix}4z^{2n} - 8\alpha z^{2n+1} + 8\alpha^2 z^{2n+2} - 4(\alpha^3-2\beta)z^{2n+3} + \alpha(\alpha^3+16\beta)z^{2n+4} \\ -12\alpha^2\beta z^{2n+5} + 4\beta(\alpha^3+2\beta)z^{2n+6} - 12\alpha\beta^2 z^7 + 6\alpha^2\beta^2 z^8 - 4\beta^3 z^9 \\ +4\alpha\beta^3 z^{10} + \beta^4 z^{12}\end{bmatrix}\varphi(z)dz$$

$$= \frac{1}{C_2(\alpha,\beta)}[4E_N(z^{2n}) - 8\alpha E_N(z^{2n+1}) + 8\alpha^2 E_N(z^{2n+2}) - 4(\alpha^3-2\beta)E_N(z^{2n+3}) + \alpha(\alpha^3+16\beta)$$

$$E_N(z^{2n+4}) - 12\alpha^2\beta E_N(z^{2n+5}) + 4\beta(\alpha^3+2\beta)E_N(z^{2n+6}) - 12\alpha\beta^2 E_N(z^{2n+7}) + 6\alpha^2\beta^2 E_N(z^{2n+8})$$

$$-4\beta^3 E_N(z^{2n+9}) + 4\alpha\beta^3 E_N(z^{2n+10}) + \beta^4 E_N(z^{2n+12})]$$

$$= \frac{1}{C_2(\alpha,\beta)}\begin{bmatrix}4E_N(z^{2n}) + 8\alpha^2 E_N(z^{2n+2}) + \alpha(\alpha^3+16\beta)E_N(z^{2n+4}) + 4\beta(\alpha^3+2\beta)E_N(z^{2n+6}) + \\ 6\alpha^2\beta^2 E_N(z^{2n+8}) + 4\alpha\beta^3 E_N(z^{2n+10}) + \beta^4 E_N(z^{2n+12})\end{bmatrix}$$

where $E_N(.)$ is defined above and using this formula into the last equation we get the required result.

Similarly, $E(Z^{2n-1})$ can be proved in the same way which is omitted for the sake of brevity.