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Different Distributional Errors-in-Circular-Variables Models

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Abstract

The modeling of functional relationship between circular variables is gaining an increasing interest. The existing models assume the errors have same probability distributions, but the case of different distributional errors is not yet investigated. This paper considers the modeling of functional relationship for circular variables with different distributional errors. Two functional relationship models are proposed by assuming a combination of von Mises and wrapped Cauchy errors, with a distinction between known and unknown ratio of error concentrations. Parameters of the proposed models are estimated using the maximum likelihood method based on numerical iterative procedures. The properties of parameters' estimators are investigated via an extensive simulation study. The results show a direct relationship between the performance of parameters' estimates and the sample size, as well as the concentration parameters. For illustration, the proposed models are applied on wind directions data in two main cities in the Gaza Strip, Palestine.

Key Words: Bessel function, circular distance, circular mean, regression.

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1. Introduction

Adcock (1877, 1878) explored Errors-in-variables model (EIVM) as a practical statistical approach for modelling different problem (Gillard, 2010). EIVM is known as the measurement error or random regression model.

The main difference between ordinary regression and EIVM is that the response and explanatory variables in EIVM have no distinction; both are measured with errors, unlike in the regression model, where only the response variable is measured with errors. EIVM has two types, namely, functional and structural relationship (see Kendall 1951, 1952).

In the context of circular data, where variables are measured by angles and distributed on the circumference of the unit circle, the extension of EIVM is gaining an increasing interest.

Hussin (1997) proposed the first un-replicated linear functional relationship model for two circular variables. Hussin and Chik (2003) improved the earlier model by assuming that the ratio between error concentrations is known.

Caires and Wyatt (2003) discussed a simple version of (Hussin, 1997) model by assuming that the slope parameter is fixed at unity.

Other versions of the circular functional relationship model were proposed. Hussin (2005) extended his earlier model to the replicated observations case. Ibrahim (2013) and Satari et al. (2014) developed new versions of the functional relationship model on the basis of Sarma and Jammalamadaka's (1993) and Downs and Mardia's (2002) circular-circular regression models, respectively.

All of these previous models assumed that errors follow von Mises distribution with mean 0 and a constant concentration parameter, while Abuzaid et al. (2018) proposed a simple functional relationship model based on Abuzaid and Allahham's (2015) circular regression model by assuming that errors follow the wrapped Cauchy distribution. Recently, Mokhtar, et al (2019) proposed outlie-detection procedure in functional circular relationship model based on COVRATIO statistics, and Md. Al Mamun (2020) discussed the identification of high leverage points in linear functional relationship model.

The common proposed assumption that both variables have errors come from the same distribution is not guaranteed in the practice. Moreover, no published work has discussed the problem of different distributional errors in EIVM. Thus, this paper proposes a new version of the functional relationship model by assuming that errors are originated from hybrid distributions, for instance von Mises and wrapped Cauchy distributions.

The rest of this paper is organized as follows: Section 2 proposes the circular functional relationship model with different distributional errors and known ratio of concentrations; Section 3 presents the model formulation with the unknown ratio of concentration. Section 4 investigates the properties of models' parameters via an extensive simulation study. Section 5 discusses an illustrative example, while Section 6 gives the main conclusions.

2. Circular functional relationship model with different distributional errors and known ratio of concentrations

This section proposes a new version of the circular functional relationship models by assuming that the errors are originated from von Mises and wrapped Cauchy distributions with known ratio of error concentrations, λ .

2.1 Model formulation

Suppose that x_i and y_i are observed values of two circular variables X and Y , where $0 < X_i, Y_i \leq 2\pi$, for $i = 1,...,n$. The circular variables X_i and Y_i are the true values corresponding to x_i and y_i , respectively. Assume there is a linear relationship between these two circular variables with unknown slope β for any fixed X_i and Y_i . We assume that the observations x_i and y_i are measured with error δ_{1i} and ε_{1i} , respectively. Thus, the proposed model can be formulated as follows:

$$x_i = X_i + \delta_{1i} \, , \, y_i = Y_i + \varepsilon_{1i}$$

and
$$Y_i = \alpha + \beta X_i \, (\text{mod}\,2\pi), \qquad (2.1)$$

where δ_{1i} and ε_{1i} are independently distributed from von Mises and wrapped Cauchy distributions with 0 means and concentration parameters κ and ρ , respectively, $\delta_{1i} \sim \text{vM}(0,\kappa)$, $\varepsilon_{1i} \sim WC(0,\rho)$, and the ratio between concentration is $\lambda = \kappa/\rho$.

The probability density function of errors in the model (2.1) is given by:

$$f(x_i, y_i; \kappa, \rho, \alpha, \beta, X_i) = \frac{\exp(\kappa \cos(x_i - X_i))}{2\pi I_0(\kappa)} \frac{(1-\rho^2)}{2\pi[1+\rho^2 - 2\rho\cos(y_i - \alpha - \beta X_i)]} ,$$

since $\kappa = \lambda\rho$, then

$$f(x_i, y_i; \kappa, \rho, \alpha, \beta, X_i) = \frac{1}{(2\pi)^2 I_0(\lambda\rho)} \exp(\lambda\rho\cos(x_i - X_i)) \left(\frac{1-\rho^2}{1+\rho^2}\right) \left[1 - \frac{2\rho}{(1+\rho^2)}\cos(y_i - \alpha - \beta X_i)\right]^{-1} , \qquad (2.2)$$

let $\alpha_1 = \frac{2\rho}{(1+\rho^2)}\cos\alpha$, $\alpha_2 = \frac{2\rho}{(1+\rho^2)}\sin\alpha$ and $c = \left(\frac{1-\rho^2}{1+\rho^2}\right)$; thus, Equation (2.2) becomes

$$f(x_i, y_i; \kappa, \rho, \alpha, \beta, X_i) = \frac{1}{(2\pi)^2 I_0(\lambda\rho)} \exp(\lambda\rho\cos(x_i - X_i)) \frac{1}{c^{-1}} \left[\frac{1}{1 - \alpha_1\cos(y_i - \beta X_i) - \alpha_2\sin(y_i - \beta X_i)}\right]. \qquad (2.3)$$

Let, $\eta_1 = \frac{\alpha_1}{c}$ and $\eta_2 = \frac{\alpha_2}{c}$, then, $c^{-1} = \sqrt{1 + \eta_1^2 + \eta_2^2}$.

Thus, Equation (2.3) becomes

$$f(x_i, y_i; \lambda, \rho, \alpha, \beta, X_i) = \frac{1}{(2\pi)^2 I_0(\lambda\rho)} \exp(\lambda\rho\cos(x_i - X_i)) \frac{1}{\frac{1}{c} - \eta_1\cos(y_i - \beta X_i) - \eta_2\sin(y_i - \beta X_i)}.$$

The likelihood function of the probability density function in Equation (2.3) is given by

$$L(\lambda, \rho, \beta, X_i, \eta_1, \eta_2; x_i, y_i) = \frac{1}{(2\pi)^{2n}} \left[\frac{1}{I_0(\lambda\rho)}\right]^n \exp\sum_{i=1}^{n} \lambda\rho\cos(x_i - X_i) \frac{1}{\prod_{i=1}^{n}\left[\frac{1}{c} - \eta_1\cos(y_i - \beta X_i) - \eta_2\sin(y_i - \beta X_i)\right]}.$$

Then, the corresponding log-likelihood function is given as follows:

$$\ell = \log L(\lambda, \rho, \beta, X_i, \eta_1, \eta_2; x_i, y_i) = -2n\log 2\pi - n\log I_0(\lambda\rho) + \sum_{i=1}^{n} \lambda\rho\cos(x_i - X_i)$$
$$- \sum_{i=1}^{n} \log\left[\frac{1}{c} - \eta_1\cos(y_i - \beta X_i) - \eta_2\sin(y_i - \beta X_i)\right]. \qquad (2.4)$$

2.2 Parameters estimation:

The maximum likelihood estimates of model's parameters are obtained as follows:

a) Estimation of α :

To estimate the intercept parameter, α we derive the log likelihood in Equation (2.4) with respect to η_1 and η_2, and equating them by 0 as follows:

$$\frac{\partial\ell}{\partial\eta_1} = c\sum_{i=1}^{n}\left[\cos(y_i - \beta X_i) - \alpha_1\right]\omega_i = 0, \text{ and } \frac{\partial\ell}{\partial\eta_2} = c\sum_{i=1}^{n}\left[\sin(y_i - \beta X_i) - \alpha_2\right]\omega_i = 0,$$

where $\omega_i = \frac{1}{1 - \alpha_1\cos(y_i - \beta X_i) - \alpha_2\sin(y_i - \beta X_i)}$. By solving the previous two equations we have

$$\hat{\alpha}_1 = \left.\sum_{i=1}^{n}\omega_i\cos(y_i - \hat{\beta}\hat{X}_i)\right/\sum_{i=1}^{n}\omega_i, \text{ and } \hat{\alpha}_2 = \left.\sum_{i=1}^{n}\omega_i\sin(y_i - \hat{\beta}\hat{X}_i)\right/\sum_{i=1}^{n}\omega_i .$$

Thus; the MLE of α is expanded to the following form:

$$\hat{\alpha} = \begin{cases} \tan^{-1}\left(\hat{\alpha}_2/\hat{\alpha}_1\right), & \text{if} \quad \hat{\alpha}_1 > 0, \ \hat{\alpha}_2 > 0, \\ \pi/2, & \text{if} \quad \hat{\alpha}_1 > 0, \hat{\alpha}_2 = 0, \\ \tan^{-1}\left(\hat{\alpha}_2/\hat{\alpha}_1\right)+\pi, & \text{if} \quad \hat{\alpha}_2 < 0, \\ \tan^{-1}\left(\hat{\alpha}_2/\hat{\alpha}_1\right)+2\pi, & \text{if} \quad \hat{\alpha}_1 < 0, \hat{\alpha}_2 \geq 0, \\ \text{undefined}, & \text{if} \quad \hat{\alpha}_1 = 0, \ \hat{\alpha}_2 = 0. \end{cases} \qquad (2.5)$$

b) Estimation of X_i :

The first partial derivative of the log likelihood function in (2.4) with respect to X_i is given by

$$\frac{\partial \ell}{\partial X_i} = \sum_{i=1}^{n} \lambda \rho \sin\left(x_i - X_i\right) - c \sum_{i=1}^{n} \omega_i \left[\beta \eta_2 \cos\left(y_i - \beta X_i\right) - \beta \eta_1 \sin\left(y_i - \beta X_i\right)\right]. \qquad (2.6)$$

After equating Equation (2.6) by 0, the solution is obtained iteratively for a given initial guesses of X_i. Suppose $\hat{X}_{i0}$ is an initial estimate for X_i such as,

$$x_i - X_i = x_i - X_i + X_{i0} - X_{i0} = x_i - X_{i0} - \Delta_{1i}, \text{ where } \Delta_{1i} = X_i - X_{i0}.$$

For linear relationship between X and Y, the slope parameter β should be close to 1, and for small Δ_{1i}, then $\cos\beta\Delta_{1i} \approx 1$ and $\sin\beta\Delta_{1i} \approx \Delta_{1i}$. Thus, we can get the following reformulations:

$$\sin\left(x_i - X_i\right) = \sin\left[\left(x_i - X_{i0}\right) - \Delta_{1i}\right] = \sin\left(x_i - X_{i0}\right) - \Delta_{1i}\cos\left(x_i - X_{i0}\right).$$
$$\cos\left(y_i - \beta X_i\right) = \cos\left[\left(y_i - \beta X_{i0}\right) - \beta\Delta_{1i}\right] = \cos\left(y_i - \beta X_{i0}\right) + \Delta_{1i}\sin\left(y_i - \beta X_{i0}\right).$$
$$\sin\left(y_i - \beta X_i\right) = \sin\left[\left(y_i - \beta X_{i0}\right) - \beta\Delta_{1i}\right] = \sin\left(y_i - \beta X_{i0}\right) - \Delta_{1i}\cos\left(y_i - \beta X_{i0}\right).$$

Then, Equation (2.6) becomes

$$\frac{\partial \ell}{\partial X_i} = \sum_{i=1}^{n} \lambda \rho \sin\left(x_i - X_{io}\right) - c\sum_{i=1}^{n} \omega_i \beta \eta_2 \cos\left(y_i - \beta X_{i0}\right) + c\sum_{i=1}^{n} \omega_i \beta \eta_1 \sin\left(y_i - \beta X_{io}\right)$$
$$+ \Delta_{1i}\left[-\sum_{i=1}^{n} \lambda \rho \cos\left(x_i - X_{io}\right) - c\sum_{i=1}^{n} \omega_i \beta \eta_2 \sin\left(y_i - \beta X_{io}\right) + c\sum_{i=1}^{n} \omega_i \beta \eta_1 \cos\left(y_i - \beta X_{i0}\right)\right] = 0.$$

Thus, the MLE of X_i is obtained iteratively as follows:

$$\hat{X}_i = \hat{X}_{i0} + \frac{\sum_{i=1}^{n} \lambda \rho \sin\left(x_i - \hat{X}_{i0}\right) - c\sum_{i=1}^{n} \omega_i \hat{\beta}\hat{\eta}_2 \cos\left(y_i - \hat{\beta}\hat{X}_{i0}\right) + c\sum_{i=1}^{n} \omega_i \hat{\beta}\hat{\eta}_1 \sin\left(y_i - \hat{\beta}\hat{X}_{i0}\right)}{\sum_{i=1}^{n} \lambda \rho \cos\left(x_i - \hat{X}_{i0}\right) + c\sum_{i=1}^{n} \omega_i \hat{\beta}\hat{\eta}_2 \sin\left(y_i - \hat{\beta}\hat{X}_{i0}\right) - c\sum_{i=1}^{n} \omega_i \hat{\beta}\hat{\eta}_1 \cos\left(y_i - \hat{\beta}\hat{X}_{i0}\right)}. \qquad (2.7)$$

Possible initial guesses for iteration are $\hat{X}_{i0} = x_i$ for $i = 1,...,n$.

c) Estimation of β :

The first partial derivative of the log likelihood function in (2.4) with respect to β is given by:

$$\frac{\partial \ell}{\partial \beta} = -\frac{1}{c^{-1}} \sum_{i=1}^{n} \frac{-\eta_1 \sin\left(y_i - \beta X_i\right)X_i + \eta_2 \cos\left(y_i - \beta X_i\right)X_i}{1 - \alpha_1 \cos\left(y_i - \beta X_i\right) - \alpha_2 \sin\left(y_i - \beta X_i\right)}. \qquad (2.8)$$

After equating Equation (2.8) by 0, then its solution is obtained iteratively for a given initial guess of β_0, The following component is reformulated as follow:

$$y_i - \beta X_i = y_i - (\beta - \beta_0) X_i - \beta_0 X_i = y_i - \beta_0 X_i - \Delta_2 X_i, \text{ where } \Delta_2 = \beta - \beta_0.$$

For small Δ_2, we have $\cos \Delta_2 X_i \approx 1$ and $\sin \Delta_2 X_i \approx \Delta_2 X_i$, and then the following components reformulated by:

$$\cos(y_i - \beta X_i) = \cos(y_i - \beta_0 X_i) + \sin(y_i - \beta_0 X_i) \Delta_2 X_i,$$
$$\text{and } \sin(y_i - \beta X_i) = \sin(y_i - \beta_0 X_i) - \cos(y_i - \beta_0 X_i) \Delta_2 X_i.$$

Hence, Equation (2.8) can be rewritten as

$$\frac{\partial \ell}{\partial \beta} = -c \sum_{i=1}^n \omega_i \eta_2 \cos(y_i - \beta_0 X_i) X_i + c \sum_{i=1}^n \omega_i \eta_1 \sin(y_i - \beta_0 X_i) X_i$$
$$- \Delta_2 \left[c \sum_{i=1}^n \omega_i \eta_2 \sin(y_i - \beta_0 X_i) X_i^2 + c \sum_{i=1}^n \omega_i \eta_1 \cos(y_i - \beta_0 X_i) X_i^2 \right].$$

Then, the MLE of $\hat\beta$ is given by:

$$\hat\beta = \hat\beta_0 - \frac{\sum_{i=1}^n \omega_i \hat\eta_2 \cos\left(y_i - \hat\beta_0 \hat X_i\right) \hat X_i - \sum_{i=1}^n \omega_i \hat\eta_1 \sin\left(y_i - \hat\beta_0 \hat X_i\right) \hat X_i}{\sum_{i=1}^n \omega_i \hat\eta_2 \sin\left(y_i - \hat\beta_0 \hat X_i\right) \hat X_i^2 + \sum_{i=1}^n \omega_i \hat\eta_1 \cos\left(y_i - \hat\beta_0 \hat X_i\right) \hat X_i^2} \tag{2.9}$$

One possible initial guess for iteration is $\hat\beta_0 = 1$; this is sensible since both X and Y are assumed to be linearly dependent.

d) Estimation of λ :

The first partial derivative of the log likelihood function (2.4) with respect to λ is given by:

$$\frac{\partial \ell}{\partial \lambda} = -nA(\lambda\rho)\rho + \sum_{i=1}^n \rho \cos(x_i - X_i), \tag{2.10}$$

where $A(\lambda\rho) = \dfrac{I_1(\lambda\rho)}{I_0(\lambda\rho)}$, I_0 is the modified Bessel function of the first kind and order zero, I_1 is the first derivative of I_0. One possible expansion of the ratio $A(\lambda\rho)$ is given by (Oldham et al., 2010).

$$A(\lambda\rho) = 1 - \frac{1}{2\lambda\rho} - \frac{1}{8\lambda^2\rho^2}.$$

Equating Equation (2.10) by 0, and after few direct mathematical operations, the MLE of λ is given by

$$\hat\lambda = \frac{1 - \sqrt{1 + 2(1 - C_n)}}{4\rho(1 - C_n)}, \tag{2.11}$$

where $C_n = \sum_{i=1}^n \cos(x_i - X_i)$.

e) Estimation of ρ :

From the definitions of α_1 and α_2, we have $\alpha_1^2 + \alpha_2^2 = \dfrac{4\rho^2}{(1 + \rho^2)^2}$, by reformulate it into a quadratic

equation form then the MLE of ρ is given by $\hat\rho = \dfrac{1 - \sqrt{1 - \hat\alpha_1^2 - \hat\alpha_2^2}}{\sqrt{\hat\alpha_1^2 + \hat\alpha_2^2}}$, where $0 < \hat\rho < 1$. (2.12)

3   Circular functional relationship model with different distributional errors and unknown ratio of concentrations

In this section we assume unknown relationship between errors' concentration parameters κ and ρ. Thus, we follow the same procedures used in Section 2. However, we replace $\lambda\rho$ with κ in Equation (2.2). Therefore, the log-likelihood function of the probability for this model is given by

$$\ell = \log L(\kappa,\beta,X_i,\eta_1,\eta_2;x_i,y_i) = -2n\log 2\pi - n\log I_0(\kappa) + \sum_{i=1}^{n}\kappa\cos(x_i - X_i) - \sum_{i=1}^{n}\log\left[\frac{1}{c} - \eta_1\cos(y_i - \beta X_i) - \eta_2\sin(y_i - \beta X_i)\right].$$

The estimation of the parameters is briefly presented as follow:

a)   **MLE of α:** The MLE of α_1 and α_2 are given by $\hat{\alpha}_1 = \sum_{i=1}^{n}\omega_i\cos\left(y_i - \hat{\beta}\hat{X}_i\right)\Big/\sum_{i=1}^{n}\omega_i$, and

$\hat{\alpha}_2 = \sum_{i=1}^{n}\omega_i\sin\left(y_i - \hat{\beta}\hat{X}_i\right)\Big/\sum_{i=1}^{n}\omega_i$. Thus, the MLE of α is given

$$\hat{\alpha} = \begin{cases} \tan^{-1}\left(\hat{\alpha}_2/\hat{\alpha}_1\right), & \text{if} \quad \hat{\alpha}_1 > 0,\ \hat{\alpha}_2 > 0, \\ \pi/2, & \text{if} \quad \hat{\alpha}_1 > 0, \hat{\alpha}_2 = 0, \\ \tan^{-1}\left(\hat{\alpha}_2/\hat{\alpha}_1\right) + \pi, & \text{if} \quad \hat{\alpha}_2 < 0, \\ \tan^{-1}\left(\hat{\alpha}_2/\hat{\alpha}_1\right) + 2\pi, & \text{if} \quad \hat{\alpha}_1 < 0, \hat{\alpha}_2 \geq 0, \\ \text{undefined}, & \text{if} \quad \hat{\alpha}_1 = 0,\ \hat{\alpha}_2 = 0. \end{cases} \qquad (3.1)$$

b)   **MLE of X_i:** It is given by

$$\hat{X}_i = \hat{X}_{i0} + \frac{\sum_{i=1}^{n}\kappa\sin\left(x_i - \hat{X}_{i0}\right) - c\sum_{i=1}^{n}\omega_i\hat{\beta}\hat{\eta}_2\cos\left(y_i - \hat{\beta}\hat{X}_{i0}\right) + c\sum_{i=1}^{n}\omega_i\hat{\beta}\hat{\eta}_1\sin\left(y_i - \hat{\beta}\hat{X}_{i0}\right)}{\sum_{i=1}^{n}\kappa\cos\left(x_i - \hat{X}_{i0}\right) + c\sum_{i=1}^{n}\omega_i\hat{\beta}\hat{\eta}_2\sin\left(y_i - \hat{\beta}\hat{X}_{i0}\right) - c\sum_{i=1}^{n}\omega_i\hat{\beta}\hat{\eta}_1\cos\left(y_i - \hat{\beta}\hat{X}_{i0}\right)}. \qquad (3.2)$$

The possible initial guesses of iteration are $\hat{X}_{i0} = x_i$ for $i = 1,...,n$.

c)   **MLE of β:** It is given by

$$\hat{\beta} = \hat{\beta}_0 - \frac{\sum_{i=1}^{n}\omega_i\hat{\eta}_2\cos\left(y_i - \hat{\beta}_0\hat{X}_i\right)\hat{X}_i - \sum_{i=1}^{n}\omega_i\hat{\eta}_1\sin\left(y_i - \hat{\beta}_0\hat{X}_i\right)\hat{X}_i}{\sum_{i=1}^{n}\omega_i\hat{\eta}_2\sin\left(y_i - \hat{\beta}_0\hat{X}_i\right)\hat{X}_i^2 + \sum_{i=1}^{n}\omega_i\hat{\eta}_1\cos\left(y_i - \hat{\beta}_0\hat{X}_i\right)\hat{X}_i^2}. \qquad (3.3)$$

d)   **MLE of κ:** It is given by $\hat{\kappa} = \dfrac{1 - \sqrt{1 + 2\left(1 - C^*_n\right)}}{4\left(1 - C^*_n\right)}$, where $C^*_n = 1 - \dfrac{1}{2\kappa} - \dfrac{1}{8\kappa^2}$. $\qquad (3.4)$

e)   **MLE of ρ:** It is given by $\hat{\rho} = \dfrac{1 - \sqrt{1 - \hat{\alpha}_1^2 - \hat{\alpha}_2^2}}{\sqrt{\hat{\alpha}_1^2 + \hat{\alpha}_2^2}}$, where $0 < \hat{\rho} < 1$. $\qquad (3.5)$

The MLE estimates of the models' parameters can be numerically obtained by using any statistical software such as R. The iterative re-weighting algorithm for the maximum likelihood estimation is working as follows:

Step 1: Initialize $\alpha_1^{[0]}$, $\alpha_2^{[0]}$ with $\alpha_1^{[0]} + \alpha_2^{[0]} < 1$, $\hat{X}_{i0} = x_i$, $\hat{\beta}_0 = 1$ and $\lambda^{[0]} = 1$, then calculate $\omega_i^{[0]}$ using

$$\omega_i = \frac{1}{1 - \alpha_1 \cos(y_i - \beta X_i) - \alpha_2 \sin(y_i - \beta X_i)}.$$

Step 2: Given $\alpha_1^{[k]}$, $\alpha_2^{[k]}$, $X^{[k]}$, $\hat{\beta}_0^{[k]}$, $\lambda^{[k]}$ and $\omega_i^{[k]}$ at iteration k, calculate $\alpha_1^{[k+1]}$, $\alpha_2^{[k+1]}$, $X^{[k+1]}$, $\hat{\beta}_0^{[k+1]}$ and $\lambda^{[k+1]}$.

Step 3: Repeat step 2 until algorithm converges.

Step 4: Obtain the values of $\hat{\alpha}$ and $\hat{\rho}$ by the formulas given in Sections 2 and 3.

The following section discusses the settings and results of the simulation study on the characteristics of parameters' estimates, including bias, mean and mean square errors.

4   Simulation study

This section presents the settings and results of the simulation study to assess the accuracy and biasedness of the parameters' estimates in the two proposed models.

4.1 Settings:

The simulation results are obtained on the basis of 1,000 generated samples, where the values of the random variable X follows the von Mises distribution with mean 0.5 and concentration parameter 2, i.e. $vM(0.5, 2)$. Without the loss of generality, we fix the intercept parameter $\alpha = 0$ and the slope parameter $\beta = 1$. Six values of sample size, n are considered, viz $n = 10, 30, 50, 70, 100$ and 150 are considered.

Two random errors are generated from circular distributions as described below:

Model 1: The errors, δ_{1i}, follows von Mises distribution with mean 0 and parameter concentration , κ, $\delta_{1i} \sim vM(0, \kappa)$, and ε_{1i} follow wrapped Cauchy distribution with mean 0 and concentration parameter ρ, $\varepsilon_{1i} \sim WC(0, \rho)$ with known ratio of error concentrations, $\lambda = \kappa/\rho$. Thus, three values of λ, namely 1, 1.5 and 0.5 (i.e.) are considered. $\kappa = \rho, \kappa > \rho, \kappa < \rho$

Model 2: The errors δ_{2i} follow von Mises distribution with mean 0 and concentration parameter, κ, $\delta_{2i} \sim vM(0, \kappa)$, and ε_{2i} follow wrapped Cauchy distribution with mean 0 and concentration parameter, ρ, $\varepsilon_{2i} \sim WC(0, \rho)$ and , unknown ratio of error concentrations. Three values of κ are considered, namely 0.5, 1 and 2.

For both models, nine different choices of concentration parameters $\rho = 0.2, 0.3, 0.4, 0.5. 0.6, 0.7, 0.8, 0.9$ and 0.99 of random errors ε are generated from wrapped Cauchy distribution.

The following settings are considered to generate the data sets of the considered models

1) A random sample X of size n is generated from von Mises with mean 0.5 and concentration parameter 2.
2) According to the considered model
   i- The random variable x_i is obtained using $x_i = X_i + \delta_{ji} \pmod{2\pi}$, $j = 1$ or 2.
   ii- The random variable Y_i is obtained using Equation (2.1) as $Y_i = \alpha + \beta X_i$.
   iii- The random variable y_i is obtained using $y_i = Y_i + \varepsilon_{ji} \pmod{2\pi}$, $j = 1$ or 2.
3) Estimate the parameters for the generated data based on the derived estimators in Sections 2 and 3.

4) The simulation procedure is applied for each pair of concentration parameters ρ and sample size n, and is repeated $S = 1000$ times to ensure the convergence criterion.

Note that, in the derived models, parameter α is a circular quantity, whereas parameters, κ, ρ and β are linear, where $\kappa > 0$, $0 < \rho < 1$ and $-\infty < \beta < \infty$.

The following measures and indicators are obtained:

(a) Measures of circular parameter, α :
   i)   Circular mean of $\hat{\alpha}$.
   ii)   Bias of $\hat{\alpha}$, it can be obtained using circular distance $\pi - |\pi - |\hat{\alpha} - \bar{\hat{\alpha}}||$.
   iii)   Mean circular error, $MCE = \dfrac{1}{s}\sum_{j=1}^{s}\left(1 - \cos\left(\hat{\alpha}_j - \bar{\hat{\alpha}}\right)\right)$, (see Abuzaid, 2010).

(b) Measures of linear parameters κ, ρ and β : Let ω be a generic term for κ, ρ and β
   i)   Arithmetic mean, κ, ρ and β.
   ii)   Bias $= |\hat{\omega} - \bar{\hat{\omega}}|$.
   iii)   Mean square errors, $MSE = \sqrt{\dfrac{1}{s}\sum_{j=1}^{s}\left(\hat{\omega}_j - \bar{\hat{\omega}}\right)^2}$, where s=1,000.

The simulation study is conducted using R statistical software package.

4.2 Results :

Simulation results show a direct relationship between the goodness estimates of all considered parameters and the sample size n, as well as the concentration parameter ρ for all considered cases.

The performances of the two proposed models are compared at $\kappa = \rho$ (i.e. $\lambda = 1$). For the two considered models a homogeneity is observed in the performance of the slope parameter estimates, $\hat{\beta}$ especially for $n \geq 30$.

Figure 1 shows an inverse relationship between the *MSE* of $\hat{\beta}$ estimates and the values of concentration parameter, ρ where the performances of estimates are enhanced as the value of ρ increases.

Figure 2 presents the *MCE* of the intercept parameter estimates, where the estimates of α for Model 1 outperforms the other model for all considered concentration parameters and sample size n.

The performance of concentration parameter estimate reveals good performance for large sample size ($n \geq 30$) at any value of ρ. Moreover, the results of other performance measures are obtained and lead to similar conclusions. These results are available upon request.

5. Application

As an illustration, a total of 96 measurements of wind directions that have been collected from Gaza and Khan Younis meteorological stations in the Gaza Strip, Palestine. Badawi (2013) analyzed these data in order to establish a wind farm to reach the optimum electricity energy. Abuzaid and Allahham (2015) and Abuzaid et al. (2018) fitted the wind directions to the simple circular regression model and functional relationship models with wrapped Cauchy errors. Figure 3 displays the spoke plot (Zubairi et al. 2008) and scatter plot of wind directions data. The plots reflect a linear strong circular correlation with coefficient 0.8519.
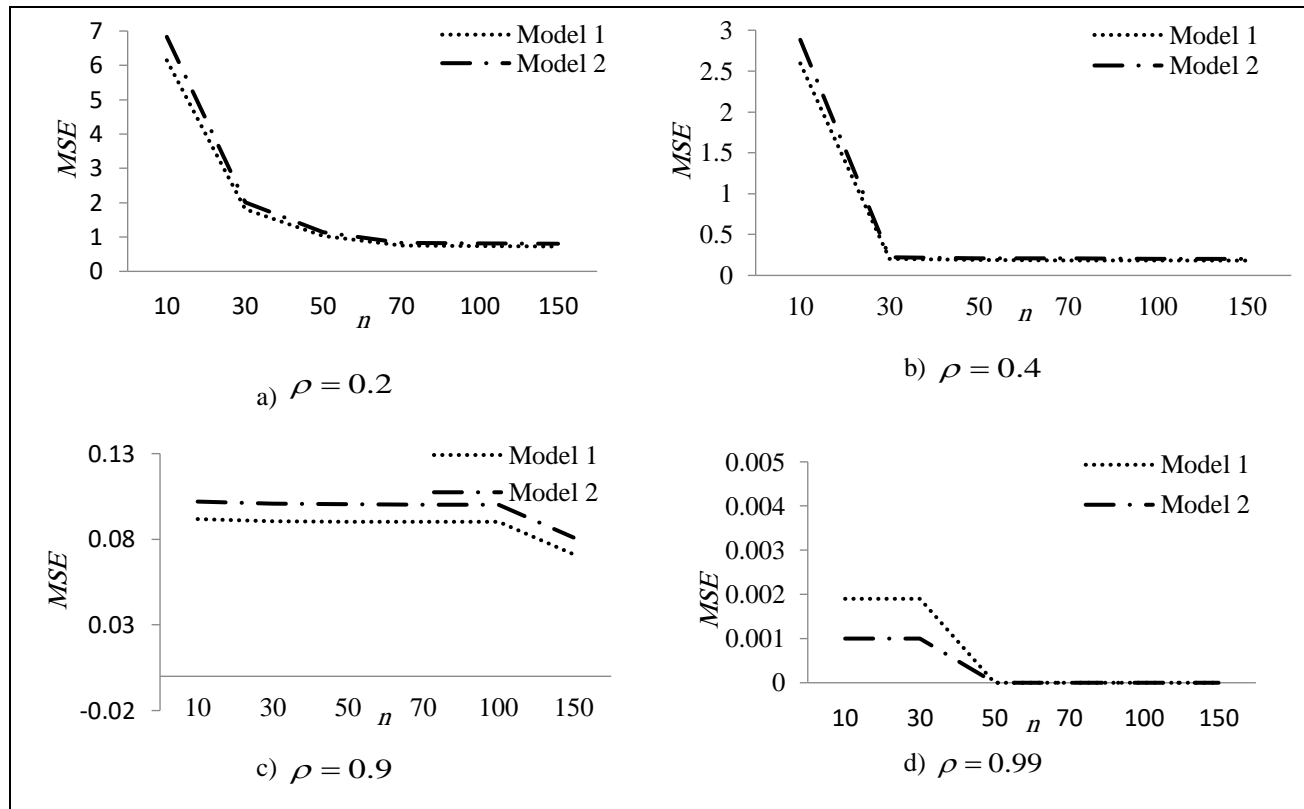
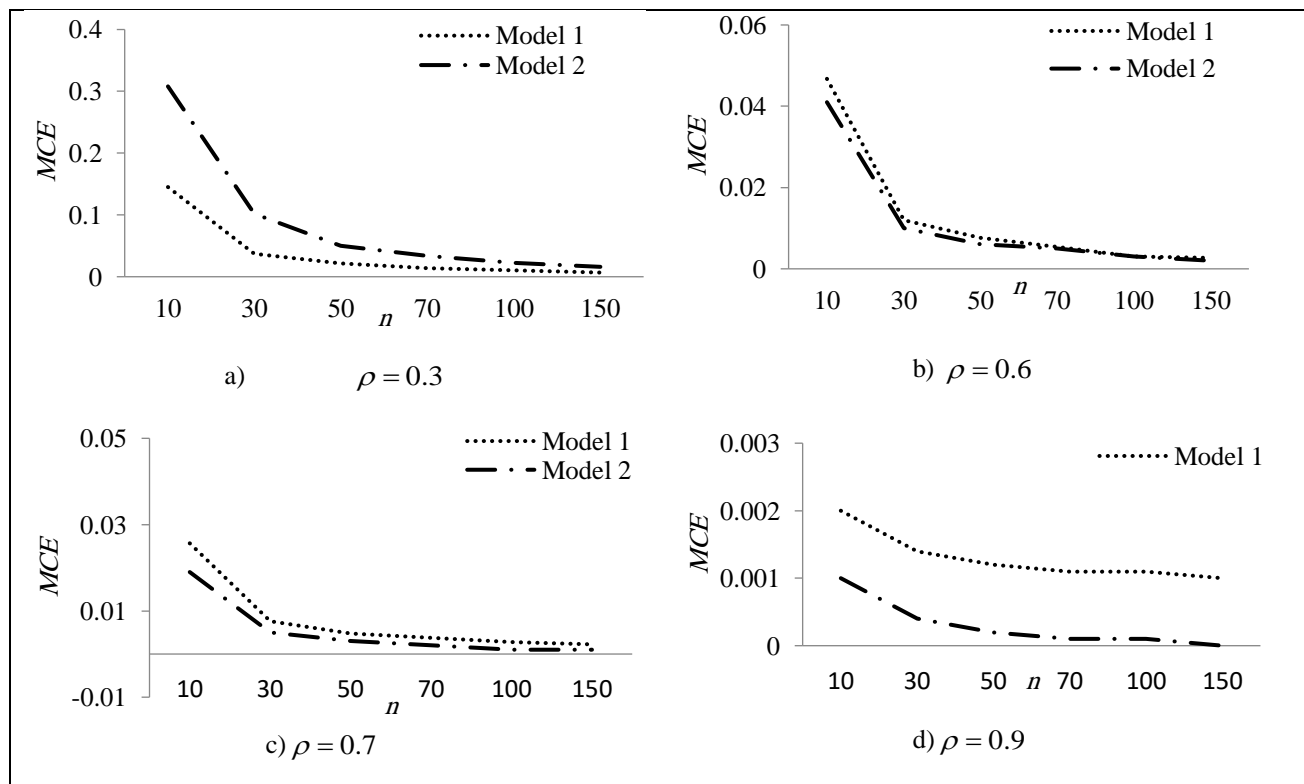Figure 1: *MSE* for the estimates of slope parameter β, for $\kappa = \rho$.

Figure 2: *MCE* for the intercept parameter $\hat{\alpha}$, for $\kappa = \rho$.

Given the reasonably linear relationship between wind directions, the proposed circular functional relationship models are suggested to fit the data. The parameters estimates are obtained by applying the iterative procedures and shown in Table 1.
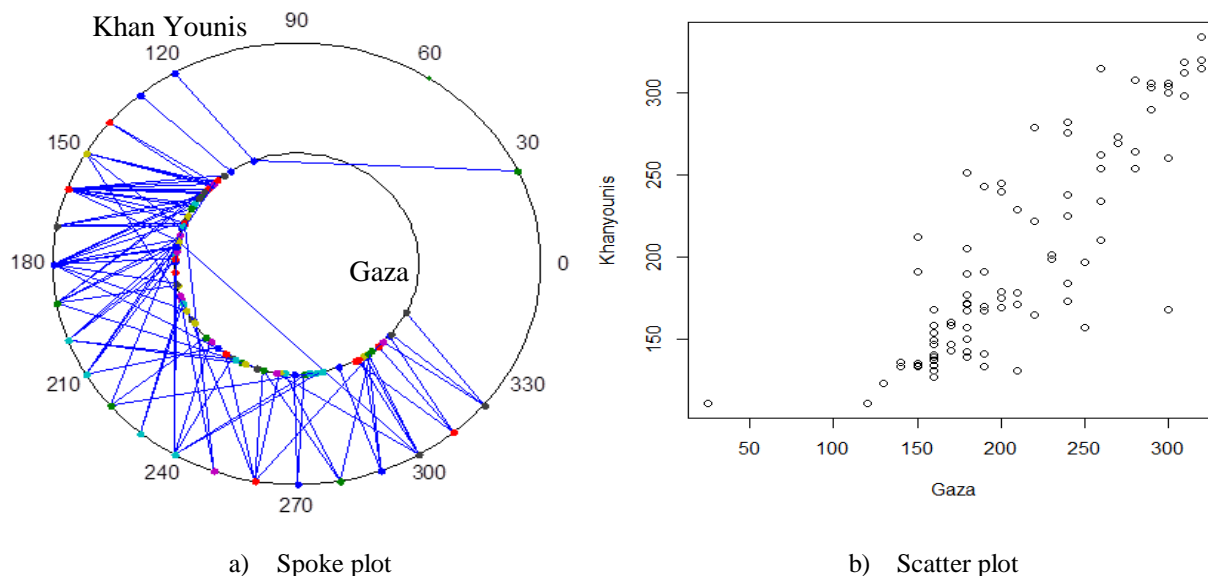

| a) Spoke plot | b) Scatter plot |

Figure 3: Spoke and scatter plots of wind direction data for Gaza and Khan Younis, *n* =96.

Table 1: Parameters estimate of circular functional relationship models.

Parameter	Model 1	Model 2
$\hat{\alpha}$	5.728 ≡ - 0.555 (mod2π)	3.558 ≡ -2.725 (mod2π)
$\hat{\rho}$	0.795	0.375
$\hat{\kappa}$	5.780	1.402
$\hat{\beta}$	1.117	3.651
$\hat{\lambda}$	7.267	–
No.of Iterations	643	219

The estimate of intercept parameter α in Model 1 is 5.724, which is equivalent to -0.555 and closed to 0. The estimate of α in Model 2 is 3.558, which is far from the 0 direction. Furthermore, the estimate of the slope parameter β in Model 1 is 1.117 which is closed to the true value one, but the estimate of β in Model 2 is 3.6512 which is far from 1. Moreover, it is an unjustified value in the circular context.

The estimate of concentration parameter ρ of error for y is 0.7954 and 0.3745 for Models 1 and 2, respectively. Moreover, the estimation of concentration parameter κ of error for x is equal to 1.2849 and 1.4019 in Models 1 and 2, respectively, which agree with the range of the concentration parameter in the von Mises distribution $(\kappa > 0)$.

For furthar diagonistic checking, Abuzaid (2010) suggested a statistic analogues to the coefficient of determination in the linear models, $A^*(\kappa) = \frac{1}{n} \sum \cos^2(y_i - \hat{y}_i)$, where $0 \le A^*(\kappa) \le 1$. Tabel 2 shows that Model 1 has the highest value of $A^*(\kappa)$.

Table 2: Goodness- of-fit for the fitted model.

Model	Model 1	Model 2
$A^*(\kappa)$	0.5881	0.4368

The plot of the circular errors in the Y variable versus the observations and its histogram in the two considered models are given in Figures 4 and 5, respectively. Both figures show that the circular means of errors in both models are close to zero. Moreover, the histogram of errors in Model 1 is symmetric with suspected outliers in its right tail. The errors of the other Model are asymmetric.
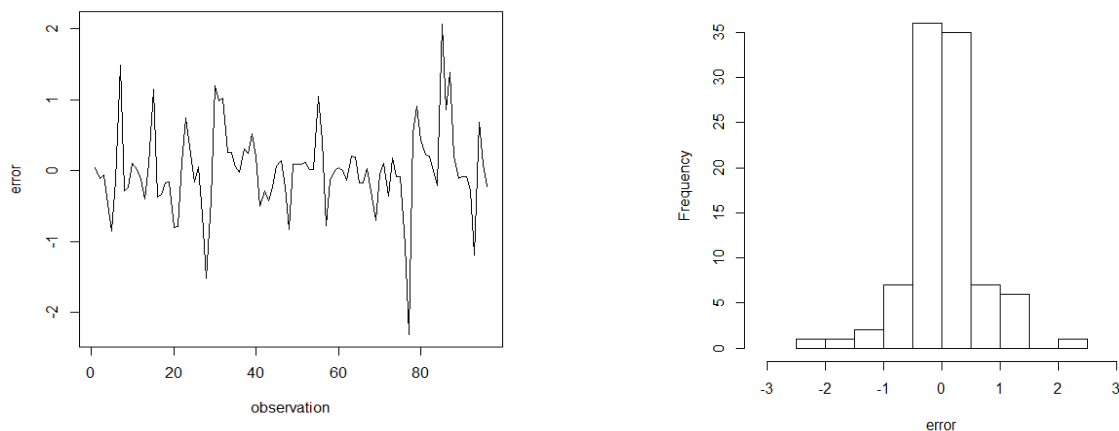

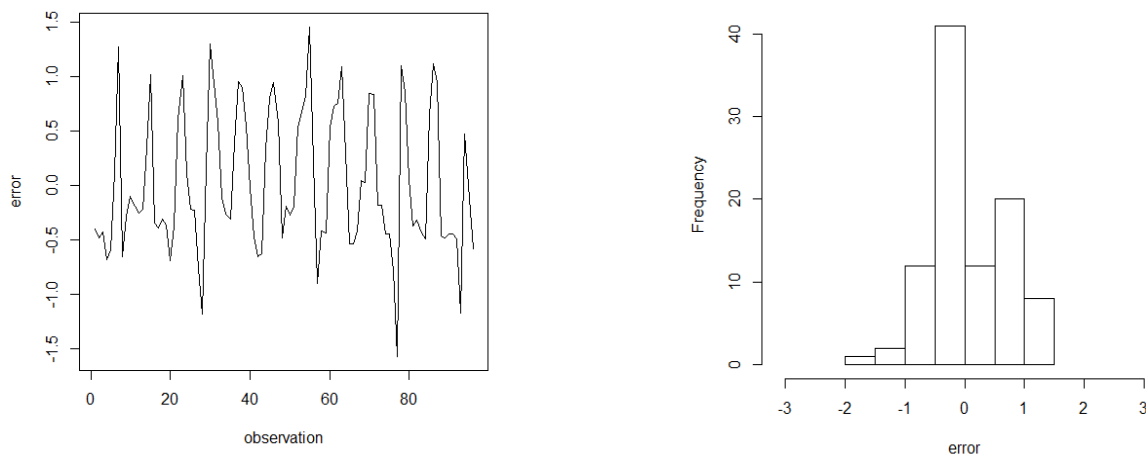Figure 4: Errors of wind directions data of KhanYounis measurement based on Model 1.

Figure 5: Errors of wind directions data of KhanYounis measurement based on Model 2

The obtained residuals were tested to follow wrapped Cauchy distribution and von Mises distribution via Kolmogorov-Simirnov test and Watson test, respectively, at 0.05 level of significance. Table 3 provides the summary.

Table 3:Test statistics for circular errors of observations.

Model	K.S. for wrapped Cauchy error ε (*p-value*)	Watson for von Mises error δ (*p-value*)
Model 1	0.0482 (0.9792)	0.3819 (0.113)
Model 2	0.1121 (0.1792)	0.1801 (0.101)

The results in Table 3 reveal that the obtained residuals from the two models follow their proposed models. Therefore, we conclude that Model 1 is the best fit between the two considered models.

6. Conclusions

This paper has addressed the problem of functional modeling of two circular variables with different distributions of errors, namely, von Mises and wrapped Cauchy distributions. The maximum likelihood estimations of parameters are obtained in both cases when the ratios of error concentrations are known and unknwn. The results of the simulation study show that the estimators perform well for large sample sizes and for large concentration parameters.

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