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The Exponentiated Half-logistic Odd Burr III-G: Model, Properties and Applications

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Abstract

In this article, we develop and study in detail a new family of distributions called exponentiated half-logistic Odd Burr III-G (EHL-OBIII-G). Some of the mathematical and statistical properties for this new family of distributions such as the hazard function, quantile function, moments, probability weighted moments, Rényi entropy and stochastic orders are derived. The model parameters are estimated using the maximum likelihood estimation method. The usefulness of the proposed family of distributions is demonstrated via extensive simulation studies. Finally the proposed model and its special case is applied to real data sets to illustrate its best fit and flexibility. The model is compared to some of the existing non-nested models having equal number of parameters and from these results, the proposed model performed better than other fitted models.

Key Words: Half-logistic distribution; Odd Burr-III distribution; Family of distributions; Maximum likelihood Estimation.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

Recently, many researchers have done more work in developing new families by adding extra shape parameters to achieve better fits and more flexibility in modelling practical data. Various families of distributions have been developed and used in the past to model data in different fields such as finance, economics, engineering, reliability analysis, environmental sciences and medical sciences. Some of the well-known families are the Marshall-Olkin-G by (20), the beta-G by (19), odd log-logistic-G by (3), the transmuted-G by (25), the gamma-G by (28), the Kumaraswamy-G by (14), the logistic-G by (27), exponentiated generalized-G by (15), T-X family by (4), the Weibull-G by (7), the exponentiated half-logistic generated family by (12) and the beta odd log-logistic generalized by (13) to mention just a few. (8) developed a system of cumulative distribution functions which have been widely extended by many researchers to generate more flexible and useful distributions. The Burr-XII models have many applications in different areas including acceptance sampling plans, reliability and failure time modeling. (1) developed a new flexible family of distributions called the Odd exponentiated half logistic-G (OEHL-G) family of distributions using the Half-Logistic

(HL) distribution as the generator. The use of these new generators of continuous distributions have attracted the attention of various authors in recent times.

In this article, we develop a new family of distributions called exponentiated half-logistic Odd Burr III-G (EHL-OBIII-G) by extending the exponentiated half-logistic generated family by (12) using the Odd Burr III-G by (2). The motivation for developing this new family of distributions arises from its ability to model failure time data with increasing, decreasing, unimodal and bathtub shaped hazard rates. The proposed models represents a good alternative to most of the failure time distributions that lacks flexibility in modeling various forms of real life data problems.

The new family of distributions exhibits reverse-J, J, symmetric, left or right-skewed shapes for the probability density function. The hazard rate function of these new family of distributions is very flexible and has bathtub, upside bathtub, reverse-J, J, increasing and decreasing shapes. Furthermore, the proposed distribution has a desirable tractability property since the distribution can be expressed as an infinite linear combination of exponentiated-G distribution. Finally, the development of this new family of distributions is necessitated by the need to model various forms of lifetime data to include, economics, engineering, survival analysis and finance with models that takes into consideration not only shape and scale but also skewness, kurtosis and tail variation.

This paper is organized as follows: Section 2, develops the proposed model, present the quantile function, series expansion of the probability density function and some of the special cases. The mathematical properties for the EHL-OBIII-G family, namely; moments, probability weighted moments, Rényi entropy, stochastic orders and maximum likelihood estimates are presented under Section 3. We run and present Simulation results under Section 4. Section 5 gives results on model applications using real life datasets to show the efficacy of the fitted model. We finally give concluding remarks under Section 6.

2. Developing the EHL-OBIII-G Model

In this section, we develop the new model, derive its quantile function, present the series expansion of the density function and study some of its special cases.

2.1. The Model

Consider a family of distributions called the exponentiated half-logistic (EHL) family by (12) derived from the gamma-generator by (28). The cumulative distribution function (CDF) of the EHL-G family of distributions is given by

$$F(x;\alpha,\lambda,\underline{\boldsymbol{\xi}}) = \left\{\frac{1-[1-G(x;\underline{\boldsymbol{\xi}})]^{\lambda}}{1+[1-G(x;\underline{\boldsymbol{\xi}})]^{\lambda}}\right\}^{\alpha} \tag{1}$$

and the corresponding probability distribution function (PDF) is given by

$$f(x;\alpha,\lambda,\underline{\boldsymbol{\xi}}) = 2\alpha\lambda g(x;\underline{\boldsymbol{\xi}})[1-G(x;\underline{\boldsymbol{\xi}})]^{\lambda-1}\frac{(1-[1-G(x;\underline{\boldsymbol{\xi}})]^{\lambda})^{\alpha-1}}{(1+[1-G(x;\underline{\boldsymbol{\xi}})]^{\lambda})^{\alpha+1}}, \tag{2}$$

where $G(x;\underline{\boldsymbol{\xi}})$ and $g(x;\underline{\boldsymbol{\xi}})$ is the CDF and PDF respectively for any baseline distribution, and $\alpha,\lambda>0$ are additional shape parameters with $\underline{\boldsymbol{\xi}}$ as the vector of parameters. (2) proposed a new family of distributions called the Odd Burr III-G (OBIII-G) family having the CDF and PDF given by

$$F(x;a,b,\underline{\boldsymbol{\psi}}) = \int_0^{\frac{G(x;\underline{\boldsymbol{\psi}})}{1-G(x;\underline{\boldsymbol{\psi}})}} abt^{-b-1}(1+t^{-a})^{-b-1}dt = \left\{1+\Big(\frac{1-G(x;\underline{\boldsymbol{\psi}})}{G(x;\underline{\boldsymbol{\psi}})}\Big)^{a}\right\}^{-b} \tag{3}$$

and

$$f(x;a,b,\underline{\boldsymbol{\psi}}) = ab\frac{[1-G(x;\underline{\boldsymbol{\psi}})]^{a-1}}{G(x;\underline{\boldsymbol{\psi}})^{a+1}}\left[1+\Big(\frac{1-G(x;\underline{\boldsymbol{\psi}})}{G(x;\underline{\boldsymbol{\psi}})}\Big)^{a}\right]^{-b-1} g(x;\underline{\boldsymbol{\psi}}), \tag{4}$$

respectively, for $a,b>0$ and the parameter vector $\underline{\boldsymbol{\psi}}$.

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If we let equation (3) to be the baseline CDF in equation (1), then the CDF of the EHL-OBIII-G family of distributions can be written as

$$F(x;a,b,\alpha,\underline{\psi}) = \left(\frac{\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right]} \right)^{\alpha}, \tag{5}$$

and the corresponding PDF is given by

$$\begin{aligned} f(x;a,b,\alpha,\underline{\psi}) &= 2\alpha ab \left(\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right)^{\alpha-1} \left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b-1} \\ &\times \left(1+\left[1-\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)} \\ &\times \left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a-1} \frac{g(x;\underline{\psi})}{\left(\overline{G}(x;\underline{\psi})\right)^{2}}, \end{aligned} \tag{6}$$

for a, b, α, $x > 0$ and parameter vector $\underline{\psi}$. The hazard rate function (HRF) is given by

$$\begin{aligned} h(x;a,b,\alpha,\underline{\psi}) &= 2\alpha ab \left(\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right)^{\alpha-1} \left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b-1} \\ &\times \left(1+\left[1-\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)} \left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a-1} \frac{g(x;\underline{\psi})}{\left(\overline{G}(x;\underline{\psi})\right)^{2}} \\ &\times \left(1-\left(\frac{\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right]} \right)^{\alpha}\right)^{-1}. \end{aligned} \tag{7}$$

2.2. Quantile Function

The quantile function for the EHL-OBIII-G family of distributions is derived by inverting the following function

$$\left(\frac{\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right]} \right)^{\alpha} = u,$$

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for $0 < u < 1$. Note that

$$\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b} \quad = \quad \frac{2u^{\frac{1}{\alpha}}}{1+u^{\frac{1}{\alpha}}},$$

that is,

$$\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})} \quad = \quad \left(\left(\frac{2u^{\frac{1}{\alpha}}}{1+u^{\frac{1}{\alpha}}}\right)^{-\frac{1}{b}}-1\right)^{-\frac{1}{a}}. \tag{8}$$

The expression further simplifies to

$$G(x;\underline{\boldsymbol{\psi}}) \quad = \quad \left(\left(\left(\frac{2u^{\frac{1}{\alpha}}}{1+u^{\frac{1}{\alpha}}}\right)^{-\frac{1}{b}}-1\right)^{-\frac{1}{a}}+1\right)^{-1},$$

and therefore, the quantile function is given by

$$Q_{X_{(n)}}(u) = G^{-1}\left[\left(\left(\left(\frac{2u^{\frac{1}{\alpha}}}{1+u^{\frac{1}{\alpha}}}\right)^{-\frac{1}{b}}-1\right)^{-\frac{1}{a}}+1\right)^{-1}\right], \tag{9}$$

which can be solved using iterative methods.

2.3. Series Expansion of the Density Function

In this section, series expansion of the PDF is presented. The linear representation of the PDF allows for useful mathematical and statistical properties to be derived. Recall that the PDF of EHL-OBIII-G family of distributions is given by

$$\begin{aligned} f(x;a,b,\alpha,\underline{\boldsymbol{\psi}}) \quad &= \quad 2\alpha ab\left(\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right)^{\alpha-1}\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b-1}\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a-1} \\ &\times \quad \left(1+\left[1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)}\frac{g(x;\underline{\boldsymbol{\psi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\psi}})\right)^{2}}, \end{aligned}$$

and using the expansion

$$\begin{aligned}\left(1+\left[1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)} &= \sum_{p=0}^{\infty}(-1)^{p}\frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \\ &\times \left[1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right]^{p} \\ &= \sum_{p,q=0}^{\infty}(-1)^{p+q}\frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!}\binom{p}{q} \\ &\times \left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-bq},\end{aligned}$$

we can write the EHL-OBIII-G PDF as

$$\begin{aligned}f(x;a,b,\alpha,\underline{\boldsymbol{\psi}}) &= 2\alpha ab\sum_{p,q=0}^{\infty}(-1)^{p+q}\frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!}\binom{p}{q} \\ &\times \left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b(q+\alpha)-1}\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a-1}\frac{g(x;\underline{\boldsymbol{\psi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\psi}})\right)^{2}}.\end{aligned}$$

Furthermore, by applying the expansion

$$\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b(q+\alpha)-1} = \sum_{l=0}^{\infty}\binom{-b(q+\alpha)-1}{l}\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-al},$$

we get

$$\begin{aligned}f(x;a,b,\alpha,\underline{\boldsymbol{\psi}}) &= 2\alpha ab\sum_{p,q,l=0}^{\infty}(-1)^{p+q}\frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!}\binom{p}{q}\binom{-b(q+\alpha)-1}{l} \\ &\times \left(G(x;\underline{\boldsymbol{\psi}})\right)^{-a(l+1)-1}\frac{g(x;\underline{\boldsymbol{\psi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\psi}})\right)^{-a(l+1)+1}}.\end{aligned}$$

Using the expansion

$$\left(G(x;\underline{\boldsymbol{\psi}})\right)^{-a(l+1)-1} = \left(1-\overline{G}(x;\underline{\boldsymbol{\psi}})\right)^{-a(l+1)-1} = \sum_{j=0}^{\infty}\binom{-a(l+1)-1}{j}(-1)^{j}\overline{G}^{j}(x;\underline{\boldsymbol{\psi}}),$$

we have

$$
\begin{aligned}
f(x;a,b,\alpha,\underline{\psi}) &= 2\alpha ab \sum_{p,q,l,j=0}^{\infty} (-1)^{p+q+j} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \binom{-b(q+\alpha)-1}{l} \\
&\times \binom{-a(l+1)-1}{j} g(x;\underline{\psi}) \left(\overline{G}(x;\underline{\psi})\right)^{a(l+1)-1+j} \\
&= 2\alpha ab \sum_{p,q,l,j,k=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \binom{-b(q+\alpha)-1}{l} \\
&\times \binom{-a(l+1)-1}{j} \binom{a(l+1)-1+j}{k} g(x;\underline{\psi}) \left(G(x;\underline{\psi})\right)^{k} \\
&= 2\alpha ab \sum_{p,q,l,j,k=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \binom{-b(q+\alpha)-1}{l} \\
&\times \binom{-a(l+1)-1}{j} \binom{a(l+1)-1+j}{k} \frac{k+1}{k+1} g(x;\underline{\psi}) \left(G(x;\underline{\psi})\right)^{k} \\
&= \sum_{k=0}^{\infty} w_k g_k(x;\underline{\psi}), \qquad (10)
\end{aligned}
$$

where $g_k(x;\underline{\psi}) = (k+1)\left(G(x;\underline{\psi})\right)^k g(x;\underline{\psi})$ is the exponentiated-G (Exp-G) distribution with power parameter k and

$$
\begin{aligned}
w_k &= \sum_{p,q,l,j=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(\alpha+1+p)}{\Gamma(\alpha+1)p!} \binom{p}{q} \binom{-b(q+\alpha)-1}{l} \\
&\times \binom{-a(l+1)-1}{j} \binom{a(l+1)-1+j}{k} \frac{2\alpha ab}{k+1}. \qquad (11)
\end{aligned}
$$

2.4. Some Special Cases

We considered some special cases by changing the baseline distribution function $G(x;\underline{\psi})$ to flexible distributions. The parameter vector space is limited to atmost 2 component vector to avoid over parametrization and redundancy.

2.4.1. EHL-OBIII-Log-Logistic (EHL-OBIII-LLoG) distribution

Consider the log-logistic distribution as the baseline distribution with parameter $\lambda > 0$ having CDF and PDF $G(x;\lambda) = 1-(1+x^{\lambda})^{-1}$ and $g(x;\lambda) = \lambda x^{\lambda-1}(1+x^{\lambda})^{-2}$, respectively. The CDF, PDF and HRF of EHL-OBIII-LLoG distribution are given by

$$
F(x;a,b,\alpha,\lambda) = \left(\frac{\left(1+\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a}\right)^{-b}\right]} \right)^{\alpha}, \qquad (12)
$$

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$$
\begin{aligned}
f(x) &= 2\alpha ab\left(\left(1+\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a}\right)^{-b}\right)^{\alpha-1}\left(1+\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a}\right)^{-b-1}\\
&\times\left(1+\left[1-\left(1+\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)}\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a-1}\\
&\times\frac{\lambda x^{\lambda-1}(1+x^{\lambda})^{-2}}{((1+x^{\lambda})^{-1})^{2}},
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
h(x) &= 2\alpha ab\left(\left(1+\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a}\right)^{-b}\right)^{\alpha-1}\left(1+\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a}\right)^{-b-1}\\
&\times\left(1+\left[1-\left(1+\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)}\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a-1}\\
&\times\frac{\lambda x^{\lambda-1}(1+x^{\lambda})^{-2}}{((1+x^{\lambda})^{-1})^{2}}\left(1-\left(\frac{\left(1+\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{1-(1+x^{\lambda})^{-1}}{(1+x^{\lambda})^{-1}}\right)^{-a}\right)^{-b}\right]}\right)^{\alpha}\right)^{-1},
\end{aligned}
$$

respectively, for a, b, $\alpha, \lambda > 0$.

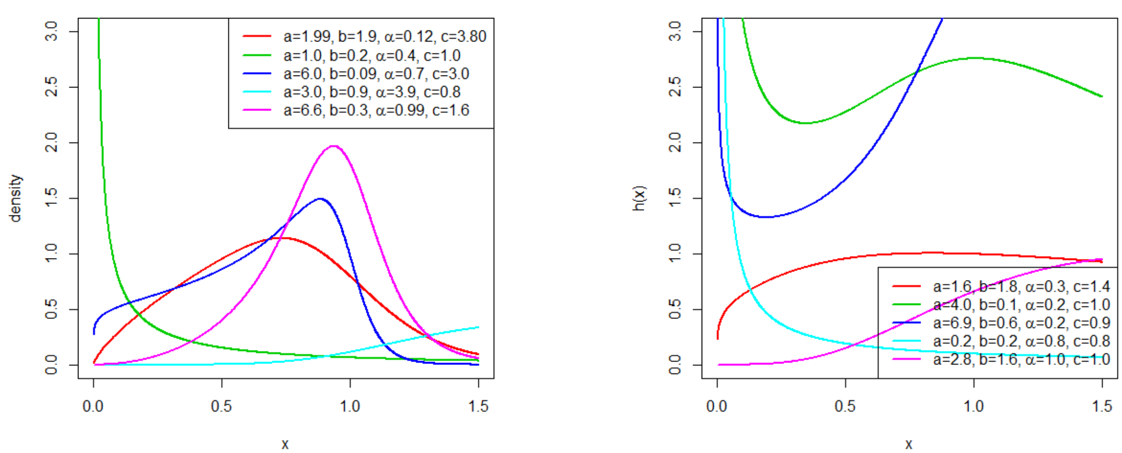

Figure 1: Density and HRF plots for EHL-OBIII-LLoG distribution

Figure 1 illustrates the flexible nature of the EHL-OBIII-LLoG distribution for selected parameter values. The PDFs of the EHL-OBIII-LLoG distribution can adopt various shapes that include reverse-J, uni-modal, left or right skewed shapes. In addition, the EHL-OBIII-LLoG distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

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2.4.2. EHL-OBIII-Exponential (EHL-OBIII-E) distribution

Let the exponential distribution be the baseline distribution with PDF and CDF given by $g(x;\lambda) = \lambda e^{-\lambda x}$ and $G(x;\lambda) = 1 - e^{-\lambda x}$, respectively, for $\lambda > 0$, then the CDF, PDF and HRF of the EHL-OBIII-E distribution are given by

$$F(x;a,b,\alpha,\lambda) = \left(\frac{\left(1+\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a}\right)^{-b}\right]} \right)^{\alpha}, \tag{14}$$

$$\begin{aligned} f(x) \;&=\; 2\alpha ab \left(\left(1+\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a}\right)^{-b}\right)^{\alpha-1} \left(1+\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a}\right)^{-b-1} \\ &\times \left(1+\left[1-\left(1+\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)} \left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a-1} \\ &\times \frac{\lambda e^{-\lambda x}}{\left(e^{-\lambda x}\right)^2} \end{aligned} \tag{15}$$

and

$$\begin{aligned} h(x) \;&=\; 2\alpha ab \left(\left(1+\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a}\right)^{-b}\right)^{\alpha-1} \left(1+\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a}\right)^{-b-1} \\ &\times \left(1+\left[1-\left(1+\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)} \left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a-1} \frac{\lambda e^{-\lambda x}}{\left(e^{-\lambda x}\right)^2} \\ &\times \left(1-\left(\frac{\left(1+\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{1-e^{-\lambda x}}{e^{-\lambda x}}\right)^{-a}\right)^{-b}\right]} \right)^{\alpha}\right)^{-1}, \end{aligned}$$

respectively, for a, b, $\alpha, \lambda > 0$.

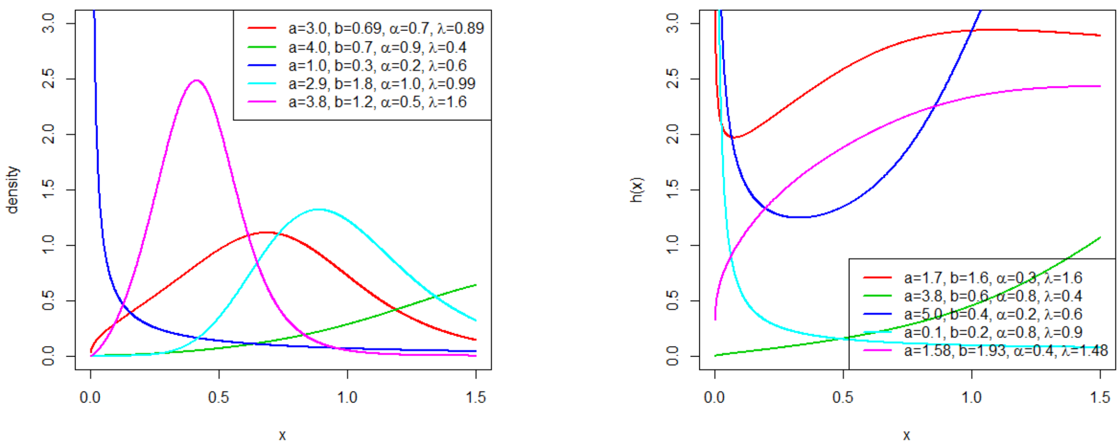

Figure 2: Density and HRF plots for EHL-OBIII-E distribution

Figure 2 demonstrate the efficacy of the EHL-OBIII-E distribution for selected parameter values. The PDFs of the EHL-OBIII-E distribution takes various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the EHL-OBIII-E distribution gives decreasing, increasing and bathtub followed by upside down bathtub shapes.

2.4.3. EHL-OBIII-Lomax (EHL-OBIII-Lx) distribution

Suppose that we take the baseline distribution to be a Lomax distribution with CDF and PDF given by $G(x;\gamma,\lambda) = 1-(1+\lambda x)^{-\gamma}$ and $g(x;\gamma,\lambda) = \gamma\lambda\,(1+\lambda x)^{-\gamma-1}$, for $\gamma,\lambda > 0$, and $x > 0$, then we obtain the EHL-OBIII-Lx distribution with CDF, PDF and HRF given by

$$F(x;a,b,\alpha,\gamma,\lambda) = \left(\frac{\left(1+\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a}\right)^{-b}\right]}\right)^{\alpha}, \tag{16}$$

$$\begin{aligned} f(x) &= 2\alpha ab\left(\left(1+\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a}\right)^{-b}\right)^{\alpha-1}\left(1+\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a}\right)^{-b-1} \\ &\times\left(1+\left[1-\left(1+\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)}\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a-1} \\ &\times\frac{\gamma\lambda\,(1+\lambda x)^{-\gamma-1}}{((1+\lambda x)^{-\gamma})^{2}} \end{aligned} \tag{17}$$

and

$$
\begin{aligned}
h(x) \;=\; & 2\alpha ab\left(\left(1+\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a}\right)^{-b}\right)^{\alpha-1}\left(1+\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a}\right)^{-b-1}\\
\times\; & \left(1+\left[1-\left(1+\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)}\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a-1}\\
\times\; & \frac{\gamma\lambda\,(1+\lambda x)^{-\gamma-1}}{\left((1+\lambda x)^{-\gamma}\right)^{2}}\left(1-\left(\frac{\left(1+\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{1-(1+\lambda x)^{-\gamma}}{(1+\lambda x)^{-\gamma}}\right)^{-a}\right)^{-b}\right]}\right)^{\alpha}\right)^{-1},
\end{aligned}
$$

respectively, for a, b, $\alpha, \gamma, \lambda > 0$.

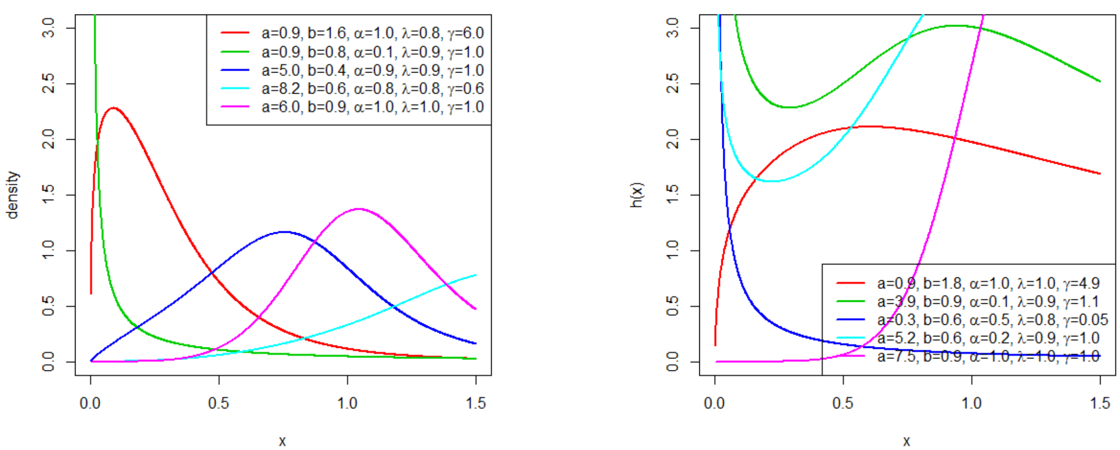

Figure 3: Density and HRF plots for EHL-OBIII-Lx distribution

From Figure 3, we note tha the EHL-OBIII-Lx distribution can adopt various flexible shapes and these include reverse-J, uni-modal, left or right skewed shapes. It can also be noted that the HRF of the EHL-OBIII-Lx distribution gives decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

2.5. EHL-OBIII-Lindley (EHL-OBIII-L) distribution

If we let Lindley distribution be the baseline distribution with PDF and CDF given by $g(x;\lambda) = \frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}$ and $G(x;\lambda) = 1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}$, respectively, for $\lambda > 0$, then the CDF, PDF and HRF of the EHL-OBIII-L distribution are given by

$$
F(x;a,b,\alpha,\lambda) = \left(\frac{\left(1+\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a}\right)^{-b}\right]}\right)^{\alpha}, \tag{18}
$$

$$
\begin{aligned}
f(x;a,b,\alpha,\lambda) &= 2\alpha ab\left(\left(1+\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a}\right)^{-b}\right)^{\alpha-1}\left(1+\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a}\right)^{-b-1} \\
&\times \left(1+\left[1-\left(1+\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)} \\
&\times \left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a-1}\frac{\frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}}{\left((1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right)^2} \qquad (19)
\end{aligned}
$$

and

$$
\begin{aligned}
h(x;a,b,\alpha,\lambda) &= 2\alpha ab\left(\left(1+\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a}\right)^{-b}\right)^{\alpha-1}\left(1+\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a}\right)^{-b-1} \\
&\times \left(1+\left[1-\left(1+\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha+1)} \\
&\times \left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a-1}\frac{\frac{\lambda^2}{(1+\lambda)}(1+x)e^{-\lambda x}}{\left((1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}\right)^2} \\
&\times \left(1-\left(\frac{\left(1+\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a}\right)^{-b}}{1+\left[1-\left(1+\left(\frac{1-(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}{(1+\frac{\lambda x}{1+\lambda})e^{-\lambda x}}\right)^{-a}\right)^{-b}\right]}\right)^{\alpha}\right)^{-1},
\end{aligned}
$$

respectively, for $a, b, \beta, \lambda > 0$.

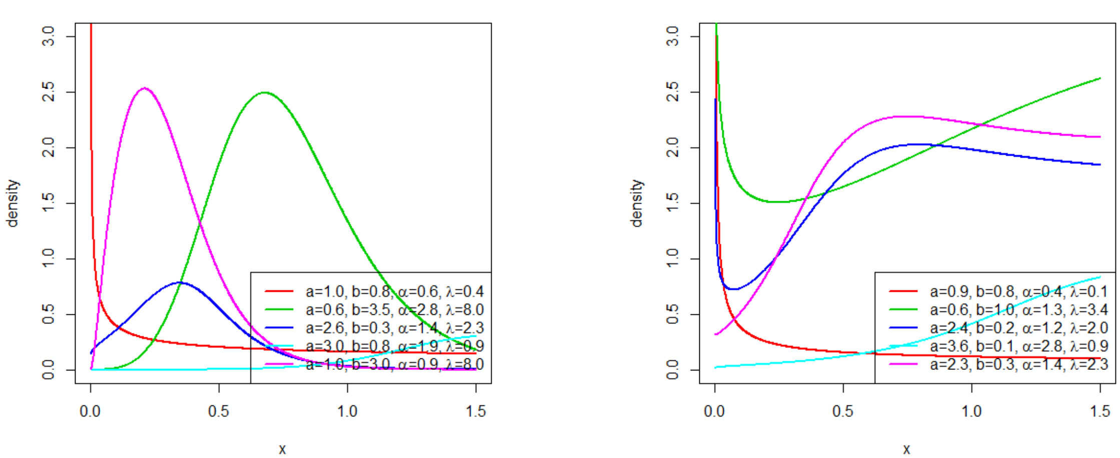

Figure 4: Density and HRF plots for EHL-OBIII-L distribution

Figure 4 reveals the flexibility of the EHL-OBIII-L distribution. The PDFs of the EHL-OBIII-L distribution can flexibly take different shapes that include reverse-J, uni-modal, left or right skewed shapes. Additionally, the EHL-OBIII-L distribution HRF plots also give decreasing, increasing, bathtub and upside down bathtub shapes and decreasing-increasing-decreasing shapes.

3. Some Mathematical Properties

In this section, we derive some useful mathematical and statistical properties for the EHL-OBIII-G family of distributions such as moments, conditional moments, Lorenz and Bonferroni curves, probability weighted moments, Rényi entropy and maximum likelihood estimates.

3.1. Moments

Let $Y_k \sim$ Exp-G(k), then using equation (10) the s^{th} raw moment, μ_s' of the EHL-OBIII-G family of distributions is obtained as

$$\mu_s' = E(X^s) \quad = \quad \int_{-\infty}^{\infty} x^s f(x)dx = \sum_{k=0}^{\infty} w_k E(Y_k^s),$$

where w_k is given by equation (11). We presnt the first five moments for the EHL-OBIII-E distibution and the corresponding standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) from selected values. These results are given under Table 2. The quantile values are also presented under Table 1.

Table 1: Quantiles for EHL-OBIII-E Distribution for selected parameter values

	(a,b,α, λ)					
u	(1.4,8.5,6.1,0.4)	(1.3,8.3,6.4,0.6)	(1.1,9.9,5.8,0.8)	(1.6,9.1,6,0.3)	(1.2,8.2,6.2,0.6)	(1.1,8.9,5.7,0.2)
0.1	3.0531	2.0120	1.7548	3.7952	2.1975	7.8777
0.2	2.3981	1.6071	1.3186	3.0811	1.6747	5.3712
0.3	2.1813	1.4615	1.1694	2.8329	1.5039	4.7265
0.4	2.0543	1.3753	1.0823	2.6859	1.4046	4.3598
0.5	1.9664	1.3154	1.0224	2.5836	1.3362	4.1103
0.6	1.9001	1.2702	0.9775	2.5062	1.2847	3.9244
0.7	1.8474	1.2341	0.9419	2.4444	1.2440	3.7779
0.8	1.8040	1.2043	0.9128	2.3933	1.2104	3.6580
0.9	1.7672	1.1791	0.8882	2.3499	1.1821	3.5572

Table 2: Moments for EHL-OBIII-E Distribution for selected parameter values

	(a,b,α, λ)				
	(0.8,1.4,0.6,1.5)	(0.3,1.7,1.1,1.8)	(0.8,2.2,0.7,2.4)	(0.4,2.3,0.7,1.7)	(0.5,1.5,0.7,0.9)
$E(X)$	0.2298	0.0843	0.3063	0.1210	0.1169
$E(X^2)$	0.1378	0.0525	0.1930	0.0746	0.0692
$E(X^3)$	0.0980	0.0384	0.1395	0.0543	0.0493
$E(X^4)$	0.0759	0.0304	0.1088	0.0427	0.0383
$E(X^5)$	0.0619	0.0252	0.0889	0.0352	0.0313
SD	0.2916	0.2131	0.3149	0.2449	0.2356
CV	1.2691	2.5277	1.0280	2.0239	2.0157
CS	1.1002	2.7252	0.6298	2.0916	2.1548
CK	2.9224	9.4740	2.0402	6.2188	6.6049

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3.2. Probability Weighted Moments (PWMs)

The $(i,\eta)^{th}$ Probability Weighted Moment (PWM) of X denoted by $\kappa_{i,\eta}$ is

$$\kappa_{i,\eta}=E(X^i(F(X))^\eta)=\int_{-\infty}^{\infty}x^i(F(x))^\eta f(x)dx.$$

Using equations (5) and (6), we can write

$$\begin{aligned}
f(x)(F(x))^\eta &= 2\alpha ab\left(\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right)^{\alpha(\eta+1)-1}\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b-1}\\
&\times \left(1+\left[1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha(\eta+1)+1)}\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a-1}\\
&\times \frac{g(x;\underline{\boldsymbol{\psi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\psi}})\right)^2}.
\end{aligned}$$

Using the series expansion

$$\begin{aligned}
\left(1+\left[1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha(\eta+1)+1)} &= \sum_{p=0}^{\infty}(-1)^p\frac{\Gamma(\alpha(\eta+1)+1+p)}{\Gamma(\alpha(\eta+1)+1)p!}\\
&\times \left[1-\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right]^p\\
&= \sum_{p,q=0}^{\infty}(-1)^{p+q}\frac{\Gamma(\alpha(\eta+1)+1+p)}{\Gamma(\alpha(\eta+1)+1)p!}\binom{p}{q}\\
&\times \left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-bq}
\end{aligned}$$

we can write

$$\begin{aligned}
f(x)(F(x))^\eta &= 2\alpha ab\sum_{p,q=0}^{\infty}(-1)^{p+q}\frac{\Gamma(\alpha(\eta+1)+1+p)}{\Gamma(\alpha(\eta+1)+1)p!}\binom{p}{q}\\
&\times \left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b(q+(\alpha(\eta+1)))-1}\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a-1}\frac{g(x;\underline{\boldsymbol{\psi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\psi}})\right)^2}.
\end{aligned}$$

Furthermore, applying the series expansion

$$\left(1+\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b(q+(\alpha(\eta+1)))-1} = \sum_{l=0}^{\infty}\binom{-b(q+(\alpha(\eta+1)))-1}{l}\left(\frac{G(x;\underline{\boldsymbol{\psi}})}{\overline{G}(x;\underline{\boldsymbol{\psi}})}\right)^{-al},$$

we get

$$
\begin{aligned}
f(x)(F(x))^{\eta} &= 2\alpha ab \sum_{p,q,l=0}^{\infty} (-1)^{p+q} \frac{\Gamma(\alpha(\eta+1)+1+p)}{\Gamma(\alpha(\eta+1)+1)p!} \binom{p}{q} \binom{-b(q+(\alpha(\eta+1)))-1}{l} \\
&\times \left(G(x;\underline{\boldsymbol{\psi}})\right)^{-a(l+1)-1} \frac{g(x;\underline{\boldsymbol{\psi}})}{\left(\overline{G}(x;\underline{\boldsymbol{\psi}})\right)^{-a(l+1)+1}}.
\end{aligned}
$$

Using the expansion

$$
\left(G(x;\underline{\boldsymbol{\psi}})\right)^{-a(l+1)-1} = \left(1-\overline{G}(x;\underline{\boldsymbol{\psi}})\right)^{-a(l+1)-1} = \sum_{j=0}^{\infty} \binom{-a(l+1)-1}{j} (-1)^j \overline{G}^j(x;\underline{\boldsymbol{\psi}}),
$$

we have

$$
\begin{aligned}
f(x)(F(x))^{\eta} &= 2\alpha ab \sum_{p,q,l,j=0}^{\infty} (-1)^{p+q+j} \frac{\Gamma(\alpha(\eta+1)+1+p)}{\Gamma(\alpha(\eta+1)+1)p!} \binom{p}{q} \binom{-b(q+(\alpha(\eta+1)))-1}{l} \\
&\times \binom{-a(l+1)-1}{j} g(x;\underline{\boldsymbol{\psi}}) \left(\overline{G}(x;\underline{\boldsymbol{\psi}})\right)^{a(l+1)-1+j} \\
&= 2\alpha ab \sum_{p,q,l,j,k=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(\alpha(\eta+1)+1+p)}{\Gamma(\alpha(\eta+1)+1)p!} \binom{p}{q} \binom{-b(q+(\alpha(\eta+1)))-1}{l} \\
&\times \binom{-a(l+1)-1}{j} \binom{a(l+1)-1+j}{k} g(x;\underline{\boldsymbol{\psi}}) \left(G(x;\underline{\boldsymbol{\psi}})\right)^{k} \\
&= \sum_{w=0}^{\infty} w_k^* g_k(x;\underline{\boldsymbol{\psi}}), \qquad (20)
\end{aligned}
$$

where $g_k(x;\underline{\boldsymbol{\psi}}) = (k+1)\left(G(x;\underline{\boldsymbol{\psi}})\right)^k g(x;\underline{\boldsymbol{\psi}})$ is the exponentiated-G (Exp-G) distribution with power parameter k and

$$
\begin{aligned}
w_k^* &= \sum_{p,q,l,j=0}^{\infty} (-1)^{p+q+j+k} \frac{\Gamma(\alpha(\eta+1)+1+p)}{\Gamma(\alpha(\eta+1)+1)p!} \binom{p}{q} \binom{-b(q+(\alpha(\eta+1)))-1}{l} \\
&\times \binom{-a(l+1)-1}{j} \binom{a(l+1)-1+j}{k} \frac{2\alpha ab}{k+1}. \qquad (21)
\end{aligned}
$$

Finally, the PWMs of the EHL-OBIII-G family of distributions can be written as

$$
\kappa_{i,\eta} = \int_0^{\infty} x^i \sum_{k=0}^{\infty} w_k^* g_k(x;\underline{\boldsymbol{\psi}}) dx \sum_{k=0}^{\infty} w_k^* \int_{-\infty}^{\infty} x^i g_k(x;\underline{\boldsymbol{\psi}}) dx,
$$

which shows that the $(i,\eta)^{th}$ PWMs of EHL-OBIII-G family of distributions can be obtained from the moments of the E-G distribution.

3.2.1. Rényi Entropy

The Rényi entropy ((22)), is an extension of Shannon Entropy ((24)) and is defined as

$$
I_R(v) = \frac{1}{1-v} \log\left(\int_0^{\infty} [f(x;a,b,\alpha,\underline{\boldsymbol{\psi}})]^v dx\right), v \neq 1, v > 0. \qquad (22)
$$

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Note that as $v \to 1$, the Rényi entropy tends to Shannon entropy and $I_R(v)$ for the EHL-OBIII-G distribution can be written as

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^\infty \left[(2\alpha ab)^v \left(\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right)^{v\alpha-v} \left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-vb-v}\right.\right.$$
$$\left.\left.\times\left(1+\left[1-\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right]\right)^{-v(\alpha+1)} \left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-va-v} \frac{g^v(x;\underline{\psi})}{\left(\overline{G}(x;\underline{\psi})\right)^{2v}}\right]dx\right).$$

Using the generalized binomial series expansion

$$\begin{aligned}\left(1+\left[1-\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right]\right)^{-v(\alpha+1)} &= \sum_{p=0}^{\infty}(-1)^p\frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!} \\ &\times \left[1-\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right]^p \\ &= \sum_{p,q=0}^{\infty}(-1)^{p+q}\frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!}\binom{p}{q} \\ &\times \left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-bq},\end{aligned}$$

we can write

$$\begin{aligned}f^v(x;a,b,\alpha,\underline{\psi}) &= (2\alpha ab)^v\sum_{p,q=0}^{\infty}(-1)^{p+q}\frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!}\binom{p}{q} \\ &\times \left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b(q+v+(v\alpha-v))-1}\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-va-v}\frac{g^v(x;\underline{\psi})}{\left(\overline{G}(x;\underline{\psi})\right)^{2v}}.\end{aligned}$$

Furthermore, applying the expansion

$$\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b(q+v+(v\alpha-v))-1} = \sum_{l=0}^{\infty}\binom{-b(q+v+(v\alpha-v))-1}{l}\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-al},$$

we get

$$\begin{aligned}f^v(x;a,b,\alpha,\underline{\psi}) &= (2\alpha ab)^v\sum_{p,q=0}^{\infty}(-1)^{p+q}\frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!}\binom{p}{q}\binom{-b(q+v+(v\alpha-v))-1}{l} \\ &\times \left(G(x;\underline{\psi})\right)^{-a(l+v)-v}\frac{g(x;\underline{\psi})}{\left(\overline{G}(x;\underline{\psi})\right)^{-a(l+v)+v}}.\end{aligned}$$

Using the generalized binomial series expansion

$$\big(G(x;\underline{\boldsymbol{\psi}})\big)^{-a(l+v)-v}=\big(1-\overline{G}(x;\underline{\boldsymbol{\psi}})\big)^{-a(l+v)-v}=\sum_{j=0}^{\infty}\binom{-a(l+v)-v}{j}(-1)^j\overline{G}^j(x;\underline{\boldsymbol{\psi}}),$$

we have

$$\begin{aligned} f^v(x;a,b,\alpha,\underline{\boldsymbol{\psi}}) &= (2\alpha ab)^v\sum_{p,q,j=0}^{\infty}(-1)^{p+q+j}\frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!}\binom{p}{q}\binom{-b(q+v+(v\alpha-v))-1}{l}\\ &\times \binom{-a(l+v)-v}{j}g^v(x;\underline{\boldsymbol{\psi}})\big(\overline{G}(x;\underline{\boldsymbol{\psi}})\big)^{a(l+v)-v+j}\\ &= (2\alpha ab)^v\sum_{p,q,j,k=0}^{\infty}(-1)^{p+q+j+k}\frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!}\binom{p}{q}\binom{-b(q+v+(v\alpha-v))-1}{l}\\ &\times \binom{a(l+v)-v+j}{k}\binom{-a(l+v)-v}{j}g^v(x;\underline{\boldsymbol{\psi}})\,\big(G(x;\underline{\boldsymbol{\psi}})\big)^k. \end{aligned} \tag{23}$$

Finally, we can write the Rényi entropy for the EHL-OBIII-G family of distributions as

$$\begin{aligned} I_R(v) &= \frac{1}{1-v}\log\Bigg((2\alpha ab)^v\sum_{p,q,j,k=0}^{\infty}(-1)^{p+q+j+k}\frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!}\binom{p}{q}\binom{-b(q+v+(v\alpha-v))-1}{l}\\ &\times\binom{a(l+v)-v+j}{k}\binom{-a(l+v)-v}{j}\frac{1}{\left[1+\frac{k}{v}\right]^v}\int_0^\infty\left(\left[1+\frac{k}{v}\right]G(x;\underline{\boldsymbol{\psi}})^{\frac{k}{v}}g(x;\underline{\boldsymbol{\psi}})dx\right)^v\Bigg)\\ &= \frac{1}{1-v}\log\left[\sum_{k=0}^{\infty}w_k^{**}\exp((1-v)I_{REG})\right], \end{aligned}$$

for $v>0$, $v\neq 1$, where $I_{REG}=\frac{1}{1-v}\log\left[\int_0^\infty\left(\left[\frac{k}{v}+1\right]\left(G(x;\underline{\boldsymbol{\psi}})\right)^{\frac{k}{v}}g(x;\underline{\boldsymbol{\psi}})\right)^v dx\right]$ is the Rényi entropy of Exp-G distribution with power parameter $\left(\frac{k}{v}+1\right)$, and

$$\begin{aligned} w_k^{**} &= \frac{1}{1-v}\log\Bigg((2\alpha ab)^v\sum_{p,q,j=0}^{\infty}(-1)^{p+q+j+k}\frac{\Gamma(v(\alpha+1)+p)}{\Gamma(v(\alpha+1))p!}\binom{p}{q}\binom{-b(q+v+(v\alpha-v))-1}{l}\\ &\times\binom{a(l+v)-v+j}{k}\binom{-a(l+v)-v}{j}\frac{1}{\left[1+\frac{k}{v}\right]^v}. \end{aligned}$$

3.3. Stochastic Ordering

The stochastic ordering for random variables is a common and widely used concept. The usual stochastic order, the hazard rate order and the likelihood ratio order are the commonly known orders of distribution functions. These three orders are defined as below.

Let X and Y be the two random variables with the CDFs $F_x(t)$ and $F_y(t)$, respectively, and $\overline{F}_x(t)=1-F_x(t)$ is the reliability or survival function. The random variable X is said to be stochastically smaller than the random variable Y if $\overline{F}_x(t)\leq\overline{F}_y(t)$ for all t or $F_x(t)\geq F_y(t)$ for all t. This is denoted by $X<_{st}Y$. The hazard rate order and likelihood ratio order are stronger and are given by $X<_{hr}Y$ if $h_x(t)\geq h_y(t)$ for all t, and $X<_{\ell_r}Y$ if $\frac{f_x(t)}{f_y(t)}$ is decreasing in t. It is well established that $X<_{\ell_r}Y \implies X<_{hr}Y \implies X<_{st}Y$, (see (23) for additional details).

In this section, the likelihood ratio ordering is presented as follows. If we let X_1 and X_2 be the two independent random variables following $EHL-OBIII-G(a,b,\alpha_1,\underline{\varphi})$ and $EHL-OBIII-G(a,b,\alpha_2,\underline{\varphi})$ distributions, then the PDFs are given by

$$\begin{aligned} f_1(x) &= 2\alpha_1 ab\left(\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right)^{\alpha_1-1}\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b-1} \\ &\times \left(1+\left[1-\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha_1+1)}\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a-1}\frac{g(x;\underline{\psi})}{\left(\overline{G}(x;\underline{\psi})\right)^2} \end{aligned}$$

and

$$\begin{aligned} f_2(x) &= 2\alpha_2 ab\left(\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right)^{\alpha_2-1}\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b-1} \\ &\times \left(1+\left[1-\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}\right]\right)^{-(\alpha_2+1)}\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a-1}\frac{g(x;\underline{\psi})}{\left(\overline{G}(x;\underline{\psi})\right)^2}. \end{aligned}$$

The ratio, $\frac{f_1(x)}{f_2(x)}$ takes the form

$$\frac{f_1(x)}{f_2(x)} = \frac{\alpha_1}{\alpha_2}z^{\alpha_1-\alpha_2}\left(1+[1-z]\right)^{-(\alpha_1-\alpha_2+2)}, \tag{24}$$

where

$$z=\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b}.$$

If we differentiate equation (24) with respect to x, we get

$$\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{\alpha_1}{\alpha_2}\left(\alpha_1-\alpha_2\right)z^{\alpha_1-\alpha_2}\left(1+[1-z]\right)^{-(\alpha_1-\alpha_2+2)}\left[\frac{1}{z}+\frac{(1+2(\alpha_1-\alpha_2))}{(1+[1-z])}\right],$$

where $z' = ab\left(1+\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a}\right)^{-b-1}\left(\frac{G(x;\underline{\psi})}{\overline{G}(x;\underline{\psi})}\right)^{-a-1}\frac{g(x;\underline{\psi})}{\left(\overline{G}(x;\underline{\psi})\right)^2}$, and finally if $\alpha_2<\alpha_1$, then $\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right)<0$, and therefore, the likelihood ratio order $X_1 <_{\ell_r} X_2$ exists. Consequently, $X_1 <_{st} X_2$, since $X_1 <_{\ell_r} X_2 \implies X_1 <_{hr} X_2 \implies X_1 <_{st} X_2$, where $X_1 <_{hr} X_2$ and $X_1 <_{st} X_2$ denote hazard rate order and stochastic order, respectively.

3.4. Maximum Likelihood Estimation

In this section, the method of maximum likelihood estimation (MLE) is used to estimate the model parameters. Let $X \sim EHL-OBIII-G(a,b,\alpha,\underline{\psi})$ and $\mathbf{\Delta}=(a,b,\alpha,\underline{\psi})^T$ be the vector of model parameters. The log-likelihood function $\ell_n=\ell_n(\mathbf{\Delta})$ based on a random sample of size n from the EHL-OBIII-G family of distributions is given by

$$
\begin{aligned}
\ell_n(\Delta) &= n\ln(2\alpha ab) + (\alpha-1)\sum_{i=1}^{n}\ln\left(\left(1+\left(\frac{G(x_i;\underline{\psi})}{\overline{G}x_i;\underline{\psi})}\right)^{-a}\right)^{-b}\right)(-b-1)\sum_{i=1}^{n}\ln\left(1+\left(\frac{G(x_i;\underline{\psi})}{\overline{G}(x_i;\underline{\psi})}\right)^{-a}\right)\\
&+ (-a-1)\sum_{i=1}^{n}\ln\left(\frac{G(x_i;\underline{\psi})}{\overline{G}(x_i;\underline{\psi})}\right)(\alpha+1)\sum_{i=1}^{n}\ln\left(1+\left[1-\left(1+\left(\frac{G(x_i;\underline{\psi})}{\overline{G}(x_i;\underline{\psi})}\right)^{-a}\right)^{-b}\right]\right)\\
&+ \sum_{i=1}^{n}\ln\left(g(x_i;\underline{\psi})\right) - 2\sum_{i=1}^{n}\ln\left(\overline{G}(x_i;\underline{\psi})\right).
\end{aligned}
$$

The elements of the score vector are given in the **Appendix** A. Note that these system of non-linear equations have no closed form. To obtain the estimates of model parameters denoted by $\hat{\mathbf{\Delta}}$, the Newton-Raphson procedure is used to solve the system of non-linear equations $(\frac{\partial\ell_n}{\partial a}, \frac{\partial\ell_n}{\partial b}, \frac{\partial\ell_n}{\partial\alpha}, \frac{\partial\ell_n}{\partial\underline{\psi}_k})^T = \mathbf{0}$. The multivariate normal distribution $N_{q+3}(\underline{0}, J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}} = (0,0,0,\underline{0})^T$ and $J(\hat{\mathbf{\Delta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\mathbf{\Delta}}$, which is critical in approximating confidence intervals for model parameters.

4. Monte Carlo Simulations

Inorder to evaluate consistency of the MLEs, simulation experiments were conducted using the EHL-OBIII-E distribution. The MLEs are computed for each sample by generationg N=1000 random samples of size n obtaining average bias and Root Mean Square Error (RMSE) from each sample. The results are presented under Table 3. We note that as the sample size n increase the values of average bias and RMSE decreases which satisfies convergence propoerties of consistent and unbiased estimators.

Table 3: Simulation Results for EHL-OBIII-E Distribution; Mean, Average Bias and RMSE

Parameter	Sample Size	Set I: a=1.4, b=0.6 , α=1.8 , λ=2.1			Set II: a=2.6, b=0.9, α=1.4 , λ=0.7			Set III: a=1.8, b=1.3 , α=0.8 , λ=2.3		
		Mean	Bias	RMSE	Mean	Bias	RMSE	Mean	Bias	RMSE
a	50	4.3848	2.9848	0.5194	4.1738	1.5738	3.0836	3.7459	1.9459	0.3548
	100	4.2985	2.8985	0.3322	3.4377	0.8377	2.2148	3.3726	1.5726	0.2216
	200	3.6489	2.2489	0.2997	3.1527	0.5527	1.8854	2.7388	0.9388	0.1970
	400	3.4375	2.0375	0.2510	2.9376	0.3376	1.6376	2.4628	0.6628	0.1744
	800	2.3385	0.9385	0.2250	2.8365	0.2365	0.6049	1.9285	0.1285	0.1377
	1200	1.6028	0.2028	0.1488	2.6838	0.0838	0.4307	1.8827	0.0827	0.1164
	1800	1.4204	0.0204	0.1325	2.6184	0.0184	0.2819	1.8327	0.0327	0.0895
b	50	1.7489	1.1489	0.4658	1.9264	1.0264	0.7609	2.6374	1.3374	0.5840
	100	1.3385	0.7385	0.4335	1.7256	0.8256	0.6576	2.3787	1.0787	0.4755
	200	1.2538	0.6538	0.3983	1.2737	0.3737	0.6272	1.8266	0.5266	0.4579
	400	1.1439	0.5439	0.3655	1.1828	0.2828	0.5864	1.5218	0.2218	0.4353
	800	0.8399	0.2399	0.3536	1.1029	0.2029	0.5499	1.4375	0.1375	0.3847
	1200	0.7039	0.1039	0.2977	0.9938	0.0938	0.5116	1.3848	0.0848	0.3586
	1800	0.6298	0.0298	0.2723	0.9204	0.0204	0.4485	1.3265	0.0265	0.2073
α	50	3.1939	1.3939	0.2710	2.1654	0.7654	0.4236	1.6267	0.8267	0.4197
	100	2.8399	1.0399	0.2481	2.0918	0.6918	0.3992	1.4878	0.6878	0.3319
	200	2.8028	1.0028	0.2114	1.8928	0.4928	0.3795	1.2528	0.4528	0.3095
	400	2.2739	0.4739	0.1622	1.7628	0.3628	0.3431	1.0266	0.2266	0.2652
	800	2.0177	0.2177	0.1496	1.5277	0.1277	0.3158	0.9827	0.1827	0.2232
	1200	1.9274	0.1274	0.1012	1.4829	0.0829	0.2917	0.8928	0.0928	0.1876
	1800	1.8199	0.0199	0.0812	1.4593	0.0593	0.2508	0.8478	0.0478	0.1623
λ	50	3.6488	1.5488	0.5090	1.6749	0.9749	0.7456	5.3737	3.0737	0.4352
	100	3.5388	1.4388	0.4269	1.5784	0.8784	0.7172	3.2668	0.9668	0.3372
	200	2.9366	0.8366	0.4106	1.5187	0.8187	0.6986	2.8266	0.5266	0.2883
	400	2.6313	0.5313	0.3990	1.1739	0.4739	0.6665	2.6527	0.3527	0.2617
	800	2.2885	0.1885	0.3664	0.9277	0.2277	0.5775	2.4726	0.1726	0.2074
	1200	2.1938	0.0938	0.2909	0.7828	0.0828	0.3208	2.3744	0.0744	0.1627
	1800	2.1084	0.0084	0.2330	0.7256	0.0256	0.2767	2.3018	0.0018	0.1108

5. Applications

We present applications based on the two real-life data sets to evaluate the efficacy of the proposed model and measure its performance in comparison to other non-nested models. The first data set is reported by (6) and relates to the survival times (in years) of a group of patients given chemotherapy treatment alone. The data consists of survival times (in years) for 46 patients and is as follows; 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033. The second data set is a real life example presented by (11) also reported by (21), and consists of 101 data points representing the stress-rupture life of kevlar 49/epoxy strands that are subjected to constant sustained pressure at the 90% stress level. The data set is as follows; 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

The EHL-OBIII-E distribution is compared to other known non-nested models namely, the Kumaraswamy Odd Lindley-log logistic (KOLLLoG) distribution by (10), Weibull-Lomax (WLx) distribution by (26), the Beta Generalized Exponential (BGE) distribution by (5), the Exponentiated Modified Weibull (EMW) distribution by (18), the Kumaraswamy Weibull (KW) distribution by (16) and Generalized Weibull Log-Logistic (GWLLoG) distribution by (17). To evaluate and measure the model performances, we used the goodness-of-fit statistics namely, -2 log-likelihood $(-2\ln(L))$, Akaike Information Criterion $(AIC = 2p - 2\ln(L))$, Bayesian Information Criterion $(BIC = p\ln(n) - 2\ln(L))$ and Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\boldsymbol{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. We computed results on the Cramer-von Mises (W^*) and Anderson-Darling Statistics (A^*) proposed by (9), including the Kolmogorov-Smirnov (K-S) statistic and the associated P-values. Note that smaller values are preferred for the log-likelihood function at its maximum (ℓ_n) and similarly for AIC, AICC, BIC, and the goodness-of-fit statistics W^*, A^* and K-S, smaller values are also preferred. The results from two real lifetime data sets are presented under tables 4 and 5. The R software was used to compute estimates for model parameters and run goodness-of-fit tests. The PDFs of the non-nested models used for comparisons are;

Kumaraswamy Odd Lindley-log-logistic (KOLLLoG) distribution

$$\begin{aligned}
f(x;a,b,\lambda,c) &= ab\left[\frac{\lambda^2}{(1+\lambda)}\frac{cx^{c-1}}{(1+x^c)^{-1}}\exp\left\{-\lambda\frac{1-(1+x^c)^{-1}}{(1+x^c)^{-1}}\right\}\right]\\
&\times \left[1-\frac{\lambda+(1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}}\exp\left\{-\lambda\frac{(1-(1+x^c)^{-1})}{(1+x^c)^{-1}}\right\}\right]^{a-1}\\
&\times \left(1-\left[1-\frac{\lambda+(1+x^c)^{-1}}{(1+\lambda)(1+x^c)^{-1}}\exp\left\{-\lambda\frac{(1-(1+x^c)^{-1})}{(1+x^c)^{-1}}\right\}\right]^{a}\right)^{b-1},
\end{aligned}$$

for $a, b, \lambda, c > 0$ and $x > 0$,

Weibull-Lomax (WLx) distribution

$$\begin{aligned}
f(x;a,b,\alpha,\beta) &= \frac{ab\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{b\alpha-1}\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1}\\
&\times \exp\left\{-a\left\{1+\left(\frac{x}{\beta}\right)^{\alpha}-1\right\}^{b}\right\},
\end{aligned}$$

for $a, b, c, \lambda > 0$ and $x > 0$,

Beta Generalized Exponential (BGE) distribution

$$f(x;a,b,\alpha,\lambda) = \frac{\alpha\lambda}{B(a,b)}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha a-1}\{1-(1-e^{-\lambda x})^{\alpha}\}^{b-1},$$

for $a, b, \alpha, \lambda > 0$ and $x > 0$,

Exponentiated Modified Weibull (EMW) distribution

$$f(x;\alpha,\beta,\lambda,\theta) \quad = \quad \theta(\alpha+\lambda\beta x^{\lambda-1})e^{-(\alpha x+\beta x^{\lambda})}(1-e^{\alpha x+\beta x^{\lambda}})^{\theta-1},$$

for $\alpha, \beta, \lambda, \theta > 0$ and $x > 0$,

Kumaraswamy Weibull (KW) distribution

$$\begin{aligned} f(x;a,b,c,\lambda) \quad &= \quad abc\lambda^{c}x^{c-1}\exp\{-(\lambda x)^{c}\}[1-\exp\{-(\lambda x)^{c}\}]^{a-1} \\ &\times \quad \{1-[1-\exp\{-(\lambda x)^{c}\}]^{a}\}^{b-1}, \end{aligned}$$

for $a, b, c, \lambda > 0$ and $x > 0$ and

Generalized Weibull Log-Logistic (GWLLoG) distribution.

$$\begin{aligned} f(x;\alpha,\beta,\gamma,a) \quad &= \quad \frac{\alpha\beta\gamma x^{\gamma-1}}{a^{\gamma}}\left(1+\left(\frac{x}{a}\right)^{\gamma}\right)^{-1}\left[\log\left(1+\left(\frac{x}{a}\right)^{\gamma}\right)\right]^{\beta-1} \\ &\times \quad \exp\left\{-\alpha\left[\log\left(1+\left(\frac{x}{a}\right)^{\gamma}\right)\right]^{\beta}\right\}, \end{aligned}$$

for $\alpha, \beta, \gamma, a > 0$ and $x > 0$.

The estimates of model parameters for the EHL-OBIII-E distribution and the other non-nested models (with standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W*, A*, Kolmogorov-Smirnov (K-S) and the associated P-values are given in Table 4. The plots of the fitted densities and observed probabilities are given in Figure 5.

Table 4: Model estimates for Survival times data

	Estimates				Statistics							
Model	a	b	α	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
EHL-OBIII-E	0.8594	10.4693	0.1369	1.1934	114.5943	122.5943	123.5943	129.821	0.0694	0.4713	0.1059	0.6541
	(0.2946)	(12.0444)	(0.1283)	(0.5663)								
	a	b	λ	c								
KOLLLoG	1.2521	0.1422	5.8092	0.9775	115.8898	123.8898	124.8898	131.1164	0.0735	0.4963	0.1150	0.5521
	(0.8038)	(0.0839)	(3.1117)	(0.1733)								
	a	b	α	β								
WLx	5.4516	0.9734	1.969×10^{3}	17.2670	115.9419	123.9419	124.9419	131.1685	0.0887	0.5897	0.1114	0.5924
	(1.1398)	(0.1136)	(3.8605×10^{-3})	(4.4017×10^{-4})								
	α	λ	a	b								
BGE	13.8679	0.4085	0.0617	11.0365	115.273	123.273	124.273	130.4996	0.1080	0.7146	0.12763	0.421
	(14.8578)	(0.2297)	(0.0675)	(39.0023)								
	γ	δ	λ	θ								
EMW	1.1049	0.7943	17.001	1.0×10^{-4}	116.1897	124.1897	125.1897	131.4163	0.0785	0.5268	0.1099	0.6094
	(0.2196)	(0.1511)	(1.8250×10^{-18})	(3.1034×10^{-13})								
	a	b	β	α								
KW	0.2060	3.6323	4.4705	0.1451	115.5601	123.5601	124.5601	130.7867	0.0974	0.6443	0.1157	0.5439
	(1.2986)	(16.9361)	(25.1920)	(0.4282)								
	α	β	a	θ								
GWLLoG	31.7894	0.3602	36.5270	2.9245	116.2522	124.2522	125.2522	131.4788	0.0813	0.5437	0.1094	0.6143
	(6.6608)	(1.2399)	(6.1217)	(10.0601)								

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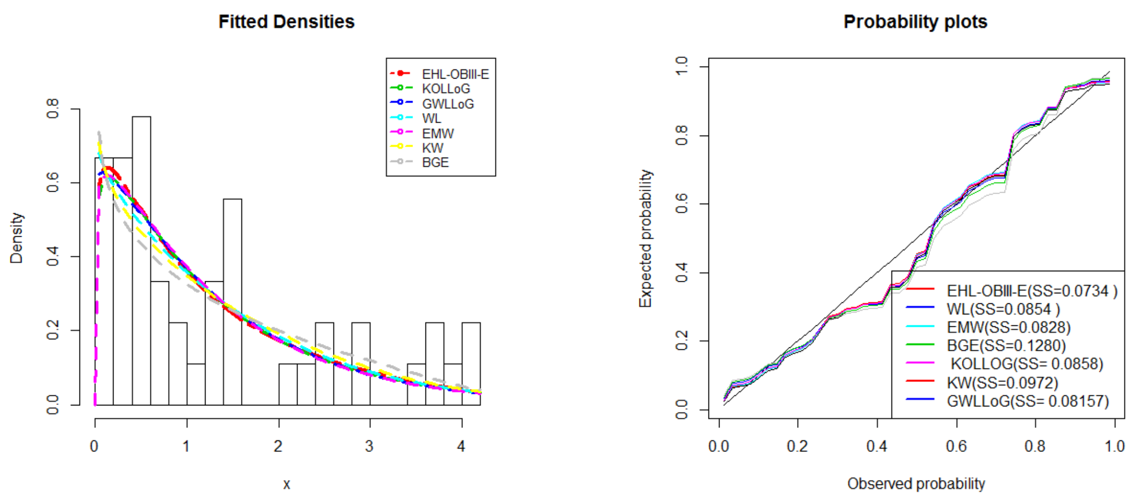

Figure 5: Fitted PDF and Observed probability Plots for Survival times data

From Table 4 and Figure 5, we note that the EHL-OBIII-E model performs better than other non-nested models looking at its smallest values of W*, A*, AIC, AICC, BIC and the higher P-value for the K-S statistic. The fitted density of the EHL-OBIII-E distribution also remain closer to the sample histogram and similarly its fitted probability plot is also closer to the empirical line. Basing on these results, the EHL-OBIII-E provides a better fit compared to these other known non-nested models used for comparison.

The model parameter estimates for the EHL-OBIII-E distribution and other non-nested models (with standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W*, A*, Kolmogorov-Smirnov (K-S) with P-values are presented under table 5. Plots of the fitted densities and observed probability are given in Figure 6.

Table 5: Model estimates for Kevlar Epoxy data

	Estimates				Statistics							
Model	a	b	α	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
EHL-OBIII-E	1.7704	0.7699	0.4634	0.5852	203.2637	211.2637	211.6804	221.7242	0.0650	0.5220	0.0648	0.7886
	(0.5205)	(0.6396)	(0.3549)	(0.1578)								
	a	b	λ	c								
KOLLLoG	0.9437	3.1448	0.6028	0.8524	205.0603	213.0603	213.477	223.5208	0.1480	0.8781	0.0786	0.5591
	(0.4861)	(12.7460)	(1.5014)	(0.4004)								
	a	b	α	β								
WLx	0.2506	0.7860	1.3580	0.3302	205.1976	213.1976	213.6143	223.6581	0.14402	0.8627	0.0787	0.5587
	(0.4172)	(0.1803)	(0.4580)	(0.6282)								
	α	λ	a	b								
BGE	2.8508	0.5761	0.2831	0.3695	205.0185	213.0185	213.4351	223.479	0.1414	0.8493	0.0775	0.5781
	(3.8184)	(0.5246)	(0.3695)	(1.8191)								
	γ	δ	λ	θ								
EMW	0.8663	0.8883	17.8940	1.0×10^{-4}	205.6399	213.6399	214.0566	224.1004	0.1786	1.0183	0.0887	0.4045
	(0.1098)	(0.1201)	(2.0310×10^{-19})	(3.6356×10^{-14})								
	a	b	α	β								
KW	1.2800	2.0691×10^{3}	2.6750×10^{-4}	0.7239	205.9592	213.9592	214.3758	224.4197	0.1991	1.1134	0.0911	0.3714
	(0.0245)	(7.6348×10^{-8})	(1.6886×10^{-4})	(0.0438)								
	α	β	a	θ								
GWLLoG	3.1309	0.1734	3.0819	5.6875	207.0122	215.0122	215.4289	225.4727	0.2240	1.2327	0.0900	0.3858
	(0.3872)	(0.0166)	(0.1552)	(0.2913)								

From the results presented under Table 5, the EHL-OBIII-E distribution gives the smallest values of W^*, A^*, K-S and higher P-value compared to other competing models which shows its superiority over the other models. Figure 6, also show that the EHL-OBIII-E distribution provides a better fit to the real life data compared with the other non-nested models.

6. Conclusion

In this article a new generalized family of distributions called exponentiated half-logistic Odd Burr III-G (EHL-OBIII-G) was developed. The structural properties of this new family of distributions have been derived and studied. Some

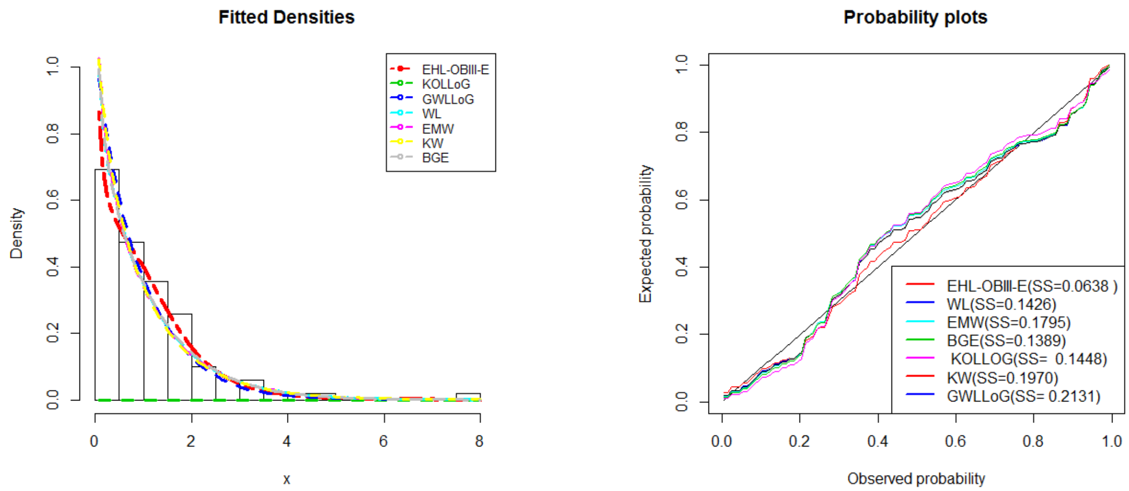

Figure 6: Fitted PDF and Observed probability Plots for Kevlar Epoxy data

of its special cases have been discussed and the EHL-OBIII-Exponential distribution is applied to two real life data examples together with the other known non-nested models for comparison. Based on the results, the model provides better fits and performs better than the other non-nested models in fitting real life data, see Tables 4 and 5 and also Figures 5 and 6 for more details. From the simulation results, the consistency of the model estimators is indicated by bias and RMSE coverging towards zero as the sample size increases. We hope that these new generated family of distributions will find wider applicability in different disciplines such financial modelling, economics, agriculture, engineering, genomics and operations research, to mention just a few.

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APPENDIX

A. Elements of the score vector

The elements of the score vector, $U(\mathbf{\Delta})$ are given by

$$\begin{aligned}
\frac{\partial \ell}{\partial a} &= \frac{n}{a} - (\alpha-1)\sum_{i=1}^{n} \frac{b\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b-1}\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\ln\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)}{\left(\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right)} \\
&+ (-b-1)\sum_{i=1}^{n} \frac{\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\ln\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)}{\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)} - \sum_{i=1}^{n}\ln\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right) \\
&- (\alpha+1)\sum_{i=1}^{n} \frac{b\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b-1}\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\ln\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)}{\left(1+\left[1-\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right]\right)},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial b} &= \frac{n}{b} + (\alpha-1)\sum_{i=1}^{n} \frac{\left(\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right)\ln\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)}{\left(\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right)} \\
&- \sum_{i=1}^{n}\ln\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right) \\
&+ (\alpha+1)\sum_{i=1}^{n} \frac{\left(\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right)\ln\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)}{\left(1+\left[1-\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right]\right)},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left(\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right) \\
&- \sum_{i=1}^{n}\ln\left(1+\left[1-\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right]\right)
\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial \ell}{\partial \underline{\boldsymbol{\psi}}_k} &= (\alpha-1)ba\sum_{i=1}^{n}\frac{\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b-1}\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a-1}\frac{\partial G(x_i;\underline{\boldsymbol{\psi}})}{\partial \underline{\boldsymbol{\psi}}_k}}{\left(\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right)\overline{G}^2(x_i;\underline{\boldsymbol{\psi}})} \\
&- a\,(-b-1)\sum_{i=1}^{n}\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a-1}\frac{\frac{\partial G(x_i;\underline{\boldsymbol{\psi}})}{\partial \underline{\boldsymbol{\psi}}_k}}{\overline{G}^2(x_i;\underline{\boldsymbol{\psi}})}+(-a-1)\sum_{i=1}^{n}\frac{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}{G(x_i;\underline{\boldsymbol{\psi}})}\frac{\frac{\partial G(x_i;\underline{\boldsymbol{\psi}})}{\partial \underline{\boldsymbol{\psi}}_k}}{\overline{G}^2(x_i;\underline{\boldsymbol{\psi}})} \\
&+ \sum_{i=1}^{n}\frac{ab\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b-1}\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a-1}\frac{\partial G(x_i;\underline{\boldsymbol{\psi}})}{\partial \underline{\boldsymbol{\psi}}_k}}{\left(1+\left[1-\left(1+\left(\frac{G(x_i;\underline{\boldsymbol{\psi}})}{\overline{G}(x_i;\underline{\boldsymbol{\psi}})}\right)^{-a}\right)^{-b}\right]\right)\overline{G}^2(x_i;\underline{\boldsymbol{\psi}})} \\
&+ \sum_{i=1}^{n}\frac{\frac{\partial g(x_i;\underline{\boldsymbol{\psi}})}{\partial \underline{\boldsymbol{\psi}}_k}}{\left(g(x_i;\underline{\boldsymbol{\psi}})\right)}-2\sum_{i=1}^{n}\frac{\frac{\partial \overline{G}(x_i;\underline{\boldsymbol{\psi}})}{\partial \underline{\boldsymbol{\psi}}_k}}{\left(\overline{G}(x_i;\underline{\boldsymbol{\psi}})\right)}.
\end{aligned}
$$