DOI: http://dx.doi.org/10.18187/pjsor.v18i3.3657

Pakistan Journal of Statistics and Operation Research

Continuous wavelet estimation for multivariate fractional Brownian motion

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Abstract

In this paper, we propose a method using continuous wavelets to study the multivariate fractional Brownian motion through the deviations of the transformed random process to find an efficient estimate of Hurst exponent using eigenvalue regression of the covariance matrix. The results of simulations experiments shown that the performance of the proposed estimator was efficient in bias but the variance get increase as signal change from short to long memory the MASE increase relatively. The estimation process was made by calculating the eigenvalues for the variance-covariance matrix of Meyer's continuous wavelet details coefficients.

Key Words: Multivariate fractional Brownian motion; Hurst exponent; Continuous wavelet transform; Meyer's wavelets, Short memory; long memory; Gaussian noise.

1. Introduction

Fractional Brownian motion (FBM) provides an appropriate modeling framework for non-stationary self-similar stochastic processes with stationary increments. It has been widely used to model random phenomena related to different research fields. On other hand, wavelet transforms provide a regularized differentiation of the processes, have a filter structure in perfect adequacy with 1/f type of spectral behavior, and may eliminate long-range dependence properties if the analyzing wavelet was properly chosen. Thus, studying multivariate fractional signals through the lens of the wavelet is indeed natural, and we expect that it will be useful as will in revealing the interaction structure between the components of the FBM.

The fractional Brownian motion was defined for the first time within Hilbert space by Kolmogorov in 1940 where he called it Wiener Helix and was studied more broadly by Yaglom in 1958 but the designation belongs to the researchers Mandelbrot and Van Ness (1968) where they explained the random integration of this process in its standard form when the Hurst parameter is 0.5, where the fractional Brownian is a generalization of this case, and unlike the standard form of this random process, the increase is not necessarily independent, also this process is a continuous time, with zero mean and variance-covariance equal to:

$$E[B_H(t)B_H(s)] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \qquad (1)$$

The problem we seek to solve in this paper is to develop a method that can be used to model and study mth order fractional Brownian motion using Meyer's continuous wavelets. The proposed procedure will rely on finding the eigenvalues for the variance-covariance matrix of the detail coefficients of the fractional Brownian motion. The performance of this estimator was validated through a simulation study using the Mean average square error (MASE).

The paper is structured as follow: Section 2 the multivariate fractional Brownian motion will be presented, in Section 3 we give an overview on continuous wavelet transform, and in Section 4 the proposed method will be briefed, Section 5 the simulation study will be conducted and finally Section 6 the conclusions.

2. Multivariate Fractional Brownian motion (m-FBM)

The multivariate formulas in the fractional Brownian motion were numerous, as Ayache et al. in (2002) proposed the Brownian Sheet, which required an estimate of a complete matrix of parameters and was studied from the wavelet perspective by Wu and Ding in (2015) , and the researchers followed a similar method for Abry et al in (2000).

In (2015) and (2018), Abry and Didier studied FBM Operator, which is the estimation of the scale parameter in a semi-diagonal matrix formula (Jordan Form). The field remains open for researchers to delve deeper in this direction in order to arrive at the best estimates that would assist in the development and modernization.

As for this paper we will consider the multivariate fractional Brownian motion m-FBM proposed by Perrin in (2001) defined by performing subtraction of the m-degree up to the limit expansion of the kernel $(t-u)^{H-1/2}$ so that the equation is as follows:

$$B_H^m(t) = \frac{1}{\Gamma\left(H+\frac{1}{2}\right)}\left(\left[\int_{-\infty}^{0}(t-u)^{H-\frac{1}{2}} - (-u)^{H-\frac{1}{2}} - \cdots - \left(H-\frac{1}{2}\right)\ldots\left(H-\frac{2m-3}{2}\right)(-u)^{H-n+\frac{1}{2}}\frac{t^{m-1}}{(n-1)!}\right] dB(u) + \int_0^t (t-u)^{H-\frac{1}{2}} dB(u)\right) \tag{2}$$

The subtraction process ensures that the scaling parameter H will remain within the range $[m-1, m]$ as well as $B_H^m(0) = 0$. It should also be noted that this function $m-1$ derivatives that satisfy the equation:

$$B_H^m(t) = \int_0^t B_{H-1}^{m-1}(u)du \tag{3}$$

Also the covariance of this function will be

$$E[B_H^m(t)B_H^m(s)] = (-1)^m \frac{c_H}{2}\left(|t-s|^{2H} - \sum_{j=0}^{m-1}(-1)^j c_{2H}^j \left[\left(\frac{t}{s}\right)^j |s|^{2H} + \left(\frac{s}{t}\right)^j |t|^{2H}\right]\right) \tag{4}$$

Where

$$c_H = \frac{1}{\Gamma(2H+1)|\sin(\pi H)|} \tag{5}$$

$$c_{2H}^j = \frac{2H(2H-1)\ldots\big(2H-(j-1)\big)}{j!}$$

These equations illustrate the non-stationary and self-similarity of the multivariate fractional Brownian motion the fractional Gaussian noise can be defined as follows

$$G_H^m(t) = \Delta_t^m B_H^m(t) \tag{6}$$

And the covariance function for it

$$E[G_H^m(t)G_H^m(t-\tau)] = (-1)^m \frac{c_H}{2}\left(\sum_{j=-m}^{m}(-1)^j c_{2m}^{m+j}[\,|\tau+jt|^{2H}]\right) \tag{7}$$

Perrin showed that this function is stationary and converge asymptotically of the order of degree $|\tau|^{2H-2m}$ when $\tau \to \infty$, and the stochastic process is non-resistant, meaning it has a short memory when $m-1 < H < m-1/2$, but if $m-1/2 < H < m$ it has a long memory.

These configurations to the fractional Brownian motion on multivariate scale will lead us directly to modify the slop value for the proposed estimation method that will be use next by adding $\mathrm{m}-1$ to the estimated Hurst exponent, for example for bivariate if the estimated exponent was $H=.8$ the result will be $H=1.8$.

From the above definitions, we can construct a model assuming the presence of a group of observations $\mathrm{y}\,(\mathrm{t})$ where $\mathrm{t}\ =\ 1, \dots\dots\dots\ ..,$ T and that the random process $\mathrm{B}_{\mathrm{H}}^{\mathrm{m}}(\mathrm{t})$ is contaminated with an additive noise, so the random process model is as follows:

$$\mathrm{y}_{(\mathrm{t})} = \mathrm{B}_{\mathrm{H}}^{\mathrm{m}}(\mathrm{t}) + \varepsilon_{\mathrm{t}} \tag{8}$$

Where $\mathrm{y}_{(\mathrm{t})}$ represents the dependent variable under study, $\mathrm{B}_{\mathrm{H}}^{\mathrm{m}}(\mathrm{t})$ is a multivariate fractional Brownian motion, contaminated by Gaussian additive noise ε_{t} with zero mean and variance σ_{ε}^{2}, and the aim is to estimate the Hurst parameter in the presence of this noise.

3. Continuous Wavelets Transform

The general formula for Continuous Wavelet Transform (CWT) (Mallat (2009)) is:

$$W(t,a,b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} X(t)\, \psi\left(\frac{t-b}{a}\right) dt, \ -\infty < t < \infty, \ a > 0 \tag{9}$$

Which is consist from the inner product of the function $X(t)$ and the function $\psi\left(\frac{t-b}{a}\right)$, sometimes called kernel of the transform and the transformation depends totally on it. For example, in Fourier transform the transform function is $\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}$ while the translation k play the role of time and the scale s is the opposite of the frequency means that the high scale have low frequency while the low scale have the high frequency and since the scale is variable so the value $|s|^{-1/2}$ is required to ensure the normalized status of the wavelet ensures that the energy of the wavelet is equal to the integer one $\|\psi\| = 1$, no matter what the value of the scale noting that the scale always greater than zero because negative scaling is undefined. When determining the appropriate wavelet function for the data under study, the so-called type of wavelet.
The wavelet function $\psi(t)$ or the mother wavelet, called this name because it acts as a Prototype, through which all the windows used in processing the series signal are generated and that the wavelet function represents the dilation or expansion of the signal in the high frequencies and is also called the Primary Wavelet. The scale function $\phi(t)$ or the father wavelet, acts as smooth slope or low frequencies and must be orthogonal to ensure that the series energy remains constant and is not affected by the translation of data. When decompose these functions (father wavelet and mother wavelet) the so-called "basis of wavelets" appears in the sense that all the other windows that are generated in the wavelets are mere basis of these two. The scale as a mathematical process dilate the series signal or compresses it. Its worthy to mention that the translation term k play as shifting in the sense that it places a position on the time function. The scale s, it places a position on the frequency function and by finding the value of the transformation equation for the number of mattresses by changing the values of k and s and aggregating together a scale-transformation representation will appear, which is the equivalent of a frequency-time representation, and this will choosing a particular wavelet function which have best representation for the data under study, which will be used in the transformation because the choice of this function will determine the shape and characteristics of the wavelet transformation (Debnath and Ahmad Shah (2015)).

The inverse of continuous wavelet transform is:

$$X(t) = \frac{1}{C_{\psi}} \int_{0}^{\infty} \int_{-\infty}^{\infty} W\, f(a,b) s^{-1/2} \psi\left(\frac{t-a}{b}\right) da\, \frac{db}{b^2} \tag{10}$$

Where C_{ψ} is defined as:

$$C_{\psi} = \int_{0}^{\infty} \frac{|\Psi(\omega)|^2}{\omega} d\omega < \infty \tag{11}$$

$\Psi(\omega)$ is the Fourier transform of the mother wavelet $\psi(t)$ and so this equation also called the admissibility condition which implies that the Fourier transform vanish at the zero frequency, i.e., This means that wavelets must have a bandpass-like spectrum, also means that the average value of the wavelet in the time domain must be zero.
The most used type of CWT specially in adaptive filters, fractal random fields and multi-fault classification is the Meyer wavelet and that because it is orthogonal and indefinitely differentiable with infinite support and also multiresolution. The Meyer scale function and wavelet are defined in the frequency domain in terms of function v by means of well-known equations (Vermehren and de Oliveira (2015))

$$\Phi_{mey}(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}} \; if \; \omega \leq \frac{2\pi}{3} \\ \frac{1}{\sqrt{2\pi}} \cos\left(\frac{\pi}{2} v\left(\frac{3\omega}{2\pi} - 1\right)\right) if \frac{2\pi}{3} \leq \omega \leq \frac{4\pi}{3} \\ 0 \quad otherwise \end{cases} \tag{12}$$

And the wavelet is given by

$$\Psi_{mey}(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}} \sin\left(\frac{\pi}{2} v\left(\frac{3|\omega|}{2\pi} - 1\right)\right) e^{\frac{j\omega}{2}} \, if \frac{2\pi}{3} \leq |\omega| \leq \frac{4\pi}{3} \\ \frac{1}{\sqrt{2\pi}} \cos\left(\frac{\pi}{2} v\left(\frac{3|\omega|}{4\pi} - 1\right)\right) e^{\frac{j\omega}{2}} \, if \frac{4\pi}{3} \leq |\omega| \leq \frac{8\pi}{3} \\ 0 \quad otherwise \end{cases} \tag{13}$$

Where, for instance other choice can be made

$$v(x) = \begin{cases} 0 \quad if \; x < 0 \\ x \; if \; 0 \leq x \leq 1 \\ 1 \quad if \quad x > 1 \end{cases} \tag{14}$$

Many implementation of this auxiliary function one is this standard adopts

$$v(x) = 35x^4 - 84x^5 + 70x^6 - 20x^7 \tag{15}$$

In order to evaluate the corresponding wave forms of the Equations in time domain, denoted by $\Phi_{mey}(t)$ and $\Psi_{mey}(\omega)$, we use the inverse Fourier transform resulting in:

$$\Phi_{mey}(t) = \frac{2}{\sqrt{2\pi}} \int_0^{\frac{4\pi}{3}} \Phi_{mey}(\omega) \cos(\omega t) d\omega \tag{16}$$

$$\Psi_{mey}(t) = 2 \int_{\frac{2\pi}{3}}^{\frac{8\pi}{3}} \Phi_{mey}\left(\frac{\omega}{2}\right) \Phi_{mey}(\omega - 2\pi) \cos\left(\omega\left(t - \frac{1}{2}\right)\right) d\omega \tag{17}$$

5. Eigenvalue estimator for fractional Brownian motion using continuous wavelet

Characteristics of second-order random processes like periodic correlation, stationary increment, harmonizability and self-similarity can be studied through wavelet transformation, and this transformation has the attention of many researchers in the practical and theoretical fields because of its ability on revealing the relationship between the signal and its transformation. As a signal membership in the functional spaces and local smoothness characteristics that can be observed through wavelet transformation decay, but there remains a problem of random signals that cannot be directly transformed because the sample space of the random process does not have finite energy. In this section, Meyer wavelets will be used for the purpose of obtaining an efficient estimate of the Hurst parameter, starting with the definition of the general form for the wavelet transformation in equation (9).

Whereas, X (t) represents a second order complex random process (which is the fractional Brownian motion in our case), that is to be transformed with scale a and the translation b. Given that it is not possible to find an estimate of the parameters of this type of processes directly, we will find it through the second moment, as the first moment is zero and by finding the variance for the result of this transformation, we will obtain the variance of the detail coefficients of the wavelet transformation function, as shown in the following theorem.

Theorem: Let $X = \{X(t), -\infty < t < \infty\}$ be a complex-valued random process which is measurable in t then the second moment of such function can be defined as $M_X(t,s) = E\{X(t)\bar{X}(s)\}$ and the second moment of its wavelet transform is:

$$E|W(t,a,b)|^2 = E\left\{\left|\frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} X(t)\, \psi\left(\frac{t-b}{a}\right) dt\right|^2\right\} \tag{18}$$

$$= \frac{1}{|a|} \iint_{-\infty}^{\infty} \left|E\{|X(t)|\}\, \psi\left(\frac{t-b}{a}\right) E\{|\bar{X}(s)|\} \bar{\psi}\left(\frac{s-c}{a}\right)\right| dt\, ds$$

For each scale $|a|$ the wavelet coefficients d form a sequence of random coefficients but Meyer's wavelet family still constitute an orthonormal system and there is no reason for the wavelet coefficients to be uncorrelated considering the details coefficients for fractional Brownian motion $B_H(t)$ is

$$W = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{\infty} B_H(t)\, \psi\left(\frac{t-b}{a}\right) dt\,,\ Give\ the\ details\ coefficients \quad (19)$$

Assuming $\psi(t)$ satisfy the admissibility condition

$$\int_{-\infty}^{\infty} \psi(t) dt = 0 \quad (20)$$

Where the second moment of the fractional Brownian motion as shown by Mandelbrot in equation (1), Which represent the variance-covariance for FBM, from this equation and since translation not affected by increases, its straightforward calculation yields

$$E(W)^2 = \frac{1}{a} \int_{-\infty}^{\infty} E(B_H(t))\, a\, \psi(at-b)\, dt \int_{-\infty}^{\infty} E(B_H(s))\, a\, \psi(as-c)\, ds$$

$$E(W)^2 = \frac{1}{a} \iint_{-\infty}^{\infty} E(B_H(t) B_H(s))\, a\, \psi(at-b)\, a\, \psi(as-c)\ dt\, ds$$

But $E(B_H(t,s)) \approx E(B_H(t)B_H(s))$

$$E(W)^2 = a \iint_{-\infty}^{\infty} E(B_H(at,as))\, \psi(at-b)\, \psi(as-c)\ dt\, ds \quad (21)$$

From the FBM variance equation taking in account the independency between details coefficients, we get

$$E(W)^2 = \frac{1}{2} a^{2H+1} \left[- \iint_{-\infty}^{\infty} (t-s)^{2H}\, \psi(at-b)\, \psi(as-c)\ dt\, ds\right]$$

$$E(W)^2 = \frac{1}{2} a^{2H+1} [- \int_{-\infty}^{\infty} A_\psi \left(a, t-(ab-c)\right) t^{2H}\, dt\,] \quad (22)$$

Where

$$A_\psi\, (\alpha, \beta) = \int_{-\infty}^{\infty} \psi(t)\, \psi\left(\frac{t-\beta}{\alpha}\right)\, dt \quad (23)$$

And A_ψ is the transformation of the wavelet itself, then the variance of this transformation is

$$E(W)^2 = \frac{1}{2}\, G_\psi\, a^{2H+1} \quad (24)$$

Where $\quad G_\psi = -\int_{-\infty}^{\infty} A_\psi \left(a, t-(ab-c)\right) t^{2H}\, dt$

The wavelet transform field is $\left(H+\frac{1}{2}\right)$ self similar and has stationary increments at all scales.

Now we can find a method to calculate the value of the Hurst parameter for fractional Brownian motion through the variance of the details coefficients of wavelet transformation and it shall be considered as a general form for such processes, but there remains a problem with continuous wavelet transformation, where the scales are have exponential increases, which makes it difficult to define the scales given the large size of the filters used according to a specific standard, and for this purpose it must be another way for obtaining an efficient estimate, and given that the variance matrix of these parameters still contains sufficient information to conduct the estimation process, it will be relied upon, but through the eigenvalues of the wavelet transform variance for the random process, instead of relying on the wavelet spectrum behavior as a function of wavelet scales, it will be relied on the eigenstructure of the wavelet spectrum, and for this purpose it is necessary to find a mathematical formula to use in calculating the parameter from the eigenvalues, and that the variance and covariance matrix for the coefficients of the wavelet transform is positive definite and symmetric, and given the characteristics that distinguish the eigenvalues in that they contain all the information without being affected with changes in the matrix (such as correlations), and from the last formula in theorem above.

$$E(W)^2 = \frac{1}{2} G_\psi a^{2H+1} \quad (25)$$

Then by taking the logarithm for both sides to find an acceptable expression for H

$$\log_{10} E(W)^2 = (2H+1)\log_{10}(a) + constant \quad (26)$$

Considering the power-low behavior of the wavelet coefficients variance we can take the eigenvalues of this matrix as a replace for the variance matrix to avoid the influence of the exponential scale in continuous wavelet

$$\log_{10}\begin{bmatrix}\lambda_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \lambda_m\end{bmatrix} = (2H+1)\log_{10}(a) + constant$$

$$\log_{10}[diag(\lambda_1, \dots, \lambda_m)] = (2H+1)\log_{10}(a) + constant \quad (27)$$

Where

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_m$$

Then the Hurst parameter can be calculated by

$$\widehat{H_\iota} = \left|\frac{\log_{10}(\lambda_i(a_i))}{2\log_{10}(a_i)}\right| \quad (28)$$

Where this formula can be used as a general form for such processes using wavelet transform.

Algorithm

1- Transform the FBM signal using Meyer continuous wavelet.
2- Find the variance covariance matrix for the detail coefficients.
3- Calculate the eigenvalues of the variance covariance matrix.
4- Divide the series by the number of the variates where $a > 1$.
5- Use equation (28) to find the Hurst exponent.

5. Simulation study

In this section, we will conduct simulation study of multivariate fractional Brownian motion, where wavelet synthesis method proposed by Sellan and Abry (1996) will be used for generating the mentioned random process, and then we will use Eigenvalue estimator for fractional Brownian motion using continuous wavlet (EECW). The length of the series to be generated will be as $n = 100{,}200$ and the number of variables $m = 4, 8{,}12$ we will replicate the calculation with $rep = 500$ for increasing the accuracy in the estimation. As for the wavelet used for the generating, $s = 2^{10}, 2^{12}$. Taking into account that the estimation process for the Hurst parameter will be for all levels $H = [0.1{:}\ 0.9]$. The mean square error and the bias of the estimator will be calculated as follows

$$MSE(\widehat{H}) = Var(\widehat{H}) + Bias^2(\widehat{H})$$

$$= \left(\frac{1}{4}\sum_j \sigma_j^2\theta_j^2\right) + (E(\widehat{H}) - H)^2 \quad (29)$$

$$Where\ \theta_j = \frac{\sum(S*j - S_j)/\sigma_j^2}{S\,S_{jj} - S_j^2}, and\ S = \sum\frac{1}{\sigma_j^2}, S_j = \sum\frac{j}{\sigma_j^2}, S_{jj} = \sum\frac{j^2}{\sigma_j^2}$$

$$Var(y_j) = \sigma_j^2 = \frac{\varsigma\left(2, \frac{N_j}{2}\right)}{ln^2(2)}, E(y_j) = 2(H+1) + constant \quad (30)$$

Where $\varsigma\left(2,\frac{N_j}{2}\right)$ is a generalized Riemann Zeta function (Power and Turvey (2010)) which is a function of a complex variable and can be defined as

$$\varsigma\left(2,\frac{N_j}{2}\right) = 2^{-\frac{N_j}{2}} \qquad (31)$$

In addition, a new random series had been generated each time before the estimation in order to get the best view of the performance of the method used and at different levels of the random process.

Table (1) of Hurst est. bias and MASE using EECW

		$s = 2^{10}$				$s = 2^{12}$			
Var. no.	H	n=100		n=200		n=100		n=200	
		Bias	MASE	Bias	MASE	Bias	MASE	Bias	MASE
m=4	.1	0.0086	0.0208	0.0188	0.0073	0.0058	0.0152	0.0136	0.0062
	.2	0.0084	0.0327	0.0068	0.0197	-0.0073	0.0353	-0.0281	0.0144
	.3	-0.0139	0.0418	-0.0088	0.0404	0.0146	0.0693	-0.0207	0.0364
	.4	0.0185	0.0810	0.0094	0.0832	-0.0025	0.0850	0.0121	0.0785
	.5	0.0227	0.1430	-0.0049	0.1192	0.0188	0.1361	-0.0237	0.1265
	.6	-0.0257	0.1668	0.0086	0.1979	0.0058	0.1927	-0.0345	0.1485
	.7	-0.0276	0.2396	0.0173	0.2835	0.0046	0.2583	-0.0655	0.1888
	.8	0.0088	0.3603	0.0260	0.3862	0.0165	0.3465	-0.0284	0.2705
	.9	0.0381	0.5014	-0.0114	0.4260	0.0399	0.4603	0.0258	0.3850
m=8	.1	0.0165	0.0145	0.0070	0.0033	0.0297	0.0403	-0.0100	0.0098
	.2	-0.0114	0.0141	0.0156	0.0161	0.0107	0.0213	0.0081	0.0287
	.3	0.0004	0.0175	0.0062	0.0218	0.0084	0.0224	-0.0276	0.0291
	.4	0.0370	0.0463	-0.0026	0.0356	0.0077	0.0334	-0.0092	0.0425
	.5	-0.0035	0.0528	0.0197	0.0854	0.0101	0.0545	0.0356	0.0793
	.6	-0.0380	0.0888	0.0056	0.0831	-0.0090	0.0698	0.0499	0.0957
	.7	0.0008	0.1383	0.0146	0.1228	-0.0047	0.0999	0.0236	0.1109
	.8	-0.0289	0.1683	-0.0122	0.1615	0.0171	0.1363	-0.0013	0.1328
	.9	0.0004	0.2283	-0.0240	0.2125	-0.0149	0.1526	-0.0213	0.1690
m=12	.1	0.0146	0.0109	0.0090	0.0098	-0.0023	0.0012	0.0260	0.0031
	.2	-0.0265	0.0109	0.0083	0.0057	-0.0085	0.0204	0.0173	0.0138
	.3	0.0080	0.0199	-0.0032	0.0110	-0.0001	0.2436	0.0168	0.0898
	.4	-0.0054	0.0555	-0.0023	0.0216	0.0004	0.0566	-0.0247	0.1378
	.5	0.0074	0.0730	0.0063	0.0745	0.0036	0.0474	-0.0179	0.0439
	.6	0.0089	0.0689	-0.0014	0.0496	-0.0132	0.0476	0.0324	0.1951
	.7	-0.0196	0.0927	0.0054	0.0706	-0.0183	0.0702	0.0039	0.1322
	.8	0.0388	0.1464	-0.0198	0.0858	0.0093	0.1080	-0.0180	0.1099
	.9	-0.0161	0.1612	0.0014	0.1158	-0.0011	0.1363	0.0057	0.1286

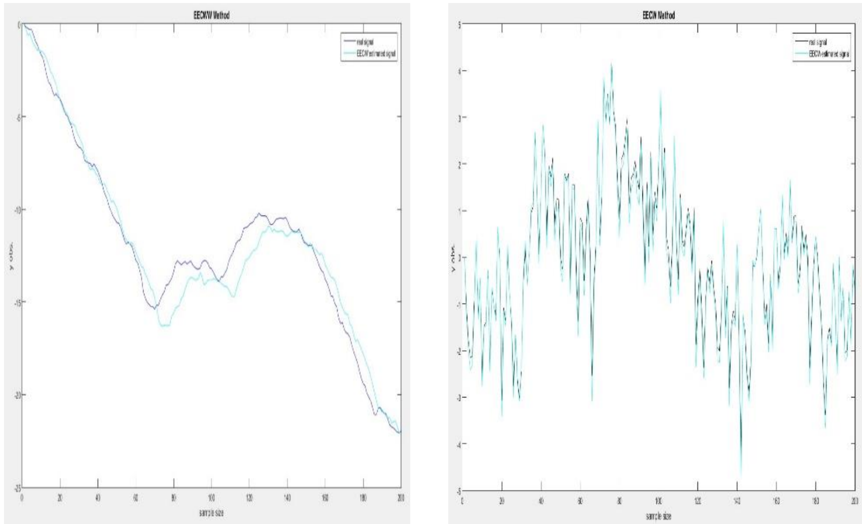

Figure (1) the left figure is $\hat{y}(t)$ for 12-variate self-similarity signal at sample size n=200 and FBM wavelet synthesis scale $s = 2^{10}$ with Hurst exponent .9 compared to EECW estimation method at $\widehat{H} = .8986$, and the figure on the right is $\hat{y}(t)$ for 4-variate self-similarity signal at sample size n=100 and FBM wavelet synthesis scale $s = 2^{12}$ with Hurst exponent .1 compared to EECW estimation method at $\widehat{H} = .0942$.

6. Conclusions

The estimation using continuous wavelet transform always suffer from the lack of compact support between the multiresolution functions sample spaces used, this problem have a great effect on the variance and so as the long range dependence parameter increases. As well as, the MASE witnessed an average increase too.

The bias on other hand was too low which reflect the best use of the proposed algorithm that's rely on the eigenvalues of the details coefficients covariance matrix to measure the slope for the wavelet basis which make the method efficient facing the long memory.

The smoothing level for the generated signal proves its effect, as its increase to $\mathbf{2^{12}}$ the bias went lower taking into account this increase in the scale might lead to misestimate for higher scales, because the simulation method will have a wrong signal estimation.

The sample size and variate numbers, also show a lower effect on the estimation procedure, because of the carful discretizing for the variates before estimation, otherwise the estimation will fail while conducting the convolution when simulating the signal.

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