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Estimation of Multicomponent Stress-strength Reliability under Inverse Topp-Leone Distribution

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Abstract

In this article, the reliability inference for a multicomponent stress-strength (MSS) model, when both stress and strength random variables follow inverse Topp-Leone distributions, was studied. The maximum likelihood and uniformly minimum variance unbiased estimates for the reliability of MSS model were obtained explicitly. The exact Bayes estimate of MSS reliability was derived under the squared error loss function. Also, the Bayes estimate was obtained using the Markov Chain Monte Carlo method for comparison with the aforementioned exact estimate. The asymptotic confidence interval was determined under the expected Fisher information matrix. Furthermore, the highest probability density credible interval was established through using the Gibbs sampling method. Monte Carlo simulations were implemented to compare the different proposed methods. Finally, a real life example was presented in support of the suggested procedures.

Key Words: Inverse Topp-Leone Distribution; Multicomponent Stress-Strength; Reliability.

Mathematical Subject Classification: 62N05, 62F10, 62F12, 62F15.

1. Introduction

In the reliability context, the stress-strength models have gained a great deal of consideration over recent decades due to its wide utilization in numerous fields. In these models, our major focus is the assessment of $R = P(Y < X)$, where X presents the random strength exposed to the random stress Y. In engineering applications, if X presents the strength of a building and Y presents the resultant of the destructive forces acting on it, such as an earthquake, then R can be interpreted as the safety factor of a building. In aquaculture, if X is the growth value of fish in a treatment group and Y is the growth value of a control group, then R shows the effectiveness of treatment. This fundamental idea was firstly studied by Birnbaum(1956). Thereafter, the problem of estimating R has been discussed by a great number of researchers. Of the recent efforts pertaining to stress-strength models, to name a few, are Al-Mutairi et al.(2013), Genc(2013) , Rezaei et al.(2015), Basirat et al.(2016), Al-Zahrani and Basloom(2016), Akgül and Şenoğlu(2017), Bai et al.(2019), Xavier and Jose(2021), Pak et al.(2022) and Jose(2022).

In recent years, inference for the reliability of MSS system has received much attention among researchers. This system contains k identical and independent strength components and it operates when at least $s\,(1 \leq s \leq k)$ of the components work properly against a common stress. It is commonly known as s-out-of-k: G system. MSS models appear in many practical situations, such as communication systems, industrial operations, military technologies and so on. For example, consider an airplane with four engines that flies when at least two engines work satisfactorily.

Thus, the airplane operation is a 2-out-of-4: G system. As another example, the kidney function in the human body is a 1-out-of-2: G system, since a person can survive with at least one healthy kidney. Assume $X_1, X_2, ..., X_k$ are independent random variables with common cdf of $F(.)$ and exposed to the common stress Y with cdf of $G(.)$. Thus, the reliability in a MSS model is given by

$$\begin{aligned} R_{s,k} &= P\left[\text{at least } s \text{ of } (X_1, X_2, ..., X_k) \text{ exceed } Y\right] \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty \left[1 - F_X(y)\right]^i \left[F_X(y)\right]^{k-i} dG(y). \end{aligned} \tag{1}$$

The mentioned model was firstly examined by Bhattacharyya and Johnson(1974). Thereafter, many authors have shown considerable interests in the MSS model. Some recent efforts regard to the issue, to mention a few, can be found in Dey et al.(2017), Kızılaslan(2017), Kızılaslan and Nadar(2018), Akgül(2019), Pak et al.(2019), Jha et al.(2019), Kohansal and Shoaee(2019), Maurya and Tripathi(2020), Kayal et al.(2020), Mahto et al.(2020), Jana and Bera(2022), Azhad et al.(2022) and Saini et al.(2022).

The Topp-Leone (TL) distribution, proposed by Topp and Leone(1955), is one of the most important lifetime distributions with finite support. However, it cannot be used for most lifetime data that have infinite support in theory. The cumulative distribution function (cdf) of the TL distribution with one positive shape parameter is specifid by

$$F_Z(z;\theta) = \left[z(2-z)\right]^\theta, \quad 0 < z < 1,$$

and corresponding probability density function (pdf) is

$$f_Z(z;\theta) = 2\theta(1-z)\left[z(2-z)\right]^{\theta-1}, \quad 0 < z < 1.$$

Recently, Hassan et al.(2020), introduced the inverse Topp-Leone (ITL) distribution defined on the domain $(0, \infty)$. The ITL distribution corresponds to the distribution of the variable $X = 1/Z - 1$, where Z has a TL distribution. The transformation $X = 1/Z - 1$, is more appropriate than the transformation $X = 1/Z$, which makes it more flexible for modelling lifetime data. The pdf and cdf of the ITL distribution are as follows, respectively:

$$f_X(x;\theta) = \frac{2\theta x}{(1+x)^3}\left[\frac{1+2x}{(1+x)^2}\right]^{\theta-1}, \quad x \geq 0, \quad \theta > 0, \tag{2}$$

$$F_X(x;\theta) = 1 - \left[\frac{1+2x}{(1+x)^2}\right]^{\theta}, \ x \geq 0, \ \theta > 0, \tag{3}$$

where θ controls the shape of the distribution. Hereafter, the short form $ITL(\theta)$ indicates the random variable X has an ITL distribution with parameter θ. Hassan et al.(2020) have discussed the several distributional properties of the ITL distribution. This distribution has a long right tail, so it will affect long term reliability predictions. Many random variables have long-tailed distributions, including traffic patterns in the internet, city population sizes, natural resource occurrences, stock price fluctuations, company sizes, income and so on.

It is important to mention that, to our knowledge, no work has been carried out on the MSS model under the ITL distribution. The focus of this article is to establish classical and Bayesian inferences on the reliability of the MSS model when the stress and the strength both follow ITL distributions. The rest of the content of this paper is organized as follows. In Section 2, the maximum likelihood estimate (MLE) of $R_{s,k}$ and associated asymptotic confidence interval (ACI) are obtained. In Section 3, the uniformly minimum variance unbiased estimate (UMVUE) of $R_{s,k}$ is investigated. The Bayes estimator of $R_{s,k}$ is determined explicitly, in Section 4. For the sake of comparison, another method of the Bayes estimate is used thought using the Markov chain Monte Carlo (MCMC) method. In addition, the highest probability density (HPD) credible interval is provided in this section. In Section 5, proposed methods are compared via Monte Carlo simulations. In Section 6, the analysis of real data sets are presented for a demonstration of the findings. Finally, concluding comments are given in Section 7.

2. MLE of $R_{s,k}$

Let $X_1, X_2, ..., X_k$ be independent strength random variables which follow $ITL(\alpha)$ and Y be stress random variable follows $ITL(\beta)$. Hence, the reliability of MSS model, using Equation (1) is obtained as

$$R_{s,k} = \sum_{i=s}^{k} \binom{k}{i} 2\beta \int_{0}^{\infty} \left[\frac{1+2y}{(1+y)^2}\right]^{\alpha i} \left[1 - \left(\frac{1+2y}{(1+y)^2}\right)^{\alpha}\right]^{k-i} \frac{y}{(1+y)^3} \left[\frac{1+2y}{(1+y)^2}\right]^{\beta-1} dy.$$

By using the change of variable $\nu = \frac{1+2y}{(1+y)^2}$, the above integral becomes

$$\begin{aligned} R_{s,k} &= \sum_{i=s}^{k} \binom{k}{i} \beta \int_{0}^{1} \nu^{\alpha i+\beta-1} (1-\nu^{\alpha})^{k-i} d\nu \\ &= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} (-1)^j \beta \int_{0}^{1} \nu^{\alpha(i+j)+\beta-1} d\nu \\ &= \sum_{i=s}^{k} \sum_{j=0}^{k-i} \binom{k}{i} \binom{k-i}{j} \frac{(-1)^j \beta}{\alpha(i+j)+\beta}. \end{aligned} \tag{4}$$

To compute the MLE of $R_{s,k}$, assume that $x_1, x_2..., x_m$ and $y_1, y_2..., y_n$ are random observations from $ITL(\alpha)$ and $ITL(\beta)$ distributions, respectively. Thus, the likelihood function takes the following form

$$\begin{aligned} L(\alpha, \beta | x, y) = 2^{m+n} \alpha^m \beta^n \exp\left[\sum_{i=1}^{m} \ln \frac{x_i}{(1+x_i)^3} + (\alpha-1) \sum_{i=1}^{m} \ln \frac{1+2x_i}{(1+x_i)^2}\right] \\ \times \exp\left[\sum_{i=1}^{n} \ln \frac{y_i}{(1+y_i)^3} + (\beta-1) \sum_{i=1}^{n} \ln \frac{1+2y_i}{(1+y_i)^2}\right], \end{aligned} \tag{5}$$

and the corresponding log-likelihood function is

$$\begin{aligned} l(\alpha, \beta | x, y) =& m \ln \alpha + n \ln \beta + \sum_{i=1}^{m} \ln \frac{x_i}{(1+x_i)^3} \\ &+ (\alpha-1) \sum_{i=1}^{m} \ln \frac{1+2x_i}{(1+x_i)^2} + \sum_{i=1}^{n} \ln \frac{y_i}{(1+y_i)^3} \\ &+ (\beta-1) \sum_{i=1}^{n} \ln \frac{1+2y_i}{(1+y_i)^2}, \end{aligned} \tag{6}$$

where the constant term is omitted from the above equation. The MLEs of unknown parameters can be computed as the solution of the following equations:

$$\begin{aligned} \frac{\partial l}{\partial \alpha} &= \frac{m}{\alpha} + \sum_{i=1}^{m} \ln \frac{1+2x_i}{(1+x_i)^2} = 0, \\ \frac{\partial l}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} \ln \frac{1+2y_i}{(1+y_i)^2} = 0. \end{aligned}$$

Consequently,

$$\hat{\alpha} = -\frac{m}{\sum_{i=1}^{m} \ln \frac{1+2x_i}{(1+x_i)^2}}, \qquad \text{and} \qquad \hat{\beta} = -\frac{n}{\sum_{i=1}^{n} \ln \frac{1+2y_i}{(1+y_i)^2}}. \tag{7}$$

By the invariance property of MLE, the MLE of $R_{s,k}$ is obtained as

$$\hat{R}_{s,k}^{MLE}=\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}\frac{(-1)^{j}\hat{\beta}}{\hat{\alpha}\left(i+j\right)+\hat{\beta}}. \tag{8}$$

Now, we derive the ACI of $R_{s,k}$ using the asymptotic distribution of $\theta=(\alpha,\beta)$. The expected Fisher information matrix of $\theta=(\alpha,\beta)$. is defined as

$$I\left(\theta\right)=-E\begin{bmatrix}\frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\beta}\\ \frac{\partial^2 l}{\partial\alpha\partial\beta} & \frac{\partial^2 l}{\partial\beta^2}\end{bmatrix}.$$

The elements of the above matrix are obtained as

$$I_{11}=\frac{m}{\alpha^2},\qquad I_{22}=\frac{n}{\beta^2},\qquad I_{12}=I_{21}=0.$$

Notice that the MLE of $R_{s,k}$ has an asymptotically normal distribution with the mean $R_{s,k}$ and variance

$$H=\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial R_{s,k}}{\partial\theta_i}\frac{\partial R_{s,k}}{\partial\theta_j}I_{ij}^{-1},$$

where I_{ij}^{-1} is the (i,j)th element of the inverse of $I\left(\theta\right)$. Also, we have

$$\begin{aligned}\frac{\partial R_{s,k}}{\partial\alpha} &= \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}\frac{(-1)^{j+1}\beta\left(i+j\right)}{\left[\alpha\left(i+j\right)+\beta\right]^2},\\ \frac{\partial R_{s,k}}{\partial\beta} &= \sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}\frac{(-1)^{j}\alpha\left(i+j\right)}{\left[\alpha\left(i+j\right)+\beta\right]^2}.\end{aligned}$$

Hence, the asymptotic variance is computed by

$$\hat{H}=\frac{\alpha^2}{m}\left(\frac{\partial R_{s,k}}{\partial\alpha}\right)^2+\frac{\beta^2}{n}\left(\frac{\partial R_{s,k}}{\partial\beta}\right)^2\Bigg|_{\left(\hat{\alpha},\hat{\beta}\right)},$$

and the 100(1-δ)% ACI of $R_{s,k}$ is obtained as follows

$$\hat{R}_{s,k}^{MLE}\pm z_{\delta/2}\sqrt{\hat{H}}, \tag{9}$$

where, $z_{\delta/2}$ is the upper $\delta/2$ th quantile of the $N\left(0,1\right)$. It should be pointed out that the confidence interval obtained from Equation (9) may not be within the interval (0,1). In this situation, we employ Logit transformation according to $h\left(R\right)=\log\left[R/(1-R)\right]$, by using the method of Ghitany et al.(2015). Thus, the 100(1-δ)% ACI for $h\left(R\right)$ takes the following form

$$\log\left(\frac{\hat{R}}{1-\hat{R}}\right)\pm z_{\frac{\delta}{2}}\frac{\sqrt{\hat{H}}}{\hat{R}\left(1-\hat{R}\right)}\equiv\left(L_1,L_2\right).$$

Finally, the 100(1-δ)% ACI of $R_{s,k}$ computed as follow

$$\left(\frac{e^{L_1}}{1+e^{L_1}},\frac{e^{L_2}}{1+e^{L_2}}\right). \tag{10}$$

3. UMVUE of $R_{s,k}$

In this section, we derive the UMVUE of $R_{s,k}$ through an unbiased estimator of $\varphi(\alpha,\beta)=(-1)^j\beta/[\alpha(i+j)+\beta]$ and a complete sufficient statistic of (α,β). According to equation (5), we see that the IT distribution belongs to the exponential family, so by result from Casella and Berger(2002) about sufficiency and completeness for the mentioned family, (U^*, V^*) is a jointly complete sufficient statistic for (α, β). Where

$$U^* = -\sum_{i=1}^{m} \ln \frac{1+2x_i}{(1+x_i)^2}, \qquad \text{and} \qquad V^* = -\sum_{i=1}^{n} \ln \frac{1+2y_i}{(1+y_i)^2}. \tag{11}$$

Furthermore, U^* and V^* follow Gamma (m, α) and Gamma (n, β) respectively. Let

$$U = -\ln \frac{1+2X_i}{(1+X_i)^2}, \qquad \text{and} \qquad V = -\ln \frac{1+2Y_i}{(1+Y_i)^2}.$$

It is easy to know that U and V come from the exponential distributions with parameters α and β respectively. Hence,

$$\psi(U,V) = \begin{cases} 1, & U > (i+j)V \\ 0, & \text{otherwise} \end{cases},$$

is an unbiased estimator of $\varphi(\alpha, \beta)$, since

$$\begin{aligned} E[\psi](U,V) &= P[u > (i+j)v] \\ &= \alpha\beta \int_{-}^{\infty} \int_{0}^{u/(i+j)} e^{-\alpha u} e^{-\beta v} \mathrm{d}v \mathrm{d}u \\ &= \alpha \int_{0}^{\infty} e^{-\alpha u} \left[1 - e^{-\frac{\beta u}{i+j}}\right] \mathrm{d}u \\ &= \alpha \left[\frac{1}{\alpha} - \frac{1}{\alpha + \beta/(i+j)}\right] \\ &= \frac{\beta}{\alpha(i+j)+\beta}. \end{aligned}$$

and so the UMVUE of $\varphi(\alpha, \beta)$ can be derived by using the Lehmann-Scheffe Theorem. Therefore,

$$\begin{aligned} \hat{\varphi}_{UM}(\alpha,\beta) &= E\left[\psi(U,V) \,|U^* = u,\ V^* = v\right] \\ &= \int_A \int f_{U|U^*=u^*}(u\,|u^*)\, f_{V|V^*=v^*}(v\,|v^*)\, du\, dv, \end{aligned} \tag{12}$$

where $A = \{(u,v):\ 0 < u < u^*,\ 0 < v < v^*,\ u > (i+j)\,v\}$. This integral can be discussed with regards to $h < 1$ and $h > 1$, where $h = (i+j)\,v^*/u^*$. When $h < 1$, the integral in Equation (12) reduces to

$$\begin{aligned} \hat{\varphi}_{UM}(\alpha,\beta) &= \int_0^{v^*} \int_{v(i+j)}^{u^*} \frac{(m-1)(n-1)}{u^* v^*} \Big(1 - \frac{u}{u^*}\Big)^{m-2} \Big(1 - \frac{v}{v^*}\Big)^{n-2} du\, dv \\ &= (n-1) \int_0^1 (1-z)^{n-2} (1-hz)^{m-1} dz, \qquad \text{where } \ z = v/v^* \\ &= \sum_{l=0}^{m-1} (-1)^l (h)^l \binom{m-1}{l} \Big/ \binom{n+l-1}{l}. \end{aligned} \tag{13}$$

Similarly, when $h > 1$, the integral in Equation (12) reduces to

$$\begin{aligned} \hat{\varphi}_{UM}(\alpha,\beta) &= \int_0^{u^*} \int_0^{u/(i+j)} \frac{(m-1)(n-1)}{u^* v^*} \Big(1 - \frac{u}{u^*}\Big)^{m-2} \Big(1 - \frac{v}{v^*}\Big)^{n-2} dv\, du \\ &= 1 - (m-1) \int_0^1 (1-z)^{m-2} \left(1 - h^{-1} z\right)^{n-1} dz, \quad \text{where } \ z = u/u^* \end{aligned}$$

$$= \quad 1-\sum_{l=0}^{n-1}(-1)^{l}(h)^{-l}\binom{n-1}{l}/\binom{m+l-1}{l}. \tag{14}$$

Thus, the $\hat{\varphi}_{UM}(\alpha,\beta)$ is obtained from Equations (13) and (14). Finally, the UMVUE of $R_{s,k}$ is determined by applying the linearity property of UMVUE as follows

$$\hat{R}_{s,k}^{UM}=\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^{j}\hat{\varphi}_{UM}(\alpha,\beta). \tag{15}$$

4. Bayes estimation of $R_{s,k}$

In this section, we compute the Bayesian estimate of $R_{s,k}$ under assumption that the prior distributions for α and β follow Gamma(a_1,b_1) and Gamma(a_2,b_2) respectively, where $a_i,b_i>0,\ i=1,2$. Thus, the joint posterior distribution of α and β becomes

$$\begin{aligned}\pi(\alpha,\beta\,|x,y) &= \frac{L(x,y\,|\alpha,\beta)\,\pi_1(\alpha)\,\pi_2(\beta)}{\int_0^\infty\int_0^\infty L(x,y\,|\alpha,\beta)\,\pi_1(\alpha)\,\pi_2(\beta)\,d\alpha d\beta}\\ &= \frac{(b_1+U^*)^{m+a_1}(b_2+V^*)^{n+a_2}}{\Gamma(m+a_1)\,\Gamma(n+a_2)}\alpha^{m+a_1-1}\beta^{n+a_2-1}\exp\left[-\alpha(b_1+U^*)-\beta(b_2+V^*)\right],\end{aligned}$$

Where U^* and V^* are shown in Equation (**??**). Then, the Bayes estimate of $R_{s,k}$ against the squared error loss function is calculated by

$$\hat{R}_{s,k}^{B}=E(R_{s,k}\,|x,y)=\sum_{i=s}^{k}\sum_{j=0}^{k-i}\binom{k}{i}\binom{k-i}{j}(-1)^{j}\int_0^\infty\int_0^\infty\frac{\beta}{\alpha(i+j)+\beta}\pi(\alpha,\beta\,|x,y)\,d\alpha d\beta.$$

Now, by using the method of Kızılaslan and Nadar(2018), the Bayes estimate of $R_{s,k}$ can be rewritten as follows

$$\hat{R}_{s,k}^{Bayes}=\begin{cases}\sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix}k\\ i\end{pmatrix}\begin{pmatrix}k-i\\ j\end{pmatrix}(-1)^{j}(1-w)^{n+a_2}\frac{n+a_2}{q}\,{}_1F_2\left(q,n+a_2+1;q+1,w\right), & |w|<1\\ \sum_{i=s}^{k}\sum_{j=0}^{k-i}\begin{pmatrix}k\\ i\end{pmatrix}\begin{pmatrix}k-i\\ j\end{pmatrix}\frac{(-1)^{j}(n+a_2)}{q(1-w)^{m+a_1}}\,{}_1F_2\left(q,m+a_1+1;q+1,\frac{w}{w-1}\right), & w<-1\end{cases}$$

where $q=m+n+a_1+a_2$ and $w=1-\frac{(b_2+V^*)(i+j)}{b_1+U^*}$. Notice that

$${}_2F_1(a,b;c,x)=\frac{1}{Beta(a,c-a)}\int_0^1 w^{a-1}(1-w)^{c-a-1}(1-xw)^{-b}dw,\quad |w|<1,$$

is the hypergeometric series, which is available in standard software such as R. Therefore, for this example, the Bayes estimate is derived in the closed form. However, we provided the Bayes estimate by using another technique, namely the MCMC method. It helps assess how efficient the approximate MCMC method compared to exact one, in terms of bias and MSE. For this purpose, we use the Gibbs sampling algorithm to determine the Bayes estimate and to establish the credible interval for $R_{s,k}$. The posterior conditional density of α and β can be derived as

$$\pi^*(\alpha\,|\beta,x,y) = \frac{(b_1+U)^{m+a_1}}{\Gamma(m+a_1)}\alpha^{m+a_1-1}\exp\left[-\alpha(b_1+U)\right], \tag{16}$$

$$\pi^*(\beta\,|\alpha,x,y) = \frac{(b_2+V)^{n+a_2}}{\Gamma(n+a_2)}\beta^{n+a_2-1}\exp\left[-\beta(b_2+V)\right]. \tag{17}$$

We see that the posterior pdfs of α and β given in Equations (16) and (17) have Gamma distribution. Thus, we generate random sample from α and β by using the Gibbs sampling algorithm steps as follows:
Step 1: Set $p=1$.

Step 2: Generate $\alpha^{(p)}$ from Gamma$(m + a_1, b_1 + U)$.
Step 3: Generate $\beta^{(p)}$ from Gamma $(n + a_2, b_2 + V)$.
Step 4: Compute $R_{s,k}^{(p)}$ from Equation (8) at $\left(\alpha^{(p)}, \beta^{(p)}\right)$.
Step 5: Set $p = p + 1$.
Step 6: Repeat steps 2-5, N times, and obtain $R_{s,k}^{(p)}$ for $p = 1, 2, ..., N$.
The Bayes estimate of $R_{s,k}$, based on the MCMC method is calculated by

$$\hat{R}_{s,k}^{B-MC} = \frac{1}{N}\sum_{p=1}^{N} R_{s,k}^{(p)}.$$

Also, through the above procedure, the 100(1-δ)% HPD credible interval for $R_{s,k}$ can be computed using the method of Chen and Shao(1999), by minimizing

$$\left(R_{s,k}{}^{([(1-\delta)N]+i)} - R_{s,k}{}^{(i)}\right), \qquad 1 \le i \le \delta N,$$

where, [.] denotes the largest integer function and the values of $R_{s,k}$ are ranked in ascending order from 1 to N.

5. Simulation study

In this section, we performed Monte Carlo simulations to compare the performances of different estimates of $R_{s,k}$ by using the classical and Bayesian methods. In this regard, we generated random samples from stress and strength variables for different combinations of parameters and sample sizes 10, 30 and 50. We estimated the reliability of the MSS model in two cases $(s, k) = (1, 4)$ and (2,5) .The criteria of mean square error (MSE), bias, average length (AL) as well as coverage probability (CP) at confidence level of 95%, were used to evaluate the simulation results. To investigate the Bayes estimations, non-informative and informative priors were considered and had been dubbed Prior 1 and Prior 2, respectively. We had also derived the Bayes estimates using the MCMC method. All of the computations were done by using R 3.4.4 based on 10,000 replications. Furthermore, the Bayes estimate along with its credible interval, were calculated using 1,000 sampling. Table 1 represents the details of the simulations. All of results are reported in Tables 2-5. The following findings can be drawn from Tables 2-5.

- As anticipated, the biases and MSEs of all the estimators decrease as sample sizes increase.
- As expected, the MSEs of all estimators are close to each other as the sample size increases.
- The biases of the estimates of $R_{s,k}$ have the general order as follows, where $\hat{R}_{s,k}^{B-P_1}$ and $\hat{R}_{s,k}^{B-P_2}$ are Bayes estimates based on Prior 1 and Prior 2, respectively.

$$bias\left(\hat{R}_{s,k}^{UM}\right) < bias\left(\hat{R}_{s,k}^{MLE}\right) < bias\left(\hat{R}_{s,k}^{B-P2}\right) < bias\left(\hat{R}_{s,k}^{B-P1}\right)$$

- The MSEs of estimates are generally in the following order when $R_{s,k}$ is close to extreme values.

$$MSE\left(\hat{R}_{s,k}^{B-P2}\right) < MSE\left(\hat{R}_{s,k}^{UM}\right) < MSE\left(\hat{R}_{s,k}^{MLE}\right) < MSE\left(\hat{R}_{s,k}^{B-P1}\right)$$

- In most cases, the MSEs of estimates are in the following order when $R_{s,k}$ is close to o.55.

$$MSE\left(\hat{R}_{s,k}^{B-P2}\right) < MSE\left(\hat{R}_{s,k}^{B-P1}\right) < MSE\left(\hat{R}_{s,k}^{MLE}\right) < MSE\left(\hat{R}_{s,k}^{UM}\right)$$

- The reliability of the Bayes and ML estimates are biased negatively when $R_{s,k} > 0.55$. In the case of UMVUE, the biases are negligible.
- The MSEs of all the estimates are large when the true value of MSS reliability is about 0.55 and they are small when the true value of MSS reliability is close to extreme values.
- As anticipated, the ALs of the intervals of $R_{s,k}$ tend to shrink as the sample size increases.

- In general, The ALs of the interval estimates of $R_{s,k}$ are ordered as

$$AL\left(\hat{R}_{s,k}^{B-P2}\right) < AL\left(\hat{R}_{s,k}^{B-P1}\right) < AL\left(\hat{R}_{s,k}^{MLE}\right)$$

- The biases and MSEs of the exact Bayes estimates are almost identical to the Bayes estimates which are computed from the MCMC method.
- The CPs of the interval estimates are relatively well, however, most of these values are lower than the predefined nominal level of 95%.
- The CPs of the asymptotic confidence intervals are more appropriate than HPD credible intervals except in a few cases for n=10 .

6. Real example

To display the application of the different approaches developed in this paper, we have considered lifetime data sets reported in Nelson(2003). The data represent the length of times (in minute) to the breakdown of an insulating fluid at seven voltage levels, ranging from 26 to 38 kilovolts (kV). These data sets are briefly referred to as the breakdown time data. Here, we consider the time to the breakdown of 34 kV and 36 kV. These data sets are reported as follows:

X: 34 kV 0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.15, 4.67, 4.85, 6.50, 7.35, 8.01, 8.27, 12.06, 31.75, 32.52, 33.91, 36.71, 72.89.

Y: 36 kV 0.35, 0.59, 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, 3.67, 3.99, 5.35, 13.77, 25.50.

The validity of the ITL distribution for the considered data sets has been checked by using the Kolmogorov-Smirnov (K-S) test. It was observed that for X, the K-S distance and the corresponding p-value are 0.1899 and 0.4451, respectively. Also, for Y, the K-S distance and the corresponding p-value are 0.1826 and 0.6345, respectively. Based on p-values, it is clear that the ITL distribution provides reasonable satisfaction for both of the data sets. Furthermore, in this example the ITL distribution was compared to some well-known lifetime distributions, namely Chen, Gompertz, generalized Rayleigh (GR), Burr type XII (B-XII) and generalized inverted exponential (GIE). The pdfs of these distributions are listed below:

$$\begin{aligned}
\text{Chen} &: \quad f(z;\alpha,\theta) = \alpha\theta z^{\theta-1}\exp\left[\alpha\left\{1-\exp\left(z^{\theta}\right)\right\}+z^{\theta}\right],\\
\text{Gompertz} &: \quad f(z;\alpha,\theta) = \alpha\exp\left[\theta z-\frac{\alpha}{\theta}\left\{\exp(\theta z)-1\right\}\right],\\
\text{GR} &: \quad f(z;\alpha,\theta) = 2\alpha\theta\exp\left(-\theta z^2\right)\left[1-\exp\left(-\theta z^2\right)\right]^{\alpha-1},\\
\text{B-XII} &: \quad f(z;\alpha,\theta) = \alpha\theta z^{\theta-1}\left(1+z^{\theta}\right)^{-(\alpha+1)},\\
\text{GIE} &: \quad f(z;\alpha,\theta) = \frac{\alpha\theta}{z^2}\exp\left(-\frac{\theta}{z}\right)\left[1-\exp\left(-\frac{\theta}{z}\right)\right]^{\alpha-1},
\end{aligned}$$

where $\alpha,\ \theta$ and z are positive. For the purpose of comparison of the above distributions with the ITL distribution, we used several criteria including, the Akaike information criterion (AIC), the finite sample-corrected AIC (AICc), the Bayesian information criterion (BIC) and Hannon and Quinn information criterion (HQIC). The statistic value of AIC, AICc, BIC and HQIC is reported in Table 6 for the X and Y data sets. Based on Table 6, it was claimed that the ITL distribution has the lowest goodness of fit statistics compared to the other competitive models. So, it can be selected as the suitable model. Now, we obtain the reliability of the MSS model through classical and Bayesian methods for (s,k)=(1,3), (2,4), (1,4) and (2,5). First, from the above data sets, the ML estimates of α and β were computed as $\hat{\alpha} = 0.6570$ and $\hat{\beta} = 1.1886$, respectively. Then, the MLE of $R_{s,k}$ along with its ACI were obtained from Equations (8) and (10), respectively. Also, the UMVUE of $R_{s,k}$ was determined in Equation (15). To analyze the data from the Bayesian view, because there does not exist any prior knowledge of unknown parameters, we had taken non-informative priors with the sets of hyperparameters as (a_i, b_i) =$(0.0001, 0.0001)$, $i = 1, 2$. Table 7 gives the point and interval estimates of $R_{s,k}$. It was observed that the Bayes estimates obtained by using the MCMC method were the same as the exact Bayes estimates. Moreover, the HPD credible intervals of $R_{s,k}$, were shorter than the ACIs.

7. Concluding comments

In this article, we considered the reliability of the MSS model under the assumption that the stress and strength random variables were taken from ITL distributions. The reliability of the MSS model was obtained using the ML, UMVU and Bayes estimates, explicitly. The asymptotic and HPD intervals were constructed, respectively, through the Fisher information matrix and the MCMC algorithm.

The simulation results showed that the bias and MSE of $R_{s,k}$ decrease as the sample size increases. Also, the average lengths of interval estimates get shorter when the sample size increases. The biases of the UMVU estimates were lesser than that of the other estimates in all cases. According to the MSE and AL values, the Bayesian estimators under the informative priors had the best performances among the estimators. Moreover, the MSEs and ALs of the all estimators were small when $R_{s,k}$ tends to the extreme value and they were large when $R_{s,k}$ tends to 0.55. Comparing the different estimators in terms of the CPs indicated that the asymptotic confidence intervals generally worked better. Furthermore, the estimates of $R_{s,k}$ obtained from the MCMC and exact Bayes methods were almost identical.

Table 1: Different combinations of stress and strength parameters along with the true values of MSS reliability for Monte Carlo simulations.

	True values of $R_{s,k}$		Bayesian framework		
	$R_{s,k}$		Prior 1	Prior 2	
(α,β)	(1,4)	(2,5)	(a_1,b_1)=(a_2,b_2)	(a_1,b_1)	(a_2,b_2)
(3,0.1544)	0.10	0.063	(0.0001,0.0001)	(3,1)	(0.154,1)
(3,0.3333)	0.20	0.130	(0.0001,0.0001)	(3,1)	(0.333,1)
(3,0.5444)	0.30	0.202	(0.0001,0.0001)	(3,1)	(0.544,1)
(3,0.7993)	0.40	0.279	(0.0001,0.0001)	(3,1)	(0.799,1)
(3,1.1169)	0.50	0.361	(0.0001,0.0001)	(3,1)	(1.117,1)
(3,1.3085)	0.55	0.406	(0.0001,0.0001)	(3,1)	(1.309,1)
(3,1.5304)	0.60	0.452	(0.0001,0.0001)	(3,1)	(1.530,1)
(3,2.1047)	0.70	0.552	(0.0001,0.0001)	(3,1)	(2.105,1)
(3,3)	0.80	0.667	(0.0001,0.0001)	(3,1)	(3,1)
(3,4.7869)	0.90	0.803	(0.0001,0.0001)	(3,1)	(4.787,1)

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Table 2: The Bias and MSE (presented in parenthesis) of the estimates of $R_{1,4}$.

			Classical		Bayesian			
					MCMC		Exact	
(α,β)	$R_{1,4}$	n	MLE	UMVUE	Prior1	Prior2	Prior1	Prior2
		10	0.0085 (0.0023)	-0.0001 (0.0020)	0.0169 (0.0028)	0.0153 (0.0020)	0.0170 (0.0028)	0.0155 (0.0020)
(3,0.1544)	0.10	30	0.0027 (0.0006)	0.0001 (0.0006)	0.0054 (0.0007)	0.0054 (0.0006)	0.0056 (0.0007)	0.0058 (0.0006)
		50	0.0016 (0.0004)	0.0001 (0.0004)	0.0033 (0.0004)	0.0031 (0.0004)	0.0035 (0.0004)	0.0034 (0.0004)
		10	0.0126 (0.0070)	-0.0003 (0.0066)	0.0246 (0.0076)	0.0212 (0.0057)	0.0243 (0.0075)	0.0206 (0.0057)
(3,0.3333)	0.20	30	0.0043 (0.0021)	-0.0002 (0.0020)	0.0075 (0.0021)	0.0077 (0.0020)	0.0080 (0.0022)	0.0084(0.0020)
		50	0.0025 (0.0012)	0.0001 (0.0012)	0.0043 (0.0013)	0.0051 (0.0012)	0.0045 (0.0012)	0.0038 (0.0011)
		10	0.0131 (0.0118)	-0.0009 (0.0121)	0.0229 (0.0116)	0.0212 (0.0086)	0.0227 (0.0115)	0.0220 (0.0090)
(3,0.5444)	0.30	30	0.0043 (0.0038)	-0.0003 (0.0038)	0.0088 (0.0039)	0.0090 (0.0034)	0.0084 (0.0038)	0.0075(0.0034)
		50	0.0026 (0.0023)	0.0003 (0.0023)	0.0059 (0.0023)	0.0053 (0.0021)	0.0057 (0.0023)	0.0053 (0.0021)
		10	0.0101 (0.0158)	-0.0007 (0.0171)	0.0195 (0.0152)	0.0146 (0.0109)	0.0183 (0.0148)	0.0159 (0.0108)
(3,0.7993)	0.40	30	0.0031 (0.0053)	0.0003 (0.0055)	0.0061 (0.0052)	0.0065 (0.0047)	0.0079 (0.0052)	0.0052 (0.0047)
		50	0.0021 (0.0032)	-0.0002 (0.0033)	0.0041 (0.0032)	0.0041 (0.0030)	0.0040 (0.0032)	0.0045 (0.0030)
		10	0.0045 (0.0180)	-0.0003 (0.0206)	0.0092 (0.0166)	0.0103 (0.0117)	0.0069 (0.0162)	0.0079 (0.0114)
(3,1.1169)	0.50	30	0.0014 (0.0064)	0.0004 (0.0066)	0.0035 (0.0062)	0.0043 (0.0054)	0.0052 (0.0061)	0.0042 (0.0054)
		50	0.0012 (0.0039)	0.0001 (0.0039)	0.0029 (0.0038)	0.0030 (0.0035)	0.0032 (0.0038)	0.0019 (0.0035)
		10	0.0011 (0.0185)	-0.0001 (0.0214)	0.0023 (0.0168)	0.0016 (0.0115)	0.0021 (0.0163)	0.0033 (0.0115)
(3,1.3085)	0.55	30	0.0005 (0.0065)	0.0002 (0.0069)	0.0022 (0.0063)	0.0013 (0.0056)	0.0024 (0.0062)	0.0011 (0.0055)
		50	0.0003 (0.0040)	0.0001 (0.0041)	0.0012 (0.0039)	0.0006 (0.0036)	0.0017 (0.0039)	0.0004 (0.0036)
		10	-0.0013 (0.0186)	0.0005 (0.0213)	-0.0028 (0.0161)	-0.0042 (0.0112)	-0.0032 (0.0163)	-0.0014 (0.0113)
(3,1.5304)	0.60	30	-0.0002 (0.0066)	0.0001 (0.0069)	-0.0013 (0.0063)	-0.0006 (0.0057)	-0.0004 (0.0063)	-0.0015(0.0054)
		50	-0.0002 (0.0040)	0.0005 (0.0041)	-0.0005 (0.0039)	0.0002 (0.0036)	-0.0008 (0.0038)	-0.0004 (0.0035)
		10	-0.0096 (0.0169)	0.0003 (0.0195)	-0.0170 (0.0156)	-0.0106 (0.0096)	-0.0159 (0.0149)	-0.0114 (0.0097)
(3,2.1047)	0.70	30	-0.0031 (0.0060)	-0.0001 (0.0063)	-0.0050 (0.0058)	-0.0043 (0.0049)	-0.0060 (0.0056)	-0.0065 (0.0048)
		50	-0.0019 (0.0036)	0.0007 (0.0038)	-0.0032 (0.0035)	-0.0024 (0.0032)	-0.0037 (0.0036)	-0.0031 (0.0032)
		10	-0.0163 (0.0130)	0.0004 (0.0146)	-0.0286 (0.0121)	-0.0201 (0.0072)	-0.0285 (0.0123)	-0.0205 (0.0071)
(3,3)	0.80	30	-0.0052 (0.0043)	0.0003 (0.0046)	-0.0099 (0.0043)	-0.0102 (0.0037)	-0.0105 (0.0043)	-0.0095 (0.0035)
		50	-0.0033 (0.0026)	0.0001 (0.0027)	-0.0075 (0.0026)	-0.0066 (0.0024)	-0.0065 (0.0026)	-0.0060 (0.0023)
		10	-0.0185 (0.0071)	0.0003 (0.0068)	-0.0349 (0.0078)	-0.0221 (0.0037)	-0.0348 (0.0077)	-0.0231 (0.0038)
(3,4.7869)	0.90	30	-0.0063 (0.0021)	0.0003 (0.0020)	-0.0133 (0.0023)	-0.0101 (0.0017)	-0.0116 (0.0021)	-0.0103(0.0017)
		50	-0.0037 (0.0012)	-0.0003 (0.0012)	-0.0078 (0.0013)	-0.0073 (0.0011)	-0.0078 (0.0012)	-0.0071 (0.0010)

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Table 3: The Bias and MSE (presented in parenthesis) of the estimates of $R_{2,5}$.

			Classical		Bayesian			
					MCMC		Exact	
(α,β)	$R_{2,5}$	n	MLE	UMVUE	Prior1	Prior2	Prior1	Prior2
		10	0.0060 (0.0010)	-0.0001 (0.0008)	0.0119 (0.0013)	0.0104 (0.0008)	0.0120 (0.0014)	0.0102 (0.0008)
(3,0.1544)	0.063	30	0.0019 (0.0003)	0.0001 (0.0003)	0.0039 (0.0003)	0.0033 (0.0003)	0.0040 (0.0003)	0.0036(0.0003)
		50	0.0011 (0.0002)	0.0001 (0.0002)	0.0022 (0.0002)	0.0021 (0.0002)	0.0021 (0.0002)	0.0021 (0.0002)
		10	0.0103 (0.0036)	0.0002 (0.0033)	0.0208 (0.0043)	0.0173 (0.0031)	0.0206 (0.0042)	0.0168 (0.0031)
(3,0.3333)	0.130	30	0.0034 (0.0010)	0.0001 (0.0010)	0.0065 (0.0011)	0.0062 (0.0010)	0.0068 (0.0011)	0.0064(0.0010)
		50	0.0020 (0.0006)	0.0001 (0.0006)	0.0041 (0.0006)	0.0038 (0.0006)	0.0042 (0.0006)	0.0038 (0.0006)
		10	0.0130 (0.0071)	-0.0001 (0.0068)	0.0257 (0.0080)	0.0200 (0.0055)	0.0262 (0.0078)	0.0203 (0.0056)
(3,0.5444)	0.202	30	0.0042 (0.0022)	0.0001 (0.0021)	0.0079 (0.0022)	0.0077 (0.0200)	0.0084 (0.0023)	0.0076(0.0020)
		50	0.0025 (0.0013)	0.0001 (0.0012)	0.0052 (0.0013)	0.0057 (0.0012)	0.0049 (0.0013)	0.0046 (0.0012)
		10	0.0133 (0.0111)	-0.0007 (0.0109)	0.0252 (0.0111)	0.0290 (0.0080)	0.0257 (0.0113)	0.0293 (0.0081)
(3,0.7993)	0.279	30	0.0042 (0.0035)	-0.0001 (0.0034)	0.0100 (0.0036)	0.0093 (0.0033)	0.0103 (0.0036)	0.0094(0.0033)
		50	0.0026 (0.0021)	0.0001 (0.0021)	0.0057 (0.0021)	0.0053 (0.0020)	0.0058 (0.0021)	0.0054 (0.0020)
		10	0.0113 (0.0146)	-0.0008 (0.0155)	0.0194 (0.0139)	0.0191 (0.0099)	0.0199 (0.0140)	0.0186 (0.0096)
(3,1.1169)	0.361	30	0.0037 (0.0049)	-0.0008 (0.0049)	0.0088 (0.0049)	0.0079 (0.0043)	0.0086 (0.0047)	0.0074(0.0043)
		50	0.0025 (0.0029)	-0.0001 (0.0029)	0.0048 (0.0030)	0.0048 (0.0027)	0.0047 (0.0029)	0.0045 (0.0027)
		10	0.0095 (0.0162)	-0.0012 (0.0176)	0.0203 (0.0152)	0.0137 (0.0105)	0.0194 (0.0150)	0.0141 (0.0105)
(3,1.3085)	0.406	30	0.0034 (0.0055)	-0.0001 (0.0056)	0.0060 (0.0053)	0.0056 (0.0047)	0.0065 (0.0054)	0.0062 (0.0046)
		50	0.0020 (0.0033)	-0.0001 (0.0034)	0.0042 (0.0033)	0.0045 (0.0031)	0.0039 (0.0033)	0.0040 (0.0029)
		10	0.0084 (0.0177)	0.0005 (0.0197)	0.0157 (0.0164)	0.0121 (0.0110)	0.0147 (0.0160)	0.0118 (0.0112)
(3,1.5304)	0.452	30	0.0031 (0.0061)	0.0002 (0.0063)	0.0054 (0.0059)	0.0044 (0.0051)	0.0050 (0.0059)	0.0040(0.0050)
		50	0.0018 (0.0037)	0.0001 (0.0037)	0.0033 (0.0036)	0.0029 (0.0033)	0.0035 (0.0036)	0.0025 (0.0033)
		10	0.0013 (0.0190)	-0.0004 (0.0216)	0.0035 (0.0168)	0.0040 (0.0110)	0.0030 (0.0167)	0.0040 (0.0108)
(3,2.1047)	0.552	30	0.0008 (0.0068)	-0.0001 (0.0071)	0.0013 (0.0064)	0.0017 (0.0056)	0.0014 (0.0064)	0.0018(0.0055)
		50	0.0005 (0.0041)	-0.0001 (0.0042)	0.0010 (0.0040)	0.0012 (0.0035)	0.0020 (0.0040)	0.0014 (0.0036)
		10	-0.0074 (0.0180)	0.0006 (0.0212)	-0.0121 (0.0159)	-0.0094 (0.0099)	-0.0124 (0.0160)	-0.0090 (0.0096)
(3,3)	0.667	30	-0.0020 (0.0064)	0.0002 (0.0068)	-0.0040 (0.0062)	-0.0029 (0.0052)	-0.0039 (0.0061)	-0.0026 (0.0051)
		50	-0.0013 (0.0040)	-0.0001 (0.0040)	-0.0010 (0.0038)	-0.0020 (0.0033)	-0.0009 (0.0039)	-0.0022 (0.0035)
		10	-0.0160 (0.0133)	-0.0002 (0.0150)	-0.0297 (0.0125)	-0.0184 (0.0066)	-0.0286 (0.0124)	-0.0189 (0.0066)
(3,4.7869)	0.803	30	-0.0055 (0.0045)	-0.0005 (0.0046)	-0.0110 (0.0045)	-0.0092 (0.0034)	-0.0117 (0.0044)	-0.0092 (0.0035)
		50	-0.0031 (0.0027)	-0.0001 (0.0028)	-0.0052 (0.0027)	-0.0054 (0.0023)	-0.0057 (0.0026)	-0.0051 (0.0022)

Table 4: ALs of the interval estimates of $R_{1,4}$ and their corresponding CPs (presented in parenthesis).

				HPD	
(α,β)	$R_{1,4}$	n	Asymptotic	Prior1	Prior2
(3,0.1544)	0.10	10	0.1813 (0.9456)	0.1742 (0.9277)	0.1615 (0.9482)
		30	0.0982 (0.9483)	0.0939 (0.9301)	0.0917 (0.9361)
		50	0.0748 (0.9487)	0.0718 (0.9275)	0.0705 (0.9352)
(3,0.3333)	0.20	10	0.3045 (0.9475)	0.3001 (0.9250)	0.2786 (0.9471)
		30	0.1756 (0.9482)	0.1692 (0.9316)	0.1653 (0.9319)
		50	0.1357 (0.9478)	0.1304 (0.9231)	0.1288 (0.9360)
(3,0.5444)	0.30	10	0.3893 (0.9489)	0.3863 (0.9228)	0.3619 (0.9494)
		30	0.2346 (0.9487)	0.2280 (0.9245)	0.2225 (0.9388)
		50	0.1831 (0.9502)	0.1773 (0.9263)	0.1745 (0.9363)
(3,0.7993)	0.40	10	0.4448 (0.9491)	0.4419 (0.9143)	0.4136 (0.9420)
		30	0.2760 (0.9506)	0.2681 (0.9236)	0.2612 (0.9331)
		50	0.2170 (0.9509)	0.2101 (0.9261)	0.2065 (0.9338)
(3,1.1169)	0.50	10	0.4753 (0.9504)	0.4692 (0.9093)	0.4408 (0.9428)
		30	0.2999 (0.9501)	0.2918 (0.9187)	0.2842 (0.9320)
		50	0.2370 (0.9507)	0.2297 (0.9272)	0.2254 (0.9309)
(3,1.3085)	0.55	10	0.4823 (0.9505)	0.4724 (0.9079)	0.4436 (0.9428)
		30	0.3054 (0.9499)	0.2967 (0.9245)	0.2878 (0.9309)
		50	0.2416 (0.9502)	0.2338 (0.9283)	0.2290 (0.9330)
(3,1.5304)	0.60	10	0.4839 (0.9507)	0.4691 (0.9114)	0.4395 (0.9447)
		30	0.3062 (0.9501)	0.2963 (0.9266)	0.2870 (0.9305)
		50	0.2423 (0.9491)	0.2342 (0.9265)	0.2291 (0.9311)
(3,2.1047)	0.70	10	0.4693 (0.9494)	0.4413 (0.9076)	0.4101 (0.9498)
		30	0.2928 (0.9489)	0.2795 (0.9179)	0.2699 (0.9342)
		50	0.2310 (0.9496)	0.2210 (0.9246)	0.2163 (0.9298)
(3,3)	0.80	10	0.4258 (0.9460)	0.3840 (0.9204)	0.3472 (0.9557)
		30	0.2554 (0.9481)	0.2383 (0.9261)	0.2295 (0.9370)
		50	0.1992 (0.9499)	0.1884 (0.9230)	0.1843 (0.9302)
(3,4.7869)	0.90	10	0.3365 (0.9431)	0.2796 (0.9237)	0.2373 (0.9631)
		30	0.1818 (0.9460)	0.1634 (0.9255)	0.1530 (0.9434)
		50	0.1377 (0.9487)	0.1269 (0.9286)	0.1221 (0.9315)

Table 5: ALs of the interval estimates of $R_{2,5}$ and their corresponding CPs (presented in parenthesis).

				HPD	
(α,β)	$R_{2,5}$	n	Asymptotic	Prior1	Prior2
(3,0.1544)	0.063	10	0.1231 (0.9451)	0.1167 (0.9291)	0.1074 (0.9475)
		30	0.0648 (0.9489)	0.0618 (0.9307)	0.0598 (0.9319)
		50	0.0491 (0.9492)	0.0469 (0.9301)	0.0461 (0.9365)
(3,0.3333)	0.130	10	0.2247 (0.9460)	0.2185 (0.9275)	0.2004 (0.9507)
		30	0.1243 (0.9491)	0.1191 (0.9294)	0.1161 (0.9385)
		50	0.0952 (0.9497)	0.0913 (0.9303)	0.0897 (0.9360)
(3,0.5444)	0.202	10	0.3088 (0.9494)	0.3040 (0.9213)	0.2797 (0.9499)
		30	0.1778 (0.9491)	0.1718 (0.9318)	0.1671 (0.9363)
		50	0.1375 (0.9485)	0.1325 (0.9309)	0.1306 (0.9343)
(3,0.7993)	0.279	10	0.3765 (0.9482)	0.3736 (0.9197)	0.3457 (0.9478)
		30	0.2251 (0.9494)	0.2187 (0.9262)	0.2128 (0.9319)
		50	0.1752 (0.9509)	0.1695 (0.9328)	0.1663 (0.9361)
(3,1.1169)	0.361	10	0.4293 (0.9493)	0.4262 (0.9163)	0.3978 (0.9489)
		30	0.2643 (0.9500)	0.2579 (0.9233)	0.2495 (0.9373)
		50	0.2072 (0.9498)	0.2009 (0.9283)	0.1969 (0.9311)
(3,1.3085)	0.406	10	0.4508 (0.9513)	0.4488 (0.9135)	0.4153 (0.9455)
		30	0.2803 (0.9507)	0.2731 (0.9289)	0.2643 (0.9368)
		50	0.2205 (0.9515)	0.2136 (0.9263)	0.2097 (0.9323)
(3,1.5304)	0.452	10	0.4672 (0.9506)	0.4637 (0.9108)	0.4312 (0.9452)
		30	0.2934 (0.9498)	0.2857 (0.9226)	0.2766 (0.9394)
		50	0.2314 (0.9503)	0.2244 (0.9290)	0.2199 (0.9316)
(3,2.1047)	0.552	10	0.4874 (0.9501)	0.4775 (0.9154)	0.4424 (0.9525)
		30	0.3091 (0.9501)	0.3002 (0.9250)	0.2898 (0.9333)
		50	0.2447 (0.9485)	0.2366 (0.9260)	0.2310 (0.9349)
(3,3)	0.667	10	0.4826 (0.9484)	0.4590 (0.9114)	0.4213 (0.9527)
		30	0.3039 (0.9510)	0.2920 (0.9238)	0.2799 (0.9375)
		50	0.2403 (0.9497)	0.2302 (0.9268)	0.2248 (0.9351)
(3,4.7869)	0.803	10	0.4315 (0.9461)	0.3861 (0.9174)	0.3397 (0.9637)
		30	0.2581 (0.9493)	0.2404 (0.9193)	0.2278 (0.9402)
		50	0.2013 (0.9486)	0.1890 (0.9231)	0.1828 (0.9340)

Table 6: Goodness of statistics for breakdown time data.

Data	Model	AIC	AICc	BIC	HQIC
X	Chen	143.71	144.46	145.60	144.03
	Gompertz	143.24	143.99	145.13	143.56
	GR	143.20	143.95	145.09	143.52
	Burr	146.90	147.65	148.79	147.22
	GIE	148.87	149.62	150.76	149.19
	ITL	**142.55**	**142.79**	**143.49**	**142.71**
Y	Chen	84.18	85.18	85.59	84.16
	Gompertz	79.82	80.82	81.24	79.80
	GR	84.33	85.33	85.75	84.32
	Burr	76.47	77.47	77.89	76.46
	GIE	76.13	77.13	77.55	76.12
	ITL	**75.15**	**75.46**	**75.86**	**75.14**

Table 7: Point and interval estimates of $R_{s,k}$ for breakdown time data.

	Point estimates				Interval estimates			
	Classic		Bayes		Asymptotic interval		HPD interval	
(s,k)	MLE	UMVUE	Exact	MCMC	Interval	AL	Interval	AL
(1,3)	0.8834	0.8937	0.8718	0.8717	(0.7067,0.9597)	0.2530	(0.7461,0.9765)	0.2303
(2,4)	0.7745	0.7823	0.7642	0.7642	(0.5542,0.9046)	0.3505	(0.5862,0.9260)	0.3398
(1,4)	0.9197	0.9308	0.9077	0.9077	(0.7519,0.9774)	0.2255	(0.7988,0.9906)	0.1918
(2,5)	0.8344	0.8453	0.8219	0.8219	(0.6126,0.9414)	0.3288	(0.6588,0.9634)	0.3046

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