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Estimation for Flexible Weibull Extension-Burr XII Distribution under Adaptive Type-II Progressive Censoring Scheme

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Abstract

In this paper, based on an adaptive Type-II progressive censoring scheme, estimation of flexible Weibull extension-Burr XII distribution is discussed. Maximum likelihood estimation and asymptotic confidence intervals of the unknown parameters are obtained. The adaptive Metropolis (AM) method is applied to carry out a Bayesian estimation procedure under symmetric and asymmetric loss functions and calculate the credible intervals. A simulation study is carried out to assess the performance of the estimators. Finally, a real life data set is used for illustration purpose.

Key Words: Additive distribution; Adaptive Type-II progressive censoring; Maximum likelihood estimation; Bayesian estimation.

1. Introduction

Usually, in life-testing experiments the failure time of any experimental unit may be attributable to more than one cause or risk factor. These risk factors in some sense compete for the failure of the experimental unit. In such cases, the competing risks model can be used for analysis. Recently, (Kamal and Ismail, (2020)) proposed a new lifetime distribution, which is the additive model of flexible Weibull extension and Burr XII distributions (FWBXII distribution). This new model is very flexible since it has different shapes of failure rate, which are increasing, bathtub, modified bathtub and bi-bathtub shapes. These shapes can be used to analyze various types of lifetime data sets.

The additive Flexible Weibull-Burr XII lifetime model was derived by the sum of the hazard rates of Flexible Weibull and Burr XII distributions as follows

$$h(x) = h_1(x) + h_2(x), \qquad (1)$$

where the hazard rate function of the first component (Flexible Weibull extension) is given by

$$h_1(x;\alpha,\beta) = \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}}, x > 0,$$

and the hazard rate function of the second component (Burr XII) is given by

$$h_2(x;k,c) = ck\, x^{c-1}[1 + x^c]^{-1}, x > 0.$$

So, the probability density function (PDF) of this distribution is given by

$$f(x) = h(x) \exp\left(-\int_0^x h(u)\, du\right),$$

$$f(x) = \left[\left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} + ck\, x^{c-1}[1 + x^c]^{-1}\right] exp\left[-e^{\alpha x - \frac{\beta}{x}}\right][1 + x^c]^{-k}. \qquad (2)$$

The corresponding cumulative density function (CDF) $F(x)$ and reliability function $R(x)$ of this additive model are given by

$$F(x) = 1 - exp\left[-e^{\alpha x - \frac{\beta}{x}}\right][1 + x^c]^{-k}, \tag{3}$$

$$R(x) = exp\left[-e^{\alpha x - \frac{\beta}{x}}\right][1 + x^c]^{-k}. \tag{4}$$

In lifetime experiments, there are two types of sampling, complete sampling and censored sampling. In complete sampling, the data consists of the exact failure times of all experimental units. However in practice, due to time and cost constraints, the experimenters are not able to observe the failure times of all the units. In such cases, the experimenter desires to have data with incomplete information of the failures. Sampling of this type is known as censored sampling. Censoring is very common in reliability analysis. The two most common censoring types namely Type-I and Type-II censoring are widely used in the life-testing and reliability studies where the former is censored at a fixed time and the latter is censored at a fixed number of failures, (see Lawless, (2003)).

However, Type-I and Type-II censoring do not allow for removal of units at points until the terminal point of the experiment. A type of censoring known as the progressive censoring, which has this advantage, has become very popular in the last years. There are two types of progressive censoring, progressively Type-I and progressively Type-II, for more information see (Balakrishnan and Cramer, (2014)). In progressively Type-I censoring, it is assumed that there are m fixed censoring times $T_1, \ldots, T_m$, at each censoring time T_i, R_i of the remaining units are randomly removed. On the other hand, in progressively Type-II censoring, the number of failures ($m<n$) is predetermined. Also, the progressive censoring scheme ($R_1, R_2, \ldots,$) is prefixed such that at each censoring time X_i, R_i of the remaining units are randomly removed.

One of the drawbacks of the progressive Type-II censoring, is that the experiment may last a long time before the m^{th} failure occurs. For this reason, (Ng et al., (2009)) suggested an adaptive Type-II progressive censoring scheme (AT-II PCS) in which the effective number of failures m is fixed in advance and the progressive censoring scheme ($R_1, R_2, \ldots, R_m$) is provided. The values of the R_i may be change accordingly during the experiment with the objective of the terminating the test as soon as possible if a preassigned time T is reached before the failure time of the m^{th} observation. So, the main advantage of this type of censoring is that it speeds up the experiment when the experiment duration exceeds the predetermined time T and assures us to get the effective number of failures.

Over the last years, many authors have studied statistical inferences on the parameters of lifetime models using the AT-II PCS. For example, Weibull distribution (see Lin et al., (2009)), log-normal distribution (see Hemmati and Khorram, (2013)), Pareto distribution (see Mahmoud et al., 2013), exponentiated Weibull distribution (see Al-Sobhi and Soliman, (2015)), Burr XII distribution (see Amein, (2017)), generalized exponential distribution (see Mohie El-Din et al., (2017)) and Weibull-Exponential distribution (see EL-Sagheer et al., (2019)). The major purpose of this paper is to develop statistical estimation, classical and Bayesian, for the parameters of FWBXII distribution by considering an adaptive Type-II progressive censoring scheme. To the best of our knowledge, no previous work has been done regarding adaptive Type-II progressive censoring scheme in the case of competing risks models in which the different risk factors have different distributions. The rest of the paper is organized as follows. In Section 2, the description of the AT-II PCS is introduced. Maximum likelihood estimation and asymptotic confidence intervals for FWBXII based on AT-II PCS are considered in Section 3. In Section 4, using the adaptive Metropolis (AM) algorithm, Bayes estimates are computed based on squared error and LINEX loss functions. Monte Carlo simulation results of different estimation methods are presented in Section 5. Analysis of real data set is presented in Section 6 to illustrate the proposed methods. Finally, concluding remarks are given in Section 7.

2. An adaptive Type-II Progressive Censoring Scheme

Suppose that n identical units are put on life test, the observed number of failures ($m<n$) is fixed in advance, and the experimenter provides a time T, which is an ideal total test, but the experimental time is allowed to run over time T. If the m-th progressive censored observed failure occurs before time T (i.e. $X_m< T$), the experiment stops at this time X_m, and a usual progressive Type-II censoring scheme is adopted with the prefixed progressive censoring scheme ($R_1, \ldots, R_m$). Otherwise, once the experimental time passes T but the number of observed failures has not reached m, then the number of items progressively removed from the experiment upon failure is adopted with the objective of terminating the experiment as soon as possible by setting $R_{J+1}, R_{J+2}, \ldots, R_{m-1} = 0$ and $R_m = n - m - \sum_{i=1}^{J} R_i$, where $X_J< T< X_{J+1}$, and X_J is the J-th failure time occurring before time T and J+1<m. Thus the effectively applied scheme is $R_1, \ldots, R_J, 0, \ldots, 0, n-m-\sum_{i=1}^{J} R_i$. This formulation leads to terminate the experiment as soon as possible if the *(J+1)*th failure time is greater than *T*, and the total test time will not be too far away from the time *T*.

When $T \to \infty$, the adaptive censoring scheme reduces to a progressive Type-II censoring one with censoring scheme $(R_1, \ldots, R_m)$. On the other hand, if $T=0$, then $R_1 = \cdots = R_{m-1} = 0$ and $R_m = n-m$, which leads to a conventional Type-II censoring scheme.

To obtain an adaptive Type-II progressive censored sample of size *m* from the FWBXII distribution, an algorithm given in (Ng et al., (2009)) has been modified as follows:

Step 1: Generate an ordinary Type-II progressive censored sample $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ with censoring scheme $(R_1, R_2, \ldots, R_m)$ as follows:

1. Generate *m* progressively Type-II censored order statistics Y_{1i} for $i = 1, 2, \ldots, m$ from the first component (the flexible Weibull extension distribution).

2. Generate *m* progressively Type-II censored order statistics Y_{2i} for $i = 1, 2, \ldots, m$ from the second component (Burr XII distribution).

Then $X_i = \min(Y_{1i}, Y_{2i})$, for $i = 1, 2, \ldots, m$, is the required progressive Type-II censored sample.

Step2: Determine the value of J, where $X_{J:m:n} < T < X_{J+1:m:n}$, and discard the sample $X_{j+2:m:n}, \ldots, X_{m:m:n}$.

Step 3: Generate the first $m-j-1$ order statistics from a truncated FWBXII distribution $f(x)/[1-F(x_{j+1:m:n})]$ with sample size $(n-\sum_{i=1}^{j} R_i - j - 1)$ as $X_{j+2:m:n}, X_{j+3:m:n}, \ldots, X_{m:m:n}$, as follows:

1. Generate $n-\sum_{i=1}^{j} R_i - j - 1$ independent Uniform (0, l) observations, $\grave{U}_{1\iota}$ for $i = 1, 2, \ldots, n-\sum_{i=1}^{j} R_i - j - 1$, then use them to generate a random variable $\grave{Y}_{1\iota}$ from a truncated distribution $f_1(x)/[1-F_1(x_{j+1:m:n})]$, where $f_1(.)$ and $F_1(.)$ are the pdf and cdf of the flexible Weibull extension distribution, respectively.

2. Generate $n-\sum_{i=1}^{j} R_i - j - 1$ independent Uniform (0, l) observations, $\grave{U}_{2\iota}$ for $i = 1, 2, \ldots, n-\sum_{i=1}^{j} R_i - j - 1$, then use them to generate a random variable $\grave{Y}_{2\iota}$ from a truncated distribution $f_2(x)/[1-F_2(x_{j+1:m:n})]$, where $f_2(.)$ and $F_2(.)$ are the pdf and cdf of the Burr XII distribution, respectively.

Then $X_i = \min(\grave{Y}_{1\iota}, \grave{Y}_{2\iota})$, for $i = 1, 2, \ldots, n-\sum_{i=1}^{j} R_i - j - 1$. The first $m-j-1$ ordered observations from this sample is the required Type-II censored sample.

3. Maximum Likelihood Estimation

In this section, maximum likelihood estimation and the approximate confidence intervals of the parameters of the FWBXII distribution are discussed under an adaptive Type-II progressive censored sample.

3.1 Point Estimation

Using Equations (2) and (3), the likelihood function of (α, β, c, k) is obtained as

$$L(\underline{\theta}|\underline{X}) = B\left[\prod_{i=1}^{m}\left(\left(\alpha+\frac{\beta}{x_i^{2}}\right) e^{\alpha x_i-\frac{\beta}{x_i}} + ck\, x_i^{c-1}[1+x_i^{c}]^{-1}\right)\right] exp\left[-\sum_{i=1}^{m} e^{\alpha x_i-\frac{\beta}{x_i}}\right]$$
$$\times\left[\prod_{i=1}^{m}[1+x_i^{c}]^{-k}\right]\left[exp\left[-\sum_{i=1}^{j} R_i\, e^{\alpha x_i-\frac{\beta}{x_i}}\right]\left[\prod_{i=1}^{j}[1+x_i^{c}]^{-R_i k}\right]\right.$$
$$\times\left[exp\left[-e^{\alpha x_m-\frac{\beta}{x_m}}\right][1+x_m^{c}]^{-k}\right]^{\left(n-m-\sum_{i=1}^{j} R_i\right)}, \tag{5}$$

where $B = \prod_{i=1}^{m}\left(n-i+1-\sum_{k=1}^{\max(i-1,\ J)} R_k\right)$.

The log-likelihood function is

$$\mathcal{L}(\underline{\theta}|\underline{X}) = \ln B + \sum_{i=1}^{m} \ln\left[\left(\alpha + \frac{\beta}{{x_i}^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}} + ck\, {x_i}^{c-1}[1+{x_i}^c]^{-1}\right] - \sum_{i=1}^{m} e^{\alpha x_i - \frac{\beta}{x_i}} - k\sum_{i=1}^{m}\ln(1+{x_i}^c)$$
$$-\sum_{i=1}^{j} R_i\, e^{\alpha x_i - \frac{\beta}{x_i}} - k\sum_{i=1}^{j} R_i \ln(1+{x_i}^c) - \left(n-m-\sum_{i=1}^{j} R_i\right) e^{\alpha x_m - \frac{\beta}{x_m}}$$
$$-k\left(n-m-\sum_{i=1}^{j} R_i\right)\ln(1+{x_m}^c), \qquad (6)$$

The maximum likelihood estimators of the parameters are obtained by differentiating the log-likelihood function with respect to the parameters (α, β, c, k) and setting the result to zero, as follows

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \sum_{i=1}^{m} \frac{h_\alpha(x_i;\underline{\theta})}{h(x_i;\underline{\theta})} - \sum_{i=1}^{m} a_1(x_i;\underline{\theta}) - \sum_{i=1}^{j} R_i\, a_1(x_i;\underline{\theta}) - \left(n-m-\sum_{i=1}^{j} R_i\right) a_1(x_m;\underline{\theta}) = 0, (7)$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \sum_{i=1}^{m} \frac{h_\beta(x_i;\underline{\theta})}{h(x_i;\underline{\theta})} + \sum_{i=1}^{m} a_2(x_i;\underline{\theta}) + \sum_{i=1}^{j} R_i\, a_2(x_i;\underline{\theta}) + \left(n-m-\sum_{i=1}^{j} R_i\right) a_2(x_m;\underline{\theta}) = 0, (8)$$

$$\frac{\partial \mathcal{L}}{\partial c} = \sum_{i=1}^{m} \frac{h_c(x_i;\underline{\theta})}{h(x_i;\underline{\theta})} - \sum_{i=1}^{m} b_1(x_i;\underline{\theta}) - \sum_{i=1}^{j} R_i\, b_1(x_i;\underline{\theta}) - \left(n-m-\sum_{i=1}^{j} R_i\right) b_1(x_m;\underline{\theta}) = 0, (9)$$

$$\frac{\partial \mathcal{L}}{\partial k} = \sum_{i=1}^{m} \frac{h_k(x_i;\underline{\theta})}{h(x_i;\underline{\theta})} - \sum_{i=1}^{m} b_2(x_i;\underline{\theta}) - \sum_{i=1}^{j} R_i\; b_2(x_i;\underline{\theta}) - \left(n-m-\sum_{i=1}^{j} R_i\right) b_2(x_m;\underline{\theta}) = 0, (10)$$

where:

$$a_1(x_i;\underline{\theta}) = x_i e^{\alpha x_i - \frac{\beta}{x_i}},\; a_2(x_i;\underline{\theta}) = \tfrac{1}{x_i} e^{\alpha x_i - \frac{\beta}{x_i}},$$

$$b_1(x_i;\underline{\theta}) = k\,\tfrac{{x_i}^c \ln(x_i)}{(1+{x_i}^c)},\; b_2(x_i;\underline{\theta}) = \ln(1+{x_i}^c),$$

$$h_\alpha(x_i;\underline{\theta}) = \frac{\partial h(x_i;\underline{\theta})}{\partial \alpha} = \left(\alpha + \frac{\beta}{{x_i}^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}}\; x_i + e^{\alpha x_i - \frac{\beta}{x_i}},$$

$$h_\beta(x_i;\underline{\theta}) = \frac{\partial h(x_i;\underline{\theta})}{\partial \beta} = \frac{1}{{x_i}^2} e^{\alpha x_i - \frac{\beta}{x_i}} - \frac{1}{x_i}\left(\alpha + \frac{\beta}{{x_i}^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}},$$

$$h_c(x_i;\underline{\theta}) = \frac{\partial h(x_i;\underline{\theta})}{\partial c} = ck\, {x_i}^{c-1}[1+{x_i}^c]^{-1}\left(\frac{1}{c} + \ln(x_i) - \frac{{x_i}^c \ln(x_i)}{(1+{x_i}^c)}\right),$$

$$h_k(x_i;\underline{\theta}) = \frac{\partial h(x_i;\underline{\theta})}{\partial k} = c\, {x_i}^{c-1}[1+{x_i}^c]^{-1}.$$

The maximum likelihood estimates (MLEs) can be obtained by solving the nonlinear Equations (7)-(10), numerically for α, β, c, and k.

3.2 Asymptotic Confidence Intervals

To obtain the confidence intervals for the parameters α, β, c, and k, we need the distributions of the maximum likelihood estimators. Since these estimators do not have a closed form, their exact distributions cannot be found. Therefore, the approximate confidence intervals are obtained using the asymptotic distribution of the maximum likelihood estimators, which is multivariate normal distribution with mean (α, β, c, k) and variance–covariance matrix which can be approximated by the inverse of the observed Fisher information matrix, I^{-1}, that is,

$$I^{-1} = \begin{bmatrix} -\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} & -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} & -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial c} & -\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial k} \\ -\frac{\partial^2 \mathcal{L}}{\partial \beta \partial \alpha} & -\frac{\partial^2 \mathcal{L}}{\partial \beta^2} & -\frac{\partial^2 \mathcal{L}}{\partial \beta \partial c} & -\frac{\partial^2 \mathcal{L}}{\partial \beta \partial k} \\ -\frac{\partial^2 \mathcal{L}}{\partial c \partial \alpha} & -\frac{\partial^2 \mathcal{L}}{\partial c \partial \beta} & -\frac{\partial^2 \mathcal{L}}{\partial c^2} & -\frac{\partial^2 \mathcal{L}}{\partial c \partial k} \\ -\frac{\partial^2 \mathcal{L}}{\partial k \partial \alpha} & -\frac{\partial^2 \mathcal{L}}{\partial k \partial \beta} & -\frac{\partial^2 \mathcal{L}}{\partial k \partial c} & -\frac{\partial^2 \mathcal{L}}{\partial k^2} \end{bmatrix}^{-1}$$

where the elements of the information matrix are given in the Appendix.

In the above matrix, the partial derivatives are evaluated at the maximum likelihood estimators

$$I^{-1} = \begin{bmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) & cov(\hat{\alpha},\hat{c}) & cov(\hat{\alpha},\hat{k}) \\ cov(\hat{\beta},\hat{\alpha}) & var(\hat{\beta}) & cov(\hat{\beta},\hat{c}) & cov(\hat{\beta},\hat{k}) \\ cov(\hat{c},\hat{\alpha}) & cov(\hat{c},\hat{\beta}) & var(\hat{c}) & cov(\hat{c},\hat{k}) \\ cov(\hat{k},\hat{\alpha}) & cov(\hat{k},\hat{\beta}) & cov(\hat{k},\hat{c}) & var(\hat{k}) \end{bmatrix}.$$

Furthermore, the (1−δ) 100% confidence intervals of the parameters (α, β, c, k) can be obtained by using variance-covariance matrix as the following forms

$$\hat{\alpha} \pm Z_{\delta/2}\sqrt{var(\hat{\alpha})}, \quad \hat{\beta} \pm Z_{\delta/2}\sqrt{var(\hat{\beta})}$$

$$\hat{c} \pm Z_{\delta/2}\sqrt{var(\hat{c})}, \quad \hat{k} \pm Z_{\delta/2}\sqrt{var(\hat{k})}$$

where $\delta > 0$, and $Z_{\delta/2}$ denotes the upper δ/2-th percentage point of the standard normal distribution.

4. Bayesian Estimation

In this section, based upon an adaptive Type-II progressive censored sample, Bayes estimates of the parameters of the FWBXII distribution using an informative prior are obtained.

Let us assume the parameters vector $\underline{\theta} = (\alpha, \beta, c, k) = (\theta_1, \theta_2, \theta_3, \theta_4)$, where $\theta_i, i = 1,2,3,4$, are independent and follow the gamma prior distributions given by

$$g_i(\theta_i) \propto \theta^{a_i - 1} e^{-b_i \theta_i}, \theta_i > 0, \quad a_i, b_i > 0, i = 1,2,3,4, \qquad (11)$$

where a_i and b_i , $i = 1,2,3,4$, are the hyperparameters.

Thus, the joint prior density function of $\underline{\theta}$ is obtained from Equation (11) and is given by

$$\pi(\underline{\theta}) \propto g_1(\theta_1) \times g_2(\theta_2) \times g_3(\theta_3) \times g_4(\theta_4) \propto \prod_{i=1}^{4} \theta_i^{a_i - 1} e^{-b_i \theta_i}, \qquad (12)$$

Based on Equations (5) and (12), the joint prior density function of $\underline{\theta}$ can be written as follows:

$$\pi(\underline{\theta}|\underline{X}) = \frac{\pi(\underline{\theta}) L(\underline{X}|\underline{\theta})}{\int_{\underline{\vartheta}}^{\infty} \pi(\underline{\theta}) L(\underline{X}|\underline{\theta}) d\underline{\theta}}. \qquad (13)$$

In this Section, the squared error loss function (SELF) and the linear-exponential loss function (LINEX) are considered for the Bayesian estimation of the parameters of the proposed additive model.

Under the squared error loss function, the Bayes estimate of the parameter is the posterior mean and the corresponding posterior risk is equal to the posterior variance,

$$\hat{\theta}^{BS} = E_{\theta|\underline{X}}(\theta), \qquad (14)$$

$$PR^S = E_{\theta|\underline{X}}(\theta^2) - \left[E_{\theta|\underline{X}}(\theta)\right]^2 = V_{\theta|\underline{X}}(\theta). \qquad (15)$$

Under the LINEX loss function, the Bayes estimator is given by

$$\hat{\theta}^{BL} = -\frac{1}{\lambda}\ln\left(E\left(e^{-\lambda\theta}\middle|\underline{X}\right)\right), \tag{16}$$

where $\boldsymbol{\lambda}$ is a constant. The posterior risk corresponding to this loss function is given by

$$PR^{L} = d\left(E_{\theta|\underline{X}}\left(e^{\lambda\Delta}\right) - E_{\theta|\underline{X}}(\lambda\Delta) - 1\right), \tag{17}$$

where the expectations in Equations (14) and (16) are with respect to the marginal posterior density function of θ.

In our case, the marginal posterior density functions of the parameters cannot be obtained in a closed form. Hence, no analytical Bayes estimators exist for the parameters under these two loss functions. In such cases, Markov Chain Monte Carlo (MCMC) techniques are commonly used to solve the corresponding numerical solution and evaluate posterior expectations. (Metropolis et al., (1953)) proposed the Metropolis algorithm which is a general MCMC method used to generate a sequence of random variables from a probability distribution that is difficult to sample from. The efficiency of this method, that is, the speed of the convergence, is crucially affected by the choice of the proposal distribution, which proposes the next candidate of the Markov chain. A possible remedy to increase the efficiency is provided by the adaptive Metropolis (AM) algorithm introduced by (Haario et al., (2001)). The basic idea is to update the covariance structure of the proposal distribution by using the knowledge so far acquired about the target distribution from all previous states.

In this paper, the AM algorithm is used to get the Bayes estimates of the parameters. For implementing the AM algorithm, the simulation algorithm consists of the following steps.

(1) Select a vector of starting values for the $(l \times 1)$ parameter vector $\underline{\theta}^{(0)}$.

(2) For each iteration t, (where $t = 1,2,\dots,h$), draw a candidate $\underline{\theta}^{*}$ from the proposal distribution $j_t(\underline{\theta}^{*}|\underline{\theta}^{(0)},\underline{\theta}^{(1)},\dots,\underline{\theta}^{(t-1)})$. The proposal distribution employed in this algorithm is a Gaussian distribution with mean at the current point $\underline{\theta}^{(t-1)}$ and covariance $C_t = C_t(\underline{\theta}^{(0)},\underline{\theta}^{(1)},\dots,\underline{\theta}^{(t-1)})$.

(3) Compute the acceptance rate ($Acc.Rate$) (the fraction of candidate draws that are accepted),

$$Acc.Rate = \frac{\pi\left(\underline{\theta}^{*}\middle|\underline{\boldsymbol{x}}\right)}{\pi\left(\underline{\theta}^{(t-1)}\middle|\underline{\boldsymbol{x}}\right)},$$

where $\pi\left(\underline{\theta}\middle|\underline{\boldsymbol{x}}\right)$ is the target posterior distribution, ignoring the constant of proportionality.

(4) Accept $\underline{\theta}^{*}$ as $\underline{\theta}^{(t)}$ with probability *min* ($Acc.Rate$, 1). If $\underline{\theta}^{*}$ is not accepted, then $\underline{\theta}^{(t)} = \underline{\theta}^{(t-1)}$. This can be done by generating a value u from the Uniform (0,1) distribution. If $u \le Acc.Rate$, then $\underline{\theta}^{(t)} = \underline{\theta}^{*}$, otherwise, $\underline{\theta}^{(t)} = \underline{\theta}^{(t-1)}$.

(5) Repeat steps 2-4 h times, for h large enough.

(6) Use a burn-in period, to reduce the effect of initial values, in which M of the generated $\underline{\theta}$ values are discarded.

Using the AM algorithm discussed previously, the Bayesian estimates and their corresponding posterior risks (PR) of (θ_i, for $i = 1,2,3,4$) under SE and LINEX loss functions can be written, respectively, in the forms

$$\widehat{\theta_i}^{BS} = \frac{1}{h-M}\sum_{j=M+1}^{h}\theta_{ij},$$

$$PR^{S}(\theta_i) = \frac{1}{h-M}\sum_{j=M+1}^{h}\left[\theta_{ij} - \widehat{\theta_i}^{BS}\right]^2,$$

and

$$\hat{\theta}_i^{BL} = \frac{-1}{\lambda} ln\left(\frac{1}{h-M}\sum_{j=M+1}^{h} \exp\left(-\lambda\, \theta_{ij}\right)\right),$$

$$PR^{L}(\theta_i) = d\left\{e^{\lambda\hat{\theta}_i^{BL}}\left(\frac{1}{h-M}\sum_{j=M+1}^{h} \exp\left(-\lambda\, \theta_{ij}\right)\right) - \lambda\left(\hat{\theta}_i^{BL} - \hat{\theta}_i^{BS}\right) - 1\right\},$$

where $\{(\theta_{1j}, \theta_{2j}, \theta_{3j}, \theta_{4j}); j = 1,2, \dots, h\}$ are generated from the posterior PDF (using The AM algorithm) and M is the burn-in period.

For a specified value of τ, define the (1 −τ) ×100% credible interval $(L_{\theta_i}, U_{\theta_i})$ for θ_i, i=1,2,3,4 by

$$\int_{\mathrm{L}_{\theta_i}}^{\infty} \pi_i(\theta_i|\underline{X}\,)d\theta_i = 1 - \frac{\tau}{2}, \quad \int_{\mathrm{U}_{\theta_i}}^{\infty} \pi_i(\theta_i|\underline{X}\,)d\theta_i = \frac{\tau}{2}, \tag{18}$$

where $\pi_i(\theta_i|\underline{X}\,)$, $i = 1,2,3,4$ are the marginal density functions of (α, β, c, k), respectively. It is very difficult to obtain an explicit expression for the marginal probability density function from the joint posterior distribution. However, based on the generated vector $\{(\theta_{1j}, \theta_{2j}, \theta_{3j}, \theta_{4j}); j = 1,2, \dots, h\}$ the credible intervals are readily obtained from the AM computer output.

5. Monte Carlo Simulation and Comparisons

In this section, a simulation study, carried out by the R statistical software, is performed to illustrate the performance of the different estimators of (α, β, c, k) by considering ((n, m) = (50, 30), (50, 40), (60, 30), (60, 40)), (60, 50)), different values of predetermined time T = (0.3, 0.9) and by choosing true parameters values (α=0.2, β=0.5, c=0.5, k=0.2) and (α=1, β=0.3, c=1.2, k=0.8). In addition, three progressive censoring schemes are considered into account, Scheme I is the censoring scheme, in which $n - m$ items are removed from the experiment at the first stages, while Scheme III is the opposite of Scheme I, that is, $n - m$ items are removed at the last stages and at Scheme II, $n - m$ items are removed at some middle stages. More specifically, the used schemes in this section are as follows:

Scheme I: At the first ten stages, $R's = \frac{n-m}{10}$, otherwise $R's = 0$.

Scheme II: At the middle ten stages, $R's = \frac{n-m}{10}$, otherwise $R's = 0$.

Scheme III: At the last ten stages, $R's = \frac{n-m}{10}$, otherwise $R's = 0$.

The Bayes estimates are all computed by considering the priors explained in Table (1). It is noted that, prior 1 and prior 2 are selected in such way that prior means are the same with different variances, while the prior means are the same for priors 3 and 4 with different variances. Depending on the prior belief of the experimenter in terms of location and variability of the prior distribution for each parameter, the hyperparameters are determined.

Table 1: Prior Exception and Prior Variance (in the brackets) of the parameters of (α, β, c, k).

Parameter	***Prior 1***	***Prior 2***	***Prior 3***	***Prior 4***
α	0.2000 (0.1000)	0.2000 (0.2000)	1.0000 (0.0070)	1.0000 (0.0100)
β	0.5000 (0.0100)	0.5000 (0.0200)	0.3000 (0.0070)	0.3000 (0.0100)
c	0.5000 (0.1000)	0.5000 (0.2000)	1.2000 (0.0500)	1.2000 (0.1000)
k	0.2000 (0.0100)	0.2000 (0.0200)	0.8000 (0.0500)	0.8000 (0.1000)

In each setting we obtain the MLEs and Bayes estimates under SE and LINEX (with $\boldsymbol{\lambda}$= 0.5, 1) loss functions. We replicate the process 1000 times. At each time, $\underline{\theta}$ = (α, β, c, k) is generated from the joint prior density (13) based on pre-specified (a_i and b_i , $i = 1,2,3,4$), then the adaptive Type-II progressive censored sample is generated from the FWBXII as described in Section (2). Based on this sample, the MLEs of $\underline{\theta}$ = (α, β, c, k) are computed by solving the nonlinear equations (7)-(10) by using a numerical method such as the Newton-Raphson method. The squared deviations $(\theta^* - \theta)^2$ are computed for different sizes n, where (*) stands for a MLE and θ stands for each parameter in $\underline{\theta}$. In addition, the Bayes estimates and their corresponding PRs for $\underline{\theta}$ = (α, β, c, k) are computed using the aforementioned algorithm described in Section (4) with h=22000 and M=2000. Then, the estimated mean squared error (MSE) values are computed by averaging the squared deviations and the expected posterior risks (ER) values are computed by averaging the PRs. Finally, 95% confidence/ credible interval width and coverage probabilities are obtained for the purpose of comparison.

When using MCMC techniques within the Bayesian paradigm and before proceeding to make any inference, the convergence of all parameters to the stationary distribution should be checked. For illustration, the convergence of one of the generated chains for an arbitrary chosen combination ($\alpha = 0.2, \beta = 0.5, c = 0.5, k = 0.2, n = 60, m = 30$) under Scheme I and T=0.9 is shown in Figure (1-3). These figures show the trace plots, the corresponding autocorrelation plots of this chain and the histogram of simulated (α, β, c, k). Based on these figures, the chain is considered convergent. This observation is confirmed by the Geweke's diagnostic which provides Z-scores (-0.9282, -0.3568, 0.8281, 0.4733) for (α, β, c, k), respectively.

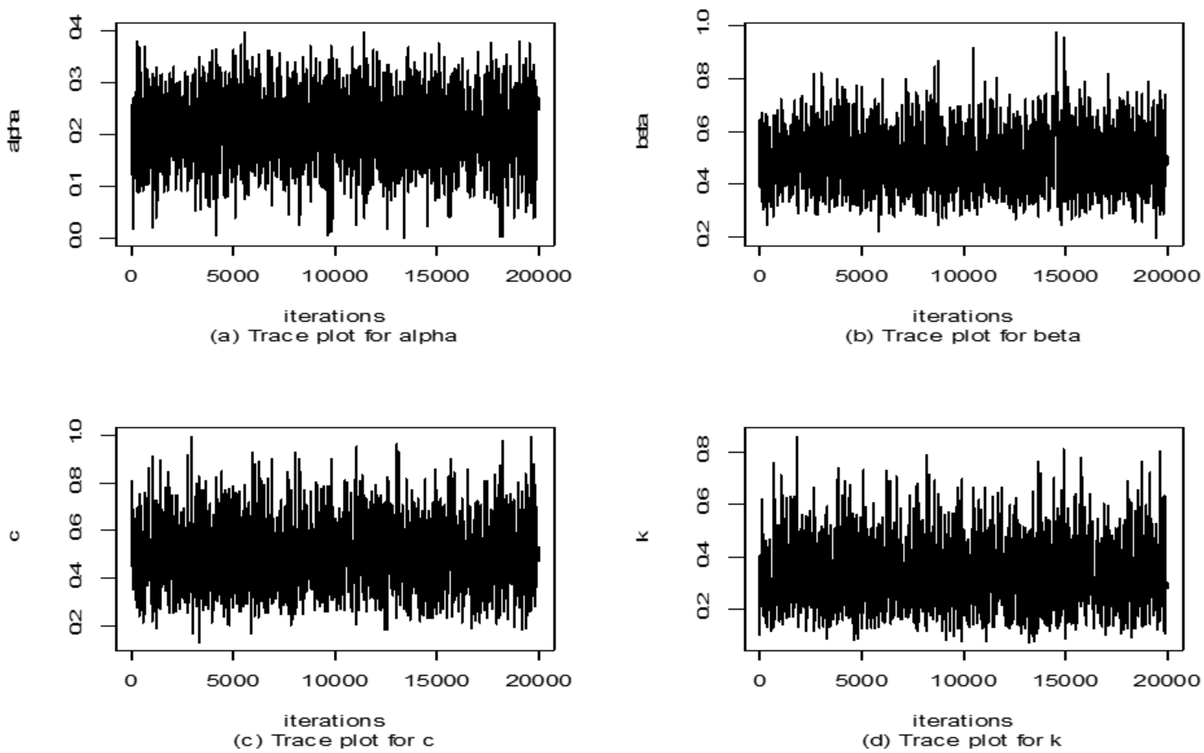

Figure 1: Trace plots of α, β, c, and k

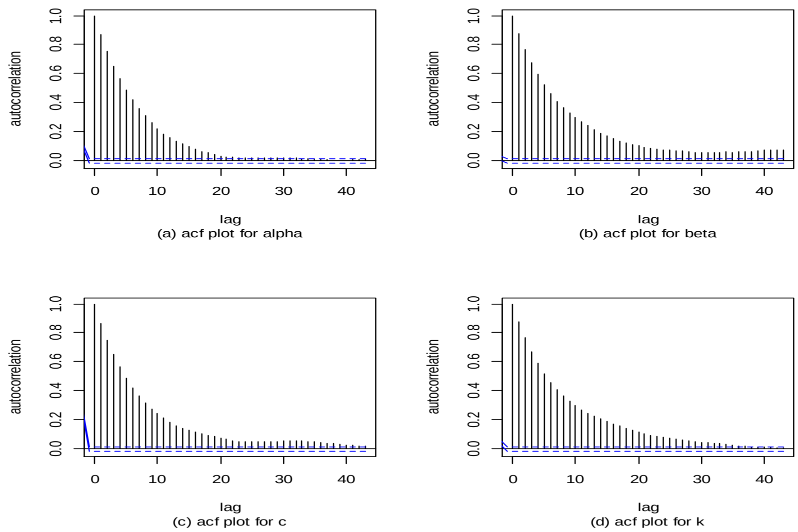

Figure 2: Autocorrelation plots of α, β, c, and k

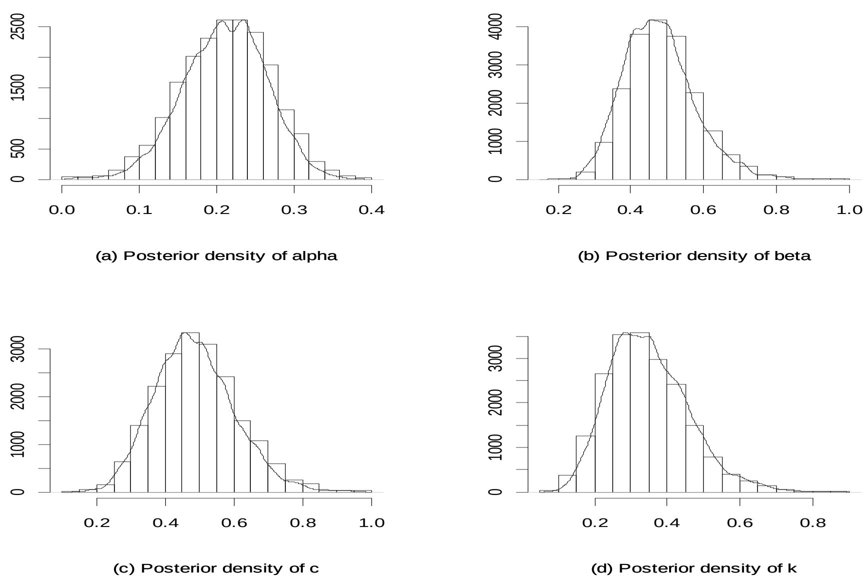

Figure 3: Histogram and Posterior density plots of α, β, c, and k

Based on the above mentioned values, the numerical results, the average estimates, the MSEs, the average Bayes estimates and the ERs, based on the AT-II-PC samples are obtained and shown in Tables 2-11. Further, the average 95% confidence width of approximate confidence intervals and credible intervals and their corresponding coverage probabilities are displayed in Tables 12-17.

From the simulation results, it is observed that as the effective sample size m increases, the MSEs decrease and Bayes estimates have the smaller ERs in most cases considered. The ERs under LINEX loss function with $\boldsymbol{\lambda} = 0.5$ are quite close to their corresponding ERs under the SE loss function. In addition, the Bayes estimates under LINEX loss function with $\boldsymbol{\lambda} = 0.5$ have the minimum ERs among all other Bayes estimates, except in some cases. For all the results, it is easy to see the Bayes estimates based on priors with small variances perform much better than the estimates for priors with large variances in the sense of having smaller ERs and width of the credible intervals. Comparing the performance of the three considered censoring schemes, it is found that Scheme I is better than Scheme III in terms of smaller MSEs/ ERs average width, except in some cases. Furthermore, we can note that for fixed T and n while m increases, the average width of approximate confidence/credible intervals becomes narrower.

Table 2: Average estimates and MSEs (in brackets) under AT-II PCS for true parameters (α=0.2, β=0.5, c=0.5, k=0.2).

n	***m***	**Est.**	**CS**					
			T=0.3			***T=0.9***		
			I	***II***	***III***	***I***	***II***	***III***
50	**30**	$\hat{\alpha}$	0.2485 (0.2148)	0.3256 (0.1878)	0.3573 (0.4417)	0.2140 (0.0080)	0.2406 (0.1569)	0.2586 (0.0893)
		$\hat{\beta}$	0.7414 (0.3438)	0.7687 (0.4354)	0.7703 (0.8126)	0.7646 (0.4239)	0.7259 (0.1826)	0.7606 (0.4595)
		$\hat{c}$	0.5399 (0.0720)	0.5363 (0.0723)	0.5256 (0.0725)	0.5145 (0.0612)	0.5147 (0.0614)	0.5308 (0.0696)
		$\hat{k}$	0.2261 (0.0363)	0.2283 (0.0335)	0.2177 (0.0302)	0.2097 (0.0268)	0.2090 (0.0272)	0.2158 (0.0315)
50	**40**	$\hat{\alpha}$	0.2115 (0.0031)	0.2201 (0.0141)	0.2203 (0.0148)	0.2087 (0.0017)	0.2265 (0.1410)	0.2125 (0.0074)
		$\hat{\beta}$	0.7291 (0.2735)	0.7068 (0.3857)	0.7257 (0.5413)	0.7765 (0.3965)	0.7230 (0.1382)	0.7267 (0.2348)
		$\hat{c}$	0.5293 (0.0621)	0.5282 (0.0704)	0.5270 (0.0693)	0.5078 (0.0545)	0.5193 (0.0660)	0.5259 (0.0662)
		$\hat{k}$	0.2184 (0.0300)	0.2134 (0.0263)	0.2120 (0.0279)	0.2101 (0.0259)	0.2076 (0.0246)	0.2077 (0.0261)
60	**30**	$\hat{\alpha}$	0.2473 (0.0597)	0.4042 (0.3031)	0.4528 (0.3096)	0.2160 (0.0031)	0.2286 (0.0098)	0.3062 (0.2703)
		$\hat{\beta}$	0.7577 (0.3133)	0.7818 (0.4827)	0.7798 (0.8987)	0.7695 (0.2753)	0.7461 (0.3482)	0.7861 (0.5501)
		$\hat{c}$	0.5327 (0.0629)	0.5422 (0.0680)	0.5363 (0.0732)	0.5307 (0.0614)	0.5335 (0.0616)	0.5444 (0.0721)
		$\hat{k}$	0.2274 (0.0304)	0.2278 (0.0292)	0.2222 (0.0296)	0.2230 (0.0272)	0.2165 (0.0253)	0.2271 (0.0323)
60	**40**	$\hat{\alpha}$	0.2109 (0.0043)	0.2364 (0.0247)	0.2505 (0.0395)	0.2065 (0.0015)	0.2159 (0.0043)	0.2222 (0.0310)
		$\hat{\beta}$	0.7353 (0.2970)	0.6996 (0.0773)	0.7206 (0.3736)	0.7516 (0.2091)	0.7283 (0.2794)	0.7540 (0.3509)
		$\hat{c}$	0.5433 (0.0677)	0.5295 (0.0640)	0.5280 (0.0681)	0.5210 (0.0528)	0.5234 (0.0584)	0.5265 (0.0616)
		$\hat{k}$	0.2264 (0.0298)	0.2169 (0.0238)	0.2171 (0.0266)	0.2165 (0.0253)	0.2070 (0.0230)	0.2155 (0.0242)
60	**50**	$\hat{\alpha}$	0.2094 (0.0017)	0.2059 (0.0051)	0.2061 (0.0052)	0.2077 (0.0013)	0.2094 (0.0020)	0.2119 (0.0124)
		$\hat{\beta}$	0.7080 (0.1558)	0.7125 (0.1968)	0.7119 (0.1967)	0.7325 (0.1549)	0.7304 (0.2496)	0.7231 (0.2511)
		$\hat{c}$	0.5331 (0.0590)	0.5392 (0.0622)	0.5376 (0.0620)	0.5169 (0.0484)	0.5201 (0.0537)	0.5121 (0.0544)
		$\hat{k}$	0.2142 (0.0203)	0.2162 (0.236)	0.2159 (0.0236)	0.2049 (0.0183)	0.2023 (0.0207)	0.2018 (0.0207)

Table 3: Average estimates and MSEs (in brackets) under AT-II PCS for true parameters (α=1, β=0.3, c=1.2, k=0.8).

n	*m*	Est.	CS					
			T=0.3			*T=0.9*		
			I	*II*	*III*	*I*	*II*	*III*
50	30	$\hat{\alpha}$	1.2060 (3.4470)	1.4562 (3.0082)	1.5267 (4.4868)	1.2450 (2.6590)	1.1911 (1.8154)	1.3041 (3.9797)
		$\hat{\beta}$	0.5260 (1.0860)	0.6858 (3.5957)	0.6222 (2.6882)	0.5310 (2.0620)	0.5543 (2.8634)	0.5589 (2.3448)
		$\hat{c}$	1.0420 (0.2050)	1.0222 (0.2097)	1.1784 (1.2624)	0.9400 (0.1910)	0.9447 (0.1947)	1.0098 (0.6421)
		$\hat{k}$	0.8060 (1.0900)	0.8013 (1.1558)	0.9294 (1.7336)	0.5310 (0.6640)	0.5533 (0.6568)	0.6645 (1.0221)
50	40	$\hat{\alpha}$	1.1450 (0.7450)	1.2384 (2.0631)	1.2327 (2.1492)	1.1010 (0.3540)	1.0980 (0.4200)	1.1255 (1.2271)
		$\hat{\beta}$	0.5190 (1.5380)	0.5921 (2.6857)	0.5346 (2.2544)	0.5000 (1.1550)	0.5005 (1.6492)	0.5038 (1.2600)
		$\hat{c}$	1.0460 (0.2220)	1.0199 (0.1994)	0.9866 (0.1967)	0.9150 (0.1910)	0.8981 (0.1979)	0.9087 (0.1893)
		$\hat{k}$	0.7730 (0.9790)	0.8022 (1.0348)	0.7105 (0.8844)	0.4720 (0.5650)	0.4935 (0.5936)	0.5107 (0.6557)
60	30	$\hat{\alpha}$	1.2150 (2.0860)	1.4898 (4.6309)	1.6366 (7.2162)	1.1950 (1.8400)	1.1829 (2.5300)	1.2596 (5.8519)
		$\hat{\beta}$	0.5960 (3.9370)	0.6371 (4.2189)	0.7910 (5.5291)	0.5390 (1.6520)	0.5138 (2.4740)	0.6157 (4.6060)
		$\hat{c}$	1.0520 (0.1820)	1.0322 (0.1932)	1.2083 (0.8923)	0.9590 (0.1820)	0.9530 (0.1814)	1.0971 (0.6745)
		$\hat{k}$	0.8210 (1.0460)	0.7882 (1.0679)	1.0565 (2.1207)	0.5690 (0.6440)	0.5738 (0.6716)	0.8221 (1.4350)
60	40	$\hat{\alpha}$	1.1690 (1.9050)	1.2594 (2.2390)	1.4771 (5.2463)	1.1260 (0.9620)	1.0969 (0.3979)	1.1117 (4.4606)
		$\hat{\beta}$	0.5230 (1.1450)	0.5149 (1.3436)	0.6144 (2.3774)	0.4820 (0.9430)	0.4688 (0.8976)	0.4912 (2.3103)
		$\hat{c}$	1.0710 (0.1910)	1.0256 (0.2032)	1.0938 (0.7133)	0.9290 (0.1870)	0.9183 (0.1868)	0.9634 (0.4419)
		$\hat{k}$	0.8500 (0.9130)	0.7797 (0.9889)	0.8650 (1.3015)	0.4900 (0.5470)	0.4965 (0.5427)	0.6019 (0.7095)
60	50	$\hat{\alpha}$	1.1010 (0.2670)	1.1597 (1.3161)	1.2131 (2.1221)	1.1230 (0.4990)	1.0863 (0.4268)	1.0753 (1.3180)
		$\hat{\beta}$	0.5070 (0.9520)	0.5091 (0.9036)	0.4926 (0.6926)	0.4960 (0.8380)	0.4380 (0.4792)	0.4773 (1.0752)
		$\hat{c}$	1.0650 (0.1900)	1.0329 (0.1973)	1.0355 (0.1906)	0.9010 (0.1820)	0.9004 (0.1794)	0.9356 (0.1828)
		$\hat{k}$	0.7990 (0.8160)	0.8199 (0.9252)	0.7795 (0.9240)	0.4630 (0.5130)	0.4534 (0.5023)	0.5296 (0.5821)

Table 4: Bayes estimates and ERs (in brackets) based on different sample sizes and censoring schemes under AT-II PCS with Prior 1 and T=0.3.

n	m	Est.	***BS***			***BL*** **(*λ*=**0.5)			***BL*** **(*λ*=**1)		
			I	II	III	I	II	III	I	II	III
50	30	$\hat{\alpha}$	0.2377 (0.0099)	0.2031 (0.0263)	0.2052 (0.0346)	0.2352 (0.0100)	0.1966 (0.0262)	0.1967 (0.0341)	0.2327 (0.0400)	0.1901 (0.1041)	0.1885 (0.1341)
		$\hat{\beta}$	0.5614 (0.0087)	0.5638 (0.0291)	0.5637 (0.0267)	0.5594 (0.0079)	0.5605 (0.0129)	0.5607 (0.0122)	0.5576 (0.0298)	0.5588 (0.0396)	0.5590 (0.0379)
		$\hat{c}$	0.4950 (0.0517)	0.4891 (0.0509)	0.4955 (0.0523)	0.4824 (0.0499)	0.4769 (0.0489)	0.4829 (0.0502)	0.4705 (0.1934)	0.4656 (0.1883)	0.4713 (0.1936)
		$\hat{k}$	0.1831 (0.0058)	0.1854 (0.0057)	0.1827 (0.0057)	0.1817 (0.0057)	0.1839 (0.0056)	0.1813 (0.0056)	0.1803 (0.0225)	0.1826 (0.0223)	0.1799 (0.0222)
50	40	$\hat{\alpha}$	0.2632 (0.0058)	0.2220 (0.0090)	0.2188 (0.0096)	0.2618 (0.0058)	0.2198 (0.0090)	0.2166 (0.0096)	0.2603 (0.0231)	0.2175 (0.0362)	0.2142 (0.0387)
		$\hat{\beta}$	0.5647 (0.0066)	0.5660 (0.0206)	0.5681 (0.0168)	0.5631 (0.0066)	0.5635 (0.0102)	0.5657 (0.0094)	0.5615 (0.0261)	0.5618 (0.0337)	0.5641 (0.0320)
		$\hat{c}$	0.4926 (0.0509)	0.4918 (0.0506	0.4896 (0.0511)	0.4752 (0.0482)	0.4778 (0.0483)	0.4774 (0.0489)	0.4668 (0.1841)	0.4664 (0.1873)	0.4661 (0.1884)
		$\hat{k}$	0.1895 (0.0057)	0.1880 (0.0057)	0.1902 (0.0064)	0.1881 (0.0057)	0.1862 (0.0056)	0.1886 (0.0061)	0.1867 (0.0224)	0.1848 (0.0223)	0.1872 (0.0236)
60	30	$\hat{\alpha}$	0.2177 (0.0096)	0.1993 (0.0388)	0.2297 (0.0569)	0.2153 (0.0096)	0.1905 (0.0353)	0.2170 (0.0505)	0.2129 (0.0385)	0.1823 (0.1364)	0.2057 (0.1918)
		$\hat{\beta}$	0.5643 (0.0067)	0.5742 (0.0104)	0.5982 (0.0082)	0.5626 (0.0066)	0.5722 (0.0081)	0.5964 (0.0070)	0.5609 (0.0264)	0.5705 (0.0300)	0.5949 (0.0267)
		$\hat{c}$	0.4915 (0.0483)	0.4712 (0.0513)	0.4681 (0.0523)	0.4796 (0.0495)	0.4589 (0.0494)	0.4555 (0.0504)	0.4685 (0.1874)	0.4474 (0.1905)	0.4438 (0.1943)
		$\hat{k}$	0.1892 (0.0081	0.1880 (0.0055)	0.1841 (0.0057)	0.1872 (0.0085)	0.1866 (0.0055)	0.1827 (0.0057)	0.1857 (0.0328)	0.1853 (0.0215)	0.1813 (0.0224)
60	40	$\hat{\alpha}$	0.2359 (0.0052)	0.2026 (0.0160)	0.1993 (0.0172)	0.2346 (0.0052)	0.1988 (0.0160)	0.1950 (0.0172)	0.2333 (0.0209)	0.1948 (0.0640)	0.1908 (0.0685)
		$\hat{\beta}$	0.5707 (0.0065)	0.5714 (0.0059)	0.5788 (0.0091)	0.5691 (0.0064)	0.5699 (0.0059)	0.5771 (0.0071)	0.5675 (0.0256)	0.5685 (0.0233)	0.5755 (0.0266)
		$\hat{c}$	0.4924 (0.0465)	0.4969 (0.0485)	0.4906 (0.0466)	0.4834 (0.0461)	0.4852 (0.0469)	0.4790 (0.0489)	0.4725 (0.1792)	0.4742 (0.1819)	0.4684 (0.1828)
		$\hat{k}$	0.1851 (0.0054)	0.1840 (0.0053)	0.1889 (0.0054)	0.1838 (0.0054)	0.1827 (0.0052)	0.1876 (0.0053)	0.1825 (0.0212)	0.1814 (0.0206)	0.1863 (0.0210)
60	50	$\hat{\alpha}$	0.2549 (0.0035)	0.2151 (0.0057)	0.2276 (0.0067)	0.2541 (0.0035)	0.2137 (0.0057)	0.2260 (0.0067)	0.2532 (0.0139)	0.2122 (0.0229)	0.2243 (0.0268)
		$\hat{\beta}$	0.5696 (0.0061)	0.5732 (0.0059)	0.5724 (0.0059)	0.5681 (0.0060)	0.5717 (0.0059)	0.5709 (0.0058)	0.5666 (0.0241)	0.5703 (0.0235)	0.5695 (0.0233)
		$\hat{c}$	0.4942 (0.0463)	0.4932 (0.0475)	0.4965 (0.0467)	0.4865 (0.0453)	0.4817 (0.0458)	0.4888 (0.0468)	0.4758 (0.1761)	0.4710 (0.1775)	0.4778 (0.1817)
		$\hat{k}$	0.1846 (0.0052)	0.1875 (0.0053)	0.1875 (0.0054)	0.1833 (0.0051)	0.1862 (0.0052)	0.1861 (0.0053)	0.1821 (0.0204)	0.1849 (0.0206)	0.1849 (0.0210)

Table 5: Bayes estimates and ERs (in brackets) based on different sample sizes and censoring schemes under AT-II PCS with Prior 2 and T=0.3.

n	m	Est.	***BS***			***BL*** ($\boldsymbol{\lambda}$=0.5)			***BL*** ($\boldsymbol{\lambda}$=1)		
			I	II	III	I	II	III	I	II	III
50	30	$\hat{\alpha}$	0.3007 (0.0168)	0.1887 (0.0274)	0.2006 (0.0538)	0.2964 (0.0169)	0.1820 (0.0268)	0.1906 (0.0400)	0.2922 (0.0679)	0.1755 (0.1060)	0.1812 (0.1569)
		$\hat{\beta}$	0.5859 (0.0106)	0.6016 (0.0229)	0.5937 (0.0318)	0.5833 (0.0105)	0.5976 (0.0447)	0.5898 (0.0223)	0.5807 (0.0418)	0.5945 (0.1584)	0.5872 (0.0652)
		$\hat{c}$	0.5266 (0.1114)	0.4655 (0.1431)	0.4636 (0.1002)	0.5007 (0.1037)	0.4400 (0.1017)	0.4409 (0.0906)	0.4780 (0.3885)	0.4209 (0.3563)	0.4217 (0.3347)
		$\hat{k}$	0.1904 (0.0098)	0.2042 (0.0240)	0.1879 (0.0101)	0.1880 (0.0096)	0.2005 (0.0148)	0.1856 (0.0094)	0.1857 (0.0376)	0.1979 (0.0498)	0.1835 (0.0356)
50	40	$\hat{\alpha}$	0.3399 (0.0102)	0.2504 (0.0129)	0.2772 (0.0175)	0.3373 (0.0102)	0.2474 (0.0130)	0.2729 (0.0175)	0.3348 (0.0409)	0.2442 (0.0521)	0.2685 (0.0702)
		$\hat{\beta}$	0.5886 (0.0100)	0.5965 (0.0167)	0.5899 (0.0098)	0.5862 (0.0099)	0.5929 (0.0283)	0.5875 (0.0097)	0.5837 (0.0393)	0.5902 (0.0780)	0.5851 (0.0383)
		$\hat{c}$	0.5219 (0.1101)	0.5340 (0.1184)	0.5191 (0.0994)	0.4962 (0.1026)	0.5074 (0.1062)	0.4937 (0.0978)	0.4737 (0.3852)	0.4846 (0.3945)	0.4726 (0.3805)
		$\hat{k}$	0.1957 (0.0091)	0.1930 (0.0094)	0.1950 (0.0096)	0.1934 (0.0089)	0.1907 (0.0092)	0.1927 (0.0094)	0.1913 (0.0352)	0.1885 (0.0359)	0.1899 (0.0355)
60	30	$\hat{\alpha}$	0.2655 (0.0149)	0.2066 (0.1067)	0.1900 (0.1026)	0.2618 (0.0149)	0.1953 (0.0452)	0.1743 (0.0683)	0.2580 (0.0600)	0.1860 (0.1651)	0.1616 (0.2401)
		$\hat{\beta}$	0.5892 (0.0107)	0.6054 (0.0333)	0.6008 (0.0119)	0.5864 (0.0118)	0.6020 (0.0144)	0.5979 (0.0121)	0.5836 (0.0798)	0.5995 (0.0481)	0.5956 (0.0441)
		$\hat{c}$	0.5117 (0.0911)	0.4624 (0.1380)	0.4522 (0.0919)	0.4903 (0.0855)	0.4393 (0.0923)	0.4310 (0.0848)	0.4703 (0.3229)	0.4206 (0.3343)	0.4128 (0.3151)
		$\hat{k}$	0.1883 (0.0090)	0.1931 (0.0223)	0.1807 (0.0084)	0.1846 (0.0153)	0.1872 (0.0235)	0.1787 (0.0082)	0.1823 (0.0489)	0.1837 (0.0747)	0.1767 (0.0321)
60	40	$\hat{\alpha}$	0.3175 (0.0101)	0.2252 (0.0191)	0.2116 (0.0248)	0.3150 (0.0102)	0.2206 (0.0191)	0.2055 (0.0246)	0.3124 (0.0409)	0.2158 (0.0765)	0.1994 (0.0981)
		$\hat{\beta}$	0.5943 (0.0099)	0.5977 (0.0088)	0.5904 (0.0163)	0.5918 (0.0098)	0.5955 (0.0088)	0.5820 (0.0680)	0.5894 (0.0389)	0.5933 (0.0349)	0.5790 (0.2246)
		$\hat{c}$	0.5072 (0.0904)	0.4931 (0.0884)	0.5000 (0.1001)	0.4858 (0.0849)	0.4725 (0.0822)	0.4766 (0.1037)	0.4665 (0.3208)	0.4545 (0.3090)	0.4570 (0.3695)
		$\hat{k}$	0.1885 (0.0085)	0.1900 (0.0099)	0.2150 (0.0086)	0.1841 (0.0146)	0.1878 (0.0090)	0.2130 (0.0080)	0.1818 (0.0476)	0.1858 (0.0338)	0.2110 (0.0315)
60	50	$\hat{\alpha}$	0.3558 (0.0081)	0.2797 (0.0107)	0.2914 (0.0125)	0.3538 (0.0081)	0.2770 (0.0107)	0.2883 (0.0125)	0.3517 (0.0326)	0.2743 (0.0431)	0.2851 (0.0502)
		$\hat{\beta}$	0.5911 (0.0091)	0.5916 (0.0086)	0.5955 (0.0121)	0.5888 (0.0091)	0.5898 (0.0085)	0.5930 (0.0098)	0.5866 (0.0360)	0.5877 (0.0338)	0.5908 (0.0371)
		$\hat{c}$	0.5089 (0.0890)	0.5169 (0.0884)	0.5352 (0.0958)	0.4880 (0.0835)	0.4960 (0.0835)	0.5125 (0.0899)	0.4694 (0.3157)	0.4759 (0.3168)	0.4923 (0.3395)
		$\hat{k}$	0.1891 (0.0083)	0.1935 (0.0085)	0.1958 (0.0087)	0.1871 (0.0082)	0.1914 (0.0084)	0.1936 (0.0085)	0.1851 (0.0322)	0.1894 (0.0329)	0.1916 (0.0336)

Table 6: Bayes estimates and ERs (in brackets) based on different sample sizes and censoring schemes under AT-II PCS with Prior 3 and T=0.3.

n	m	Est.	*BS*			*BL* (λ=0.5)			*BL* (λ=1)		
			I	II	III	I	II	III	I	II	III
50	30	$\hat{\alpha}$	1.0042 (0.0443)	1.0038 (0.0432)	1.0093 (0.0812)	1.0010 (0.0159)	0.9995 (0.0170)	1.0032 (0.0397)	0.9993 (0.0453)	0.9975 (0.0496)	1.0009 (0.0976)
		$\hat{\beta}$	0.3580 (0.0229)	0.3607 (0.0153)	0.3609 (0.0203)	0.3560 (0.0173)	0.3585 (0.0086)	0.3583 (0.0102)	0.3548 (0.0439)	0.3573 (0.0270)	0.3570 (0.0310)
		$\hat{c}$	1.2085 (0.0362)	1.2140 (0.0372)	1.2450 (0.0368)	1.1996 (0.0358)	1.2048 (0.0367)	1.2359 (0.0363)	1.1908 (0.1415)	1.1959 (0.1449)	1.2271 (0.1434)
		$\hat{k}$	0.7728 (0.0425)	0.7647 (0.0419)	0.7633 (0.0430)	0.7624 (0.0415)	0.7545 (0.0408)	0.7530 (0.0414)	0.7525 (0.1625)	0.7447 (0.1597)	0.7431 (0.1613)
50	40	$\hat{\alpha}$	1.0035 (0.0317)	1.0005 (0.0357)	0.9999 (0.0602)	1.0008 (0.0107)	0.9971 (0.0136)	0.9957 (0.0201)	0.9993 (0.0339)	0.9954 (0.0414)	0.9938 (0.0550)
		$\hat{\beta}$	0.3680 (0.0107)	0.3564 (0.0123)	0.3559 (0.0191)	0.3665 (0.0119)	0.3547 (0.0083)	0.3539 (0.0100)	0.3654 (0.0326)	0.3536 (0.0255)	0.3528 (0.0291)
		$\hat{c}$	1.2026 (0.0355)	1.2125 (0.0363)	1.2331 (0.0367)	1.1938 (0.0351)	1.2035 (0.0359)	1.2241 (0.0362)	1.1853 (0.1388)	1.1947 (0.1420)	1.2152 (0.1432)
		$\hat{k}$	0.7721 (0.0407)	0.7657 (0.0406)	0.7629 (0.0415)	0.7621 (0.0399)	0.7558 (0.0398)	0.7528 (0.0402)	0.7525 (0.1566)	0.7462 (0.1559)	0.7432 (0.1571)
60	30	$\hat{\alpha}$	1.0008 (0.0164)	1.0076 (0.0416)	1.0132 (0.0856)	0.9985 (0.0094)	1.0038 (0.0232)	1.0063 (0.0565)	0.9968 (0.0320)	1.0018 (0.0621)	1.0037 (0.1334)
		$\hat{\beta}$	0.3528 (0.0173)	0.3601 (0.0189)	0.3658 (0.0179)	0.3509 (0.0076)	0.3580 (0.0083)	0.3633 (0.0102)	0.3498 (0.0239)	0.3569 (0.0255)	0.3619 (0.0311)
		$\hat{c}$	1.2054 (0.0342)	1.2091 (0.0346)	1.2845 (0.0352)	1.1969 (0.0338)	1.2005 (0.0342)	1.2758 (0.0347)	1.1887 (0.1339)	1.1921 (0.1354)	1.2673 (0.1369)
		$\hat{k}$	0.7740 (0.0428)	0.7672 (0.0422)	0.7607 (0.0472)	0.7636 (0.0417)	0.7570 (0.0410)	0.7500 (0.0425)	0.7537 (0.1630)	0.7472 (0.1603)	0.7402 (0.1636)
60	40	$\hat{\alpha}$	0.9987 (0.0098)	0.9997 (0.0297)	1.0001 (0.0453)	0.9969 (0.0075)	0.9971 (0.0212)	0.9961 (0.0213)	0.9953 (0.0275)	0.9953 (0.0565)	0.9941 (0.0588)
		$\hat{\beta}$	0.3547 (0.0145)	0.3622 (0.0142)	0.3600 (0.0110)	0.3530 (0.0067)	0.3607 (0.0077)	0.3582 (0.0071)	0.3520 (0.0217)	0.3597 (0.0231)	0.3571 (0.0231)
		$\hat{c}$	1.2113 (0.0346)	1.2121 (0.0346)	1.2340 (0.0351)	1.2027 (0.0342)	1.2036 (0.0342)	1.2253 (0.0346)	1.1943 (0.1353)	1.1952 (0.1352)	1.2169 (0.1368)
		$\hat{k}$	0.7657 (0.0405)	0.7653 (0.0408)	0.7649 (0.0449)	0.7558 (0.0396)	0.7553 (0.0399)	0.7544 (0.0421)	0.7462 (0.1556)	0.7457 (0.1563)	0.7446 (0.1621)
60	50	$\hat{\alpha}$	1.0031 (0.0068)	1.0031 (0.0178)	0.9988 (0.0090)	1.0016 (0.0062)	1.0008 (0.0091)	0.9969 (0.0075)	1.0001 (0.0241)	0.9992 (0.0312)	0.9953 (0.0281)
		$\hat{\beta}$	0.3670 (0.0049)	0.3622 (0.0074)	0.3599 (0.0044)	0.3659 (0.0041)	0.3610 (0.0050)	0.3589 (0.0040)	0.3650 (0.0157)	0.3600 (0.0175)	0.3580 (0.0151)
		$\hat{c}$	1.2125 (0.0345)	1.2138 (0.0348)	1.2191 (0.0350)	1.2040 (0.0341)	1.2052 (0.0344)	1.2104 (0.0346)	1.1956 (0.1350)	1.1968 (0.1361)	1.2020 (0.1369)
		$\hat{k}$	0.7651 (0.0397)	0.7644 (0.0394)	0.7623 (0.0390)	0.7554 (0.0388)	0.7547 (0.0386)	0.7527 (0.0383)	0.7461 (0.1525)	0.7454 (0.1516)	0.7435 (0.1506)

Table 7: Bayes estimates and ERs (in brackets) based on different sample sizes and censoring schemes under AT-II PCS with Prior 4 and *T*=0.3.

n	m	Est.	*BS*			*BL* (λ=0.5)			*BL* (λ=1)		
			I	II	III	I	II	III	I	II	III
50	30	$\hat{\alpha}$	1.0005 (0.0918)	1.0153 (0.1237)	1.0172 (0.2364)	0.9960 (0.0231)	1.0097 (0.0757)	1.0066 (0.1178)	0.9938 (0.0643)	1.0072 (0.1716)	1.0036 (0.2598)
		$\hat{\beta}$	0.3528 (0.0939)	0.3722 (0.0359)	0.3688 (0.0502)	0.3497 (0.0260)	0.3702 (0.0324)	0.3652 (0.0271)	0.3482 (0.0634)	0.3687 (0.0769)	0.3634 (0.0685)
		$\hat{c}$	1.2088 (0.0630)	1.2089 (0.0630)	1.2496 (0.0634)	1.1934 (0.0616)	1.1935 (0.0616)	1.2342 (0.0620)	1.1787 (0.2412)	1.1787 (0.2412)	1.2193 (0.2426)
		$\hat{k}$	0.7537 (0.0754)	0.7566 (0.0831)	0.7480 (0.0834)	0.7355 (0.0727)	0.7371 (0.0779)	0.7288 (0.0767)	0.7185 (0.2814)	0.7197 (0.2950)	0.7116 (0.2910)
50	40	$\hat{\alpha}$	1.0001 (0.0102)	1.0061 (0.0334)	1.0059 (0.0742)	0.9978 (0.0089)	1.0030 (0.0400)	1.0005 (0.0669)	0.9958 (0.0343)	1.0006 (0.0987)	0.9980 (0.1539)
		$\hat{\beta}$	0.3567 (0.0113)	0.3616 (0.0327)	0.3676 (0.0320)	0.3550 (0.0066)	0.3595 (0.0287)	0.3650 (0.0225)	0.3537 (0.0235)	0.3581 (0.0680)	0.3635 (0.0567)
		$\hat{c}$	1.1976 (0.0615)	1.2122 (0.0630)	1.2420 (0.0619)	1.1826 (0.0601)	1.1968 (0.0616)	1.2269 (0.0606)	1.1682 (0.2356)	1.1820 (0.2411)	1.2124 (0.2373)
		$\hat{k}$	0.7514 (0.0706)	0.7521 (0.0736)	0.7446 (0.0707)	0.7342 (0.0686)	0.7343 (0.0709)	0.7276 (0.0683)	0.7180 (0.2667)	0.7178 (0.2741)	0.7116 (0.2646)
60	30	$\hat{\alpha}$	1.0030 (0.0718)	1.0070 (0.0571)	1.0328 (0.2508)	0.9985 (0.0267)	1.0027 (0.0626)	1.0212 (0.1778)	0.9962 (0.0722)	1.0002 (0.1456)	1.0180 (0.3817)
		$\hat{\beta}$	0.3573 (0.0668)	0.3658 (0.1103)	0.3725 (0.0851)	0.3544 (0.0318)	0.3630 (0.0340)	0.3675 (0.0302)	0.3530 (0.0752)	0.3616 (0.0794)	0.3657 (0.0749)
		$\hat{c}$	1.2168 (0.0601)	1.2016 (0.0588)	1.3029 (0.0604)	1.2021 (0.0588)	1.1872 (0.0575)	1.2882 (0.0588)	1.1879 (0.2306)	1.1734 (0.2255)	1.2742 (0.2297)
		$\hat{k}$	0.7568 (0.0799)	0.7581 (0.0763)	0.7616 (0.2148)	0.7379 (0.0757)	0.7397 (0.0734)	0.7353 (0.1052)	0.7206 (0.2898)	0.7227 (0.2834)	0.7171 (0.3557)
60	40	$\hat{\alpha}$	0.9972 (0.0146)	0.9995 (0.0554)	1.0098 (0.2158)	0.9948 (0.0194)	0.9960 (0.0845)	1.0021 (0.1229)	0.9927 (0.0554)	0.9936 (0.1882)	0.9993 (0.2682)
		$\hat{\beta}$	0.3626 (0.0228)	0.3694 (0.0559)	0.3662 (0.0666)	0.3607 (0.0160)	0.3678 (0.0412)	0.3620 (0.0285)	0.3594 (0.0423)	0.3662 (0.0953)	0.3605 (0.0693)
		$\hat{c}$	1.2029 (0.0595)	1.1966 (0.0579)	1.2738 (0.0598)	1.1883 (0.0582)	1.1824 (0.0567)	1.2592 (0.0584)	1.1743 (0.2281)	1.1688 (0.2222)	1.2453 (0.2285)
		$\hat{k}$	0.7441 (0.0700)	0.7490 (0.0720)	0.7551 (0.0787)	0.7272 (0.0677)	0.7317 (0.0694)	0.7370 (0.0832)	0.7113 (0.2629)	0.7154 (0.2687)	0.7197 (0.3051)
60	50	$\hat{\alpha}$	1.0005 (0.0143)	0.9999 (0.0458)	1.0006 (0.0480)	0.9983 (0.0106)	0.9967 (0.0127)	0.9974 (0.0306)	0.9963 (0.0371)	0.9945 (0.0428)	0.9951 (0.0794)
		$\hat{\beta}$	0.3680 (0.0086)	0.3690 (0.0187)	0.3660 (0.0109)	0.3666 (0.0111)	0.3672 (0.0072)	0.3646 (0.0164)	0.3654 (0.0319)	0.3660 (0.0238)	0.3634 (0.0421)
		$\hat{c}$	1.2083 (0.0588)	1.2040 (0.0579)	1.2011 (0.0589)	1.1939 (0.0575)	1.1899 (0.0567)	1.1867 (0.0575)	1.1801 (0.2256)	1.1762 (0.2224)	1.1730 (0.2248)
		$\hat{k}$	0.7483 (0.0685)	0.7431 (0.0682)	0.7528 (0.0688)	0.7317 (0.0664)	0.7266 (0.0661)	0.7361 (0.0668)	0.7160 (0.2582)	0.7110 (0.2565)	0.7203 (0.2594)

Table 8: Bayes estimates and ERs (in brackets) based on different sample sizes and censoring schemes under AT-II PCS with Prior 1 and T=0.9.

n	m	Est.	***BS***			***BL*** **(*λ*=0.5)**			***BL*** **(*λ*=1)**		
			I	II	III	I	II	III	I	II	III
50	30	$\hat{\alpha}$	0.2463 (0.0063)	0.2294 (0.0092)	0.1726 (0.0212)	0.2447 (0.0063)	0.2273 (0.0092)	0.1673 (0.0211)	0.2431 (0.0253)	0.2249 (0.0372)	0.1621 (0.0840)
		$\hat{\beta}$	0.5536 (0.0068)	0.5727 (0.0340)	0.5663 (0.0131)	0.5519 (0.0067)	0.5690 (0.0148)	0.5642 (0.0083)	0.5502 (0.0269)	0.5672 (0.0444)	0.5626 (0.0298)
		$\hat{c}$	0.4958 (0.0526)	0.4924 (0.0522)	0.4764 (0.0486)	0.4831 (0.0508)	0.4798 (0.0504)	0.4648 (0.0465)	0.4712 (0.1968)	0.4680 (0.1948)	0.4540 (0.1792)
		$\hat{k}$	0.1861 (0.0058)	0.1894 (0.0057)	0.1807 (0.0053)	0.1847 (0.0058)	0.1880 (0.0056)	0.1794 (0.0052)	0.1833 (0.0228)	0.1866 (0.0224)	0.1781 (0.0208)
50	40	$\hat{\alpha}$	0.2401 (0.0043)	0.2406 (0.0051)	0.2224 (0.0086)	0.2390 (0.0043)	0.2394 (0.0051)	0.2205 (0.0086)	0.2380 (0.0171)	0.2381 (0.0204)	0.2183 (0.0347)
		$\hat{\beta}$	0.5599 (0.0066)	0.5703 (0.0066)	0.5686 (0.0089)	0.5582 (0.0066)	0.5686 (0.0066)	0.5667 (0.0074)	0.5566 (0.0263)	0.5670 (0.0262)	0.5651 (0.0280)
		$\hat{c}$	0.4810 (0.0479)	0.4898 (0.0504)	0.4789 (0.0483)	0.4694 (0.0464)	0.4777 (0.0487)	0.4673 (0.0467)	0.4585 (0.1799)	0.4663 (0.1885)	0.4568 (0.1786)
		$\hat{k}$	0.1813 (0.0054)	0.1836 (0.0053)	0.1841 (0.0055)	0.1800 (0.0053)	0.1823 (0.0053)	0.1827 (0.0055)	0.1787 (0.0211)	0.1810 (0.0209)	0.1839 (0.0216)
60	30	$\hat{\alpha}$	0.2304 (0.0058)	0.2313 (0.0085)	0.1681 (0.0337)	0.2289 (0.0058)	0.2292 (0.0086)	0.1604 (0.0385)	0.2275 (0.0233)	0.2270 (0.0344)	0.1532 (0.1344)
		$\hat{\beta}$	0.5599 (0.0066)	0.5760 (0.0088)	0.5779 (0.0108)	0.5583 (0.0066)	0.5732 (0.0726)	0.5758 (0.0120)	0.5566 (0.0264)	0.5712 (0.1750)	0.5742 (0.0373)
		$\hat{c}$	0.4729 (0.0441)	0.4907 (0.0464)	0.4630 (0.0510)	0.4622 (0.0428)	0.4795 (0.0448)	0.4508 (0.0491)	0.4521 (0.1663)	0.4690 (0.1737)	0.4394 (0.1893)
		$\hat{k}$	0.1846 (0.0054)	0.2364 (0.0051)	0.1836 (0.0052)	0.1832 (0.0053)	0.2351 (0.0051)	0.1824 (0.0051)	0.1819 (0.0210)	0.2339 (0.0201)	0.1811 (0.0202)
60	40	$\hat{\alpha}$	0.2624 (0.0051)	0.2330 (0.0056)	0.1880 (0.0140)	0.2611 (0.0051)	0.2316 (0.0056)	0.1847 (0.0141)	0.2598 (0.0205)	0.2301 (0.0226)	0.1812 (0.0564)
		$\hat{\beta}$	0.5666 (0.0065)	0.5783 (0.0063)	0.5725 (0.0246)	0.5656 (0.0065)	0.5771 (0.0153)	0.5697 (0.0113)	0.5640 (0.0260)	0.5754 (0.0448)	0.5680 (0.0355)
		$\hat{c}$	0.4802 (0.0440)	0.4907 (0.0472)	0.4913 (0.0469)	0.4695 (0.0427)	0.4793 (0.0457)	0.4800 (0.0453)	0.4593 (0.1660)	0.4681 (0.1774)	0.4694 (0.1754)
		$\hat{k}$	0.1864 (0.0053)	0.1852 (0.0052)	0.1837 (0.0052)	0.1859 (0.0053)	0.1831 (0.0050)	0.1824 (0.0051)	0.1842 (0.0210)	0.1818 (0.0200)	0.1811 (0.0203)
60	50	$\hat{\alpha}$	0.2421 (0.0029)	0.2424 (0.0037)	0.2396 (0.0063)	0.2413 (0.0029)	0.2415 (0.0037)	0.2381 (0.0063)	0.2406 (0.0116)	0.2405 (0.0149)	0.2365 (0.0253)
		$\hat{\beta}$	0.5716 (0.0063)	0.5791 (0.0062)	0.5784 (0.0062)	0.5700 (0.0063)	0.5776 (0.0061)	0.5769 (0.0062)	0.5684 (0.0250)	0.5760 (0.0244)	0.5754 (0.0246)
		$\hat{c}$	0.4709 (0.0431)	0.4856 (0.0456)	0.4868 (0.0461)	0.4605 (0.0418)	0.4746 (0.0442)	0.4757 (0.0446)	0.4506 (0.1624)	0.4642 (0.1715)	0.4652 (0.1732)
		$\hat{k}$	0.1843 (0.0050)	0.1817 (0.0049)	0.1850 (0.0051)	0.1830 (0.0050)	0.1805 (0.0048)	0.1837 (0.0051)	0.1818 (0.0197)	0.1793 (0.0191)	0.1825 (0.0202)

Table 9: Bayes estimates and ERs (in brackets) based on different sample sizes and censoring schemes under AT-II PCS with Prior 2 and T=0.9.

n	m	Est.	***BS***			***BL* (*λ*=0.5)**			***BL* (*λ*=1)**		
			I	II	III	I	II	III	I	II	III
50	30	$\hat{\alpha}$	0.3126 (0.0110)	0.2744 (0.0146)	0.1788 (0.0537)	0.3098 (0.0111)	0.2707 (0.0147)	0.1713 (0.0300)	0.3070 (0.0444)	0.2670 (0.0591)	0.1646 (0.1133)
		$\hat{\beta}$	0.5725 (0.0106)	0.6027 (0.0143)	0.5984 (0.0289)	0.5699 (0.0105)	0.6001 (0.0234)	0.5945 (0.0377)	0.5673 (0.0416)	0.5973 (0.0690)	0.5913 (0.1332)
		$\hat{c}$	0.5012 (0.1043)	0.5069 (0.1088)	0.4678 (0.1059)	0.4769 (0.0969)	0.4819 (0.1000)	0.4449 (0.0918)	0.4558 (0.3630)	0.4604 (0.3726)	0.4259 (0.3357)
		$\hat{k}$	0.1845 (0.0090)	0.1822 (0.0136)	0.1884 (0.0149)	0.1823 (0.0088)	0.1796 (0.0103)	0.1857 (0.0108)	0.1802 (0.0347)	0.1775 (0.0375)	0.1836 (0.0378)
50	40	$\hat{\alpha}$	0.3314 (0.0080)	0.3205 (0.0100)	0.2379 (0.0126)	0.3294 (0.0080)	0.3180 (0.0100)	0.2348 (0.0127)	0.3274 (0.0321)	0.3155 (0.0403)	0.2316 (0.0509)
		$\hat{\beta}$	0.5880 (0.0105)	0.5947 (0.0101)	0.5896 (0.0106)	0.5854 (0.0104)	0.5922 (0.0100)	0.5870 (0.0103)	0.5829 (0.0412)	0.5896 (0.0780)	0.5845 (0.0407)
		$\hat{c}$	0.4917 (0.0985)	0.4834 (0.1067)	0.5117 (0.1012)	0.4688 (0.0915)	0.4589 (0.0980)	0.4879 (0.0940)	0.4488 (0.3432)	0.4379 (0.3650)	0.4674 (0.3522)
		$\hat{k}$	0.1887 (0.0085)	0.1797 (0.0079)	0.1917 (0.0087)	0.1866 (0.0083)	0.1778 (0.0078)	0.1893 (0.0085)	0.1845 (0.0328)	0.1759 (0.0305)	0.1872 (0.0335)
60	30	$\hat{\alpha}$	0.3364 (0.0133)	0.2802 (0.0168)	0.1779 (0.0554)	0.3330 (0.0134)	0.2760 (0.0170)	0.1649 (0.0528)	0.3296 (0.0539)	0.2716 (0.0685)	0.1546 (0.1881)
		$\hat{\beta}$	0.5793 (0.0105)	0.6137 (0.1160)	0.6124 (0.0263)	0.5767 (0.0104)	0.6078 (0.0489)	0.6089 (0.0211)	0.5742 (0.0412)	0.6049 (0.1211)	0.6063 (0.0630)
		$\hat{c}$	0.4685 (0.0847)	0.5155 (0.1078)	0.4530 (0.0897)	0.4486 (0.0793)	0.4909 (0.0982)	0.4324 (0.0825)	0.4311 (0.2990)	0.4698 (0.3656)	0.4147 (0.3066)
		$\hat{k}$	0.1895 (0.0087)	0.1839 (0.0274)	0.1820 (0.0078)	0.1873 (0.0085)	0.1809 (0.0121)	0.1801 (0.0076)	0.1853 (0.0335)	0.1788 (0.0406)	0.1783 (0.0298)
60	40	$\hat{\alpha}$	0.3061 (0.0075)	0.2857 (0.0093)	0.2544 (0.0239)	0.3043 (0.0076)	0.2833 (0.0094)	0.2485 (0.0238)	0.3024 (0.0303)	0.2810 (0.0377)	0.2426 (0.0951)
		$\hat{\beta}$	0.5992 (0.0101)	0.6181 (0.0238)	0.6050 (0.0121)	0.5967 (0.0101)	0.6145 (0.0353)	0.6020 (0.0175)	0.5942 (0.0400)	0.6116 (0.0943)	0.5995 (0.0547)
		$\hat{c}$	0.4934 (0.0878)	0.5090 (0.1001)	0.4830 (0.0889)	0.4729 (0.0821)	0.4857 (0.0932)	0.4619 (0.0880)	0.4547 (0.3100)	0.4654 (0.3495)	0.4434 (0.3238)
		$\hat{k}$	0.1854 (0.0080)	0.1887 (0.0080)	0.1890 (0.0073)	0.1834 (0.0078)	0.1867 (0.0078)	0.1865 (0.0114)	0.1815 (0.0309)	0.1848 (0.0308)	0.1846 (0.0377)
60	50	$\hat{\alpha}$	0.3809 (0.0090)	0.3057 (0.0072)	0.2652 (0.0098)	0.3786 (0.0090)	0.3039 (0.0072)	0.2628 (0.0098)	0.3764 (0.0362)	0.3021 (0.0290)	0.2603 (0.0394)
		$\hat{\beta}$	0.5895 (0.0091)	0.6095 (0.0093)	0.6087 (0.0094)	0.5872 (0.0090)	0.6072 (0.0093)	0.6064 (0.0093)	0.5850 (0.0358)	0.6049 (0.0736)	0.6041 (0.0371)
		$\hat{c}$	0.4962 (0.0897)	0.5127 (0.0964)	0.5262 (0.0975)	0.4745 (0.0841)	0.4903 (0.0896)	0.5028 (0.0912)	0.4553 (0.3174)	0.4707 (0.3363)	0.4824 (0.3434)
		$\hat{k}$	0.1778 (0.0075)	0.1852 (0.0073)	0.1841 (0.0073)	0.1760 (0.0074)	0.1828 (0.0070)	0.1823 (0.0072)	0.1742 (0.0290)	0.1811 (0.0277)	0.1803 (0.0284)

Table 10: Bayes estimates and ERs (in brackets) based on different sample sizes and censoring schemes under AT-II PCS with Prior 3 and T=0.9.

n	m	Est.	*BS*			*BL* (λ=0.5)			*BL* (λ=1)		
			I	II	III	I	II	III	I	II	III
50	30	$\hat{\alpha}$	0.9862 (0.0195)	0.9900 (0.0329)	0.9946 (0.0594)	0.9838 (0.0110)	0.9866 (0.0179)	0.9903 (0.0239)	0.9822 (0.0347)	0.9848 (0.0502)	0.9883 (0.0631)
		$\hat{\beta}$	0.3546 (0.0395)	0.3575 (0.0344)	0.3665 (0.0592)	0.3518 (0.0140)	0.3549 (0.0203)	0.3633 (0.0126)	0.3506 (0.0377)	0.3537 (0.0504)	0.3621 (0.0351)
		$\hat{c}$	1.1944 (0.0359)	1.1979 (0.0358)	1.2511 (0.0369)	1.1855 (0.0355)	1.1890 (0.0354)	1.2420 (0.0365)	1.1769 (0.1402)	1.1804 (0.1400)	1.2331 (0.1442)
		$\hat{k}$	0.7418 (0.0395)	0.7472 (0.0401)	0.7382 (0.0401)	0.7322 (0.0385)	0.7374 (0.0391)	0.7285 (0.0389)	0.7230 (0.1507)	0.7281 (0.1530)	0.7192 (0.1522)
50	40	$\hat{\alpha}$	0.9805 (0.0124)	0.9905 (0.0082)	0.9806 (0.0258)	0.9787 (0.0106)	0.9890 (0.0099)	0.9778 (0.0154)	0.9772 (0.0328)	0.9875 (0.0317)	0.9761 (0.0443)
		$\hat{\beta}$	0.3564 (0.0241)	0.3840 (0.0120)	0.3618 (0.0545)	0.3546 (0.0136)	0.3826 (0.0119)	0.3586 (0.0128)	0.3535 (0.0359)	0.3815 (0.0324)	0.3574 (0.0349)
		$\hat{c}$	1.1970 (0.0362)	1.1945 (0.0358)	1.1906 (0.0359)	1.1885 (0.0358)	1.1859 (0.0354)	1.2560 (0.0354)	1.1798 (0.1414)	1.1773 (0.1400)	1.2474 (0.1398)
		$\hat{k}$	0.7250 (0.0365)	0.7308 (0.0365)	0.7254 (0.0369)	0.7160 (0.0359)	0.7218 (0.0359)	0.7164 (0.0361)	0.7074 (0.1411)	0.7132 (0.1410)	0.7078 (0.1415)
60	30	$\hat{\alpha}$	1.0002 (0.0573)	0.9922 (0.0259)	1.0066 (0.1601)	0.9968 (0.0137)	0.9897 (0.0133)	0.9991 (0.0297)	0.9952 (0.0399)	0.9880 (0.0399)	0.9971 (0.0760)
		$\hat{\beta}$	0.3719 (0.0225)	0.3683 (0.0157)	0.3719 (0.0242)	0.3703 (0.0148)	0.3666 (0.0104)	0.3695 (0.0098)	0.3692 (0.0387)	0.3656 (0.0292)	0.3683 (0.0293)
		$\hat{c}$	1.1963 (0.0342)	1.1996 (0.0345)	1.2986 (0.0351)	1.1879 (0.0338)	1.1911 (0.0341)	1.2899 (0.0347)	1.1796 (0.1336)	1.1828 (0.1347)	1.2814 (0.1369)
		$\hat{k}$	0.7472 (0.0396)	0.7453 (0.0402)	0.7454 (0.0583)	0.7376 (0.0386)	0.7355 (0.0391)	0.7350 (0.0413)	0.7284 (0.1509)	0.7262 (0.1528)	0.7257 (0.1573)
60	40	$\hat{\alpha}$	0.9883 (0.0296)	0.9870 (0.0223)	0.9919 (0.0478)	0.9858 (0.0099)	0.9846 (0.0096)	0.9892 (0.0145)	0.9843 (0.0318)	0.9831 (0.0314)	0.9875 (0.0426)
		$\hat{\beta}$	0.3720 (0.0142)	0.3752 (0.0109)	0.3744 (0.0178)	0.3705 (0.0117)	0.3741 (0.0089)	0.3727 (0.0085)	0.3694 (0.0323)	0.3731 (0.0258)	0.3717 (0.0255)
		$\hat{c}$	1.1923 (0.0339)	1.1988 (0.0346)	1.2423 (0.0347)	1.1839 (0.0335)	1.1903 (0.0342)	1.2337 (0.0343)	1.1757 (0.1326)	1.1819 (0.1352)	1.2253 (0.1356)
		$\hat{k}$	0.7318 (0.0367)	0.7299 (0.0372)	0.7265 (0.0383)	0.7228 (0.0360)	0.7208 (0.0364)	0.7173 (0.0368)	0.7141 (0.1415)	0.7121 (0.1429)	0.7086 (0.1432)
60	50	$\hat{\alpha}$	0.9826 (0.0126)	0.9795 (0.0100)	0.9861 (0.0259)	0.9810 (0.0066)	0.9779 (0.0090)	0.9837 (0.0095)	0.9796 (0.0239)	0.9764 (0.0295)	0.9822 (0.0309)
		$\hat{\beta}$	0.3769 (0.0050)	0.3692 (0.0147)	0.3801 (0.0151)	0.3758 (0.0059)	0.3677 (0.0084)	0.3784 (0.0068)	0.3747 (0.0202)	0.3667 (0.0250)	0.3774 (0.0217)
		$\hat{c}$	1.1874 (0.0339)	1.1898 (0.0341)	1.1880 (0.0343)	1.1790 (0.0335)	1.1813 (0.0337)	1.1795 (0.0339)	1.1708 (0.1327)	1.1731 (0.1334)	1.1712 (0.1341)
		$\hat{k}$	0.7172 (0.0342)	0.7192 (0.0356)	0.7130 (0.0335)	0.7088 (0.0336)	0.7104 (0.0348)	0.7047 (0.0329)	0.7006 (0.1325)	0.7021 (0.1366)	0.6967 (0.1297)

Table 11: Bayes estimates and ERs (in brackets) based on different sample sizes and censoring schemes under AT-II PCS with Prior 4 and T=0.9.

n	m	Est.	*BS*			*BL* (λ=0.5)			*BL* (λ=1)		
			I	II	III	I	II	III	I	II	III
50	30	$\hat{\alpha}$	0.9807 (0.0434)	0.9867 (0.0573)	0.9869 (0.1067)	0.9902 (0.0181)	0.9826 (0.0214)	0.9873 (0.1011)	0.9881 (0.0528)	0.9804 (0.0608)	0.9846 (0.2232)
		$\hat{\beta}$	0.3608 (0.0685)	0.3648 (0.0508)	0.3707 (0.0841)	0.3735 (0.0218)	0.3619 (0.0228)	0.3715 (0.0424)	0.3720 (0.0558)	0.3605 (0.0571)	0.3699 (0.0982)
		$\hat{c}$	1.1766 (0.0609)	1.1887 (0.0620)	1.2296 (0.0637)	1.1617 (0.0596)	1.1735 (0.0606)	1.2140 (0.0621)	1.1474 (0.2333)	1.1590 (0.2373)	1.1992 (0.2431)
		$\hat{k}$	0.7084 (0.0658)	0.7126 (0.0673)	0.7090 (0.0709)	0.6925 (0.0638)	0.6964 (0.0652)	0.6922 (0.0672)	0.6775 (0.2475)	0.6810 (0.2529)	0.6767 (0.2583)
50	40	$\hat{\alpha}$	0.9803 (0.0385)	0.9768 (0.0110)	0.9740 (0.0237)	0.9767 (0.0142)	0.9746 (0.0089)	0.9711 (0.0695)	0.9747 (0.0444)	0.9727 (0.0589)	0.9688 (0.1574)
		$\hat{\beta}$	0.3651 (0.0243)	0.3664 (0.0116)	0.3693 (0.0257)	0.3629 (0.0205)	0.3647 (0.0070)	0.3674 (0.0343)	0.3614 (0.0531)	0.3634 (0.0245)	0.3658 (0.0813)
		$\hat{c}$	1.1779 (0.0616)	1.1885 (0.0627)	1.1817 (0.0625)	1.1625 (0.0602)	1.1732 (0.0613)	1.1664 (0.0611)	1.1481 (0.2355)	1.1585 (0.2398)	1.1518 (0.2391)
		$\hat{k}$	0.6924 (0.0615)	0.6872 (0.0613)	0.6844 (0.0618)	0.6775 (0.0597)	0.6723 (0.0593)	0.6694 (0.0598)	0.6634 (0.2320)	0.6584 (0.2302)	0.6554 (0.2315)
60	30	$\hat{\alpha}$	0.9892 (0.0711)	0.9863 (0.0306)	0.9958 (0.2180)	0.9855 (0.0150)	0.9832 (0.0157)	0.9879 (0.1114)	0.9834 (0.0465)	0.9810 (0.0490)	0.9850 (0.2461)
		$\hat{\beta}$	0.3568 (0.0215)	0.3672 (0.0227)	0.3753 (0.0691)	0.3697 (0.0163)	0.3652 (0.0108)	0.3709 (0.0318)	0.3683 (0.0439)	0.3640 (0.0317)	0.3692 (0.0769)
		$\hat{c}$	1.1780 (0.0581)	1.1923 (0.0592)	1.2388 (0.0607)	1.1638 (0.0568)	1.1778 (0.0579)	1.2240 (0.0593)	1.1502 (0.2224)	1.1640 (0.2270)	1.2098 (0.2324)
		$\hat{k}$	0.7143 (0.0657)	0.7144 (0.0685)	0.7152 (0.0719)	0.6984 (0.0636)	0.6979 (0.0660)	0.6980 (0.0689)	0.6834 (0.2471)	0.6825 (0.2550)	0.6820 (0.2652)
60	40	$\hat{\alpha}$	0.9835 (0.0418)	0.9803 (0.0238)	0.9776 (0.1296)	0.9804 (0.0126)	0.9776 (0.0106)	0.9733 (0.0731)	0.9785 (0.0407)	0.9756 (0.0372)	0.9710 (0.1649)
		$\hat{\beta}$	0.3763 (0.0145)	0.3668 (0.0371)	0.3685 (0.0221)	0.3746 (0.0139)	0.3647 (0.0083)	0.3668 (0.0243)	0.3732 (0.0389)	0.3635 (0.0260)	0.3655 (0.0591)
		$\hat{c}$	1.1755 (0.0581)	1.1821 (0.0582)	1.2350 (0.0595)	1.1613 (0.0569)	1.1679 (0.0569)	1.2205 (0.0582)	1.1477 (0.2228)	1.1543 (0.2231)	1.2066 (0.2279)
		$\hat{k}$	0.6874 (0.0593)	0.6973 (0.0616)	0.6873 (0.0614)	0.6730 (0.0577)	0.6823 (0.0598)	0.6725 (0.0595)	0.6594 (0.2244)	0.6682 (0.2328)	0.6585 (0.2307)
60	50	$\hat{\alpha}$	0.9773 (0.0149)	0.9770 (0.0073)	0.9718 (0.0415)	0.9750 (0.0090)	0.9752 (0.0073)	0.9681 (0.0191)	0.9732 (0.0324)	0.9734 (0.0291)	0.9660 (0.0551)
		$\hat{\beta}$	0.3720 (0.0091)	0.3693 (0.0048)	0.3704 (0.0197)	0.3706 (0.0077)	0.3681 (0.0048)	0.3688 (0.0159)	0.3693 (0.0258)	0.3669 (0.0191)	0.3675 (0.0423)
		$\hat{c}$	1.1707 (0.0580)	1.1752 (0.0586)	1.2071 (0.0588)	1.1568 (0.0568)	1.1609 (0.0573)	1.1927 (0.0575)	1.1432 (0.2225)	1.1472 (0.2243)	1.1790 (0.2251)
		$\hat{k}$	0.6751 (0.0559)	0.6790 (0.0566)	0.6681 (0.0554)	0.6615 (0.0544)	0.6652 (0.0551)	0.6546 (0.0539)	0.6486 (0.2123)	0.6521 (0.2148)	0.6418 (0.2099)

Table 12: 95% Approximate CIs (in brackets) along with their width and coverage probability (in square brackets) with true parameters (α=0.2, β=0.5, c=0.5, k=0.2).

n	*m*	Est.	CS					
			T=0.3			*T=0.9*		
			I	*II*	*III*	*I*	*II*	*III*
50	30	$\hat{\alpha}$	(0.0329,0.4380) 0.4050 [0.9530]	(-0.2932,1.1320) 1.4251 [0.9423]	(-0.2910,1.1780) 1.4690 [0.9391]	(0.1141,0.3140) 0.2000 [0.9510]	(0.0880,0.3923) 0.3044 [0.9598]	(-0.2649,1.1909) 1.4558 [0.9563]
		$\hat{\beta}$	(0.2770,1.7500) 1.4730 [0.9250]	(0.2009,1.6657) 1.4648 [0.8454]	(0.2466,1.6830) 1.4364 [0.8309]	(0.2971,1.5370) 1.2400 [0.9040]	(0.3098,1.6632) 1.3534 [0.8713]	(0.0796,2.4662) 2.3866 [0.8558]
		$\hat{c}$	(0.0246,1.1210) 1.0960 [0.9240]	(-0.0465,1.1556) 1.2021 [0.9198]	(-0.0332,1.332) 1.1664 [0.9120]	(0.0323,1.0560) 1.0240 [0.9050]	(0.0155,1.0657) 1.0502 [0.9269]	(-0.0324,1.1353) 1.1677 [0.9342]
		$\hat{k}$	(-0.1120,0.6320) 0.7440 [0.9290]	(-0.1581,0.6427) 0.8008 [0.9140]	(-0.1549,0.6521) 0.8070 [0.9266]	(-0.0834,0.5630) 0.6470 [0.9350]	(-0.0874,0.5583) 0.6456 [0.9053]	(-0.1369,0.6128) 0.7497 [0.9089]
50	40	$\hat{\alpha}$	(0.1099,0.3110) 0.2010 [0.9540]	(0.0128,0.4276) 0.4148 [0.9496]	(0.0081,0.4313) 0.4232 [0.9513]	(0.1275,0.2910) 0.1630 [0.9500]	(0.1057,0.3169) 0.2112 [0.9600]	(-0.1052,1.2136) 1.3188 [0.9659]
		$\hat{\beta}$	(0.3560,1.1730) 0.8170 [0.8780]	(0.3134,1.2267) 0.9124 [0.8663]	(0.2825,1.3017) 1.0192 [0.8617]	(0.3615,1.2730) 0.9110 [0.8690]	(0.3281,1.2877) 0.9596 [0.8478]	(0.3203,1.9891) 1.6688 [0.8641]
		$\hat{c}$	(0.0420,1.0690) 1.0270 [0.9190]	(0.0093,1.0945) 1.0852 [0.9234]	(0.0062,1.0993) 1.0931 [0.9329]	(0.0507,1.0520) 1.0010 [0.8950]	(0.0282,1.0559) 1.0277 [0.9190]	(0.0075,1.0956) 1.0881 [0.9180]
		$\hat{k}$	(-0.0764,0.5690) 0.6450 [0.9280]	(-0.1068,0.5816) 0.6884 [0.9186]	(-0.1055,0.5818) 0.6874 [0.9212]	(-0.0637,0.5440) 0.6080 [0.9120]	(-0.0708,0.5278) 0.5985 [0.9268]	(-0.0960,0.5598) 0.6558 [0.9161]
60	30	$\hat{\alpha}$	(0.0166,0.6000) 0.5840 [0.9460]	(-0.5264,2.2791) 2.8055 [0.9373]	(-1.0712,4.8064) 5.8776 [0.9389]	(0.1160,0.3170) 0.2010 [0.9570]	(0.0939,0.3635) 0.2696 [0.9462]	(-0.2016,0.8547) 1.0563 [0.9623]
		$\hat{\beta}$	(0.0950,2.8500) 2.7550 [0.8880]	(0.1596,2.2177) 2.0582 [0.7979]	(-0.0087,3.0913) 3.1000 [0.8177]	(0.3219,1.2810) 0.9590 [0.8990]	(0.3921,1.1522) 0.7601 [0.9170]	(0.3822,1.2139) 0.8318 [0.8058]
		$\hat{c}$	(0.0451,1.0530) 1.0080 [0.9440]	(0.0019,1.1085) 1.1067 [0.938]	(-0.0204,1.1516) 1.1720 [0.9333]	(0.0532,1.0450) 0.9920 [0.9190]	(0.0511,1.0456) 0.9945 [0.9179]	(0.0086,1.0815) 1.0729 [0.9360]
		$\hat{k}$	(-0.0874,0.5850) 0.6720 [0.8970]	(-0.1202,0.6036) 0.7238 [0.9103]	(-0.1507,0.6483) 0.7990 [0.9154]	(-0.0756,0.5600) 0.6350 [0.9090]	(-0.0692,0.5429) 0.6121 [0.9104]	(-0.1115,0.5894) 0.7009 [0.9171]
60	40	$\hat{\alpha}$	(0.1012,0.3220) 0.2210 [0.9600]	(-0.3281,1.7732) 2.1013 [0.9515]	(-0.3382,1.8765) 2.2147 [0.9440]	(0.1217,0.3680) 0.2460 [0.9620]	(0.1144,0.3151) 0.2007 [0.9407]	(-0.0659,0.5302) 0.5960 [0.9730]
		$\hat{\beta}$	(0.3501,1.2040) 0.8540 [0.8670]	(0.2065,2.1337) 1.9273 [0.8040]	(0.1517,2.4542) 2.3025 [0.8226]	(0.3639,1.2050) 0.8410 [0.8650]	(0.4050,1.0564) 0.6514 [0.8136]	(0.3860,1.1960) 0.8101 [0.8173]
		$\hat{c}$	(0.0698,1.074) 1.0040 [0.9150]	(0.0250,1.0802) 1.0552 [0.9174]	(0.0098,1.1111) 1.1013 [0.9307]	(0.0542,1.0050) 0.9510 [0.9120]	(0.0450,1.0110) 0.9659 [0.9247]	(0.0106,1.0806) 1.0700 [0.9329]
		$\hat{k}$	(-0.0677,0.5640) 0.6310 [0.9140]	(-0.1044,0.5915) 0.6959 [0.9069]	(-0.1234,0.6053) 0.7287 [0.9099]	(-0.0528,0.5090) 0.5620 [0.8970]	(-0.0601,0.4948) 0.5549 [0.9171]	(-0.0998,0.5499) 0.6498 [0.9068]
60	50	$\hat{\alpha}$	(0.1252,0.2870) 0.1620 [0.9600]	(0.0501,0.3620) 0.3120 [0.9650]	(0.0517,0.3827) 0.3311 [0.9471]	(0.1381,0.2780) 0.1390 [0.9660]	(0.1212,0.3043) 0.1831 [0.9572]	(0.0641,0.3656) 0.3015 [0.9583]
		$\hat{\beta}$	(0.3592,1.106) 0.7470 [0.8460]	(0.4045,1.0690) 0.6650 [0.8260]	(0.3118,1.3803) 1.0685 [0.8064]	(0.3890,1.1350) 0.7470 [0.8510]	(0.3613,1.2680) 0.9067 [0.8017]	(0.3152,1.3110) 0.9958 [0.7981]
		$\hat{c}$	(0.0638,1.0190) 0.9550 [0.9170]	(0.0536,1.0620) 1.0080 [0.9380]	(0.0470,1.0333) 0.9863 [0.9292]	(0.0651,1.0040) 0.9390 [0.9280]	(0.0568,1.0158) 0.9590 [0.9134]	(0.0404,1.0182) 0.9778 [0.9213]
		$\hat{k}$	(-0.0531,0.5020) 0.5550 [0.9050]	(-0.0795,0.5540) 0.6340 [0.9200]	(-0.0726,0.5325) 0.6051 [0.9216]	(-0.0492,0.4940) 0.5430 [0.9200]	(-0.0516,0.4887) 0.5404 [0.9097]	(-0.0679,0.5031) 0.5709 [0.8995]

Table 13: 95% Approximate CIs (in brackets) along with their width and coverage probability (in square brackets) with true parameters (α=1, β=0.3, c=1.2, k=0.8).

n	m	Est.	CS					
			Approximate CI (T=0.3)			***Approximate CI (T=0.9)***		
			I	***II***	***III***	***I***	***II***	***III***
50	**30**	$\hat{\alpha}$	(-0.0532,3.3520) 3.4052[0.9232]	(-1.0864,5.2320) 6.3185[0.9289]	(-3.5859,9.2466) 12.8325[0.9645]	(0.2848,2.1242) 1.8393[0.9719]	(0.1586,2.9527) 2.7941[0.9711]	(-1.3708,4.1508) 5.5216[0.9786]
		$\hat{\beta}$	(-0.1739,2.4567) 2.6306[0.9664]	(-0.3947,2.9593) 3.3540[0.9564]	(-0.6946,3.5377) 4.2324[0.9565]	(-0.0368,1.6182) 1.6550[0.9710]	(-0.0121,2.1090) 2.1211[0.9656]	(-0.0648,1.7662) 1.8310[0.9778]
		$\hat{c}$	(0.1755,2.1012) 1.9257[0.9225]	(0.0410,2.1912) 2.1502[0.9200]	(-0.0869,2.5899) 2.6768[0.9211]	(0.0798,1.9402) 1.8604[0.9044]	(0.0921,1.9622) 1.8701[0.9149]	(-0.0488,2.2270) 2.2758[0.9201]
		$\hat{k}$	(-1.0773,3.3199) 4.3972[0.8348]	(-1.5148,3.7361) 5.2509[0.8150]	(-2.3755,5.0085) 7.3840[0.8099]	(-0.7849,2.3711) 3.1560[0.7725]	(-0.8833,2.6052) 3.4886[0.7948]	(-1.5297,3.5053) 5.0350[0.7947]
50	**40**	$\hat{\alpha}$	(0.0165,2.6216) 2.6051[0.9448]	(-0.8762,3.9928) 4.8690[0.9149]	(-1.7707,6.0828) 7.8535[0.9344]	(0.4318,1.8475) 1.4157[0.9704]	(0.2442,2.4625) 2.2183[0.9760]	(-0.5355,4.0245) 4.5600[0.9829]
		$\hat{\beta}$	(-0.1881,2.1223) 2.3104[0.9805]	(-0.4841,2.5344) 3.0185[0.9653]	(-0.6617,3.1009) 3.7626[0.9584]	(0.0771,1.2721) 1.1950[0.9758]	(-0.0889,1.9058) 1.9947[0.9680]	(-0.0410,1.5509) 1.5919[0.9783]
		$\hat{c}$	(0.2314,2.1150) 1.8835[0.9165]	(0.1268,2.1434) 2.0166[0.9329]	(0.0441,2.1629) 2.1188[0.9175]	(0.1066,1.8618) 1.7552[0.9122]	(0.1083,1.9174) 1.8091[0.9043]	(0.0521,1.9955) 1.9434[0.9167]
		$\hat{k}$	(-0.9326,3.2005) 4.1330[0.8418]	(-1.3380,3.6146) 4.9525[0.8383]	(-1.5154,3.7513) 5.2668[0.8025]	(-0.6492,2.1184) 2.7676[0.7679]	(-0.7217,2.2989) 3.0203[0.7644]	(-1.1858,2.9247) 4.1105[0.7825]
60	**30**	$\hat{\alpha}$	(-0.2583,6.5613) 6.8196[0.9327]	(-1.5391,8.2175) 9.7565[0.9437]	(-3.8573,10.8649) 14.7222[0.9686]	(0.2097,3.0316) 2.8219[0.9758]	(-0.3058,6.1741) 6.4799[0.9709]	(-2.9825,7.8840) 10.8665[0.9720]
		$\hat{\beta}$	(-0.3886,5.3761) 5.7647[0.9744]	(-0.2789,2.7319) 3.0108[0.9638]	(-1.683,5.9789) 7.1472[0.9497]	(-0.1460,3.3161) 3.4621[0.9725]	(-0.1746,2.1405) 2.3151[0.9783]	(-0.2907,3.7706) 4.0613[0.9631]
		$\hat{c}$	(0.2038,2.0623) 1.8585[0.9086]	(0.0712,2.1454) 2.0742[0.9102]	(-0.0732,2.5510) 2.6242[0.8988]	(0.1068,1.9187) 1.8118[0.9196]	(0.1117,1.9650) 1.8532[0.9190]	(-0.2585,2.6324) 2.8909[0.9096]
		$\hat{k}$	(-1.0274,3.2395) 4.2669[0.8355]	(-2.1113,4.4000) 6.5114[0.7846]	(-2.5168,5.0934) 7.6102[0.7775]	(-0.7479,2.3055) 3.0534[0.7695]	(-0.9330,2.6737) 3.6067[0.7926]	(-2.4581,4.7329) 7.1910[0.7890]
60	**40**	$\hat{\alpha}$	(-0.0444,4.5397) 4.5841[0.9227]	(-1.2002,5.5626) 6.7628[0.9283]	(-2.2635,6.2651) 8.5286[0.9325]	(0.4687,1.7990) 1.3303[0.9812]	(0.3114,2.1598) 1.8484[0.9734]	(-1.3205,5.9307) 7.2512[0.9873]
		$\hat{\beta}$	(-0.1789,3.7966) 3.9755[0.9763]	(-0.2442,2.4623) 2.7065[0.9598]	(-0.3862,2.8545) 3.2407[0.9457]	(0.0997,1.2468) 1.1471[0.9698]	(0.0172,1.5869) 1.5697[0.9609]	(-0.1869,2.6929) 2.8798[0.9711]
		$\hat{c}$	(0.2517,2.0652) 1.8136[0.9142]	(0.1195,2.2238) 2.1043[0.9265]	(-0.1084,2.4308) 2.5392[0.9123]	(0.1072,1.8517) 1.7445[0.9102]	(0.0968,1.9172) 1.8204[0.9170]	(-0.0209,2.1271) 2.1480[0.9095]
		$\hat{k}$	(-0.8809,3.0856) 3.9666[0.8190]	(-1.3628,3.5862) 4.9490[0.8318]	(-2.6273,5.0604) 7.6877[0.7985]	(-0.6237,2.0076) 2.6313[0.7371]	(-0.7141,2.1692) 2.8834[0.7519]	(-1.2405,2.9225) 4.1630[0.7902]
60	**50**	$\hat{\alpha}$	(0.1513,2.3801) 2.2288[0.9386]	(-0.5066,2.9733) 3.4798[0.9193]	(-1.1696,4.7270) 5.8966[0.9293]	(0.4690,1.9410) 1.4720[0.9764]	(0.4705,1.9048) 1.4343[0.9784]	(-0.2056,2.6953) 2.9009[0.9862]
		$\hat{\beta}$	(-0.0475,1.8946) 1.9421[0.9727]	(-0.0111,1.2948) 1.3060[0.9687]	(-0.1997,2.1860) 2.3857[0.9589]	(0.0291,1.6659) 1.6368[0.9650]	(0.0553,1.4560) 1.4007[0.9752]	(-0.0494,1.6565) 1.7059[0.9781]
		$\hat{c}$	(0.2876,2.0608) 1.7732[0.9184]	(0.1373,2.0767) 1.9393[0.9196]	(0.1158,2.1452) 2.0294[0.9239]	(0.1185,1.8413) 1.7228[0.9148]	(0.1370,1.8189) 1.6819[0.9049]	(0.0735,1.9288) 1.8554[0.9175]
		$\hat{k}$	(-0.7535,2.9429) 3.6964[0.8278]	(-1.1630,3.2192) 4.3822[0.8002]	(-1.3638,3.5567) 4.9205[0.8118]	(-0.5383,1.8527) 2.3910[0.7330]	(-0.5470,1.8984) 2.4454[0.7346]	(-0.8282,2.2862) 3.1144[0.7565]

Table 14: 95% Credible Intervals (in brackets) along with their width and coverage probability (in square brackets) with T=0.3, Prior 1 and Prior 2.

n	m	Est.	CS					
			Prior 1			***Prior 2***		
			I	***II***	***III***	***I***	***II***	***III***
50	**30**	$\hat{\alpha}$	(0.1055,0.3824) 0.2769[0.9471]	(0.0186,0.4697) 0.4512[0.9411]	(0.0087,0.5018) 0.4931[0.9380]	(0.1353,0.4306) 0.2953[0.9206]	(0.0562,0.5109) 0.4547[0.8746]	(0.0277,0.5288) 0.5011[0.8627]
		$\hat{\beta}$	(0.4106,0.7446) 0.3340[0.9134]	(0.4198,0.7409) 0.3211[0.8502]	(0.4176,0.7241) 0.3064[0.8442]	(0.4089,0.7999) 0.3910[0.8750]	(0.4207,0.8256) 0.4049[0.8206]	(0.4279,0.8646) 0.4367[0.7858]
		$\hat{c}$	(0.1737,0.9701) 0.7964[0.0.9435]	(0.1726,0.9403) 0.7677[0.9512]	(0.1614,0.9444) 0.7830[0.9497]	(0.1191,1.0953) 0.9762[0.9426]	(0.1189,1.0492) 0.9303[0.9408]	(0.1145,1.0482) 0.9337[0.9316]
		$\hat{k}$	(0.0699,0.3688) 0.2989[0.9380]	(0.0706,0.3577) 0.2871[0.9386]	(0.0689,0.3579) 0.2890[0.9349]	(0.0539,0.4082) 0.3543[0.9450]	(0.0580,0.3995) 0.3414[0.9320]	(0.0628,0.4034) 0.3406[0.9311]
50	**40**	$\hat{\alpha}$	(0.1471,0.3219) 0.1749[0.9460]	(0.0904,0.3534) 0.2630[0.9424]	(0.0919,0.3641) 0.2722[0.9531]	(0.1781,0.3922) 0.2141[0.9491]	(0.1201,0.4169) 0.2968[0.9207]	(0.1173,0.4072) 0.2899[0.9373]
		$\hat{\beta}$	(0.4167,0.7385) 0.3218[0.8696]	(0.4227,0.7287) 0.3060[0.8552]	(0.4229,0.7309) 0.3080[0.8611]	(0.4096,0.7836) 0.3740[0.8352]	(0.4221,0.7963) 0.3742[0.8120]	(0.4195,0.7898) 0.3703[0.8026]
		$\hat{c}$	(0.1748,0.9491) 0.7742[0.9479]	(0.1710,0.9323) 0.7613[0.9500]	(0.1761,0.9515) 0.7754[0.9512]	(0.1244,1.0951) 0.9707[0.9604]	(0.1237,1.1132) 0.9895[0.9588]	(0.1289,1.0907) 0.9617[0.9500]
		$\hat{k}$	(0.0720,0.3584) 0.2864[0.9385]	(0.0711,0.3576) 0.2865[0.9418]	(0.0715,0.3596) 0.2881[0.9459]	(0.0603,0.4111) 0.3508[0.9405]	(0.0600,0.4100) 0.3501[0.9424]	(0.0623,0.4092) 0.3469[0.9357]
60	**30**	$\hat{\alpha}$	(0.0873,0.3528) 0.2655[0.9439]	(0.0110,0.5480) 0.5370[0.9551]	(0.0021,0.6275) 0.6254[0.9522]	(0.1237,0.4193) 0.2955[0.9268]	(0.0366,0.5841) 0.5476[0.8677]	(0.0195,0.6843) 0.6648[0.8645]
		$\hat{\beta}$	(0.4192,0.7378) 0.3186[0.8692]	(0.4328,0.7382) 0.3054[0.8071]	(0.4335,0.7369) 0.3034[0.8140]	(0.4182,0.8715) 0.4533[0.8513]	(0.4391,0.8155) 0.3764[0.7772]	(0.4381,0.8064) 0.3683[0.7551]
		$\hat{c}$	(0.1816,0.9112) 0.7296[0.9551]	(0.1714,0.9063) 0.7349[0.9466]	(0.1696,0.8996) 0.7300[0.9545]	(0.1270,1.1066) 0.9796[0.9522]	(0.1219,1.0349) 0.9130[0.9383]	(0.1243,1.0045) 0.8801[0.9274]
		$\hat{k}$	(0.0744,0.3639) 0.2895[0.9467]	(0.0739,0.3538) 0.2799[0.9380]	(0.0723,0.3515) 0.2792[0.9205]	(0.0576,0.4131) 0.3555[0.9357]	(0.0606,0.3812) 0.3207[0.9248]	(0.0614,0.3830) 0.3217[0.9172]
60	**40**	$\hat{\alpha}$	(0.1302,0.3089) 0.1787[0.9625]	(0.0458,0.3949) 0.3491[0.9531]	(0.0369,0.3975) 0.3606[0.9499]	(0.1842,0.4269) 0.2427[0.9410]	(0.0740,0.4260) 0.3520[0.9010]	(0.0698,0.4568) 0.3870[0.8944]
		$\hat{\beta}$	(0.4202,0.7276) 0.3074[0.8658]	(0.4333,0.7315) 0.2982[0.8174]	(0.4316,0.7328) 0.3013[0.8245]	(0.4253,0.8322) 0.4070[0.8277]	(0.4388,0.7913) 0.3525[0.7745]	(0.4327,0.7788) 0.3461[0.7647]
		$\hat{c}$	(0.1846,0.9132) 0.7286[0.9481]	(0.1781,0.9111) 0.7330[0.9595]	(0.1790,0.9138) 0.7347[0.9381]	(0.1278,1.0756) 0.9478[0.9459]	(0.1295,1.0232) 0.8937[0.9425]	(0.1315,1.0114) 0.8799[0.9303]
		$\hat{k}$	(0.0715,0.3526) 0.2811[0.9361]	(0.0732,0.3501) 0.2768[0.9428]	(0.0726,0.3508) 0.2782[0.9323]	(0.0613,0.4103) 0.3490[0.9533]	(0.0609,0.3822) 0.3213[0.9276]	(0.0628,0.3845) 0.3218[0.9412]
60	**50**	$\hat{\alpha}$	(0.1837,0.3488) 0.1651[0.9406]	(0.1140,0.3262) 0.2122[0.9510]	(0.1168,0.3335) 0.2167[0.9469]	(0.1943,0.3733) 0.1791[0.9534]	(0.1659,0.4387) 0.2727[0.9290]	(0.1418,0.3963) 0.2544[0.9489]
		$\hat{\beta}$	(0.4292,0.7340) 0.3048[0.8526]	(0.4359,0.7331) 0.2972[0.8074]	(0.4336,0.7306) 0.2970[0.7975]	(0.4272,0.7807) 0.3534[0.8142]	(0.4358,0.7853) 0.3495[0.7730]	(0.4332,0.7785) 0.3453[0.7860]
		$\hat{c}$	(0.1827,0.8961) 0.7134[0.9549]	(0.1827,0.9099) 0.7272[0.9510]	(0.1833,0.8840) 0.7007[0.9551]	(0.1429,1.0878) 0.9449[0.9442]	(0.1407,1.0595) 0.9188[0.9275]	(0.1514,1.0575) 0.9061[0.9530]
		$\hat{k}$	(0.0742,0.3516) 0.2775[0.9444]	(0.0744,0.3522) 0.2779[0.9323]	(0.0764,0.3543) 0.2779[0.9400]	(0.0623,0.3895) 0.3272[0.9540]	(0.0619,0.3989) 0.3370[0.9386]	(0.0632,0.3975) 0.3355[0.9407]

Table 15: 95% Credible Intervals (in brackets) along with their width and coverage probability (in square brackets) with T=0.3, Prior 3 and Prior 4.

n	m	Est.	CS					
			Prior 3			Prior 4		
			I	II	III	I	II	III
50	30	$\hat{\alpha}$	(0.8491, 1.1592) 0.3102 [0.9388]	(0.8436, 1.1648) 0.3212 [0.9419]	(0.8413, 1.1732) 0.3319 [0.9502]	(0.8239, 1.1862) 0.3624 [0.9574]	(0.8176, 1.2130) 0.3955 [0.9478]	(0.8127, 1.2241) 0.4113 [0.9618]
		$\hat{\beta}$	(0.2369, 0.4812) 0.2443 [0.8934]	(0.2493, 0.4886) 0.2393 [0.8649]	(0.2506, 0.4925) 0.2419 [0.8525]	(0.2318, 0.5050) 0.2732 [0.9069]	(0.2458, 0.5232) 0.2774 [0.8504]	(0.2454, 0.5180) 0.2726 [0.8551]
		$\hat{c}$	(0.8704, 1.6090) 0.7386 [0.9457]	(0.8719, 1.6107) 0.7388 [0.9520]	(0.8704, 1.6156) 0.7452 [0.9491]	(0.7837, 1.7414) 0.9578 [0.9574]	(0.7827, 1.7412) 0.9585 [0.9528]	(0.7843, 1.7578) 0.9735 [0.9467]
		$\hat{k}$	(0.4242, 1.2152) 0.7910 [0.9506]	(0.4206, 1.2080) 0.7874 [0.9329]	(0.4196, 1.2348) 0.8152 [0.9420]	(0.3206, 1.3663) 1.0457 [0.9366]	(0.3191, 1.3701) 1.0511 [0.9341]	(0.3136, 1.3617) 1.0481 [0.9356]
50	40	$\hat{\alpha}$	(0.8512, 1.1548) 0.3036 [0.9497]	(0.8464, 1.1601) 0.3137 [0.9583]	(0.8448, 1.1624) 0.3176 [0.9475]	(0.8282, 1.1795) 0.3513 [0.9475]	(0.8214, 1.1964) 0.3750 [0.9508]	(0.8185, 1.1946) 0.3761 [0.9571]
		$\hat{\beta}$	(0.2445, 0.4882) 0.2438 [0.8569]	(0.2505, 0.4909) 0.2404 [0.8451]	(0.2486, 0.4856) 0.2370 [0.8642]	(0.2424, 0.5139) 0.2715 [0.8545]	(0.2427, 0.5091) 0.2664 [0.8623]	(0.2445, 0.5097) 0.2652 [0.8385]
		$\hat{c}$	(0.8720, 1.6106) 0.7386 [0.9474]	(0.8692, 1.6034) 0.7342 [0.9444]	(0.8753, 1.6183) 0.7431 [0.9425]	(0.7857, 1.7481) 0.9624 [0.9584]	(0.7863, 1.7441) 0.9578 [0.9474]	(0.7809, 1.7332) 0.9522 [0.9501]
		$\hat{k}$	(0.4241, 1.2036) 0.7794 [0.9447]	(0.4254, 1.2014) 0.7760 [0.9364]	(0.4231, 1.1995) 0.7764 [0.9415]	(0.3174, 1.3299) 1.0125 [0.9267]	(0.3197, 1.3478) 1.0282 [0.9292]	(0.3251, 1.3503) 1.0251 [0.9212]
60	30	$\hat{\alpha}$	(0.8496, 1.1635) 0.3139 [0.9409]	(0.8433, 1.1642) 0.3209 [0.9483]	(0.8401, 1.1817) 0.3416 [0.9430]	(0.8227, 1.1884) 0.3657 [0.9453]	(0.8171, 1.2032) 0.3861 [0.9604]	(0.8113, 1.2434) 0.4322 [0.9522]
		$\hat{\beta}$	(0.2430, 0.4866) 0.2436 [0.8740]	(0.2545, 0.4833) 0.2288 [0.8350]	(0.2578, 0.4940) 0.2361 [0.8207]	(0.2391, 0.5142) 0.2751 [0.8786]	(0.2509, 0.5104) 0.2595 [0.8457]	(0.2607, 0.5309) 0.2702 [0.8059]
		$\hat{c}$	(0.8788, 1.5987) 0.7199 [0.9508]	(0.8784, 1.5990) 0.7207 [0.9563]	(0.8753, 1.6017) 0.7263 [0.9285]	(0.7844, 1.7037) 0.9193 [0.9642]	(0.7905, 1.7168) 0.9262 [0.9664]	(0.7870, 1.7159) 0.9289 [0.9583]
		$\hat{k}$	(0.4265, 1.2200) 0.7935 [0.9449]	(0.4230, 1.2102) 0.7873 [0.9433]	(0.4197, 1.2861) 0.8664 [0.9461]	(0.3215, 1.3587) 1.0372 [0.9383]	(0.3211, 1.3694) 1.0483 [0.9496]	(0.3130, 1.3606) 1.0476 [0.9370]
60	40	$\hat{\alpha}$	(0.8520, 1.1576) 0.3055 [0.9556]	(0.8467, 1.1651) 0.3185 [0.9385]	(0.8430, 1.1715) 0.3285 [0.9533]	(0.8286, 1.1854) 0.3568 [0.9554]	(0.8201, 1.2065) 0.3864 [0.9450]	(0.8146, 1.2063) 0.3917 [0.9482]
		$\hat{\beta}$	(0.2471, 0.4853) 0.2382 [0.8718]	(0.2596, 0.4878) 0.2282 [0.8611]	(0.2576, 0.4894) 0.2318 [0.8024]	(0.2464, 0.5170) 0.2706 [0.8423]	(0.2564, 0.5210) 0.2646 [0.8045]	(0.2507, 0.5073) 0.2566 [0.7949]
		$\hat{c}$	(0.8842, 1.6110) 0.7268 [0.9527]	(0.8810, 1.6003) 0.7193 [0.9524]	(0.8722, 1.5893) 0.7170 [0.9583]	(0.7907, 1.7075) 0.9168 [0.9514]	(0.7879, 1.7062) 0.9183 [.9715]	(0.7830, 1.7016) 0.9186 [0.9453]
		$\hat{k}$	(0.4240, 1.2076) 0.7836 [0.9448]	(0.4256, 1.2039) 0.7783 [0.9325]	(0.4265, 1.2429) 0.8164 [0.9384]	(0.3245, 1.3477) 1.0233 [0.9306]	(0.3226, 1.3408) 1.0181 [0.9322]	(0.3232, 1.3555) 1.0323 [0.9316]
60	50	$\hat{\alpha}$	(0.8537, 1.1542) 0.3005 [0.9529]	(0.8473, 1.1579) 0.3107 [0.9488]	(0.8472, 1.1603) 0.3130 [0.9458]	(0.8328, 1.1821) 0.3493 [0.9474]	(0.8243, 1.1867) 0.3624 [0.9519]	(0.8211, 1.1989) 0.3777 [0.9590]
		$\hat{\beta}$	(0.2541, 0.4907) 0.2366 [0.8597]	(0.2506, 0.4841) 0.2281 [0.7996]	(0.2576, 0.4833) 0.2256 [0.8148]	(0.2474, 0.5117) 0.2642 [0.8451]	(0.2576, 0.5119) 0.2543 [0.8193]	(0.2558, 0.5103) 0.2545 [0.8155]
		$\hat{c}$	(0.8814, 1.6009) 0.7195 [0.9578]	(0.8755, 1.5889) 0.7134 [0.9498]	(0.8871, 1.6138) 0.7267 [0.9468]	(0.7983, 1.7291) 0.9307 [0.9652]	(0.7944, 1.7153) 0.9209 [0.9607]	(0.7910, 1.7093) 0.9183 [0.9431]
		$\hat{k}$	(0.4257, 1.1941) 0.7684 [0.9342]	(0.4269, 1.1954) 0.7685 [0.9350]	(0.4251, 1.1887) 0.7635 [0.9291]	(0.3252, 1.3341) 1.0089 [0.9196]	(0.3231, 1.3178) 0.9948 [0.9411]	(0.3254, 1.3296) 1.0042 [0.9199]

Table 16: 95% Credible Intervals (in brackets) along with their width and coverage probability (in square brackets) with T=0.9, Prior 1 and Prior 2.

n	m	Est.	CS					
			Prior 1			Prior 2		
			I	II	III	I	II	III
50	30	$\hat{\alpha}$	(0.1413,0.3440) 0.2027[0.9380]	(0.1101,0.3696) 0.2596[0.9423]	(0.0147,0.4171) 0.4025[0.9396]	(0.1899,0.4415) 0.2517[0.9522]	(0.1359,0.4299) 0.2940[0.9453]	(0.0349,0.4128) 0.3779[0.8792]
		$\hat{\beta}$	(0.4073,0.7258) 0.3185[0.8945]	(0.4236,0.7770) 0.3534[0.8455]	(0.4225,0.7468) 0.3243[0.8428]	(0.3979,0.7902) 0.3923[0.8493]	(0.4251,0.8126) 0.3875[0.8273]	(0.4212,0.8612) 0.4400[0.8148]
		$\hat{c}$	(0.1687,0.9504) 0.7817[0.9499]	(0.1695,0.9432) 0.7737[0.9413]	(0.1690,0.9149) 0.7459[0.9419]	(0.1128,1.0414) 0.9285[0.9432]	(0.1153,1.1220) 1.0067[0.9453]	(0.1126,1.0473) 0.9347[0.9258]
		$\hat{k}$	(0.0702,0.3603) 0.2901[0.9314]	(0.0743,0.4111) 0.3367[0.9300]	(0.0735,0.3647) 0.2913[0.9365]	(0.0576,0.4071) 0.3495[0.9167]	(0.0570,0.4024) 0.3454[0.9279]	(0.0577,0.3943) 0.3366[0.9177]
50	40	$\hat{\alpha}$	(0.1547,0.3130) 0.1583[0.9680]	(0.1574,0.3575) 0.2000[0.9484]	(0.0881,0.3385) 0.2504[0.9474]	(0.1915,0.3821) 0.1906[0.9507]	(0.1759,0.3935) 0.2176[0.9329]	(0.1055,0.3659) 0.2604[0.9327]
		$\hat{\beta}$	(0.4131,0.7285) 0.3154[0.8676]	(0.4224,0.7503) 0.3279[0.8316]	(0.4231,0.7465) 0.3234[0.8571]	(0.4108,0.7942) 0.3834[0.8343]	(0.4195,0.7927) 0.3732[0.8212]	(0.4238,0.8190) 0.3952[0.7942]
		$\hat{c}$	(0.1722,0.9350) 0.7627[0.9566]	(0.1694,0.9422) 0.7728[0.9356]	(0.1714,0.9665) 0.7951[0.9474]	(0.1137,1.0283) 0.9145[0.9349]	(0.1226,1.1136) 0.9911[0.9441]	(0.1237,1.0790) 0.9554[0.9386]
		$\hat{k}$	(0.0735,0.3587) 0.2851[0.9326]	(0.0700,0.3522) 0.2822[0.9095]	(0.0716,0.3649) 0.2933[0.9215]	(0.0598,0.3952) 0.3354[0.9466]	(0.0567,0.3941) 0.3375[0.9220]	(0.0577,0.3936) 0.3359[0.9212]
60	30	$\hat{\alpha}$	(0.1478,0.3623) 0.2144[0.9568]	(0.0953,0.3570) 0.2617[0.9476]	(0.0041,0.4640) 0.4599[0.9390]	(0.1909,0.4560) 0.2651[0.9422]	(0.1200,0.4195) 0.2995[0.9404]	(0.0192,0.4968) 0.4777[0.9299]
		$\hat{\beta}$	(0.4145,0.7316) 0.3171[0.8910]	(0.4340,0.7510) 0.3170[0.8160]	(0.4343,0.7423) 0.3080[0.8185]	(0.4080,0.7982) 0.3902[0.8581]	(0.4332,0.7983) 0.3651[0.8015]	(0.4362,0.8041) 0.3679[0.7983]
		$\hat{c}$	(0.1761,0.8837) 0.7076[0.9464]	(0.1786,0.9187) 0.7400[0.9455]	(0.1803,0.8922) 0.7119[0.9421]	(0.1316,1.0339) 0.9023[0.9396]	(0.1262,1.0932) 0.9670[0.9428]	(0.1314,0.9742) 0.8428[0.9586]
		$\hat{k}$	(0.0740,0.3580) 0.2839[0.9361]	(0.0752,0.3563) 0.2811[0.9305]	(0.0740,0.3449) 0.2710[0.9143]	(0.0644,0.4132) 0.3488[0.9335]	(0.0605,0.3926) 0.3321[0.9314]	(0.0645,0.3777) 0.3132[0.9402]
60	40	$\hat{\alpha}$	(0.1662,0.3361) 0.1699[0.9441]	(0.1440,0.3502) 0.2062[0.9498]	(0.0398,0.3568) 0.3171[0.9470]	(0.1976,0.3991) 0.2015[0.9562]	(0.1631,0.3886) 0.2255[0.9449]	(0.0644,0.3851) 0.3207[0.8999]
		$\hat{\beta}$	(0.4181,0.7263) 0.3082[0.8500]	(0.4348,0.7391) 0.3042[0.8085]	(0.4353,0.7409) 0.3056[0.7976]	(0.4174,0.7937) 0.3763[0.8278]	(0.4405,0.8056) 0.3650[0.7589]	(0.4376,0.8890) 0.4515[0.7614]
		$\hat{c}$	(0.1766,0.8822) 0.7056[0.9461]	(0.1787,0.9019) 0.7231[0.9327]	(0.1758,0.8940) 0.7181[0.9530]	(0.1323,1.0042) 0.8719[0.9465]	(0.1285,1.0430) 0.9145[0.9466]	(0.1296,1.0612) 0.9316[0.9257]
		$\hat{k}$	(0.0734,0.3494) 0.2760[0.9451]	(0.0743,0.3512) 0.2769[0.9098]	(0.0751,0.3496) 0.2745[0.9231]	(0.0656,0.3995) 0.3339[0.9242]	(0.0613,0.3825) 0.3212[0.9090]	(0.0648,0.3982) 0.3335[0.9119]
60	50	$\hat{\alpha}$	(0.1759,0.3213) 0.1454[0.9404]	(0.1719,0.3434) 0.1715[0.9471]	(0.1079,0.3017) 0.1939[0.9492]	(0.1978,0.3618) 0.1640[0.9486]	(0.1865,0.3689) 0.1824[0.9464]	(0.1445,0.3903) 0.2458[0.9291]
		$\hat{\beta}$	(0.4289,0.7328) 0.3039[0.8368]	(0.4368,0.7388) 0.3021[0.8312]	(0.4400,0.7472) 0.3077[0.7879]	(0.4314,0.7968) 0.3654[0.7814]	(0.4438,0.8046) 0.3608[0.7219]	(0.4386,0.7972) 0.3586[0.7626]
		$\hat{c}$	(0.1823,0.8942) 0.7118[0.9335]	(0.1767,0.8963) 0.7196[0.9383]	(0.1810,0.8953) 0.7143[0.9387]	(0.1229,0.9935) 0.8706[0.9279]	(0.1314,1.0171) 0.8857[0.9439]	(0.1361,1.0872) 0.9511[0.9487]
		$\hat{k}$	(0.0731,0.3438) 0.2707[0.9286]	(0.0718,0.3429) 0.2711[0.9115]	(0.0752,0.3452) 0.2700[0.9346]	(0.0660,0.3835) 0.3174[0.9135]	(0.0679,0.3785) 0.3106[0.9095]	(0.0619,0.3867) 0.3248[0.9261]

Table 17: 95% Credible Intervals (in brackets) along with their width and coverage probability (in square brackets) with T=0.9, Prior 3 and Prior 4.

n	m	Est.	CS					
			Prior 3			***Prior 4***		
			I	***II***	***III***	***I***	***II***	***III***
50	**30**	$\hat{\alpha}$	(0.8393, 1.1467) 0.3074 [0.9478]	(0.8394, 1.1487) 0.3092 [0.9517]	(0.8343, 1.1563) 0.3220 [0.9449]	(0.8130, 1.1621) 0.3492 [0.9436]	(0.8114, 1.1742) 0.3628 [0.9390]	(0.8031, 1.1764) 0.3733 [0.9395]
		$\hat{\beta}$	(0.2362, 0.4923) 0.2561 [0.8847]	(0.2479, 0.4919) 0.2440 [0.8629]	(0.2533, 0.5010) 0.2477 [0.8457]	(0.2352, 0.5188) 0.2836 [0.8882]	(0.2462, 0.5232) 0.2771 [0.8722]	(0.2487, 0.5178) 0.2691 [0.8494]
		$\hat{c}$	(0.8614, 1.5985) 0.7371 [0.9498]	(0.8619, 1.5967) 0.7348 [0.9536]	(0.8646, 1.6089) 0.7443 [0.9449]	(0.7580, 1.6984) 0.9404 [0.9518]	(0.7603, 1.6997) 0.9393 [0.9528]	(0.7669, 1.7339) 0.9670 [0.9366]
		$\hat{k}$	(0.4112, 1.1843) 0.7731 [0.9074]	(0.4116, 1.1796) 0.7680 [0.9270]	(0.4050, 1.1695) 0.7644 [0.9138]	(0.3019, 1.2818) 0.9799 [0.9036]	(0.3058, 1.2996) 0.9938 [0.9027]	(0.2948, 1.2667) 0.9718 [0.8860]
50	**40**	$\hat{\alpha}$	(0.8385, 1.1311) 0.2926 [0.9392]	(0.8374, 1.1325) 0.2952 [0.9567]	(0.8316, 1.1394) 0.3077 [0.9516]	(0.8138, 1.1503) 0.3365 [0.9396]	(0.8123, 1.1594) 0.3471 [0.9490]	(0.8022, 1.1569) 0.3548 [0.9149]
		$\hat{\beta}$	(0.2458, 0.4945) 0.2487 [0.8862]	(0.2506, 0.4937) 0.2430 [0.8632]	(0.2510, 0.4968) 0.2457 [0.8500]	(0.2424, 0.5210) 0.2786 [0.8743]	(0.2471, 0.5222) 0.2751 [0.8548]	(0.2473, 0.5203) 0.2730 [0.8485]
		$\hat{c}$	(0.8597, 1.5981) 0.7384 [0.9617]	(0.8637, 1.6094) 0.7457 [0.9449]	(0.8538, 1.5880) 0.7342 [0.9477]	(0.7578, 1.7034) 0.9456 [0.9644]	(0.7550, 1.6937) 0.9387 [0.9607]	(0.7505, 1.6918) 0.9413 [0.9386]
		$\hat{k}$	(0.4015, 1.1395) 0.7380 [0.9048]	(0.3989, 1.1383) 0.7394 [0.8799]	(0.4022, 1.1401) 0.7379 [0.9023]	(0.2965, 1.2383) 0.9418 [0.9000]	(0.2979, 1.2421) 0.9441 [0.8852]	(0.2943, 1.2191) 0.9247 [0.8614]
60	**30**	$\hat{\alpha}$	(0.8372, 1.1395) 0.3023 [.9536]	(0.8381, 1.1486) 0.3105 [0.9449]	(0.8362, 1.1659) 0.3297 [0.9518]	(0.8119, 1.1611) 0.3492 [0.9277]	(0.8119, 1.1861) 0.3742 [0.9498]	(0.8050, 1.1842) 0.3792 [0.9505]
		$\hat{\beta}$	(0.2458, 0.4985) 0.2527 [0.8905]	(0.2586, 0.4960) 0.2374 [0.8486]	(0.2584, 0.4919) 0.2335 [0.7990]	(0.2384, 0.5144) 0.2760 [0.8772]	(0.2550, 0.5353) 0.2803 [0.8138]	(0.2579, 0.5193) 0.2614 [0.8099]
		$\hat{c}$	(0.8699, 1.5923) 0.7223 [0.9487]	(0.8708, 1.5899) 0.7192 [0.9489]	(0.8674, 1.5899) 0.7224 [0.9621]	(0.7713, 1.6913) 0.9200 [0.9574]	(0.7721, 1.6832) 0.9111 [0.9409]	(0.7749, 1.7018) 0.9269 [0.9446]
		$\hat{k}$	(0.4064, 1.1628) 0.7564 [0.9142]	(0.4098, 1.1713) 0.7615 [0.9105]	(0.4124, 1.1942) 0.7818 [0.9323]	(0.3017, 1.2776) 0.9759 [0.8960]	(0.3077, 1.3066) 0.9988 [0.8936]	(0.3035, 1.2929) 0.9894 [0.9020]
60	**40**	$\hat{\alpha}$	(0.8407, 1.1365) 0.2958 [0.9448]	(0.8370, 1.1382) 0.3012 [0.9440]	(0.8318, 1.1458) 0.3140 [0.9331]	(0.8140, 1.1507) 0.3367 [0.9564]	(0.8130, 1.1671) 0.3541 [0.9479]	(0.8018, 1.1682) 0.3664 [0.9416]
		$\hat{\beta}$	(0.2527, 0.5022) 0.2495 [0.8708]	(0.2604, 0.4929) 0.2325 [0.7944]	(0.2655, 0.5040) 0.2384 [0.8062]	(0.2469, 0.5191) 0.2722 [0.8584]	(0.2550, 0.5221) 0.2671 [0.8340]	(0.2606, 0.5217) 0.2612 [0.8183]
		$\hat{c}$	(0.8643, 1.5799) 0.7156 [0.9448]	(0.8681, 1.5864) 0.7183 [0.9533]	(0.8649, 1.5839) 0.7190 [0.9510]	(0.7709, 1.6910) 0.9201 [0.9475]	(0.7770, 1.7029) 0.9258 [0.9617]	(0.7674, 1.6874) 0.9200 [0.9574]
		$\hat{k}$	(0.4057, 1.1464) 0.7407 [0.9142]	(0.4030, 1.1445) 0.7415 [0.9076]	(0.4023, 1.1367) 0.7343 [0.8861]	(0.2956, 1.2295) 0.9339 [0.8881]	(0.3003, 1.2586) 0.9583 [0.8919]	(0.2983, 1.2298) 0.9315 [0.8723]
60	**50**	$\hat{\alpha}$	(0.8392, 1.1240) 0.2848 [0.9420]	(0.8372, 1.1266) 0.2894 [0.9529]	(0.8292, 1.1291) 0.2999 [0.9389]	(0.8169, 1.1447) 0.3279 [0.9554]	(0.8160, 1.1537) 0.3377 [0.9460]	(0.8014, 1.1474) 0.3460 [0.9396]
		$\hat{\beta}$	(0.2602, 0.5056) 0.2454 [0.8338]	(0.2627, 0.5024) 0.2397 [0.8234]	(0.2614, 0.4998) 0.2384 [0.8067]	(0.2547, 0.5222) 0.2675 [0.8356]	(0.2597, 0.5244) 0.2647 [0.8106]	(0.2579, 0.5176) 0.2597 [0.8109]
		$\hat{c}$	(0.8610, 1.5755) 0.7145 [0.9577]	(0.8623, 1.5789) 0.7166 [0.9558]	(0.8572, 1.5716) 0.7143 [0.9517]	(0.7647, 1.6792) 0.9145 [0.9406]	(0.7671, 1.6845) 0.9174 [0.9588]	(0.7680, 1.6948) 0.9268 [0.9406]
		$\hat{k}$	(0.4000, 1.1172) 0.7172 [0.8820]	(0.3974, 1.1165) 0.7191 [0.8970]	(0.3978, 1.1107) 0.7129 [0.8738]	(0.2936, 1.1988) 0.9051 [0.8673]	(0.2949, 1.2060) 0.9110 [0.8773]	(0.2890, 1.1783) 0.8893 [0.8248]

6. Illustrative Example

In this section, the applicability of the proposed model in real situation has been highlighted based on the data of times to breakdown of a type of electrical insulating fluid between electrodes that was given by Nelson (1982). Using the Kolmogorov Smirnov (*K-S*) distance test statistic and its corresponding P-value, we can say that the FWBXII model fits this real data set well, such that *K-S*=0.1355 and P-value=0.9781, so we can see that the P -value corresponding to the K–S test statistics for the introduced distribution is greater than the significance level (0.05), So the FWBXII distribution is acceptable for modeling this data set.

Now, we generate the AT-II-PC samples considering n = 19, m = 10 and R= (3,0,0,0,3,0,0,0,3,0) and different values of T (5, 15) and censoring schemes from the original measurements as:

Case 1 for (T=5): (0.19, 0.78, 0.96, 1.31, 2.78, 3.16, 4.67, 4.85, 7.35, and 31.75).

Case 2 for (T=15): (0.19, 1.31, 2.78, 3.16, 4.15, 4.85, 6.50, 7.35, 8.27, and 33.91).

The ML and Bayesian estimates, under SE and LINEX loss functions, are reported in Table 18. In addition, the 95% approximate confidence intervals and credible intervals are provided in the same table. For Bayesian estimation, it is assumed that we have an informative prior with means and variances of (α, β, c, k) equal to (0.01, 3, 1, 0.1) and (0.0001, 0.1, 0.1, 0.001), respectively. These values were used in calculating the hyperparameters. Also, the trace and their corresponding autocorrelation plots of the illustrative example in the case of T=5 are shown in Figures 4 and 5, respectively, and suggest probable convergence.

Table 18: Point and interval ML and Bayes estimates of the illustrative example data.

T	**Parameter**	***Point Est.***				***Interval Est.***	
		MLE	***BS***	***BL (λ=0.2)***	***BL(λ=2)***	***Confidence intervals***	***Credible Intervals***
5	α	0.0066	0.0072	0.0072	0.0072	(0.0000,0.0282)	(0.0003,0.0213)
	β	3.1107	3.0181	3.0088	2.9292	(0.3221,5.8993)	(2.4505,3.6654)
	c	0.8918	0.9770	0.9698	0.9105	(0.0000,2.1851)	(0.5121,1.5601)
	k	0.1076	0.0997	0.0996	0.0989	(0.0000,0.3092)	(0.0507,0.1602)
15	α	0.0326	0.0153	0.0153	0.0152	(0.0000,0.0663)	(0.0008,0.0395)
	β	7.8040	3.1534	3.1436	3.0588	(0.0000,15.7219)	(2.5713,3.8082)
	c	0.9834	0.9553	0.9479	0.8871	(0.0000,2.3567)	(0.4867,1.5570)
	k	0.1098	0.0977	0.0977	0.0969	(0.0000,0.2930)	(0.0490,0.1605)

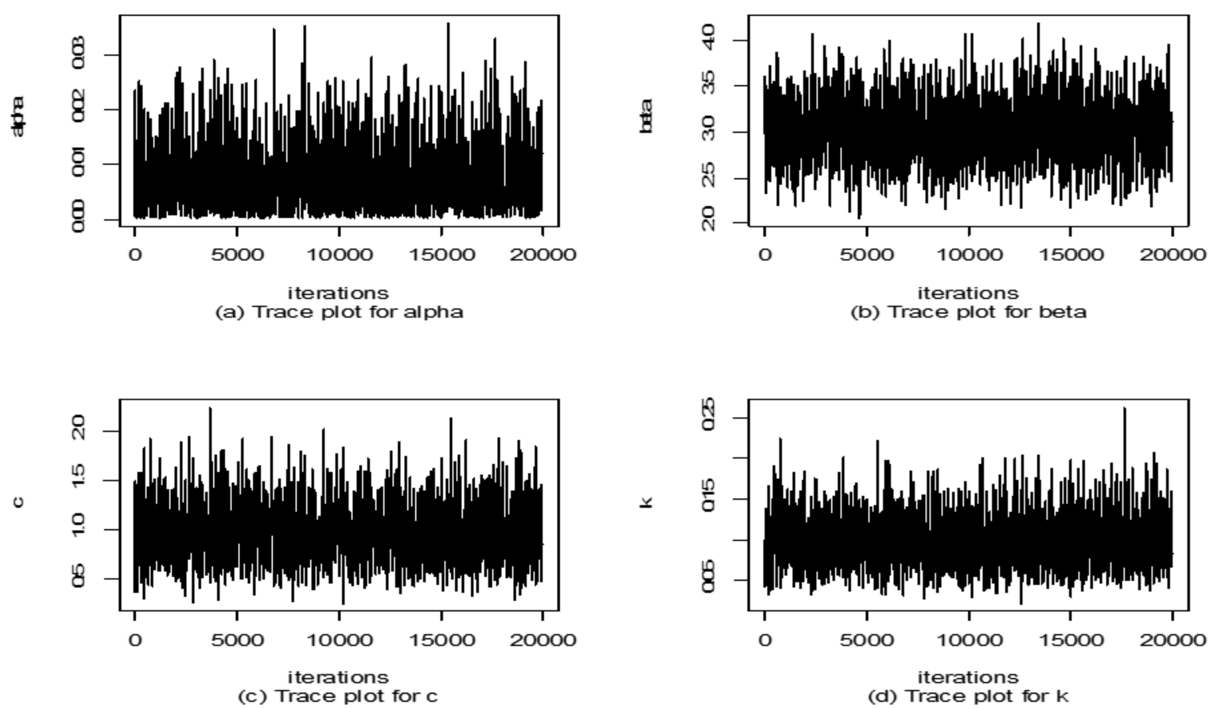

Figure 4: Trace plots of the illustrative example in the case of T=5.

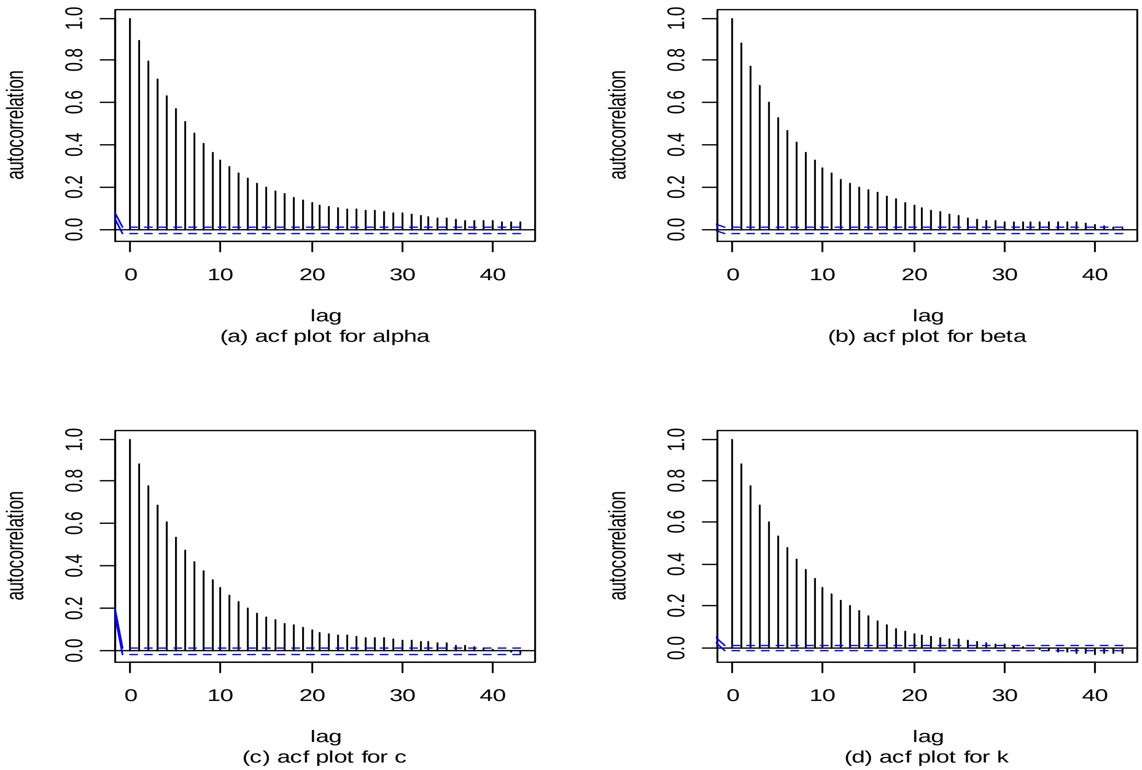

Figure 5: Autocorrelation plots of the illustrative example in the case of T=5.

7. Conclusion

In this paper, frequentist and Bayesian point and interval estimation methods were developed for the parameters of the FWBXII distribution under an adaptive Type-II progressive censoring scheme. In the Bayesian context, to estimate the unknown parameters, informative priors under SELF and LINEX were assumed. As the resultant Bayes estimators and the corresponding posterior risks cannot be derived in a closed form, the AM algorithm has been implemented. The performance of the different estimators was examined via a detailed Monte Carlo simulation study depending on different sample sizes, test termination times, parameter values, and censoring schemes. Based on the simulation results, we can conclude that the efficiency of the model increases when the effective sample size increases. In addition, the results proved that, the Bayes estimates under LINEX loss function with small shape parameter behave quite close to the estimates under the squared error loss function. Moreover, the efficiency of the Bayes estimates under LINEX loss function is better for the smaller value of the shape parameter of the loss function. Also, a real data set was used to show the applicability of the schemes in practice.

References

1. AL Sobhi, M.M. and Soliman, A.A. (2015). Estimation for the exponentiated Weibull model with adaptive Type-II progressive censored schemes. Applied Mathematical Modelling, 40, 1180–1192.
2. Amein, M.M. (2017). Estimation for Unknown Parameters of the Burr Type-XII Distribution Based on an Adaptive Progressive Type-II Censoring Scheme. Global Journal of Pure and Applied Mathematics, 13, 7709-7723.
3. Balakrishnan, N. and Cramer, E. (2014). The Art of Progressive Censoring: Applications to Reliability and Quality. Birkh$\acute{a}$user. New York.
4. EL-Sagheer, R.M., Mahmoud, M.A.W and Nagaty, H. (2019). Statistical Inference for Weibull-Exponential Distribution Using Adaptive Type-II Progressive Censoring. Journal of Statistics Applications & Probability, 2, 1-13.
5. Haario, H., Saksman, E. and Tamminen, J. (2001). An Adaptive Metropolis Algorithm. Bernoulli, 7, 223-242.

6. Hastings, W. K. (1970). Monte Carlo sampling methods using Markov chains and their applications. Biometrika, 57, 97–109.
7. Hemmati, F. and Khorram, E. (2013). Statistical Analysis of the Log-Normal Distribution under Type-II Progressive Hybrid Censoring Schemes. Communications in Statistics-Theory and Methods, 42, 52–75.
8. Kamal, R.M. and Ismail, M.A. (2020). The Flexible Weibull Extension-Burr XII Distribution: Model, Properties and Applications. Pakistan Journal of Statistics and Operation Research, 16, 447–460.
9. Lawless, J. F. (2003). Statistical Models and Methods for Lifetime Data. John Wiley and Sons. New York.
10. Lin, C. T. Ng, H. K. T. and Chan, P. S. (2009). Statistical inference of type-II progressively hybrid censored data with Weibull lifetimes. Communications in Statistics-Theory and Methods, 38, 1710-1729.
11. Mahmoud, M. A. W., Soliman, A. A., Abd Ellah, A. H. and El-Sagheer, R.M. (2013). Estimation of generalized Pareto under an adaptive type-II progressive censoring. Intelligent Information Management, 5, 73-83.
12. Metropolis, N., Rosenbluth, A., Rosenbluth, M., Teller, A., and Teller, E. (1953). Equations of state calculations by fast computing machines. Journal of Chemical Physics, 21, 1087–1092.
13. Mohie El-Din, M. M. M., Amein, M. M. and Shafay, A. R. and Mohamed, S. (2017). Estimation of generalized exponential distribution based on an adaptive progressively type-II censored sample. Journal of Statistical Computation and Simulation, 87, 1292-1304.
14. Nelson. W. (1982). Applied Life Data Analysis. John Wiley & Sons. New York. NY.
15. Ng, H.K.T., Kundu, D., and Chan, P.S. (2009). Statistical Analysis of Exponential Lifetimes under an Adaptive Type-II Progressive Censoring Scheme. Naval Research Logistics, 56, 687–698.

Appendix

Fisher Information Matrix

The likelihood function of $\underline{\theta}= (\alpha, \beta, c, k)$ based on the FWBXII distribution is given by Equation (6). By differentiating the log-likelihood function $\mathcal{L}$ with respect to the parameters, we obtain the first order derivatives of $\mathcal{L}$ as given in Equations (7)-(10). Upon differentiating these expressions once again with respect to the parameters, we obtain the partial derivatives of second order as follows:

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} = \sum_{i=1}^{m}\left(\frac{h(x_i;\underline{\theta})\, h_{\alpha\alpha}(x_i;\underline{\theta}) - \left(h_\alpha(x_i;\underline{\theta})\right)^2}{\left(h(x_i;\underline{\theta})\right)^2}\right) - \sum_{i=1}^{m} e_1\,(x_i;\underline{\theta}) - \sum_{i=1}^{j} R_i\, e_1(x_i;\underline{\theta}) - \left(n - m - \sum_{i=1}^{j} R_i\right) e_1(x_m;\underline{\theta}),$$

$$\frac{\partial^2 \mathcal{L}}{\partial \beta^2} = \sum_{i=1}^{m}\left(\frac{h(x_i;\underline{\theta})\, h_{\beta\beta}(x_i;\underline{\theta}) - \left(h_\beta(x_i;\underline{\theta})\right)^2}{\left(h(x_i;\underline{\theta})\right)^2}\right) + \sum_{i=1}^{m} e_2(x_i;\underline{\theta}) + \sum_{i=1}^{j} R_i\, e_2(x_i;\underline{\theta}) + \left(n - m - \sum_{i=1}^{j} R_i\right) e_2(x_m;\underline{\theta}),$$

$$\frac{\partial^2 \mathcal{L}}{\partial c^2} = \sum_{i=1}^{m}\left(\frac{h(x_i;\underline{\theta})\, h_{cc}(x_i;\underline{\theta}) - \left(h_c(x_i;\underline{\theta})\right)^2}{\left(h(x_i;\underline{\theta})\right)^2}\right) - \sum_{i=1}^{m} e_3(x_i;\underline{\theta}) - \sum_{i=1}^{j} R_i\, e_3(x_i;\underline{\theta}) - \left(n - m - \sum_{i=1}^{j} R_i\right) e_3(x_m;\underline{\theta}),$$

$$\frac{\partial^2 \mathcal{L}}{\partial k^2} = -\sum_{i=1}^{m}\left(\frac{\left(h_k(x_i;\underline{\theta})\right)^2}{\left(h(x_i;\underline{\theta})\right)^2}\right),$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta}=\sum_{i=1}^{m}\left(\frac{h(x_i;\underline{\theta})\,h_{\alpha\beta}(x_i;\underline{\theta})-h_{\alpha}(x_i;\underline{\theta})h_{\beta}(x_i;\underline{\theta})}{\left(h(x_i;\underline{\theta})\right)^2}\right)-\sum_{i=1}^{m}e_4\,(x_i;\underline{\theta})-\sum_{i=1}^{j}R_i\,e_4(x_i;\underline{\theta})$$
$$-\left(n-m-\sum_{i=1}^{j}R_i\right)e_4(x_m;\underline{\theta}),$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial c}=-\sum_{i=1}^{m}\left(\frac{h_{\alpha}(x_i;\underline{\theta})h_c(x_i;\underline{\theta})}{\left(h(x_i;\underline{\theta})\right)^2}\right),$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial k}=-\sum_{i=1}^{m}\left(\frac{h_{\alpha}(x_i;\underline{\theta})h_k(x_i;\underline{\theta})}{\left(h(x_i;\underline{\theta})\right)^2}\right),$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta\partial c}=-\sum_{i=1}^{m}\left(\frac{h_{\beta}(x_i;\underline{\theta})h_c(x_i;\underline{\theta})}{\left(h(x_i;\underline{\theta})\right)^2}\right),$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta\partial k}=-\sum_{i=1}^{m}\left(\frac{h_{\beta}(x_i;\underline{\theta})h_k(x_i;\underline{\theta})}{\left(h(x_i;\underline{\theta})\right)^2}\right),$$

$$\frac{\partial^2\mathcal{L}}{\partial c\partial k}=\sum_{i=1}^{m}\left(\frac{h(x_i;\underline{\theta})\,h_{ck}(x_i;\underline{\theta})-h_{c}(x_i;\underline{\theta})h_{k}(x_i;\underline{\theta})}{\left(h(x_i;\underline{\theta})\right)^2}\right)-\sum_{i=1}^{m}e_5(x_i;\underline{\theta})-\sum_{i=1}^{j}R_i\,e_5(x_i;\underline{\theta})$$
$$-\left(n-m-\sum_{i=1}^{j}R_i\right)e_5(x_m;\underline{\theta}),$$

where

$$h_{\alpha\alpha}(x_i;\underline{\theta})=\frac{\partial^2 h(x_i;\underline{\theta})}{\partial\alpha^2}=\left(\alpha+\frac{\beta}{{x_i}^2}\right)\,e^{\alpha x_i-\frac{\beta}{x_i}}\;{x_i}^2+2x_i e^{\alpha x_i-\frac{\beta}{x_i}},$$

$$h_{\beta\beta}(x_i;\underline{\theta})=\frac{\partial^2 h(x_i;\underline{\theta})}{\partial\beta^2}=-\frac{2}{{x_i}^3}e^{\alpha x_i-\frac{\beta}{x_i}}+\frac{1}{{x_i}^2}\left(\alpha+\frac{\beta}{{x_i}^2}\right)e^{\alpha x_i-\frac{\beta}{x_i}},$$

$$h_{cc}(x_i;\underline{\theta})=\frac{\partial^2 h(x_i;\underline{\theta})}{\partial c^2}$$
$$=-ck\,{x_i}^{c-1}[1+{x_i}^c]^{-1}\left(\frac{1}{c^2}+\frac{(1+{x_i}^c){x_i}^c(\ln(x_i))^2-{x_i}^{2c}(\ln(x_i))^2}{(1+{x_i}^c)^2}\right)$$
$$-k\,{x_i}^{c-1}[1+{x_i}^c]^{-1}\left(\frac{1}{c}+\ln(x_i)-\frac{{x_i}^c\ln(x_i)}{(1+{x_i}^c)}\right)(c{x_i}^c[1+{x_i}^c]^{-1}\ln(x_i)-c\ln(x_i)-1),$$

$$h_{\alpha\beta}(x_i;\underline{\theta})=\frac{\partial^2 h(x_i;\underline{\theta})}{\partial\alpha\partial\beta}=-\left(\alpha+\frac{\beta}{{x_i}^2}\right)\,e^{\alpha x_i-\frac{\beta}{x_i}},$$

$$h_{ck}(x_i;\underline{\theta})=\frac{\partial^2 h(x_i;\underline{\theta})}{\partial c\partial k}=c\,{x_i}^{c-1}[1+{x_i}^c]^{-1}\left(\frac{1}{c}+\ln(x_i)-\frac{{x_i}^c\ln(x_i)}{(1+{x_i}^c)}\right),$$

$$e_1(x_i;\underline{\theta})=\frac{\partial a_1(x_i;\underline{\theta})}{\partial\alpha}={x_i}^2 e^{\alpha x_i-\frac{\beta}{x_i}},\;e_2(x_i;\underline{\theta})=\frac{\partial a_2(x_i;\underline{\theta})}{\partial\beta}=-\frac{1}{{x_i}^2}e^{\alpha x_i-\frac{\beta}{x_i}},$$

$$e_3(x_i;\underline{\theta})=\frac{\partial b_1(x_i;\underline{\theta})}{\partial c}=k\ln(x_i)\frac{{x_i}^c\ln(x_i)\,(1+{x_i}^c)-{x_i}^{2c}\ln(x_i)}{(1+{x_i}^c)^2},$$

$$e_4(x_i;\underline{\theta}) = \frac{\partial a_1(x_i;\underline{\theta})}{\partial \beta} = -e^{\alpha x_i - \frac{\beta}{x_i}},\ e_5(x_i;\underline{\theta}) = \frac{\partial b_1(x_i;\underline{\theta})}{\partial k} = \frac{x_i{}^c \ln(x_i)}{(1+x_i{}^c)}.$$