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The Topp Leone-G Power Series Class of Distributions with Applications

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Abstract

We present a new class of distributions called the Topp-Leone-G Power Series (TL-GPS) class of distributions. This model is obtained by compounding the Topp-Leone-G distribution with the power series distribution. Statistical properties of the TL-GPS class of distributions are obtained. Maximum likelihood estimates for the proposed model were obtained. A simulation study is carried out for the special case of Topp-Leone Log-Logistic Poisson distribution to assess the performance of the maximum likelihood estimates. Finally, we apply Topp-Leone-log-logistic Poisson distribution to real data sets to illustrate the usefulness and applicability of the proposed class of distributions.

Key Words: Topp-Leone-G Distribution; Power Series Distribution; Maximum Likelihood Estimation.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

Statistical distributions are widely used to explain different types of real life events. Because of their usefulness, statistical distribution theory is extensively researched and new distributions are being developed. The Topp Leone (TL) distribution is among the distributions used within the theory and practice of statistics. It was proposed by Topp and Leone (1955) as a lifetime model. Nadarajah and Kotz (2003) studied its properties and provided its moments and the characteristic function. Numerous authors also studied the TL distribution. Ghitany et al. (2005) provided some reliability measures of the TL distribution, while Vicaria et al. (2008) introduced a two-sided generalized version of the TL distribution and Al-Zahrani (2012) derived the goodness-of-fit test for the TL distribution.

Al-Shomrani et al. (2016) proposed the Topp-Leone generated family of distributions with cumulative distribution function (cdf), probability density function (pdf), and survival function given by

$$F_{TL-G}(x;b,\psi) = [1-\bar{G}(x;\psi)^2]^b, \tag{1}$$

$$f_{TL-G}(x;b,\psi) = 2bg(x;\psi)\bar{G}(x;\psi)[1-\bar{G}(x;\psi)^2]^{b-1}, \tag{2}$$

and

$$S_{TL-G}(x;b,\psi) = 1 - F_{TLG}(x;b,\psi) = 1 - [1-\bar{G}(x;\psi)^2]^b, \tag{3}$$

respectively, for b > 0, where $G(x;\psi)$ is the baseline cdf depending on a parameter vector ψ, $g(x;\psi) = dG(x;\psi)/dx$, and $\bar{G}(x;\psi) = 1 - G(x;\psi)$ is the survival function.

Some generalizations of the Topp-Leone-G family of distributions include the Topp-Leone-Marshall-Olkin-G family by Chipepa et al. (2020), Type II power Topp-Leone generated family by Bantan et al. (2020), Topp-Leone-Weibull by Rezaei et al. (2016), Topp-Leone generalized exponential Sangsanit and Bodhisuwan (2016).

In this paper, we develop a new class of distributions called the Topp-Leone-G Power Series (TL-GPS) class of distributions. We are motivated by the flexibility in data fitting obtained from the TL-GPS class of distributions and the applicability of the new class of distributions to data sets that exhibit monotonic or non-monotonic hazard rate shapes. Another motivation for developing the TL-GPS class of distributions is the applicability of the power series distributions in different fields such as finance, economics and actuarial sciences.

This paper is organized as follows. In Section 2 we introduce the new class of distributions and present its cdf and pdf. We also discuss some sub-classes of the TL-GPS distribution and present some special cases when the baseline cdf is specified. Some statistical properties of the TL-GPS distribution including moments, conditional moments, order statistics, and Rényi entropy are presented in Section 3. Maximum likelihood estimates of the unknown parameters are presented in Section 4. Monte Carlo simulations for special cases are conducted in Section 5. Applications are given in Section 6, followed by some concluding remarks.

2. The Model

In this section, we develop the TL-GPS class of distributions and derive some statistical properties which include series expansion of the pdf, quantile and hazard functions, sub-classes and some special cases.

Suppose that the random variable X has the Topp-Leone-G distribution with cdf defined by equation (1). Given N, let $X_1,...,X_N$ be independent and identically distributed random variables from the Topp-Leone-G distribution. Let N be a discrete random variable with a power series distribution (truncated at zero) and probability mass function (pmf)

$$P(N=n) = \frac{a_n\theta^n}{C(\theta)}, \quad n = 1,2,...,$$

where $a_n \geq 0$ depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n\theta^n$ and $\theta \in (0,s)$ (s can be ∞) is chosen such that $C(\theta)$ is finite and its three derivatives with respect to θ are defined and given by $C'(.), C''(.)$ and $C'''(.)$, respectively. The power series family of distributions includes Binomial, Poisson, Geometric and Logarithmic distributions. See Johnson et al. (1994) for additional details. Let X=min$(X_1,...,X_N)$, then the conditional cdf of $X|N=n$ is given by

$$F_{X|N=n}(x) = 1 - [S_{TL-G}(x;b,\psi)]^n, \quad x > 0. \tag{4}$$

The Topp-Leone-G Power Series class of distributions is defined by the marginal cdf of X. The general form of the cdf and pdf of the Topp-Leone-G Power Series class of distributions are given by

$$F_{TL-GPS}(x;\theta,b,\psi) = 1 - \frac{C(\theta S_{TL-G}(x;b,\psi))}{C(\theta)} \tag{5}$$

and

$$f_{TL-GPS}(x;\theta,b,\psi) = \frac{\theta f_{TL-G}(x;b,\psi)C'(\theta S_{TL-G}(x;b,\psi))}{C(\theta)}, \tag{6}$$

respectively.
On the other hand, if we consider $X_{(n)} = \max(X_1,\dots,X_N)$ and conditioning upon $N = n$, then the conditional distribution of $X_{(n)}$ given $N = n$ is obtained as

$$G_{X_{(n)}|N=n}(x) = [1-\bar{G}(x;\psi)^2]^{nb},$$

which is also a Topp-Leone-G distribution. The marginal cdf of $X_{(n)}$, say F_{TL-GPS}, is given by

$$F_{TL-GPS}(x;\theta,b,\psi) = \frac{C(\theta F_{TL-G}(x;b,\psi))}{C(\theta)}.$$

The hazard rate function (hrf) is given by

$$h_{TL-GPS}(x;\theta,b,\psi) = \theta f_{TL-G}(x;b,\psi)\frac{C'(\theta S_{TL-G}(x;b,\psi))}{C(\theta S_{TL-G}(x;b,\psi))}. \quad (7)$$

Similarly, the reverse hazard rate function (rhrf) becomes

$$\tau_{TL-GPS}(x;\theta,b,\psi) = \frac{\theta f_{TL-G}(x;b,\psi)C'(\theta S_{TL-G}(x;b,\psi))}{C(\theta)-C(\theta S_{TL-G}(x;b,\psi))}. \quad (8)$$

2.1. Quantile function

The quantile function of the TL-GPS class of distributions is easily obtained by inverting equation (5), $F_{TL-GPS}(x;\theta,b,\psi)$ = u, $0 \le u \le 1$. Note that

$$1-\frac{C(\theta S_{TL-G}(x;b,\psi))}{C(\theta)} = u,$$

so that

$$C(\theta S_{TL-G}(x;b,\psi)) = C(\theta)(1-u).$$

This is equivalent to

$$C^{-1}(C(\theta)(1-u)) = \theta S_{TL-G}(x;b,\psi),$$

Therefore, we obtain the quantile values from the TL-GPS class of distributions by solving the non-linear equation

$$C^{-1}(C(\theta)(1-u)) - \theta S_{TL-G}(x;b,\psi) = 0, \quad (9)$$

using iterative methods in R, SAS or MATLAB software.

2.2. Expansion of the Density Function

Expansion of the density function of the TL-GPS class of distributions is presented in this sub-section. Equation (6) can be rewritten as

$$f_{TL-GPS}(x;\theta,b,\psi) = \sum_{n=1}^{\infty}\frac{na_n\theta^n}{C(\theta)}2bg(x;\psi)\bar{G}(x;\psi)[1-\bar{G}(x;\psi)^2]^{b-1}[1-(1-\bar{G}(x;\psi)^2)^b]^{n-1}.$$

Using the generalized binomial expansion

$$[1-(1-\bar{G}(x;\psi)^2)^b]^{n-1} = \sum_{i=0}^{\infty}(-1)^i\binom{n-1}{i}[1-\bar{G}(x;\psi)^2]^{bi},$$

the pdf of the TL-GPS class of distribution is given by

$$f_{TL-GPS}(x;\theta,b,\psi) = \sum_{i=0}^{\infty}\sum_{n=1}^{\infty}(-1)^i\binom{n-1}{1}\frac{na_n\theta^n}{C(\theta)}2b \times g(x;\psi)\bar{G}(x;\psi)[1-\bar{G}(x;\psi)^2]^{b(i+1)-1}.$$

Also, applying the generalized binomial expansion

$$[1-\bar{G}(x;\psi)^2]^{b(i+1)-1}=\sum_{j=0}^{\infty}(-1)^j\binom{b(i+1)-1}{j}\bar{G}(x;\psi)^{2j}$$

we get

$$f_{TL-GPS}(x;\theta,b,\psi)=\sum_{i,j=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{i+j}\binom{n-1}{i}\binom{b(i+1)-1}{j}\frac{na_n\theta^n}{C(\theta)}2b \times g(x;\psi)\bar{G}(x;\psi)^{2j+1}.$$

Furthermore, using the binomial expansion

$$\bar{G}(x;\psi)^{2j+1}=[1-G(x;\psi)]^{2j+1}=\sum_{k=0}^{\infty}(-1)^k\binom{2j+1}{k}G(x;\psi)^k$$

yields

$$\begin{aligned} f_{TL-GPS}(x;\theta,b,\psi) &= \sum_{i,j,k=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{i+j+k}\binom{n-1}{i}\binom{b(i+1)-1}{j}\binom{2j+1}{k}\frac{na_n\theta^n}{C(\theta)}2b \\ &\times g(x;\psi)G(x;\psi)^k \\ &= \sum_{i,j,k=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{i+j+k}\binom{n-1}{i}\binom{b(i+1)-1}{j}\binom{2j+1}{k}\frac{na_n\theta^n}{C(\theta)}2b \\ &\times \left(\frac{k+1}{k+1}\right)g(x;\psi)G(x;\psi)^k \\ &= \sum_{k=0}^{\infty}\eta_{k+1}g_{k+1}(x;\psi), \end{aligned} \tag{10}$$

where

$$g_{k+1}(x;\psi)=(k+1)g(x;\psi)G(x;\psi)^k$$

is the exponentiated-G (Exp-G) distribution with power parameter $(k+1)$, and

$$\eta_{k+1}=\sum_{i,j=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{i+j+k}\binom{n-1}{i}\binom{b(i+1)-1}{j}\binom{2j+1}{k}\frac{na_n\theta^n}{C(\theta)}\frac{2b}{k+1}. \tag{11}$$

It follows that the TL-GPS distribution can be expressed as an infinite linear combination of Exp-G densities.

2.3. Sub-classes of the TL-GPS Distribution

We derive expressions for cdfs of sub-classes of the TL-GPS class of distributions and these are presented in Table 1.

2.4. Some Special Cases of the TL-GPS Class of Distributions

In this section, we present some special cases of the TL-GPS class of distributions. We consider cases when the baseline distribution are Weibull and log-logistic distributions.

2.4.1. Topp-Leone-Weibull-Poisson Distribution

The cdf and pdf of the Topp-Leone-Weibull Poisson (TL-WP) distribution are given by

$$F_{TL-WP}(x;\theta,b,\alpha,\beta)=1-\frac{e^{\theta(1-(1-e^{-2\alpha x^{\beta}})^b)}-1}{e^{\theta}-1}$$

Table 1: Sub-Classes of the TL-GPS Distribution

Distribution	a_n	$C(\theta)$	cdf
Topp-Leone G Poisson	$(n!)^{-1}$	$e^{\theta}-1$	$1 - \frac{e^{\theta\left(1-[1-\bar{G}(x;\psi)^2]^b\right)}-1}{e^{\theta}-1}$
Topp-Leone G Geometric	1	$\theta(1-\theta)^{-1}$	$1 - \frac{\left(1-[1-\bar{G}(x;\psi)^2]^b\right)(1-\theta)}{1-\theta\left(1-[1-\bar{G}(x;\psi)^2]^b\right)}$
Topp-Leone G Logarithmic	n^{-1}	$-\log(1-\theta)$	$1 - \frac{\log\left(1-\theta\left(1-[1-\bar{G}(x;\psi)^2]^b\right)\right)}{\log(1-\theta)}$
Topp-Leone G Binomial	$\binom{m}{n}$	$(1+\theta)^m-1$	$1 - \frac{\left(1+\theta\left(1-[1-\bar{G}(x;\psi)^2]^b\right)\right)^m-1}{(1+\theta)^m-1}$

and

$$f_{TL-WP}(x;\theta,b,\alpha,\beta)=\frac{2b\theta\alpha\beta x^{\beta-1}e^{-2\alpha x^{\beta}}(1-e^{-2\alpha x^{\beta}})^{b-1}e^{\theta(1-(1-e^{-2\alpha x^{\beta}})^{b})}}{e^{\theta}-1},$$

respectively, for $\alpha,\beta,b,\theta>0$ and $x>0$. The hrf and rhrf are given by

$$h_{TL-WP}(x;\theta,b,\alpha,\beta)=2b\theta\alpha\beta x^{\beta-1}e^{-2\alpha x^{\beta}}(1-e^{-2\alpha x^{\beta}})^{b-1}\frac{e^{\theta(1-(1-e^{-2\alpha\beta x^{\beta}})^{b})}}{e^{\theta(1-(1-e^{-2\alpha x^{\beta}})^{b})}-1}$$

and

$$\tau_{TL-WP}(x;\theta,b,\alpha,\beta)=2b\theta\alpha x^{\beta-1}e^{-2\alpha x^{\beta}}(1-e^{-2\alpha x^{\beta}})^{b-1}\frac{e^{\theta(1-(1-e^{-2\alpha x^{\beta}}))}}{e^{\theta}-e^{\theta(1-(1-e^{-2\alpha x^{\beta}}))}},$$

respectively. Figure 1 shows the plots of the pdfs and hrfs for the TL-WP distribution for selected parameter values.

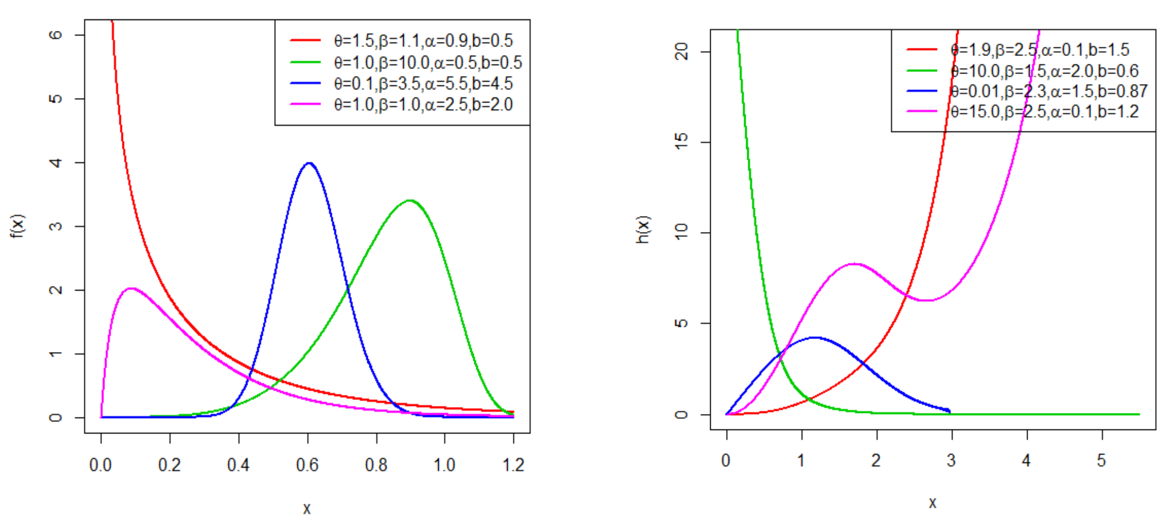

Figure 1: Pdfs and hrfs plots for the TL-WP distribution

Plots of the TL-WP pdf exhibit different shapes including almost symmetric, left-skewed, right-skewed, and reverse-J shapes. Plots of the hrf of the TL-WP distribution shows different shapes including increasing, decreasing, upside-down bathtub followed by bathtub and uni-modal shapes.

The quantile function of the TL-WP distribution is obtained by solving the non-linear equation

$$\log[(e^{\theta}-1)(1-u)+1]-\theta(1-[1-e^{-2\alpha x^{\beta}}]^{b})=0. \tag{12}$$

As such, random numbers can be generated from the TL-WP distribution by numerically solving the non-linear equation (12). Quantile values of the TL-WP distribution are given in Table 2.

Table 2: Table of Quantiles for TL-WP Distribution

	$(\theta,\alpha,\beta,$b)				
u	(0.8,2.1,0.9,1.0)	(1.5,1.2,1.8,2.1)	(3.5,1.2,5.5,0.6)	(0.5,1.0,0.3,0.1)	(2.0,3.0,0.4,1.5)
0.1	0.0112	0.3069	0.7432	0.2066	0.0011
0.2	0.0261	0.3877	0.7814	0.2806	0.0041
0.3	0.0446	0.4529	0.8082	0.3412	0.0100
0.4	0.0971	0.5751	0.8520	0.4552	0.0379
0.6	0.1357	0.6417	0.8732	0.5170	0.0688
0.7	0.1891	0.7193	0.8964	0.5885	0.1264
0.8	0.2704	0.8196	0.9245	0.6804	0.2494
0.9	0.4226	0.9764	0.9666	0.8245	0.6031

2.4.2. Topp-Leone-Weibull-Binomial Distribution

The cdf and pdf of the Topp-Leone-Weibull Binomial (TL-WB) distribution are given by

$$F_{TL-WB}(x;\theta,b,\alpha,\beta,m)=1-\frac{(1+\theta(1-[1-e^{-2\alpha x^{\beta}}]^{b}))^{m}-1}{(1+\theta)^{m}-1}$$

and

$$f_{TL-WB}(x;\theta,b,\alpha,\beta,m)=2b\theta\alpha\beta x^{\beta-1}e^{-2\alpha x^{\beta}}(1-e^{-2\alpha x^{\beta}})^{b-1}\times\frac{m(1+\theta(1-[1-e^{-2\alpha x^{\beta}}]^{b}))^{m-1}}{(1+\theta)^{m}-1},$$

respectively for $\alpha,\beta,b,\theta>0$ and $x>0$. The hrf and rhrf are given by

$$h_{TL-WB}(x;\theta,b,\alpha,\beta,m)=2b\theta\alpha\beta x^{\beta-1}e^{-2\alpha x^{\beta}}(1-e^{-2\alpha x^{\beta}})^{b-1}\times\frac{m(1+\theta(1-[1-e^{-2\alpha x^{\beta}}]^{b}))^{m-1}}{(1+\theta(1-[1-e^{-2\alpha x^{\beta}}]^{b}))^{m}-1}$$

and

$$\tau_{TL-WB}(x;\theta,b,\alpha,\beta,m)=2b\theta\alpha\beta x^{\beta-1}e^{-2\alpha x^{\beta}}(1-e^{-2\alpha x^{\beta}})^{b-1}\times\frac{m(1+\theta(1-[1-e^{-2\alpha x^{\beta}}]^{b}))^{m-1}}{(1+\theta)^{m}-(1+\theta(1-[1-e^{-2\alpha x^{\beta}}]^{b}))^{m}},$$

respectively. Figure 2 shows the plots of the pdfs and hrfs for the TL-WB distribution for selected parameters values. Plots of the TL-WB pdf exhibit different shapes including symmetric, skewed to the right, skewed to the left and reverse-J shapes. Plots of the hrf of the TL-WB distribution shows different shapes including increasing, decreasing, bathtub and uni-modal shapes.

The quantile function of the TL-WB distribution can be obtained by solving the non-linear equation

$$[((1+\theta)^{m}-1)(1-u)+1]^{\frac{1}{m}}-1-\theta(1-[1-e^{-2\alpha x^{\beta}}]^{b})=0. \tag{13}$$

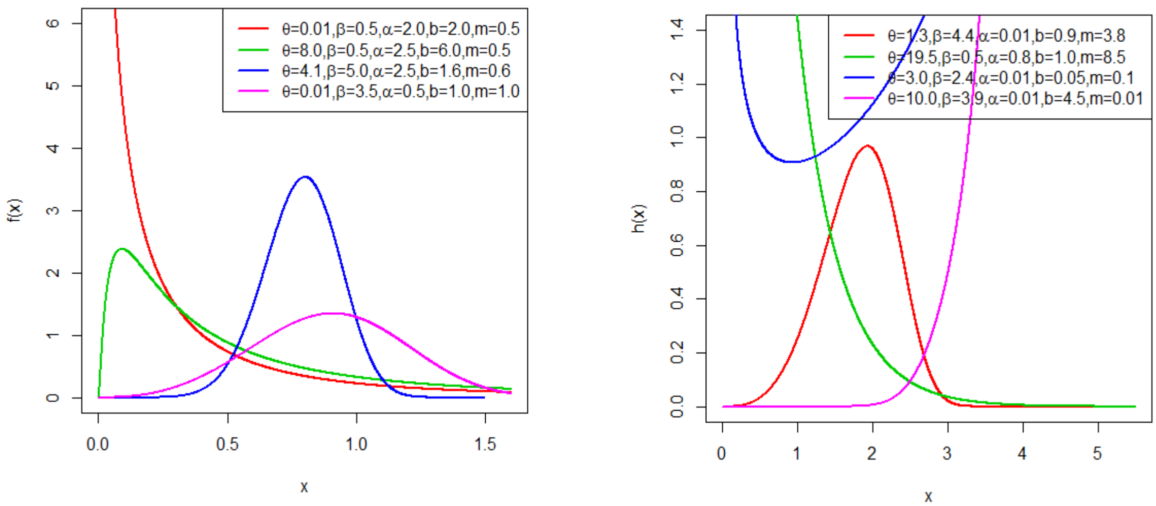

Figure 2: Pdfs and hrfs plots for the TL-WB distribution

As such, random numbers can be generated from the TL-WB power series distribution by numerically solving the non-linear equation (13). Quantile values of the TL-WB distribution are given in Table 3.

Table 3: Table of Quantiles for TL-WB Distribution

	(θ,α,β,b,m)				
u	(0.8,2.1,0.9,1.0,1.5)	(1.2,1.8,2.1,3.5,1.2)	(5.5,0.6,0.5,1.0,0.3)	(0.1,1.2,1.8,0.4,0.7)	(1.5,0.9,0.5,0.3,1.1)
0.1	0.0167	0.5307	0.0219	0.0069	0.0000
0.2	0.0382	0.5999	0.0819	0.0230	0.0001
0.3	0.0639	0.6504	0.1755	0.0500	0.0001
0.4	0.0947	0.6938	0.3022	0.0925	0.0004
0.5	0.1323	0.7346	0.4653	0.1586	0.0019
0.6	0.1795	0.7756	0.6730	0.2635	0.0082
0.7	0.2420	0.8197	0.9435	0.4393	0.0329
0.8	0.3323	0.8714	1.3200	0.7695	0.1350
0.9	0.4908	0.9435	1.9431	1.5783	0.7004

2.4.3. Topp-Leone-Log-Logistic-Poisson Distribution

The cdf and pdf of the Topp-Leone-Log-Logistic-Poisson (TL-LLP) distribution are given by

$$F_{TL-LLP}(x;\theta,b,c) = 1 - \frac{e^{\theta(1-[1-(1+x^c)^{-2}]^b)} - 1}{e^{\theta} - 1}$$

and

$$f_{TL-LLP}(x;\theta,b,c) = \frac{2\theta bcx^{c-1}(1+x^c)^{-3}(1-(1+x^c)^{-2})^{b-1}e^{\theta(1-[1-(1+x^c)^{-2}]^b)}}{e^{\theta}-1},$$

respectively for $\theta,b,c > 0$ and $x > 0$. The hrf and rhrf are given by

$$h_{TL-LLP}(x;\theta,b,c) = 2\theta bcx^{c-1}(1+x^c)^{-3}(1-(1+x^c)^{-2})^{b-1}\frac{e^{\theta(1-[1-(1+x^c)^{-2}]^b)}}{e^{\theta(1-[1-(1+x^c)^{-2}]^b)}-1}$$

and

$$\tau_{TL-LLP}(x;\theta,b,c) = \frac{2\theta bcx^{c-1}(1+x^c)^{-3}(1-(1+x^c)^{-2})^{b-1}e^{\theta(1-[1-(1+x^c)^{-2}]^b)}}{e^{\theta}-e^{\theta(1-[1-(1+x^c)^{-2}]^b)}},$$

respectively. Figure 3 shows the plots of the hrf for the TL-LLP distribution for selected parameters values. The plots shows different shapes including reverse-J, decreasing, bathtub followed by an upside-down bathtub and upside-down bathtub shapes.

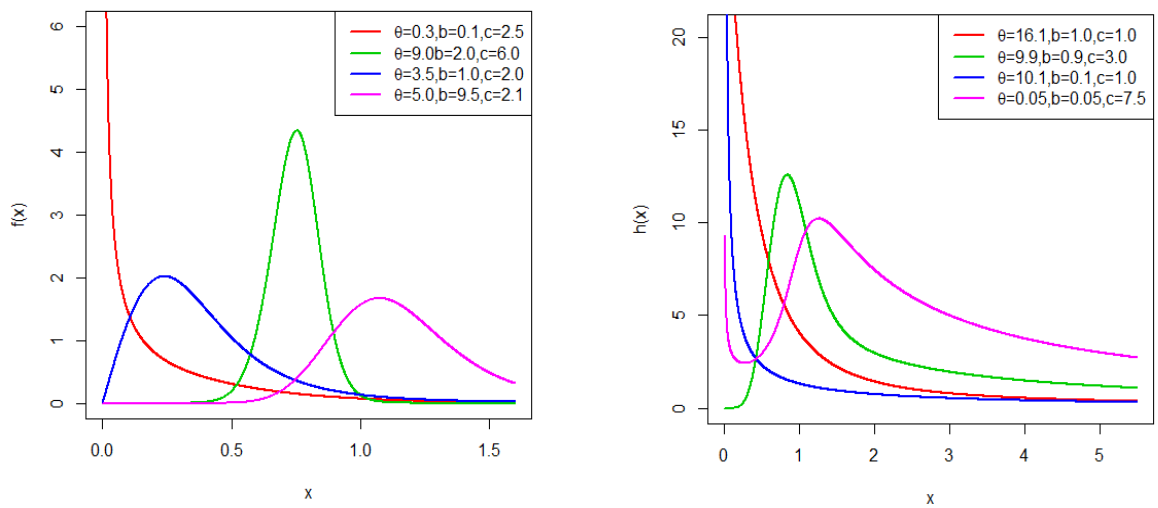

Figure 3: Pdfs and hrfs plots for the TL-LLP distribution

The quantile function of the TL-LLP distribution obtained by solving the non-linear equation

$$\ln[(e^{\theta}-1)(1-u)+1]-\theta(1-[1-(1+x^{c})^{-2}]^{b})=0. \tag{14}$$

Therefore, random numbers can be generated from the TL-LLP distribution by numerically solving the non-linear equation (14). Quantile values of the TL-LLP distribution are given in Table 4.

Table 4: Table of Quantiles for the TL-LLP Distribution

	(θ,b,c)				
u	(0.8,2.1,0.9)	(1.0,1.5,1.2	(1.8,2.1,3.5)	(1.2,5.5,0.6	(0.5,1.0,0.3)
0.1	0.1500	0.1375	0.5753	0.4053	0.0000
0.2	0.2535	0.2229	0.6551	0.6659	0.0004
0.3	0.3646	0.3085	0.7161	0.9585	0.0021
0.4	0.4950	0.4029	0.7715	1.3202	0.0078
0.5	0.6583	0.5143	0.8268	1.8012	0.0259
0.6	0.8776	0.6552	0.8867	2.4944	0.0820
0.7	1.2010	0.8496	0.9572	3.6101	0.2722
0.8	1.7550	1.1574	1.0511	5.7606	1.0841
0.9	3.0603	1.8061	1.2090	11.8717	7.3663

2.4.4. Topp-Leone-Log-Logistic Binomial Distribution

The cdf and the pdf of the Topp-Leone-Log-Logistic Binomial (TL-LLB) distribution are given by

$$F_{TL-LLB}(x;\theta,b,c,m)=1-\frac{(1+\theta(1-[1-(1+x^{c})^{-2}]^{b}))^{m}-1}{(1+\theta)^{m}-1}$$

and

$$\begin{aligned}f_{TL-LLB}(x;\theta,b,c,m)&=2\theta bcx^{c-1}(1+x^{c})^{-3}(1-(1+x^{c})^{-2})^{b-1}\\&\times\frac{m(1+\theta(1-[1-(1+x^{c})^{-2}]^{b}))^{m-1}}{(1+\theta)^{m}-1},\end{aligned}$$

respectively for $\theta, b, c > 0$ and $x > 0$. The hrf and rhrf are given by

$$h_{TL-LLB}(x;\theta,b,c,m) = 2\theta bcx^{c-1}(1+x^c)^{-3}(1-(1+x^c)^{-2})^{b-1} \times \frac{m(1+\theta(1-[1-(1+x^c)^{-2}]^b))^{m-1}}{(1+\theta(1-[1-(1+x^c)^{-2}]^b))^m - 1}$$

and

$$\tau_{TL-LLB}(x;\theta,b,c,m) = 2\theta bcx^{c-1}(1+x^c)^{-3}(1-(1+x^c)^{-2})^{b-1} \times \frac{m(1+\theta(1-[1-(1+x^c)^{-2}]^b))^{m-1}}{(1+\theta)^m - (1+\theta(1-[1-(1+x^c)^{-2}]^b))^m},$$

respectively. Figure 4 shows the plots of the pdfs and hrfs for the TL-WP distribution for selected parameters values.

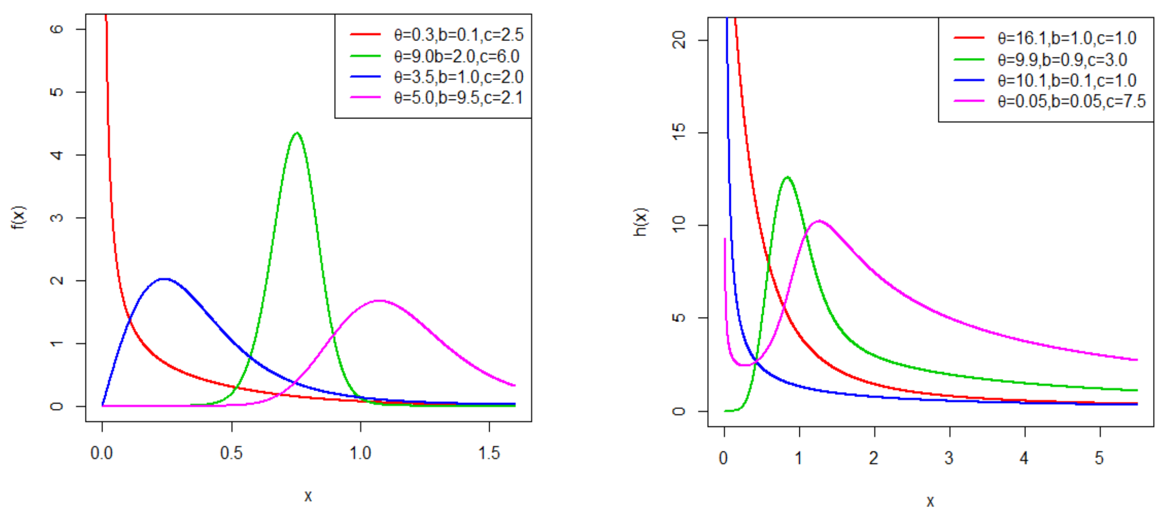

Figure 4: Pdfs and hrfs plots for the TL-LLB distribution

Plots of the TL-LLB pdf exhibit different shapes skewed to the right, skewed to the left, reverse-J and almost symmetric shapes. Plots of the hrf of the TL-LLB distribution shows different shapes including reverse-J, decreasing, bathtub followed by an upside-down bathtub and uni-modal shapes.

The quantile function obtained by solving the non-linear equation

$$[((1+\theta)^m - 1)(1-u)+1]^{\frac{1}{m}} - 1 + \theta(1-[1-(1+x^c)^{-2}]^b) = 0. \tag{15}$$

Therefore, random numbers can be generated from the TL-LLB distribution by numerically solving the non-linear equation (15). Quantile values of the TL-LLB distribution are given in Table 5.

Table 5: Table of Quantiles for the TL-LLB Distribution

	(θ,b,c,m)				
u	(0.8,2.1,0.9,1.0	(1.5,1.2,1.8,2.1)	(3.5,1.2,5.5,0.6)	(0.5,1.0,0.3,2.5)	(3.0,4.5,1.8,2.9)
0.1	3.4643	1.6754	19.8517	1.3128	1.9504
0.2	2.1130	1.3088	9.3969	1.0455	1.6754
0.3	1.5031	1.0995	5.7043	0.8869	1.5145
0.4	1.1310	0.9473	3.8018	0.7685	1.3955
0.5	0.8695	0.8223	2.6371	0.6695	1.2965
0.6	0.6689	0.7108	1.8467	0.5799	1.2070
0.7	0.5041	0.6039	1.2700	0.4933	1.1196
0.8	0.3599	0.4926	0.8222	0.4025	1.0262
0.9	0.2213	0.3598	0.4466	0.2940	0.9084

3. Moments, Conditional Moments and Mean Deviations

In this section, the r^{th} moment, conditional moments, mean deviations, Lorenz and Bonferroni curves of the TL-GPS class of distributions are presented.

3.1. Moments and Generating Function

If X follows the TL-GPS distribution and $Y \sim$ Exp-G$(k+1)$, then using equation (10), the r^{th} moment of the TL-GPS class of distributions is obtained as follows

$$\mu_r^{'} = E(X^r) = \int_0^\infty x^r \cdot f_{TL-GPS}(x;\theta,b,\psi)dx = \sum_{k=0}^{\infty} \eta_{k+1} E(Y^r),$$

where $E[Y^r]$ is the r^{th} moment of the Exp-G distribution with power parameter $(k+1)$ and η_{k+1} is given by equation (11). The moment generating function (mgf) of the TL-GPS class of distributions is given by

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) = \sum_{k=0}^{\infty} \eta_{k+1} M_Y(t),$$

where $M_Y(t)$ is the mgf of the Exp-G distribution and η_{k+1} is given by equation (11).

3.2. Conditional Moments

It is also of interest to obtain the r^{th} conditional moments. The conditional r^{th} moment of the TL-GPS distribution is given by

$$\begin{aligned} E(X^r|X>t) &= \frac{1}{\bar{F}_{TL-GPS}(t;\theta,b,\psi)} \int_t^\infty x^r \cdot f_{TL-GPS}(x;\theta,b,\psi)dx \\ &= \sum_{k=0}^{\infty} \eta_{k+1} E(Y^r I_{Y^r>t}), \end{aligned}$$

where

$$E(Y^r I_{Y^r>t}) = \int_t^\infty y^r \cdot g_{k+1}(y;\psi)dy.$$

3.3. Mean Deviations, Lorenz and Bonferroni Curves

The mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the TL-GPS class of distributions are presented in this subsection.

3.3.1. Mean Deviations

If X has the TL-GPS distribution, then we can derive the mean deviation about the mean $D(\mu)$ and the median deviation about the median D(M) as follows

$$D(\mu) = \int_0^\infty |x-\mu| f_{TL-GPS}(x;\theta,b,\psi)dx = 2\mu f_{TL-GPS}(x;\theta,b,\psi) - 2\mu + 2T(\mu)$$

and

$$D(M) = \int_0^\infty |x-M| f_{TL-GPS}(x;\theta,b,\psi)dx = -\mu + 2T(M),$$

respectively, where $\mu = E(X)$ and M = Median(X) is the median of $F_{TL-GPS}(x;\theta,b,\psi)$. Note that

$$T(\mu) = \int_\mu^\infty x \cdot f_{TL-GPS}(x;\theta,b,\psi)dx = \sum_{k=0}^{\infty} \eta_{k+1} \int_\mu^\infty y \cdot g_{k+1}(y;\psi)dy$$

and

$$T(M) = \int_M^\infty x \cdot f_{TL-GPS}(x; \theta, b, \psi)dx = \sum_{k=0}^\infty \eta_{k+1} \int_M^\infty y \cdot g_{k+1}(y; \psi)dy.$$

3.3.2. Bonferroni and Lorenz Curves

In this subsection, we present Bonferroni and Lorenz curves for TL-GPS class of distributions. The Bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu} \int_0^q x \cdot f_{TL-GPS}(x; \theta, b, \psi)dx = \frac{1}{p\mu} \sum_{k=0}^\infty \eta_{k+1} \int_0^q x \cdot g_{k+1}(x; \psi)dx$$

and

$$L(p) = \frac{1}{\mu} \int_0^q x \cdot f_{TL-GPS}(x; \theta, b, \psi)dx = \frac{1}{\mu} \sum_{k=0}^\infty \eta_{k+1} \int_0^q x \cdot g_{k+1}(x; \psi)dx,$$

respectively, where $\int_0^q x \cdot g_k(x; \psi)dx$ is the first incomplete moment of the Exp-G distribution with power parameter $(k+1)$ and η_{k+1} is given by equation (11).

3.4. Order Statistics and Rényi Entropy

In this section, we present the distribution of the order statistic and Rényi entropy of the TL-GPS class of distributions.

3.4.1. Distribution of Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from the TL-GPS distribution and let $X_{i:n}$ be the corresponding i^{th} order statistics. The pdf of the i^{th} order statistic, $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} [F(x)]^{k+i-1}, \tag{16}$$

where B(.,.) is the beta function. Substituting the pdf and cdf of the TL-GPS family of distributions, we write

$$\begin{aligned} f(x)[F(x)]^{k+i-1} &= \sum_{n=1}^\infty \frac{n a_n \theta^n}{C(\theta)} [1-(1-\bar{G}(x;\psi)^2)^b]^{n-1} 2bg(x;\psi)\bar{G}(x;\psi) \\ &\times [1-\bar{G}(x;\psi)^2]^{b-1} \left[1 - \frac{C(\theta(1-(1-\bar{G}(x;\psi)^2)^b))}{C(\theta)}\right]^{k+i-1}. \end{aligned}$$

Using the generalized binomial expansion

$$\left[1 - \frac{C(\theta(1-(1-\bar{G}(x;\psi)^2)^b))}{C(\theta)}\right]^{k+i-1} = \sum_{j=0}^\infty (-1)^j \binom{k+i-1}{j} \left[\frac{C(\theta(1-(1-\bar{G}(x;\psi)^2)^b))}{C(\theta)}\right]^j,$$

and applying the result on power series raised to a positive integer, we get

$$\begin{aligned} f(x)[F(x)]^{k+i-1} &= \sum_{j=0}^\infty \sum_{n=1}^\infty \binom{k+i-1}{j} (-1)^j \frac{n a_n \theta^{n+m}}{C^{j+1}(\theta)} b_{m,j} 2bg(x;\psi)\bar{G}(x;\psi) \\ &\times [1-(1-\bar{G}(x;\psi)^2)^b]^{m+n-1} [1-\bar{G}(x;\psi)^2]^{b-1}, \end{aligned}$$

where $b_{m,j} = (ma_0)^{-1} \sum_{l=1}^m (l(j+1)-m) a_l b_{m-l}$ and $b_{0,j} = a_0^j$ (Gradshetyn (2000)). Also, using the following generalized binomial expansion

$$[1-(1-\bar{G}(x;\psi)^2)^b]^{m+n-1} = \sum_{p=0}^\infty (-1)^p \binom{m+n-1}{p} [1-\bar{G}(x;\psi)^2]^{bp},$$

we obtain

$$f(x)[F(x)]^{k+i-1} = \sum_{j,p=0}^{\infty}\sum_{n=1}^{\infty}\binom{k+i-1}{j}\binom{m+n-1}{p}(-1)^{j+p}\frac{na_n\theta^{n+m}}{C^{j+1}(\theta)}b_{m,j} \times [1-\bar{G}(x;\psi)^2]^{b(p+1)-1}2bg(x;\psi)\bar{G}(x;\psi).$$

Furthermore, applying the generalized binomial expansion

$$[1-\bar{G}(x;\psi)^2]^{b(p+1)-1} = \sum_{q=0}^{\infty}(-1)^q\binom{b(p+1)-1}{q}\bar{G}(x;\psi)^{2q}$$

yields

$$f(x)[F(x)]^{k+i-1} = \sum_{j,p,q=0}^{\infty}\sum_{n=1}^{\infty}\binom{k+i-1}{j}\binom{m+n-1}{p}\binom{b(p+1)-1}{q} \times (-1)^{j+p+q}\frac{2bna_n\theta^{n+m}}{C^{j+1}(\theta)}b_{m,j}g(x;\psi)\bar{G}(x;\psi)^{2q+1}.$$

Also, applying the binomial expansion

$$\bar{G}(x;\psi)^{2q+1} = \sum_{r=0}^{\infty}(-1)^r\binom{2q+1}{r}G(x;\psi)^r$$

yields

$$\begin{aligned} f(x)[F(x)]^{k+i-1} &= \sum_{j,p,q,r=0}^{\infty}\sum_{n=1}^{\infty}\binom{k+i-1}{j}\binom{m+n+-1}{p}\binom{b(p+1)-1}{q}\binom{2q+1}{r} \\ &\times (-1)^{j+p+q+r}\frac{2bna_n\theta^{n+m}}{C^{j+1}(\theta)}b_{m,j}\left(\frac{r+1}{r+1}\right)g(x;\psi)G(x;\psi)^r \\ &= \sum_{r=0}^{\infty}a_{r+1}g_{r+1}(x;\psi), \end{aligned} \tag{17}$$

where $g_{r+1}(x;\psi) = (r+1)g(x;\psi)G(x;\psi)^r$ is the Exp-G distribution with power parameter $(r+1)$, and

$$a_{r+1} = \sum_{j,p,q=0}^{\infty}\sum_{n=1}^{\infty}\binom{k+i-1}{j}\binom{m+n+-1}{p}\binom{b(p+1)-1}{q}\binom{2q+1}{r} \times (-1)^{j+p+q+r}\frac{2bna_n\theta^{n+m}}{C^{j+1}(\theta)(r+1)}b_{m,j}.$$

Therefore, substituting equation (17) in (16) we obtain

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)}\sum_{r=0}^{\infty}\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}a_{r+1}g_{r+1}(x;\psi). \tag{18}$$

It follows that the distribution of the i^{th} order statistic from the TL-GPS class of distributions can be obtained directly from the distribution of the i^{th} order statistic from the Exp-G distribution.

3.4.2. Rényi Entropy

In this subsection, Rényi entropy of the TL-GPS class of distributions is derived. Entropy measures the uncertainty or variation of a random variable. Rényi entropy (Rényi (1960)) is a generalization of Shannon entropy (Shannon

(1951)). Rényi entropy of the TL-GPS class of distributions is defined as

$$I_R(\nu) = \frac{1}{1-\nu}\log\left(\int_0^\infty (f_{TL-GPS}(x;\theta,b,\psi))^\nu dx\right), \qquad \nu \neq 1, \quad \nu > 0. \tag{19}$$

Note that $f^{\nu}_{TL-GPS}(x;\theta,b,\psi)$ can be written as

$$f^{\nu}_{TL-GPS}(x;\theta,b,\psi) = \Bigg[\theta 2bg(x;\psi)\bar{G}(x;\psi)[1-\bar{G}(x;\psi)^2]^{(b-1)} \times \frac{C'(\theta(1-[1-\bar{G}(x;\psi)^2]^b))}{C(\theta)}\Bigg]^{\nu}.$$

Considering the following series expansions

$$\left[C'(\theta(1-[1-\bar{G}(x;\psi)^2]^b))\right]^{\nu} = \sum_{k=0}^{\infty} d_{k,\nu}\theta^k(1-[1-\bar{G}(x;\psi)^2]^b)^k,$$

where $d_{k,\nu} = (kb_0)^{-1}\sum_{l=1}^{k}[\nu(l+1)-k]b_l d_{k-l,\nu}$ and $d_{0,\nu} = b_0^{\nu}$,

$$(1-[1-\bar{G}(x;\psi)^2]^b)^k = \sum_{m=0}^{\infty}(-1)^m\binom{k}{m}[1-\bar{G}(x;\psi)^2]^{bm},$$

$$[1-\bar{G}(x;\psi)^2]^{b(m+\nu)-\nu} = \sum_{n=0}^{\infty}(-1)^n\binom{b(m+\nu)-\nu}{n}\bar{G}(x;\psi)^{2n},$$

and

$$\bar{G}(x;\psi)^{2n+\nu} = \sum_{q=0}^{\infty}(-1)^q\binom{2n+\nu}{q}G(x;\psi)^q,$$

we get

$$f^{\nu}_{TL-GPS}(x;\theta,b,\psi) = \sum_{k,m,n,q=0}^{\infty}\left[\frac{2b}{C(\theta)}\right]^{\nu}(-1)^{m+n+q}\binom{k}{m}\binom{b(m+\nu)-\nu}{n} \times \binom{2n+\nu}{q}d_{k,\nu}\theta^{\nu+k}g^{\nu}(x;\psi)G^q(x;\psi).$$

Therefore, the Rényi entropy of the TL-GPS class of distributions is given by

$$I_R(\nu) = \frac{1}{1-\nu}\log\left(\sum_{q=0}^{\infty}\eta^*_{q+1}e^{(1-\nu)I_{REG}}\right), \tag{20}$$

where

$$\eta^*_{q+1} = \sum_{k,m,n=0}^{\infty}\left[\frac{2b}{C(\theta)}\right]^{\nu}(-1)^{m+n+q}\binom{k}{m}\binom{b(m+\nu)-\nu}{n}\binom{2n+\nu}{q}d_{k,\nu}\frac{\theta^{\nu+k}}{[\frac{q}{\nu}+1]^{\nu}}. \tag{21}$$

and

$$I_{REG} = \frac{1}{1-\nu}\int_0^\infty\left(\left[\frac{q}{\nu}+1\right]g(x;\psi)G(x;\psi)^{\frac{q}{\nu}}\right)^{\nu}dx$$

is the Rényi entropy of the Exp-G distribution with parameter $\left(\frac{q}{\upsilon}+1\right)$. As such, we can directly derive the Rényi entropy of the TL-GPS family of distributions from the Rényi entropy of the Exp-G distribution.

4. Estimation

In this section, we derive the maximum likelihood estimates of the parameter vector $(\theta, b, \psi)^T$ of the TL-GPS class of distributions. Let $X_i \sim$ TL-GPS(θ, b, ψ) and $\boldsymbol{\Delta} = (\theta, b, \psi)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\boldsymbol{\Delta})$ based on a random sample of size n is given by

$$\ell(\boldsymbol{\Delta}) = n\ln(2b\theta) + (b-1)\sum_{i=1}^{n}\ln[1-\bar{G}(x_i;\psi)^2] + \sum_{i=1}^{n}\ln[\bar{G}(x_i;\psi)] + \sum_{i=1}^{n}\ln[g(x_i;\psi)] - n\ln(C(\theta)) + \sum_{i=1}^{n}\ln\left(C'\left(\theta\left(1-[1-\bar{G}(x_i;\psi)^2]^b\right)\right)\right).$$

The elements of the score vector are given by

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \frac{nC'(\theta)}{C(\theta)} + \sum_{i=1}^{n}\frac{\left(C''\left(\theta\left(1-[1-\bar{G}(x_i;\psi)^2]^b\right)\right)\right)\left(1-\left(1-\bar{G}(x_i;\psi)^2\right)^b\right)}{C'\left(\theta\left(1-[1-\bar{G}(x_i;\psi)^2]^b\right)\right)},$$

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\ln[1-\bar{G}(x_i;\psi)^2] + \sum_{i=1}^{n}\frac{\theta\left(C''\left(\theta\left(1-[1-\bar{G}(x_i;\psi)^2]^b\right)\right)\right)}{C'\left(\theta\left(1-[1-\bar{G}(x_i;\psi)^2]^b\right)\right)} \times [1-\bar{G}(x_i;\psi)^2]^b\ln[1-\bar{G}(x_i;\psi)^2],$$

and

$$\frac{\partial \ell}{\partial \psi_k} = (b-1)\sum_{i=1}^{n}\frac{1}{(1-\bar{G}(x_i;\psi)^2)}\frac{\partial[1-\bar{G}(x_i;\psi)^2]}{\partial \psi_k} + \sum_{i=1}^{n}\frac{\frac{\partial \bar{G}(x_i;\psi)}{\partial \psi_k}}{\bar{G}(x_i;\psi)} + \sum_{i=1}^{n}\frac{\frac{\partial g(x_i;\psi)}{\partial \psi_k}}{g(x_i;\psi)} + \sum_{i=1}^{n}\frac{\left(C''\left(\theta\left(1-(1-\bar{G}(x_i;\psi)^2)^b\right)\right)\right)}{C'\left(\theta\left(1-(1-\bar{G}(x_i;\psi)^2)^b\right)\right)}2b\theta\left[1-\bar{G}(x_i;\psi)^2\right]^{b-1}\bar{G}(x_i;\psi)\frac{\partial \bar{G}(x_i;\psi)}{\partial \psi_k}.$$

The equations obtained by setting the partial derivatives equal to zero are not in closed form. The maximum likelihood estimates of the parameters denoted by $\hat{\boldsymbol{\Delta}}$ are obtained by solving the non-linear equation $\left(\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \psi_k}\right)^T = 0$ using numerical methods such as the Newton-Raphson procedure. The multivariate normal distribution $N(\underline{\mathbf{0}}, J^{-1}(\hat{\boldsymbol{\Delta}}))$, where the mean vector $\underline{\mathbf{0}} = (0,0,\mathbf{0})^{\mathbf{T}}$ and $J^{-1}(\hat{\boldsymbol{\Delta}})$ is the observed Fisher information matrix evaluated at $\hat{\boldsymbol{\Delta}}$ can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

5. Simulation Study

In this section, a simulation study was conducted to assess consistency of the maximum likelihood estimators. We considered a special case of the TL-LLP distribution. We simulated for the sample sizes n= 25,50, 100, 200, 400, 800, and 1000, for N=1000 for each sample. We estimate the mean, root mean square error (RMSE), and average bias. The bias and RMSE for the estimated parameter, say, $\hat{\Delta}$, are given by

$$Bias(\hat{\Delta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Delta}_i - \Delta), \quad \text{and} \quad RMSE(\hat{\Delta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\Delta}_i - \Delta)^2}{N}},$$

respectively. We consider simulations for the following sets of initial parameters values (I: $\theta = 0.5, b = 1.5, c = 1.0$), (II: $\theta = 1.5, b = 1.5, c = 0.5$), (III: $\theta = 0.5, b = 1.0, c = 1.5$), and (IV: $\theta = 1.0, b = 1.5, c = 0.5$). If the model performs better, we except the mean to approximate the true parameter values, the RMSE, and bias to decay toward zero for an increase in sample size. From the results in Table 6, the mean values approximate the true parameter values, RMSE and bias decay towards zero for all the parameter values.

6. Applications

In this section, we present examples to illustrate the usefulness and applicability of the TL-GPS class of distributions. This is achieved by applying the special case of Topp-Leone-Log-Logistic Poisson to two real data sets and comparing it to several equal-parameter non-nested models. Model parameters were estimated via the maximum likelihood estimation technique using the R software. The performance of the models were assessed using the following several goodness-of-fit statistics; -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*) and Andersen-Darling (A^*) (as described by Chen and Balakrishnan (1995)), Kolmogorov-Smirnov (K-S) statistic and its p-value. The model that has smaller values of these above mentioned goodness-of-fit statistics and larger p-value of the K-S statistics is deemed as the best model.

Tables 7 and 8 show the model parameters estimates (standard errors in parenthesis) and the goodness-of-fit-statistics for the two data sets considered. Plots of the fitted densities, the histogram of the data and probability plots (Chambers et al. (1983)) are also presented to show how well our model fits the observed data set compared to the selected non-nested models.

The non-nested models considered are the Weibull-Poisson (Mahmoudi and Seahdar (2013)), Topp-Leone generalized exponential (TL-GE) (Sangsanit and Bodhisuwan (2016)), alpha power Weibull (APW) (Nassar et al. (2016)), Marshall-Olkin Extended Weibull (MOEW) (Cordeiro and Lemonte (2013)) and Marshall-Olkin log-logistic (MOLL) (Gui (2013)), Topp-Leone Weibull-Lomax (WLx) (Jamal et al. (2019)), Transmutted Weibull (TW) (Ahmad et al. (2015)) distributions. The pdfs of the non-nested models are as follows:

$$f_{MOLL}(x;\alpha,\beta,\gamma)=\alpha^{\beta}\beta\gamma\frac{x^{\beta-1}}{(x^{\beta}+\gamma\alpha^{\beta})^{2}},$$

for $\alpha,\beta,\gamma>0$, and $x>0$,

$$f_{WP}(x;\theta,\beta,\gamma)=\frac{\theta\gamma\beta^{\gamma}x^{\gamma-1}\exp(-(\beta x)^{\gamma})\exp(\theta(1-\exp(-(\beta x)^{\gamma})))}{\exp(\theta)-1},$$

for $\theta,\beta,\gamma>0$, and $x>0$,

$$f_{TL-GE}(x;\alpha,\beta,\lambda)=2\alpha\beta\lambda e^{-\lambda x}(1-(1-e^{-\lambda x})^{\beta}(1-e^{-\lambda x})^{\beta\alpha-1}(2-(1-e^{-\lambda x})^{\beta}))^{\alpha-1},$$

for $\alpha,\beta,\lambda>0$, and $x>0$,

$$f_{APW}(x;\alpha,\beta,\theta)=\frac{\log(\alpha)}{(\alpha-1)}\beta\theta x^{\beta-1}e^{-\theta x^{\beta}}\alpha^{1-e^{-\theta x^{\beta}}},$$

for $\alpha,\beta,\theta>0$ and $x>0$,

$$F_{MOEW}(x;\alpha,\lambda,\gamma)=\frac{\alpha\gamma\lambda x^{\gamma-1}e^{-\lambda x^{\gamma}}}{(1-(1-\alpha)e^{-\lambda x^{\gamma}})^{2}},$$

for $\alpha,\lambda,\gamma>0$, and $x>0$,

$$f_{WLx}(x;a,b,\alpha)=\alpha ab(1+bx)^{a\alpha-1}(1-(1+bx)^{-a})^{\alpha-1}e^{-\left(\frac{1-(1+bx)^{-a}}{(1+bx)^{-1}}\right)^{\alpha}},$$

and

$$F_{TW}(x;\lambda,\beta,\alpha)=\alpha\beta x^{\beta-1}e^{-\lambda x^{\beta}}(1-\lambda+2\lambda e^{-\lambda x^{\beta}}),$$

for $\lambda,\beta,\alpha>0$ and $x>0$.

Table 6: Monte Carlo Simulation Results for TL-LLP Distribution: Mean, RMSE and Average Bias

		I: $\theta=0.5, b=1.5, c=1.0$			II: $\theta=1.5, b=1.5, c=0.5$		
	samplesize	Mean	RMSE	Bias	Mean	RMSE	Bias
θ	25	1.5971	1.6131	1.0971	2.2845	1.9846	0.7845
	50	1.4729	1.5153	0.9729	2.2459	1.6083	0.7459
	100	1.2283	1.3343	0.7283	2.0671	1.3136	0.5671
	200	0.8945	0.8672	0.3945	1.9149	1.1782	0.4149
	400	0.6985	0.4912	0.1985	1.8147	0.9015	0.3147
	800	0.5767	0.3278	0.0767	1.7387	0.6583	0.2387
	1000	0.5582	0.2816	0.0582	1.7169	0.3868	0.2169
b	25	2.1429	1.0851	0.6429	1.8348	0.9946	0.3348
	50	2.0972	1.0600	0.5972	1.9127	0.9588	0.4127
	100	1.9571	0.9382	0.4571	1.8495	0.8420	0.3495
	200	1.7545	0.5979	0.2545	1.7862	0.7787	0.2862
	400	1.6238	0.3148	0.1238	1.7139	0.5848	0.2139
	800	1.5459	0.1938	0.0459	1.6514	0.4170	0.1514
	1000	1.5336	0.1627	0.0336	1.6269	0.2284	0.1269
c	25	0.9534	0.2776	-0.0466	0.5475	0.2675	0.0475
	50	0.9260	0.2255	-0.0740	0.4895	0.1456	-0.0105
	100	0.9364	0.1784	-0.0636	0.4813	0.1069	-0.0187
	200	0.9588	0.1296	-0.0412	0.4815	0.0950	-0.0185
	400	0.9774	0.0818	-0.0226	0.4815	0.0714	-0.0185
	800	0.9915	0.0570	-0.0085	0.4841	0.0506	-0.0159
	1000	0.9946	0.0505	-0.0054	0.4856	0.0370	-0.0144
		III: $\theta=0.5, b=1.0, c=1.5$			IV: $\theta=1.0, b=1.5, c=0.5$		
θ	25	1.7985	2.7021	1.2985	1.8872	1.7102	0.8872
	50	1.3213	1.3605	0.8213	1.8396	1.5185	0.8396
	100	1.0529	1.0967	0.5529	1.5930	1.2897	0.5930
	200	0.7617	0.6139	0.2617	1.3665	0.9905	0.3665
	400	0.6205	0.3775	0.1205	1.2081	0.6073	0.2081
	800	0.5378	0.2792	0.0378	1.1668	0.4165	0.1668
	1000	0.5307	0.2437	0.0307	1.1387	0.3297	0.1387
b	25	1.5638	1.1103	0.5638	1.9438	0.9746	0.4438
	50	1.3771	0.7673	0.3771	1.9898	1.0018	0.4898
	100	1.2557	0.6358	0.2557	1.8732	0.8563	0.3732
	200	1.1193	0.3231	0.1193	1.7573	0.6714	0.2573
	400	1.0549	0.1742	0.0549	1.6424	0.3878	0.1424
	800	1.0168	0.1206	0.0168	1.6062	0.2516	0.1062
	1000	1.0126	0.1029	0.0126	1.5836	0.1891	0.0836
c	25	1.4752	0.6269	-0.0248	0.5130	0.1866	0.0130
	50	1.4338	0.3859	-0.0662	0.4776	0.1297	-0.0224
	100	1.4436	0.2663	-0.0564	0.4776	0.0965	-0.0224
	200	1.4684	0.1842	-0.0316	0.4806	0.0768	-0.0194
	400	1.4827	0.1224	-0.0173	0.4885	0.0535	-0.0115
	800	1.4971	0.0884	-0.0029	0.4887	0.0372	-0.0113
	1000	1.4989	0.0789	-0.0011	0.4930	0.0296	-0.0070

6.1. Growth Hormone Data

The data consists of the estimated time since growth hormone medication until the children reached the targeted age. This data was used by Alizadeh et al. (2017) and are as follows: 2.15, 2.20, 2.55, 2.56, 2.63, 2.74, 2.81, 2.90, 3.05, 3.41, 3.43, 3.43, 3.84, 4.16, 4.18, 4.36, 4.42, 4.51, 4.60, 4.61, 4.75, 5.03, 5.10, 5.44, 5.90, 5.96, 6.77, 7.82, 8.00, 8.16, 8.21, 8.72, 10.40, 13.20, 13.70. The estimated variance-covariance matrix for the TL-LLP model on growth hormone

Table 7: Parameter estimates and goodness-of-fit statistics for various fitted models for growth hormone data set

	Estimates			Statistics							
Model	θ	b	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
TL-LLP	6.5265×10^{-5}	5.2422×10	1.3853	155.4571	161.4571	162.2313	166.1231	0.0382	0.2660	0.0849	0.9624
	(0.1033)	(17.693)	(0.1549)								
	θ	β	γ								
WP	8.8353×10^{-9}	0.159	1.9932	165.0	170.9772	171.7514	175.6432	0.1639	1.0262	0.1454	0.4501
	(0.0117)	(0.0149)	(0.2438)								
	α	β	λ								
APW	6.4700×10^{4}	0.8444	0.7494	158.4884	164.4884	165.2626	169.1544	0.0710	0.4808	0.0924	0.9262
	(1.4803×10^{-7})	(0.1031)	(0.1365)								
	α	γ	λ								
MOEW	0.0280	3.4077	0.0002	158.1646	164.1646	164.9388	168.8307	0.0616	0.4330	0.0972	0.8953
	(0.0430)	(0.0026)	(0.0002)								
	a	b	α								
WLx	0.2548	2.4039	4.2177	162.1874	168.1874	168.9616	172.8535	0.1265	0.8062	0.1329	0.5663
	(0.1825)	(5.0117)	(2.4857)								
	α	β	λ								
TL-GE	2.3206×10^{-3}	2.7741×10^{3}	0.4786	158.1108	164.1108	164.885	168.7769	0.0724	0.4876	0.1025	0.8561
	(7.5165×10^{-4})	(3.1617×10^{-7})	(7.2079×10^{-2})								
	λ	β	α								
TW	0.6092	2.1722	0.0140	163.4343	169.4343	170.2085	174.1004	0.1437	0.9030	0.1320	0.5758
	(0.3488)	(0.2648)	(0.0084)								
	α	β	γ								
MOLL	3.3212	3.5219	3.0700	158.5828	164.5828	165.357	169.2488	0.0560	0.4061	0.0978	0.8911
	(0.2844)	(0.4909)	(0.0874)								

data is

$$\begin{bmatrix} 0.0107 & 0.0001 & -0.0013 \\ 0.0001 & 322.661 & 2.4121 \\ -0.0013 & 2.4121 & 0.0240 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\theta \in [(6.5265\times10^{-5})\pm0.2025]$, b $\in [5.2422\times 10\pm35.2075]$ and c $\in [1.3853\pm0.3036]$. Based on the results shown in Table 7, we observe that the TL-LLP model

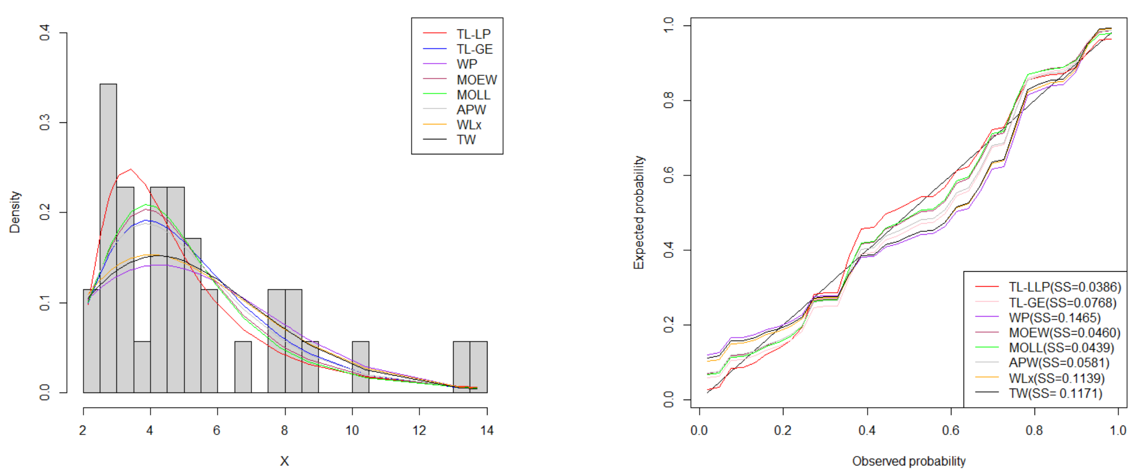

Figure 5: Fitted Densities and Probability Plots for the Growth Hormone Data

has the smallest values of all the goodness-of-fit statistics and bigger value for the K-S p-value. We therefore conclude that the TL-LLP distribution performs better than the several models considered in this paper. The fitted densities and probability plots in Figure 5 also shows that the TL-LLP model fit the growth hormone data set better than the selected non-nested models.

6.2. Repair Lifetimes Data

The second data set represents maintenance on active repair times (in hours) for an airborne communication transceiver reported by Leiva et al. (2009) and Chhikara and Folks (1977) and are as follows: 0.2, 0.3,0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5. The estimated variance-covariance matrix for the TL-LLP model

Table 8: Parameter estimates and goodness of fit statistics for various fitted models for repair lifetimes data set

	Estimates			Statistics							
Model	θ	b	c	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-value$
TL-LLP	0.0001	4.4081	0.8102	200.1734	206.1734	206.7448	211.6593	0.0547	0.3276	0.0882	0.8667
	(0.2035)	(0.6894)	(0.0905)								
	θ	β	γ								
WP	3.9697×10^{-8}	0.2953	0.8978	208.9395	214.9395	215.5110	220.4255	0.1297	0.9003	0.1209	0.5125
	(0.0219)	(0.0516)	(0.0957)								
	a	b	α								
WLx	0.2222	7.0310	2.1228	202.2696	208.2696	208.841	213.7555	0.0697	0.4745	0.1059	0.6802
	(0.1093)	(11.2084)	(0.9099)								
	α	β	λ								
APW	0.0295	1.1011	0.0924	204.9274	210.9274	211.4988	216.4133	0.0986	0.6520	0.1111	0.6209
	(0.0566)	(0.1201)	(0.0521)								
	α	γ	λ								
MOEW	0.0332	1.4861	0.0126	201.7121	207.7121	208.2836	213.1981	0.0700	0.4330	0.0922	0.8289
	(0.0540)	(0.2084)	(0.0208)								
TL-GE	2.1603×10^{-4}	4.4317×10^{3}	0.2689	209.9544	215.9544	216.5258	221.4403	0.1455	1.0116	0.1520	0.2381
	(4.2209×10^{-5})	(2.8118×10^{-8})	(5.2662×10^{-2})								
	λ	β	α								
TW	0.6735	0.9792	0.1986	206.9540	212.954	213.5254	218.4399	0.1150	0.7817	0.1161	0.5650
	(0.3038)	(0.1047)	(0.0602)								
	α	β	γ								
MOLL	0.9457	1.5439	2.8641	202.3421	208.3421	208.9135	213.828	0.0750	0.4519	0.0939	0.8122
	(114.6974)	(0.1858)	(536.30783)								

on repair times data set is

$$\begin{bmatrix} 0.0414 & 0.0459 & -0.0030 \\ 0.0459 & 0.4753 & -0.0073 \\ -0.0030 & -0.0073 & 0.0082 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\theta \in [0.0001 \pm 0.3988]$, b $\in [4.4081 \pm 1.3512]$ and c $\in [0.8102 \pm 0.1773]$. Based on the results shown in Table 8, we observe that the TL-LLP model has the smallest

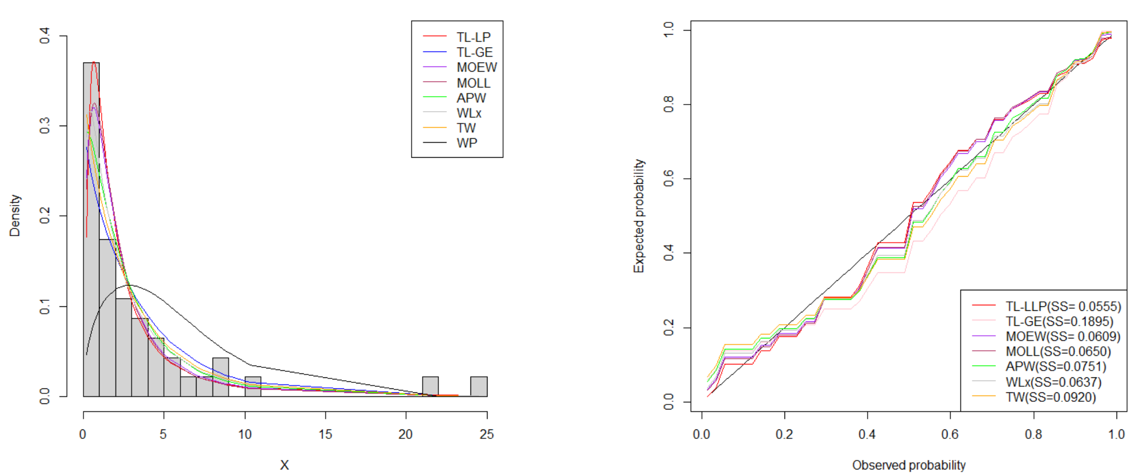

Figure 6: Fitted Densities and Probability Plots for the Repair Lifetimes Data

values of all the goodness-of-fit statistics and bigger value for the K-S p-value. We therefore, conclude that the TL-

LLP distribution performs better than the several models considered in this paper. The fitted densities and probability plots in Figure 6 also shows that the TL-LLP model fit the repair times data set better than the selected models.

7. Concluding Remarks

We developed a new class of distributions, called the Topp-Leone-G Power Series (TL-GPS) class of distributions. We presented some sub-classes and some special cases of the new proposed distribution. Structural properties were also derived including moments, mean deviations, distribution of order statistics, Rényi entropy, and maximum likelihood estimates. We also presented two real data examples to show the usefulness of the new class of distributions. The proposed model performs better than the several models on the selected data sets.

References

Ahmad, K., Ahmad, S. and Ahmed, A. (2015). Structural Properties of Transmuted Weibull Distribution. *Journal of Modern Applied Statistical Methods*, 14(2), 141-158.

Alizadeh, M., Bagheri, S., Bahrami, S. E., Ghobadi, S. and Nadarajah, S. (2017). Exponentiated Power Lindley Power Series Class of Distribution: Theory and Applications. *Communications in Statistics-Simulation and Computation*, 47(9), 2499-2531, DOI: 10.1080/03610918.2017.1350270.

Al-Shomrani, A., Arif, O., Shawky, A., Hanif, S. and Shahbaz, M. Q. (2016). Topp Leone Family of Distributions: Some Properties and Application. *Pakistan Journal of Statistics and Operation Research*, 12(3), 443-451.

Al-Zahrani, B. (2012). Goodness-of-fit for the Topp-Leone Distribution with Unknown Parameters. *Applied Mathematical Sciences*, 6(128).

Bantan, R. A. R., Jamal, F., Chesneau, C. and Elgarhy, M. (2020). Type II Power Topp-Leone Generated Family of Distributions with Statistical Inference and Applications. *Symmetry*, 12(1), 75; https://doi.org/10.3390/sym12010075.

Chambers, J., Cleveland, W., Kleiner, B. and Tukey, P. (1983). Graphical Methods for Data Analysis, *The Wadsworth Statistics/Probability Series, Boston*, MA: Duxury.

Chen, G. and Balakrishnan, N. (1995). A General Purpose Approximate Goodness-of-Fit Test. *Journal of Quality Technology*, 27, 154-161.

Chhikara, R. S and Folks J. L. (1977). The Inverse Gaussian Distribution as a Lifetime Model. *Technometrics*, Vol. 19, 461-468.

Chipepa, F., Oluyede, B. and Makubate, B. (2020). The Topp-Leone Marshall-Olkin-G Family of Distributions With Applications. *International Journal of Statistics and Probability*, 9(4). https://doi.org/10.5539/ijsp.v9n4p15.

Cordeiro, G. M., and Lemonte, A. J. (2013). On the Marshall-Olkin Extended Weibull Distribution. *Statistical papers*, 54(2), 333-353.

Jamal, F., Reyad, H. M., Nasir, M., Chesneau, A. C., Shah, M. A. A. and Ahmed, S. O. (2019). Topp-Leone Weibull-Lomax distribution: Properties, Regression Model and Applications. hal-02270561.

Ghitany, M. E., Kotz, S., and Xie, M. (2005). On Some Reliability Measures and their Stochastic Orderings for the Topp-Leone Distributio., *Journal of Applied Statistics*, 32(7), 715-722.

Gradshetyn, I. S. and Ryzhik, I. M. (2000). Tables of Integrals, Series and Products Sixth Edition, Academic Press, San Diego.

Gui, W. (2013). Marshall-Olkin Extended Log-Logistic Distribution and its Application in Minification Processes. *Applied Mathematical Sciences*, 7, 3947-3961.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1994). Continuous Distributions, Volume 1, John Wiley and Sons, New York, NY.

Leiva, V., Barros M., Paula, G. A. (2009). Generalized Birnbaum-Saunders Models using R. *www.victorleiva.cl/archivos/books/leiva*

Mahmoudi, E. and Seahdar, A. (2013). Exponentiated Weibull-Poisson Distribution: Model, Properties and Applications. *Mathematics and Computers in Simulation*, 92(2013), 76-97.

Marshall, A. N., and Olkin I. (1997). A New Method for Adding a Parameter to a Family of Distributions with Applications to the Exponential and Weibull Families. *Biometrika*, 84, 641-652.

Nadarajah, S. and Kotz, S. (2003). Moments of Some J-Shaped Distributions. *Journal of Applied Statistics*, 30(3), 311-317.

Nassar, M., Alzaatreh, A., Mead, M. and Abo-Kasem, O. E. (2016). Alpha Power Weibull Distribution: Properties and Applications. *Communication in Statistics-Theory and Methods*, 46(20), 10236-10252, DOI:10.1080/03610926.2016.1231816.

Rényi, A., (1960). On Measures of Entropy and Information. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, 1, 547 - 561.

Rezaei, S., Sadr, B. B., Alizadeh, M. and Nadarajah, S. (2016). Topp-Leone Generated Family of Distributions: Properties and Applications. *Communications in Statistics-Theory and Methods*, 46(6), 2893-2909.

Sangsanit, Y. and Bodhisuwan, W. (2016). The Topp-Leone Generator of Distributions: Properties and Inferences. *Songklanakarin Journal of Science and Technology*, 38(5), 537-548.

Shannon, C. E. (1951). Prediction and Entropy of Printed English. *The Bell System Technical Journal*, 30, 50-64.

Topp, C. W. and Leone, F. C. (1955). A Family of J-shaped Frequency Functions. *Journal of the American Statistical Association*, 50(269), 209-219.

Vicaria, D., Dorp, J. R. V., and Kotz, S. (2008). Two-Sided Generalized Topp and Leone (TS-GTL) Distributions. *Journal of Applied Statistics*, 35(10), 1115-1129.