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A Discrete Analog of the Inverted Kumaraswamy Distribution: Properties and Estimation with Application to COVID-19 Data

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Abstract

In recent years, researchers focused on introducing discrete type distributions which satisfy the necessary demand to model the complex performance of the real data sets. In this paper, a discrete inverted Kumaraswamy distribution, which is a discrete version of the continuous inverted Kumaraswamy distribution, is derived using the general approach of discretization of a continuous distribution. The new discrete inverted Kumaraswamy distribution can be applied efficiently in discrete lifetime and count data. Some important distributional and reliability properties of discrete inverted Kumaraswamy distribution such as hazard rate, moments, quantiles, order statistics and some transformations are obtained. Maximum likelihood and Bayesian approaches are applied under Type-II censored samples for estimating the parameters, survival, hazard rate and alternative hazard rate functions. Confidence and credible intervals for the parameters are obtained. A simulation study is carried out to illustrate the theoretical results of the maximum likelihood and Bayesian estimation. Finally, the performance of the new distribution is compared with some distributions using three real data sets to illustrate the suitability and flexibility of the proposed model.

Key Words: Inverted Kumaraswamy distribution; Discrete lifetime models; Survival, hazard and alternative hazard rate functions; Order statistics; Type II censored data; Maximum likelihood and Bayesian estimation; Squared error loss function; Linear exponential loss function; Confidence and credible intervals.

Mathematical Subject Classification: 62F10, 62F15, 62N01, 62N02, 62N05

1. Introduction

It is well known that the life length in the real world may be associated with continuous non-negative lifetime distributions; however, sometimes it is difficult to get samples from a continuous distribution in real life. The observed values are discrete because they are usually measured to only a finite number of decimal places and can't really constitute all points in a continuum. Even if the measures are taken on a continuous (ratio or interval) scale, the observations may be recorded in a way making discrete model more appropriate. Therefore, it is reasonable to consider the observations as coming from a discretized distribution generated from the continuous model.

In many practical situations, the reliability data are measured in terms of the numbers of runs, cycles or shocks the device sustains before it fails. For example, the number of times that the devices are switched on/off, the lifetime of the switch is a *discrete random variable* (**DRV**). Also, the number of voltages fluctuations, which an electrical or electronic item can withstand before its failure, is a **DRV**, the life of equipment is measured by the number of completed cycles or the number of times it operated before failure, or the life of weapon is measured by the number of rounds fired prior to failure. Similarly, in survival analysis the *survival function* (SF) may be a function of **DRV** that is considered as a discrete version of the analogue *continuous random variable* (**CRV**). Such as the length of stay in observation ward; when it is measured by the number of days, or the survival time that the leukemia patients survived since therapy may be counted by number of days or weeks.

Many discrete distributions are available to model such mentioned situations, for example, the geometric and negative binomial distributions which are the discrete versions for the exponential and gamma distributions, respectively, but it is well known that they have monotonic hazard rate functions and thus they are unsuitable for some situations.

On the other hand, there are few discrete distributions which can provide accurate models for both count and times. As Poisson distribution, is used to model counts but not times. Also, the binomial distribution is not considered to be popular model for reliability, failure times, count, etc. Beside that it can be approximated to Poisson distribution under suitable conditions. In addition to that, these discrete distributions only cater to positive integers along with zero, but in some analysis the variable of interest can take either zero, positive or negative values. In many situations the interest may be in the difference of two **DRV**s each having integer support $(0, \infty)$. The resulting difference will be another **DRV** with integer support $(-\infty, \infty)$, see Chakraborty and Chakravorty (2016). Thus, there is a need to derive appropriate discrete distributions by discretizing the continuous distributions to fit various types of data. Therefore, the study of the discretization of continuous is meaningful.

There are several methods to construct discrete distributions from the continuous ones, for example discrete analogue of the Pearson system of continuous distributions, discretizing using the *probability density function* (PDF). The distribution generated using this method may not always have a compact form due to the normalizing constant. Also, discretizing can be by shifting the *cumulative distribution function* (CDF), discretizing using *hazard rate function* (HRF), discretizing using SF and two composite methods. For a comprehensive review on this topic, see Bracquemond and Gaudoin (2003) and Chakraborty (2015).

Many researchers studied the general approach of discretization of some known continuous distributions for use as lifetime distributions. For example, Nakagawa and Osaki (1975) proposed a discrete Weibull distribution.

Khan *et al.* (1989) discussed two discrete Weibull distributions and they presented a simple method to estimate the parameters for one of them. They compared this method with the method of moments, and they concluded that the estimates appear to have almost similar properties.

Roy (2003) derived a discrete normal distribution and elaborated its application for evaluating the reliability of complex systems as an alternative to simulation method. Roy (2004) proposed a discrete Rayleigh distribution as a particular case of the discrete Weibull.

Inusah and Kozubowski (2006) obtained a discrete version of the Laplace (double exponential) distribution and discussed some of its statistical properties and statistical issues of estimation under the discrete Laplace model.

Krishna and Pundir (2009) presented the discrete Burr XII distribution and applied it to fit the reliability in series system and a set of real data. Also, they derived the discrete Pareto distribution as a special case of the discrete Burr distribution.

Jazi *et al.* (2010) introduced *discrete inverse Weibull* (DIW) distribution and they studied four methods of estimation (the heuristic algorithm, the inverse Weibull probability paper plot, the method of moments and the method of proportions).

Gomez-Deniz and Calderin-Ojeda (2011) constructed the discrete version of Lindley distribution and used it as an alternative to Poisson distribution to model automobile claim frequency data. Nekoukhou and *et al.* (2012) presented a new version of the discrete generalized exponential distribution, which can be viewed as different generalization of the geometric distribution, some of its distributional and moment properties were discussed. AL-Huniti and AL-

Dayian (2012) proposed the discrete Burr Type III distribution, they discussed some important properties and estimated the parameters based on the maximum likelihood and Bayesian approaches.

Lekshmi and Sebastian (2014) introduced the skewed generalized discrete Laplace distribution which arises as the difference of two independently distributed count variables; they discussed some properties of the distribution and illustrated a real data set. Para and Jan (2014) presented a discrete generalized Burr Type XII distribution. Hussain and Ahmad (2014) proposed the discrete inverse Rayleigh distribution. Hussain *et al*. (2016) obtained the *two-parameter discrete Lindley* (TDL) distribution. Alamatsaz *et al*. (2016) derived the discrete generalized Rayleigh distribution. Para and Jan (2016) obtained the discrete three parameter of Burr Type XII and discrete Lomax distributions. Chakraborty and Chakravorty (2016) proposed the discrete logistic distribution and applied it to model real life count data.

Sarhan (2017) introduced the two-parameter discrete distribution with bathtub hazard shape; he discussed some statistical properties of the distribution. Also, he used three different methods to estimate the parameters and used the bootstrap method to estimate the distributions of these point estimators. Borah and Hazarika (2017) presented the discrete Shanker distribution. Hegazy *et al*. (2018) introduced the discrete Gompertz distribution.

Migdadi (2014) used Bayesian inference to estimate the scale parameter of discrete Rayleigh distribution based on *squared error* (SE) and general entropy loss functions. This study also involved prediction for the future ordered observation. Kamari *et al.* (2015) studied Bayesian analysis of discrete Burr distribution; they used the Metropolis-Hastings method for numerical parameters estimate with two loss functions, SE and absolute error loss functions.

The rest of the paper is organized as follows: *discrete inverted Kumaraswamy* (DIKum) distribution is introduced, and some statistical properties are given in Section 2. Some relationships between DIKum distribution and other well-known distributions are provided in Section 3. While, in Section 4, *maximum likelihood* (ML) and Bayesian estimation are derived. Simulation study and results are presented. In Section 5, a real data set is analyzed showing applicability and flexibility of DIKum distribution.

2. Discretizing a Continuous Distribution

The general approach of discretizing a continuous variable can be used to construct a discrete model by introducing a grouping on the time axis see Roy (2003, 2004). If the **CRV** X has the SF, $S(x) = P(X \geq x)$and times are grouped into unit intervals so that the **DRV** of X denoted by$dX = \lfloor X \rfloor$; which is the largest integer less than or equal to x, will have the *probability mass function* (PMF)

$$P(dX = x) = P(x) = P[x \leq X < x + 1] \\ = S(x) - S(x+1), \quad x = 0,1,2 \ldots \quad (1)$$

The PMF of the **DRV**, dX, can be viewed as discrete concentration of pdf of X. So, given any continuous distribution it is possible to construct corresponding discrete distribution using (1).

One of the advantages of applying this approach of discretizing is that the SF for discrete distributions has the same functional form of the SF for the continuous distributions; as a result, many reliability characteristics and properties remain unchanged. Thus, discretization of a continuous lifetime model according to this approach is an interesting and simple approach to derive a discrete lifetime model corresponding to the continuous one.

2.1 Construction of discrete inverted Kumaraswamy distribution

Gupta *et al.* (1998) introduced two-parameter distribution as a generalization of the standard Pareto of second kind, called the *exponentiated Pareto* (EP) distribution. Also, Abd AL-Fattah *et al.* (2017) derived IKum distribution using special transformation, which has the same PDF of EP distribution. This distribution is important in a wide range of applications; for example, engineering, medical research, stress-strength analysis and lifetime problems. Also, in reliability and biological studies, IKum distribution may be used to model failure rates. Gupta *et al.* (1998) proved that EP distribution is effective in analyzing many lifetime data. EP distribution has failure rates that take decreasing and upside-down bathtub shapes depending on the value of the shape parameters similarly to *exponentiated Weibull* (EW) distribution presented by Mudholkar *et al*. (1995). They observed that exponential distribution, generalized exponential distribution, Weibull distribution, beta distribution, Gamma distribution, uniform distribution, exponentiated exponential distribution, exponentiated Gamma distribution and other distributions can be obtained as special cases of EP distribution. IKum distribution has several applications in different fields, due to its expected wide applicability. Many researchers studied a generalization and multivariate of this distribution (See Iqbal *et al*. (2017), AL-Dayian *et al*. (2020), Usman and Ahsan ul Haq (2020), Abdul Hammed *et al*. (2020) and Aly and Abuelamayem (2020)). The PDF of IKum distribution is given by

$$g(x) = \alpha\beta(1+x)^{-(\alpha+1)}(1-(1+x)^{-\alpha})^{\beta-1}, \quad x > 0;\ \alpha,\beta > 0, \tag{2}$$

where α and β are shape parameters and should be positive.

The corresponding CDF and SF are, respectively, given by

$$G(x) = (1-(1+x)^{-\alpha})^{\beta}, \qquad x > 0;\ \alpha,\beta > 0, \tag{3}$$

and

$$S(x) = 1-(1-(1+x)^{-\alpha})^{\beta}, \qquad x > 0;\ \alpha,\beta > 0. \tag{4}$$

IKum distribution has a long right tail; compared with other commonly used distributions. Thus, it will affect long term reliability predictions, producing optimistic predictions of rare events occurring in the right tail of the distribution compared with other distributions. Also, IKum distribution provides a good fit to several data in literature.

Using (1) *discrete X* (*DX*) can be viewed as the discrete analogue to the continuous IKum variable *X*, and is commonly said to have DIKum distribution with two parameters α and β, denoted by DIKum (α, β) distribution, where the corresponding PMF of *DX* can be written as

$$p(x) = (1-(2+x)^{-\alpha})^{\beta} - (1-(1+x)^{-\alpha})^{\beta}, \quad x = 0,1,2,\dots, \quad \alpha,\beta > 0, \tag{5}$$

and the CDF, SF and HRF are as follows:

$$F(x) = P(X \le x) = 1 - S(x) + P(X = x) = (1-(2+x)^{-\alpha})^{\beta},\ x = 0,1,2,\dots, \tag{6}$$

$$S(x) = P(X \ge x) = 1 - F(x) + P(X = x) = 1-(1-(1+x)^{-\alpha})^{\beta}, \quad x = 0,1,2\dots, \tag{7}$$

and

$$h(x) = \frac{P(x)}{S(x)} = \frac{(1-(2+x)^{-\alpha})^{\beta}-(1-(1+x)^{-\alpha})^{\beta}}{1-(1-(1+x)^{-\alpha})^{\beta}}, \qquad x = 0,1,2,\dots, \quad \alpha,\beta > 0. \tag{8}$$

There are some problems associated with the definition of $h(x)$, three of the more notable ones are given below:

a. $h(x)$ is not additive for series system.
b. The cumulative HRF, $H(x) = \sum h(x) \neq -\ln S(x)$.
c. $h(x) \le 1$ and it has the interpretation of a probability. [For more details, see Xie *et al.* (2002) and Lai (2013) and (2014)].

Therefore, it was necessary to find an alternative definition that is consistent with its continuous counterpart. Roy and Gupta (1992) provide an excellent alternative definition of a discrete HRF denoted by $ah(x)$:

$$ah(x) = \ln\left[\frac{S(x)}{S(x+1)}\right] = \ln\left[\frac{1-(1-(1+x)^{-\alpha})^{\beta}}{=1-(1-(2+x)^{-\alpha})^{\beta}}\right], \qquad x = 0,1,2,\dots,\ \alpha,\beta > 0. \tag{9}$$

There is a relationship between $ah(x)$ and $h(x)$, given by:

$$h(x) = 1 - e^{-ah(x)}. \tag{10}$$

The two concepts $h(x)$ and $ah(x)$ have the same monotonic property, i.e., $ah(x)$ is increasing (decreasing) if and only if $h(x)$ is increasing (decreasing).

Plots of PMF, HRF and *alternative HRF* (AHRF) of DIKum distribution are presented, respectively, in Figures 1-3, for some selected values of the parameters.

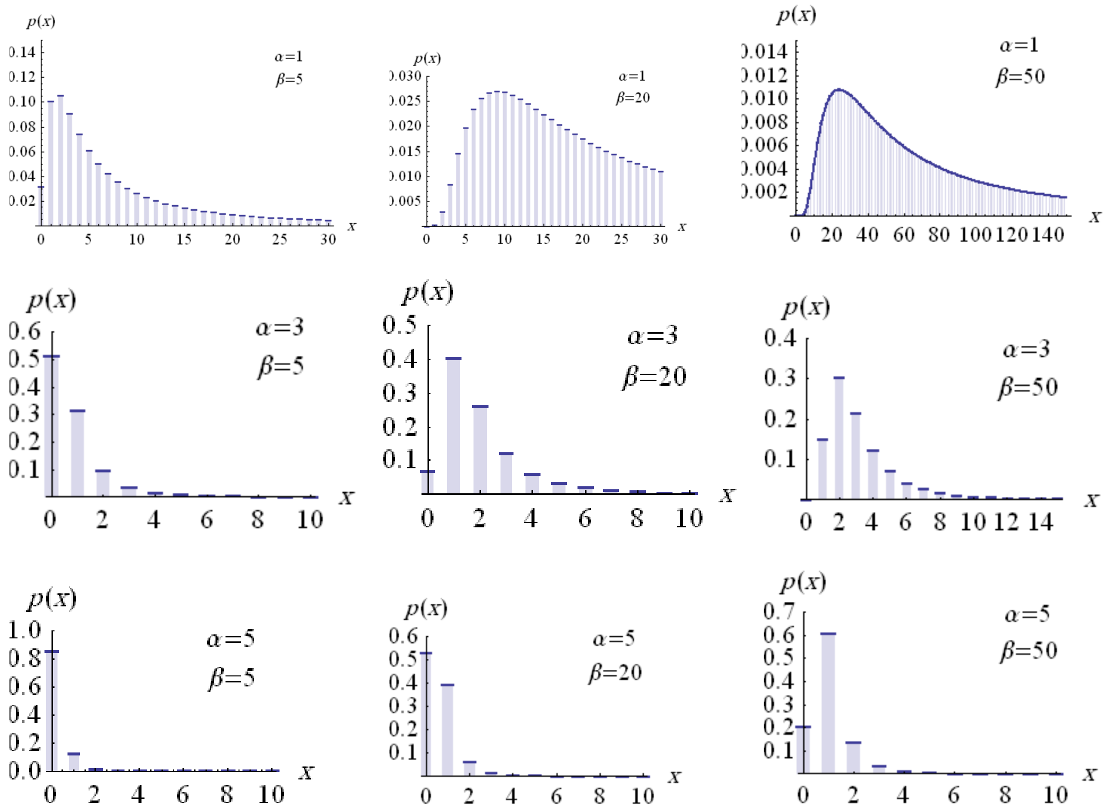


Figure: 1
Plots of the probability mass function

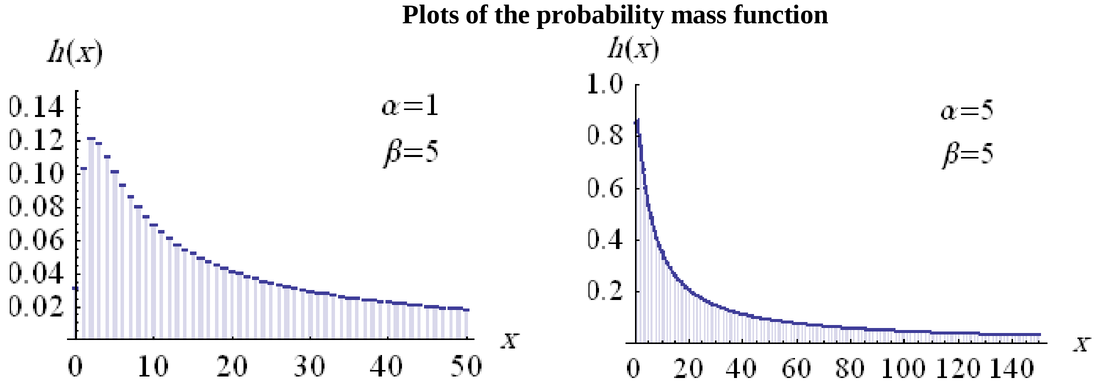


Figure: 2
Plots of the hazard rate function

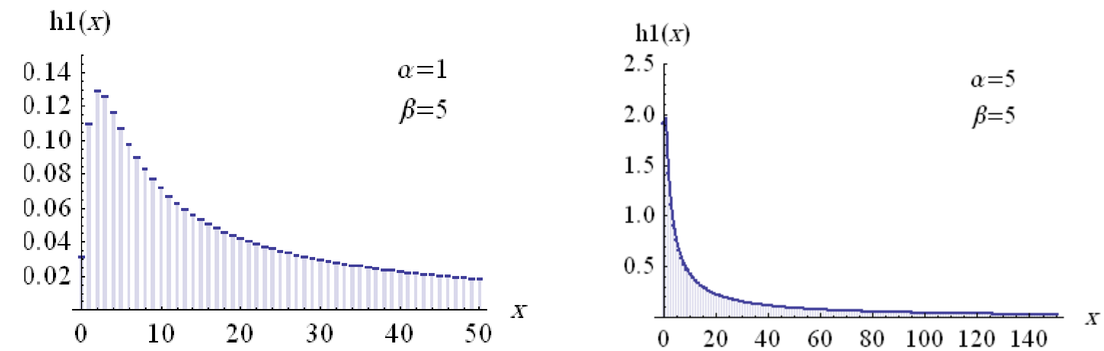


Figure: 3
Plots of the alternative hazard rate function

Figure 1 shows that the PMF of DIKum distribution can be unimodal and right skewed according to the selected values of the parameters. For some values of parameters, the PMF is decreasing over $(0,\infty)$ and the mode is at zero. While for other values of the parameters, it indicates that the PMF is increasing on $(0, x_{mode})$ and reaches the maximum at x_{mode}, then decreases to the zero on (x_{mode},∞); in this case, $x_{mode} = \left\lfloor\left(\frac{\alpha+1}{\alpha\beta+1}\right)^{-\frac{1}{\alpha}} - 1\right\rfloor$. Plots of PMF, HRF and AHRF show that DIKum distribution exhibits a long right tail compared with other commonly used distributions. Thus, it will affect long term reliability predictions, producing optimistic predictions of rare events occurring in the right tail of the distribution compared with other distributions.

Figures 2 and 3 indicate that although the HRF and AHRF of DIKum distribution are decreasing and upside-down bathtub shapes depending on the value of the shape parameters. One can observe that the HRF is less than 1.

2.2 Some properties of discrete inverted Kumaraswamy distribution

This subsection is devoted to obtaining some important distributional properties of DIKum (α,β) distribution, such as the mode, quantiles, r^{th} moments and order statistics.

2.2.1 Mode of discrete inverted Kumaraswamy distribution

The mode of DIKum distribution is at $x_{mode} = \left\lfloor\left(\frac{\alpha+1}{\alpha\beta+1}\right)^{-\frac{1}{\alpha}} - 1\right\rfloor,\ \alpha,\beta > 0.$

This can be easily verified with PMF plots given in Figure 1.

2.2.2 Quantiles of discrete inverted Kumaraswamy distribution

The u^{th} quantile of a **DRV** X, x_u, satisfies

$P(X \le x_u) \ge u$ and $P(X \ge x_u) \ge 1-u$, i.e., $F(x_u - 1) < u \le F(x_u)$. [For more details see Rohatgi and Saleh (2001)].

The u^{th}quantile x_u , of DIKum (α,β) distribution is given by

$$x_u = \left\lceil\left\{\left(1-(u)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}} - 1\right\} - 2\right\rceil, \qquad 0 < u < 1. \tag{11}$$

where $\lceil x\rceil$ denotes the smallest integer greater than or equal to x and $0 < u < 1$.

Proof

$P(X \leq x_u) \geq u$, from (6)

$(1-(2+x)^{-\alpha})^{\beta} \geq u$, hence

$$x_u \geq \left\{\left(1-(u)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}-1\right\}-2. \tag{12}$$

Similarly, if $P(X \geq x_u) \geq 1-u$, one obtains

$$x_u \leq \left\{\left(1-(u)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}-1\right\}-1. \tag{13}$$

Combining (12) and (13), one gets,

$$\left\{\left(1-(u)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}-1\right\}-2 \leq x_u \leq \left\{\left(1-(u)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}-1\right\}-1.$$

Hence, x_u is an integer value given by

$$x_u = \left\lceil\left\{\left(1-(u)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}-1\right\}-2\right\rceil. \tag{14}$$

Thus, the median of DIKum (α, β) distribution can be computed from (14) as follows:

$$x_{0.5} = \left\lceil\left\{\left(1-(0.5)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}-1\right\}-2\right\rceil. \tag{15}$$

2.2.3 The moments of discrete inverted Kumaraswamy distribution

a. The non-central moments of discrete inverted Kumaraswamy distribution

The non-central moments of DIKum distribution can be obtained using (5) as given below

$$\acute{\mu}_r = E(X^r) = \sum_{x=0}^{\infty} x^r\, p(x) \tag{16}$$

$$= \sum_{x=0}^{\infty} x^r \left[(1-(2+x)^{-\alpha})^{\beta} - (1-(1+x)^{-\alpha})^{\beta}\right], \quad r = 1, 2, \dots . \tag{17}$$

In particular, the mean (μ) of DIKum distribution is given by

$$\acute{\mu}_1 = \mu = \sum_{x=0}^{\infty} x \left[(1-(2+x)^{-\alpha})^{\beta} - (1-(1+x)^{-\alpha})^{\beta}\right]. \tag{18}$$

b. The central moments of discrete inverted Kumaraswamy distribution

The central moments can be derived using the relation between the central and non-central moments as given below

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} (-1^j)\mu^j \acute{\mu}_{r-j}, r = 1,2, \dots, \tag{19}$$

thus, the *variance* (var) of DIKum distribution is

$$\mu_2 = \sum_{x=0}^{\infty} X^2 \left[(1-(2+x)^{-\alpha})^{\beta} - (1-(1+x)^{-\alpha})^{\beta}\right] - \left[\sum_{x=0}^{\infty} X \left[(1-(2+x)^{-\alpha})^{\beta} - (1-(1+x)^{-\alpha})^{\beta}\right]\right]^2. \tag{20}$$

c. The standard moments of discrete inverted Kumaraswamy distribution

The r^{th} standard moments can be obtained as follows:

$$\alpha_r = E\left(\frac{X-\mu}{\sigma}\right)^r. \quad (21)$$

The skewness and kurtosis of DIKum distribution are given by, respectively,

$\alpha_3 = \frac{\mu_3}{\mu_2^{\frac{3}{2}}}$ and $\alpha_4 = \frac{\mu_4}{\mu_2^2}$, where $\mu_r = E(X-\mu)^r$, $r = 1,2,\dots$.

2.2.4 The order statistic of discrete inverted Kumaraswamy distribution

Let $F_i(x;\alpha,\beta)$; the cdf of the i^{th} order statistic for a random sample $X_1, X_2, \dots, X_n$, from DIKum (α,β), is given by

$$F_i(x;\alpha,\beta) = \sum_{r=i}^{n}\binom{n}{r}\,[F(x;\alpha,\beta)]^r\,[1-F(x;\alpha,\beta)]^{n-r}. \quad (22)$$

Using the binomial expansion for $[1-F_i(x;\alpha,\beta)]^{n-r}$ and substituting (6) in (22),
where

$$F_i(x;\alpha,\beta) = \sum_{r=i}^{n}\binom{n}{r}\,[F(x;\alpha,\beta)]^r \sum_{j=0}^{n-r}\binom{n-r}{j}\,(-1)^j[F(x;\alpha,\beta)]^j.$$

$$= \sum_{r=i}^{n}\binom{n}{r}\sum_{j=0}^{n-r}\binom{n-r}{j}\,(-1)^j\left[(1-(2+x)^{-\alpha})^{\beta}\right]^{r+j}. \quad (23)$$

Special cases

Case I: If $i = 1$ in (23), one can obtain the distribution function of the first order statistic, as given below

$$F_1(x;\alpha,\beta) = 1-[1-F(x;\alpha,\beta)]^n = 1-\left[1-(1-(2+x)^{-\alpha})^{\beta}\right]^n, \quad (24)$$

Case II: If $i = n$ in (23), then the distribution function of the largest order statistic, is as follows:

$$F_n(x;\alpha,\beta) = [F(x;\alpha,\beta)]^n = \left[(1-(2+x)^{-\alpha})^{\beta}\right]^n. \quad (25)$$

which is the CDF of DIKum $(\alpha, n\beta)$, and the SF of DIKum $(\alpha, n\beta)$ is

$$S(x) = 1-(1-(1+x)^{-\alpha})^{n\beta}. \quad (26)$$

Suppose that $X_1, X_2, \dots, X_n$ is a random sample from DIKum (α,β) distribution. Let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ denote the corresponding order statistics. Then, the PMF of

$X_{i:n}$, is defined by

$$P(X_{i:n} = x) = \frac{n!}{(i-1)!(n-i)!}\int_{F(x-1)}^{F(x)} v^{i-1}(1-v)^{n-i}dv. \quad (27)$$

Using the binomial expansion for $(1-v)^{n-i}$, then, the PMF in (27) is

$$P(X_{i:n} = x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\,(-1)^j\int_{F(x-1)}^{F(x)} v^{i+j-1}dv$$

$$= \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}\,(-1)^j\,\left(\frac{1}{i+j}\right)$$

$$\times\left[\left[(1-(2+x)^{-\alpha})^{\beta}\right]^{i+j}-\left[(1-(1+x)^{-\alpha})^{\beta}\right]^{i+j}\right]. \quad (28)$$

The PMF of the smallest order statistic is obtained by substituting $i = 1$ in (28) as follows:

$$P(X_{1:n} = x) = n \sum_{j=0}^{n-i} \binom{n-1}{j} (-1)^j \left(\frac{1}{1+j}\right)$$
$$\times \left[\left[(1-(2+x)^{-\alpha})^{\beta} \right]^{1+j} - \left[(1-(1+x)^{-\alpha})^{\beta} \right]^{1+j} \right], \quad (29)$$

and the PMF of the largest order statistic is obtained by substituting $i = n$ in (28) as follows:

$$P(X_{n:n} = x) = \frac{n}{n+j} \left[\left[(1-(2+x)^{-\alpha})^{\beta} \right]^{n+j} - \left[(1-(1+x)^{-\alpha})^{\beta} \right]^{n+j} \right]. \quad (30)$$

Also, (23) can be used to obtain the PMF of DIKum (α, β) distribution, (see Arnold *et al*. (2008)).

3. Some Transformations Applied to Discrete Inverted Kumaraswamy Distribution

In this section relationships between DIKum distribution and other well-known distributions are provided through using appropriate transformations, which are displayed in Table 1.

Table 1: Summary of some transformations applied to inverted Kumaraswamy distribution and DIKum distribution and the resulting distributions

Distribution	Transformation	Result	PMF
Inverted Kumaraswamy $X \sim IKum(\alpha,\beta)$	$Y_1 = \lfloor X \rfloor$	$Y_1 \sim DIKum(\alpha,\beta)$	$p(y_1) =$ $(1-(2+y_1)^{-\alpha})^{\beta} - (1-(1+y_1)^{-\alpha})^{\beta}$, $y_1 = 0,1,2,\ldots,\quad \alpha,\beta > 0.$
Discrete inverted Kumaraswamy $X_i's \sim DIKum(\alpha,\beta)$ $X_i's$ $(i = 1,2,3,\ldots n)$ be iid	$Y_2 = \max_{1\le i\le n} X_i$	$Y_2 \sim DIK\sim(\alpha, n\beta)$	$p(y_2) =$ $(1-(2+y_2)^{-\alpha})^{n\beta} - (1-(1+y_2)^{-\alpha})^{n\beta}$, $y_2 = 0,1,2,\ldots,\quad \alpha,\beta > 0.$
Discrete inverted Kumaraswamy $X_i's \sim DIKum(\alpha,\beta_i)$ $X_i's$ are independent	$Y_3 = \max_{1\le i\le n} X_i$	$Y_3 \sim DIKum(\alpha,\beta)$, $\beta = \prod_{i=1}^{n} \beta_i$.	$p(y_3) =$ $(1-(2+y_3)^{-\alpha})^{\beta} - (1-(1+y_3)^{-\alpha})^{\beta}$, $y_3 = 0,1,2,\ldots,\quad \alpha,\beta > 0.$
Discrete inverted Kumaraswamy $X \sim DIKum(\theta,1)$, where $\theta = e^{-\alpha}$	$Y_4 = \ln(2+x)$	$Y_4 \sim$ geometric distribution with $\theta = e^{-\alpha}$.	$p(y_4) = \theta^{y_4}(1-\theta)$, $y_4 = 0,1,2,\ldots,\quad 0<\theta<1.$
Discrete inverted Kumaraswamy $X \sim DIKum(\theta,\beta)$, where $\theta = e^{-\alpha}$	$Y_5 = (\ln(2+x))^{\frac{1}{2}} - 1$	$Y_5 \sim$dicrete generalized Raleigh (θ,β), $\theta = e^{-\alpha}$.	$p(y_5) =$ $\left(1-\theta^{(1+y_5)^2}\right)^{\beta} - \left(1-\theta^{y_5^2}\right)^{\beta}$, $y_5 = 0,1,2,\ldots,$ $0<\theta<1, \beta>0.$
Discrete inverted Kumaraswamy $X \sim DIKum(\theta,1)$, where $\theta = e^{-\alpha}$	$Y_6 = (\ln(2+x))^{\frac{1}{2}} - 1$	$Y_6 \sim$discrete Raleigh (θ), where $\theta = e^{-\alpha}$.	$p(y_6) =$ $\left(1-\theta^{(1+y_6)^2}\right) - \left(1-\theta^{y_6^2}\right)$, $y_6 = 0,1,2,\ldots,$ $0<\theta<1.$
Discrete inverted Kumaraswamy $X \sim DIKum(\theta,1)$, where $\theta = e^{-\alpha}$	$Y_7 = x$	$Y_7 \sim$discrete Pareto$(\theta$	$p(y_7) =$ $\left(\theta^{(1+y_7)}\right) - \left(\theta^{(2+y_7)}\right)$, $y_7 = 0,1,2,\ldots,$ $0<\theta<1.$
Exponential distribution $X \sim exp(\alpha)$	$Y_8 = e^x - 2$	$Y_8 \sim DIKum(\theta,1)$, $\theta = e^{-\alpha}$	$p(y_8) =$ $(1-(2+y_8)^{-\alpha})$-$(1-(1+y_8)^{-\alpha})$, $y_8 = 0,1,2,\ldots,\quad \alpha>0.$

4. Estimation of the Parameters of Discrete Inverted Kumaraswamy Distribution

In this section, ML and Bayesian methods are used to derive the estimators of the parameters for DIKum distribution.

4.1 Method of maximum likelihood

This subsection is devoted to estimate the vector of two parameters, $\underline{\varphi} = (\alpha, \beta)$, SF, HRF and AHRF of DIKum (α, β) distribution, based on Type II censored samples, also confidence interval of the parameters α, β, SF, HRF and AHRF are derived.

Suppose that $X_1, X_2, \ldots, X_r$ is a Type II censored sample of size r obtained from a life-test on n items whose lifetimes have DIKum (α, β) distribution. Then the likelihood function is

$$L\left(\underline{\varphi}; \underline{x}\right) \propto \left\{\prod_{i=1}^{r} p\left(x_{(i)}\right)\right\}\left[S\left(x_{(r)}\right)\right]^{n-r}, \tag{31}$$

where $p(x)$ and $S(x)$ are given, respectively, by (5) and (7). The $x_{(i)}{}'$s are ordered times for $i = 1,2, \ldots r$.

$$L\left(\underline{\varphi}; \underline{x}\right) \propto \left\{\prod_{i=1}^{r} (1-(2+x_i)^{-\alpha})^{\beta} - (1-(1+x_i)^{-\alpha})^{\beta}\right\}$$
$$\times \left[1-(1-(1+x_r)^{-\alpha})^{\beta}\right]^{n-r}. \tag{32}$$

The natural logarithm of the likelihood function is given by

$$\ell \equiv lnL\left(\underline{\varphi}; \underline{x}\right) \propto \; ln \prod_{i=1}^{r} \left[(1-(2+x_i)^{-\alpha})^{\beta} - (1-(1+x_i)^{-\alpha})^{\beta}\right]$$
$$+(n-r)\ln\left[1-(1-(1+x_r)^{-\alpha})^{\beta}\right]. \tag{33}$$

$$= \sum_{i=}^{r} ln\left[(1-(2+x_i)^{-\alpha})^{\beta} - (1-(1+x_i)^{-\alpha})^{\beta}\right]$$
$$+(n-r)\ln\left[1-(1-(1+x_r)^{-\alpha})^{\beta}\right]. \tag{34}$$

Considering the two parameters, α and β are unknown and differentiating the log likelihood function in (34), with respect to α and β, one obtains

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{r} \left\{\frac{\left[\beta(1-(2+x_i)^{-\alpha})^{\beta-1}(2+x_i)^{-\alpha}\, ln(2+x_i)\right] + \left[\beta(1-(1+x_i)^{-\alpha})^{\beta-1}(1+x_i)^{-\alpha}\, ln(1+x_i)\right]}{\left[(1-(2+x_i)^{-\alpha})^{\beta} - (1-(1+x_i)^{-\alpha})^{\beta}\right]}\right\}$$
$$+(n-r)\frac{\beta(1-(1+x_r)^{-\alpha})^{\beta-1}\;(1+x_r)^{-\alpha} ln\,(1+x_r)}{\left[1-(1-(1+x_r)^{-\alpha})^{\beta}\right]}, \tag{35}$$

and

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{r} \left\{\frac{[1-(2+x_i)^{-\alpha}]^{\beta}\ln[1-(2+x_i)^{-\alpha}] - [1-(1+x_i)^{-\alpha}]^{\beta}\ln[1-(1+x_i)^{-\alpha}]}{\left[(1-(2+x_i)^{-\alpha})^{\beta} - (1-(1+x_i)^{-\alpha})^{\beta}\right]}\right\}$$
$$-(n-r)\frac{\{1-(1+x_r)^{-\alpha}\}^{\beta}\ln\{1-(1+x_r)^{-\alpha}\}}{\left[1-\{1-(1+x_r)^{-\alpha}\}^{\beta}\right]}. \tag{36}$$

Then the ML estimators of the parameters, denoted by $\hat{\alpha}$ and $\hat{\beta}$ are derived by equating the two nonlinear likelihood (35) and (36) to zeros and solving numerically.

Depending on the invariance property, the ML estimators of $S(x)$, $h(x)$ and $ah(x)$ can be obtained by replacing α and β with their corresponding ML estimators $\hat{\alpha}$ and $\hat{\beta}$, respectively, in (7), (8) and (9), as given below

$$\hat{S}_{ML}(x) = 1 - \left(1-(1+x)^{-\hat{\alpha}}\right)^{\hat{\beta}}, \qquad x = 1,2,3, \ldots, \tag{37}$$

$$\hat{h}_{ML}(x) = \frac{\left(1-(2+x)^{-\hat{\alpha}}\right)^{\hat{\beta}} - \left(1-(1+x)^{-\hat{\alpha}}\right)^{\hat{\beta}}}{1-\left(1-(1+x)^{-\hat{\alpha}}\right)^{\hat{\beta}}}, \qquad x = 0,1,2, \ldots, \tag{38}$$

and

$$\widehat{ah}_{ML}(x) = ln\left[\frac{1-\left(1-(1+x)^{-\hat{\alpha}}\right)^{\hat{\beta}}}{1-\left(1-(2+x)^{-\hat{\alpha}}\right)^{\hat{\beta}}}\right], \qquad x = 0,1,2,\ldots \quad . \tag{39}$$

When the sample size is large and the regularity conditions are satisfied, see Lehmann and Casella (1998), the asymptotic distribution of the ML estimators is

$\underline{\hat{\varphi}} \sim Bivariate\ Normal\left(\underline{\varphi}, I^{-1}{}_{\underline{x}}\left(\underline{\varphi}\right)\right)$, where $\underline{\varphi} = (\alpha, \beta)$, $\hat{\varphi} = \left(\hat{\alpha}, \hat{\beta}\right)$, and $I^{-1}(\varphi)$ is the asymptotic variance covariance matrix of the ML estimators α and β, which is the inverse of the observed Fisher information matrix. The asymptotic observed Fisher information matrix can be obtained as follows:

$$I_{\underline{x}}(\hat{\varphi}) \approx \left[\begin{matrix} -\left(\frac{\partial^2 \ell}{d\alpha^2}\right) & -\left(\frac{\partial^2 \ell}{\partial\alpha\partial\beta}\right) \\ -\left(\frac{\partial^2 \ell}{\partial\alpha\partial\beta}\right) & -\left(\frac{\partial^2 \ell}{d\beta^2}\right) \end{matrix}\right]\Bigg|_{(\hat{\alpha},\hat{\beta})} . \tag{40}$$

The asymptotic $100(1-\alpha)\%$ confidence interval for $\alpha, \beta, S_{ML}(x), h_{ML}(x)$ and $ah_{ML}(x)$ are given, respectively, by

$$L_{\omega} = \hat{\omega} - Z_{\frac{\alpha}{2}}\,\sigma_{\hat{\omega}}, \qquad \text{and} \qquad U_{\omega} = \hat{\omega} + Z_{\frac{\alpha}{2}}\,\sigma_{\hat{\omega}}, \tag{41}$$

where L_{ω} and U_{ω} are the lower and upper bound respectively, $\hat{\omega}\ is\ \hat{\alpha}, \hat{\beta}, \hat{S}(x), \hat{h}(x)\ or\ \hat{h}_1(x)$, where Z is the100% $\left(1-\frac{\alpha}{2}\right)$th standard normal percentile, $(1-\alpha)$ is confidence coefficient and $\sigma_{\hat{\omega}}$ is the standard deviation.

4.2 Bayesian Estimation

The Bayesian approach is considered, under two types of loss functions, SE and *linear exponential* (LINEX) loss functions, to estimate the parameters, SF, HRF and AHRF of DIKum (α, β) distribution. Bayesian estimators are obtained based on Type II censored samples, using informative prior. Also, credible intervals for the parameters, SF, HRF and AHRF are obtained.

$L\left(\underline{\varphi}; \underline{x}\right)$ in (31) can be written as follows:

$$L\left(\alpha, \beta | \underline{x}\right) \propto \{\textstyle\prod_{i=1}^{r}(w_{i2} - w_{i1})\}{w_r}^{n-r}, \tag{42}$$

where

$$w_{i1} = (1-(1+x_i)^{-\alpha})^{\beta}, w_{i2} = (1-(2+x_i)^{-\alpha})^{\beta}$$

and

$$w_r = 1-(1-(1+x_r)^{-\alpha})^{\beta}. \tag{43}$$

Assuming that the parameters α and β of DIKum distribution are random variables with a joint bivariate prior density function that was used by AL-Hussaini and Jaheen (1992) as

$$\pi(\alpha, \beta) = g_1(\alpha|\beta) g_2(\beta)\ , \qquad \alpha, \beta > 0, \tag{44}$$

where

$$g_1(\alpha|\beta) = \frac{\beta^a}{\Gamma(a)}\alpha^{a-1}e^{-\beta\alpha}, \qquad a, \alpha, \beta > 0, \tag{45}$$

and

$$g_2(\beta) = \frac{b^c}{\Gamma(c)}\,\beta^{c-1}\,e^{-b\beta}, \qquad \beta, b, c > 0. \tag{46}$$

The joint prior density of α and β will be obtained by substituting (45) and (46) in (44) and it's given by

$$\pi(\alpha,\beta) \propto \alpha^{a-1}\, \beta^{a+c-1} e^{-\beta(\alpha+b)}, \qquad \alpha,\beta,a,b,c > 0. \tag{47}$$

The joint posterior distribution for α and β can be derived using (31) and (47) as follows:

$$\pi\big(\alpha,\beta|\underline{x}\big) \propto L\big(\alpha,\beta|\underline{x}\big)\, \pi(\alpha,\beta) \tag{48}$$

$$= k_1 \alpha^{a-1}\, \beta^{a+c-1} e^{-\beta(\alpha+b)} \{\textstyle\prod_{i=1}^{r}(w_{i2}-w_{i1})\} {w_r}^{n-r}, \tag{49}$$

where

$$k_1^{-1} = \int_0^\infty \beta^{a+c-1} \int_0^\infty \alpha^{a-1}\, \{\textstyle\prod_{i=1}^{r}(w_{i2}-w_{i1})\} {w_r}^{n-r} e^{-\beta(\alpha+b)} d\alpha\, d\beta, \tag{50}$$

which is a normalizing constant.

The marginal posterior distributions $\pi\big(\alpha|\underline{x}\big)$ and $\pi\big(\beta|\underline{x}\big)$ are given, respectively, by

$$\pi\big(\alpha|\underline{x}\big) = k_1\ \alpha^{a-1} \int_0^\infty \{\textstyle\prod_{i=1}^{r}(w_{i2}-w_{i1})\}\ {w_r}^{n-r} \beta^{a+c-1} e^{-\beta(\alpha+b)} d\beta, \tag{51}$$

and

$$\pi\big(\beta|\underline{x}\big) = k_1\, \beta^{a+c-1} \int_0^\infty \{\textstyle\prod_{i=1}^{r}(w_{i2}-w_{i1})\}\ {w_r}^{n-r}\ \ e^{-\beta(\alpha+b)} d\alpha, \tag{52}$$

where k_1^{-1} is a normalizing constant given in (50) and w_{i1}, w_{i2} and w_r are defined in (43).

a. Point estimation

The Bayes point estimators of the parameters, SF, HRF and AHRF are considered based on informative prior and two different loss functions: SE and LINEX loss functions.

I. Bayesian estimation under squared error loss function

Under SE loss function the Bayes estimators of the parameters α and β are given by their marginal posterior expectations using (51) and (52), respectively, as shown below

$$\alpha^*_{(SE)} = E(\alpha \mid \underline{x})$$

$$= k_1 \int_0^\infty \alpha^{a} \int_0^\infty \{\textstyle\prod_{i=1}^{r}(w_{i2}-w_{i1})\}\ {w_r}^{n-r} \beta^{a+c-1} e^{-\beta(\alpha+b)} d\beta d\alpha, \tag{53}$$

and

$$\beta^*_{(SE)} = E\big(\beta|\underline{x}\big) = k_1 \int_0^1 \beta^{a+c} \int_0^\infty \{\textstyle\prod_{i=1}^{r}(w_{i2}-w_{i1})\}\ {w_r}^{n-r}\ \ e^{-\beta(\alpha+b)} d\alpha d\beta. \tag{54}$$

Also, the Bayes estimators of the SF, HRF and AHRF under SE loss function can be obtained using (7)-(9) and (49) as follows:

$$S^*_{(SE)}(x) = E\big(S(x)|\underline{x}\big)$$

$$= 1 - k_1^{-1} \int_0^\infty \beta^{a+c-1} \int_0^\infty (1-(1+x)^{-\alpha})^{\beta}\ \alpha^{a-1}\ e^{-\beta(\alpha+b)}$$

$$\times \{\textstyle\prod_{i=1}^{r}(w_{i1}-w_{i1})\} {w_r}^{n-r} d\alpha d\beta, \tag{55}$$

$$h^*_{(SE)}(x) = E\big(h(x)|\underline{x}\big)$$

$$= k_1^{-1} \int_0^\infty \beta^{a+c-1} \int_0^\infty \frac{(1-(2+x)^{-\alpha})^{\beta} - (1-(1+x)^{-\alpha})^{\beta}}{1-(1-(1+x)^{-\alpha})^{\beta}} \alpha^{a-1}\ e^{-\beta(\alpha+b)}$$

$$\times \{\textstyle\prod_{i=1}^{r}(w_{i2}-w_{i1})\} {w_r}^{n-r} d\alpha d\beta, \tag{56}$$

and

$$ah^*_{(SE)}(x) = E\big(ah(x)\big|\underline{x}\big)$$
$$= k_1^{-1}\int_0^\infty \beta^{a+c-1}\int_0^\infty ln\left[\frac{1-(1-(1+x)^{-\alpha})^\beta}{1-(1-(2+x)^{-\alpha})^\beta}\right]\alpha^{a-1}\ e^{-\beta(\alpha+b)}$$
$$\times\{\textstyle\prod_{i=1}^r(w_{i2}-w_{i1})\}w_r{}^{n-r}d\alpha d\beta. \tag{57}$$

II. Bayesian estimation under linear exponential loss function

Under the LINEX loss function, the Bayes estimators for the parameters α and β are given, respectively, by

$$\alpha^*_{(LNX)} = \frac{-1}{\vartheta}\ln E\big(e^{-\vartheta\alpha}\big|\underline{x}\big)$$
$$= k_1\ \frac{-1}{\vartheta}\ ln\big[\int_0^\infty \alpha^{a-1}\ \ e^{-\vartheta\alpha}\int_0^\infty\{\textstyle\prod_{i=1}^r(w_{i2}-w_{i1})\}\ \ w_r{}^{n-r}\beta^{a+c-1}e^{-\beta(\alpha+b)}d\beta d\alpha\big], \tag{58}$$

and

$$\beta^*_{(LNX)} = \frac{-1}{\vartheta}\ln E\big(e^{-\vartheta\beta}\big|\underline{x}\big)$$
$$= k_1\ \frac{-1}{\vartheta}\ ln\big[\int_0^\infty \beta^{a+c-1}e^{-\vartheta\beta}\int_0^\infty \alpha^{a-1}\ \{\textstyle\prod_{i=1}^r(w_{i2}-w_{i1})\}\ \ w_r{}^{n-r}\ \ \ e^{-\beta(\alpha+b)}d\alpha d\beta, \tag{59}$$

where $\vartheta \neq 0$.

Similarly, the Bayes estimators of the SF, HRF and AHRF under LINEX loss function can be obtained from (7)-(9) and (49) as follows:

$$S^*_{(LNX)}(x) = \frac{-1}{\vartheta}\ln E\big(e^{-\vartheta S(x)}\big|\underline{x}\big)$$
$$= 1+\frac{k_1^{-1}}{\vartheta}\,ln\left[\int_0^\infty \beta^{a+c-1}\int_0^\infty e^{-\vartheta(1-(1+x)^{-\alpha})^\beta}\ \ \alpha^{a-1}\ e^{-\beta(\alpha+b)}\right.$$
$$\left.\times\{\textstyle\prod_{i=1}^r(w_{i2}-w_{i1})\}w_r{}^{n-r}\right]d\alpha d\beta, \tag{60}$$

$$h^*_{(LNX)}(x) = \frac{-1}{\vartheta}\ln E\big(e^{-\vartheta h(x)}\big|\underline{x}\big)$$
$$= \frac{-1}{\vartheta}k_1^{-1}ln\left[\int_0^\infty \beta^{a+c-1}\int_0^\infty e^{-\vartheta(1-(1+x)^{-\alpha})^\beta}\ \ \alpha^{a-1}\ e^{-\beta(\alpha+b)}\right.$$
$$\left.\times\{\textstyle\prod_{i=1}^r(w_{i2}-w_{i1})\}w_r{}^{n-r}\right]d\alpha\ d\beta, \tag{61}$$

and

$$ah^*_{(LNX)}(x) = \frac{-1}{\vartheta}\ln E\big(e^{-\vartheta ah(x)}\big|\underline{x}\big)$$
$$= \frac{-1}{\vartheta}k_1^{-1}ln\left[\int_0^\infty \beta^{a+c-1}\int_0^\infty\left[\frac{1-(1-(1+x)^{-\alpha})^\beta}{1-(1-(2+x)^{-\alpha})^\beta}\right]^{-\vartheta}\alpha^{a-1}\ e^{-\beta(\alpha+b)}\right.$$
$$\left.\times\{\textstyle\prod_{i=1}^r(w_{i2}-w_{i1})\}w_r{}^{n-r}\right]d\alpha\ d\beta. \tag{62}$$

To obtain the Bayes estimates of the parameters, SF, HRF and AHRF (53) - (62) should be evaluated numerically.

b. Credible interval for the parameters

In general, a two-sided 100(1- ω) % credible interval of φ is given by

$$P\big[L\big(\underline{x}\big) < \varphi < \ U\big(\underline{x}\big)|\underline{x}\big] = \int_{L(\underline{x})}^{U(\underline{x})}\pi(\varphi|\underline{x})\ d\varphi = 1-\omega\ ,$$

where $L\big(\underline{x}\big)$ and $U\big(\underline{x}\big)$, are the *lower limit* (LL) and *upper limit* (UL).

Since, the marginal posterior distributions are given by (51) and (52), then a 100 $(1-\omega)$ % credible interval for α; $\left(L(\underline{x}), U(\underline{x})\right)$, are given by

$$P\left[\alpha > L(\underline{x})|\underline{x}\,\right] = k_1 \int_{L(\underline{x})}^{\infty} \alpha^{a-1} \int_0^{\infty}\{\prod_{i=1}^{r}(w_{i2}-w_{i1})\}\; w_r{}^{n-r}\beta^{a+c-1}e^{-\beta(\alpha+b)}d\beta\, d\alpha = 1-\frac{\omega}{2}, \quad (63)$$

and

$$P\left[\alpha > U(x)|\underline{x}\,\right] = k_1 \int_{U(\underline{x})}^{\infty} \alpha^{a-1} \int_0^{\infty}\{\prod_{i=1}^{r}(w_{i2}-w_{i1})\}\; w_r{}^{n-r}\beta^{a+c-1}e^{-\beta(\alpha+b)}d\beta\, d\alpha = \frac{\omega}{2}, \quad (64)$$

Also, a 100 (1- ω) % credible interval for β is $\left(L(\underline{x}), U(\underline{x})\right)$ and can be obtained as follows:

$$P\left[\,\beta > L(\underline{x})|\underline{x}\,\right] = k_1 \int_{L(\underline{x})}^{\infty} \beta^{a+c-1} \int_0^{\infty}\{\prod_{i=1}^{r}(w_{i2}-w_{i1})\}\; w_r{}^{n-r}\; e^{-\beta(\alpha+b)}d\alpha d\beta = 1-\frac{\omega}{2}, \quad (65)$$

and

$$P\left[\,\beta > U(\underline{x})|\underline{x}\,\right] = k_1 \int_{U(\underline{x})}^{\infty} \beta^{a+c-1} \int_0^{\infty}\{\prod_{i=1}^{r}(w_{i2}-w_{i1})\}\; w_r{}^{n-r}\; e^{-\beta(\alpha+b)}d\alpha d\beta = \frac{\omega}{2}. \quad (66)$$

Furthermore, a 100 (1- ω) % credible interval for $S(x)$ is $\left(L(\underline{x}), U(\underline{x})\right)$, where

$$P\left[L(\underline{x}) < S(x) < U(\underline{x})|\underline{x}\right] = \int_{L(\underline{x})}^{U(\underline{x})} \pi(S|\underline{x})\, dS = 1-\omega, \quad (67)$$

where $\pi(S|\underline{x})$ is the posterior distribution of SF and $\pi(S|\underline{x}) = \int_0^{\infty}\pi(S,\mathrm{z}|\underline{x})\, dz$.

Let $S \equiv S(x) = 1-(1-(1+x)^{-\alpha})^{\beta}$ and α=z,

$\beta = \left(\frac{ln(1-S)}{ln(1-(1+x)^{-z})}\right)$, so that $\begin{vmatrix}\frac{\partial\alpha}{\partial z} & \frac{\partial\beta}{\partial z}\\ \frac{\partial\alpha}{\partial S} & \frac{\partial\beta}{\partial S}\end{vmatrix} = \frac{1}{(1-S)ln(1-(1+x)^{-z})}$.

The joint posterior distribution of S and z is

$$\pi(S,\mathrm{z}|\underline{x}) = \frac{k_1 z^{a-1}}{(1-S)ln(1-(1+x)^{-z})}\left(\frac{ln(1-S)}{ln(1-(1+x)^{-z})}\right)^{a+c-1} e^{-\left(\frac{ln(1-S)}{ln(1-(1+x)^{-z})}\right)(z+b)} \times \{\prod_{i=1}^{r}(w_{i2*}-w_{i1*})\}\, w_{r*}^{n-r}, \qquad 0<S<1, \mathrm{z}>0, \quad (68)$$

where

$$w_{i2*} = (1-(2+x_i)^{-z})^{\left(\frac{ln(1-S)}{ln\left(1-(1+x_i)^{-z}\right)}\right)}, w_{i1*} = (1-(1+x_i)^{-z})^{\left(\frac{ln(1-S)}{ln\left(1-(1+x_i)^{-z}\right)}\right)}$$

and

$$w_{r*} = 1-(1-(1+x_r)^{-z})^{\left(\frac{ln(1-S)}{ln(1-(1+x_r)^{-z})}\right)}. \quad (69)$$

Hence, the posterior density function for SF is given by

$$\pi(S|\underline{x}) = k_1\int_0^{\infty}\frac{k_1 z^{a-1}}{(1-S)ln(1-(1+x)^{-Z})}\left(\frac{ln(1-S)}{ln(1-(1+x)^{-z})}\right)^{a+c-1} e^{-\left(\frac{ln(1-S)}{ln(1-(1+x)^{-z})}\right)(z+b)} \times \{\prod_{i=1}^{r}(w_{i2*}-w_{i1*})\}w_{r*}{}^{n-r}dz, \qquad 0<S<1. \quad (70)$$

Then a 100 (1- ω) % credible interval for S is $\left(L(\underline{x}), U(\underline{x})\right)$,

$$P\left[S > L(\underline{x})|\underline{x}\right]$$
$$= k_1 \int_{L(\underline{x})}^{1} \int_0^{\infty} \frac{k_1 z^{a-1}}{(1-S)ln\left(1-(1+x)^{-Z}\right)} \left(\frac{ln(1-S)}{ln(1-(1+x)^{-z})}\right)^{a+c-1} e^{-\left(\frac{ln(1-S)}{ln(1-(1+x)^{-Z})}\right)(z+b)}$$
$$\times \{\prod_{i=1}^{r}(w_{i2*} - w_{i1*})\} {w_{r*}}^{n-r} dzdS = 1 - \frac{\omega}{2}, \quad (71)$$

and

$$P\left[S > U(\underline{x})|\underline{x}\right]$$
$$= k_1 \int_{L(\underline{x})}^{1} \int_0^{\infty} \frac{k_1 z^{a-1}}{(1-S)ln\left(1-(1+x)^{-Z}\right)} \left(\frac{ln(1-S)}{ln(1-(1+x)^{-z})}\right)^{a+c-1} e^{-\left(\frac{ln(1-S)}{ln(1-(1+x)^{-Z})}\right)(z+b)}$$
$$\times \{\prod_{i=1}^{r}(w_{i2*} - w_{i1*})\} {w_{r*}}^{n-r} dzdS = \frac{\omega}{2}. \quad (72)$$

5. Numerical Results

This section aims to investigate the precision of the theoretical results based on simulated and real data, by evaluating *relative absolute biases* (RABs) and *relative errors* (REs).

5.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML and Bayes estimates based on generated data from DIKum distribution. ML and Bayes averages of the estimates of the parameters, SF, HRF and AHRF based on Type II censoring are computed. Moreover, confidence and credible intervals for the parameters, SF, HRF and AHR are calculated. All simulation studies are performed using Mathematica 9 and R programming language. The numerical procedures are performed according to the following algorithm.

Step 1: a random sample $X_1, X_2, \dots, X_n$ of sizes (n=30, 60,120) are generated from DIKum distribution using the following transformation:

$x_i = \left\lceil \left\{ \left(1 - (u_i)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}} - 1 \right\} - 2 \right\rceil$, $i = 1,2,..,n$ and u_i are random samples from uniform $(0,1)$ and then taking the ceiling.

Step 2: Two different set values of the parameters are selected as,

Set 1 ($\alpha = 3,\ \beta = 5$) and Set 2 ($\alpha = 5,\ \beta = 50$).

Step 3: For each model parameters and for each sample size, the ML estimates are computed.

Step 4: Steps from 1 to 3 are repeated 5000 times for each sample size and for selected sets of the parameters. Then the averages, RABs, REs and variances of the estimates of the unknown parameters are computed. The RABs, REs, variances of ML and Bayes estimates of the parameters, SF, HRF and AHRF are computed as follows:

1) Averages = $\frac{\sum_{i=1}^{NR} \text{estimates}}{NR}$
2) RABs (estimates) = $\frac{|\text{bias (estimate)}|}{\text{true value}}$,
3) REs = $\frac{\textit{estimated risk}\ \text{(ER)(estimate)}}{\text{true value}}$,
4) Variances (estimate) = $\text{ER(estimate)} - \text{bias}^2\ \text{(estimate)}$,

Table 2 shows the ML averages, RABs, REs, variances, SF, HRF and AHRF estimates, also 95% confidence intervals where the population parameters values for $\alpha = 3,\ \beta = 5$ under three levels of $\frac{r}{n} \times 100$ percentage of uncensored observations Type II censoring 60%, 80% and 100%. Table 3 displays the same computational results, but for different population parameters values $\alpha = 5\ ,\ \beta = 50$, from DIKum distribution for different sample sizes where (n=30, 60 and 120) and level of Type II censoring 60%, 80% and 100% and *number of replications* (NR) = 5000.

Tables 4 and 5 present the Bayes averages of the estimates for the parameters and their RABs, REs and credible intervals under three levels of $\frac{r}{n} \times 100$ percentage of uncensored observations Type II censoring 60%, 80% and 100% for different population parameters values for $\alpha = (3,5), \beta = (5,50)$ and NR = 10000.

Table 6 displays the Bayes averages of the estimates and 95% confidence intervals of the SF, HRF and AHRF at $t_0 = 1$, from DIKum distribution based on Type II censoring for different sample sizes and NR = 10000.

Tables 7-10 present the ML, Bayes estimates and *standard errors* (SE) for the $\alpha, \beta, S(t_0),\ h(t_0)$ and $ah(t_0)$ of DIKum based on Type II censoring for the three real data sets.

5.2 Applications

This subsection aims to demonstrate how the proposed DIKum distribution can be used in practice through analyzing three real lifetime data sets.

Application 1

This real data set is obtained from Hinkley (1977). It consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul.

The data is: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, and 2.05.

Application 2

The second application is given by Murthy *et al.* (2004). The data refers to the time between failures for a repairable item

The data is: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86 and 1.17.

Application 3

The data of this application is considered by Mubarak and Almetwally (2021). This data represents a COVID-19 data which belong to the United Kingdom of 76 days, from 15 April to 30 June 2020. These data formed of drought mortality rate. **The data is**: 0.0587 0.0863 0.1165 0.1247 0.1277 0.1303 0.1652 0.2079 0.2395 0.2751 0.2845 0.2992 0.3188 0.3317 0.3446 0.3553 0.3622 0.3926 0.3926 0.4110 0.4633 0.4690 0.4954 0.5139 0.5696 0.5837 0.6197 0.6365 0.7096 0.7193 0.7444 0.8590 1.0438 1.0602 1.1305 1.1468 1.1533 1.2260 1.2707 1.3423 1.4149 1.5709 1.6017 1.6083 1.6324 1.6998 1.8164 1.8392 1.8721 1.9844 2.1360 2.3987 2.4153 2.5225 2.7087 2.7946 3.3609 3.3715 3.7840 3.9042 4.1969 4.3451 4.4627 4.6477 5.3664 5.4500 5.7522 6.4241 7.0657 7.4456 8.2307 9.6315 10.1870 11.1429 11.2019 11.4584.

To check the validity of the fitted model, *Kolmogorov-Smirnov* (K-S) goodness of fit test is performed for the three data sets. The p values are 0.799, 0.239 and 0.907, respectively. The p values show that DIKum fits the data very well.

The real data sets are provided to illustrate the flexibility and applicability of DIKum distribution. DIKum distribution is compared to some distributions such as DIW distribution introduced by Jazi *et al*. (2010), *discrete modified inverse Rayleigh* (DMIR) distribution proposed by Shahid and Raheel (2019), *exponentiated discrete inverse*

Rayleigh (EDIR) presented by Mashhadzadeh and Mirmostafaee (2020), and TDL distribution considered by Hussain *et al*. (2016).

The comparison is done by using K-S statistic, the corresponding p-value and other criteria for the purpose of model selection including *Akaike information criterion* (*AIC*), *Akaike information criterion with correction* (*AICC*) and *Bayesian information criterion* (*BIC*), where

$$AIC = 2k - 2log(L),$$

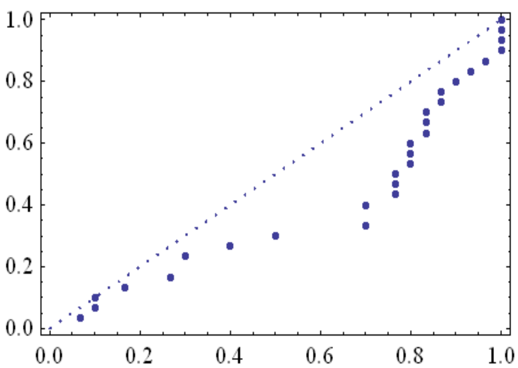
$$AICC = AIC + 2\frac{k(k+1)}{n-k-1},$$

and

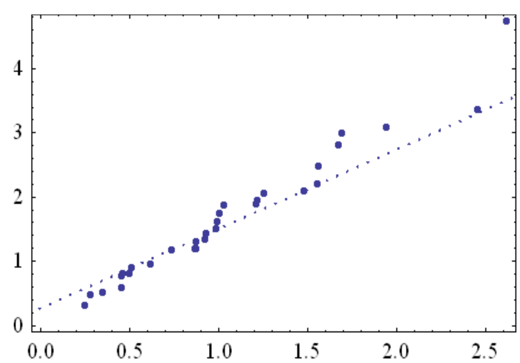
$$BIC = klog(n) - 2log(L),$$

where k denotes the number of the estimated parameters, L is the maximized value of the likelihood function for the estimated model, and n is the sample size. The distribution which has the lowest values of *AIC*, *AICC*, *BIC* and the highest p-value, fits better to the real data set.

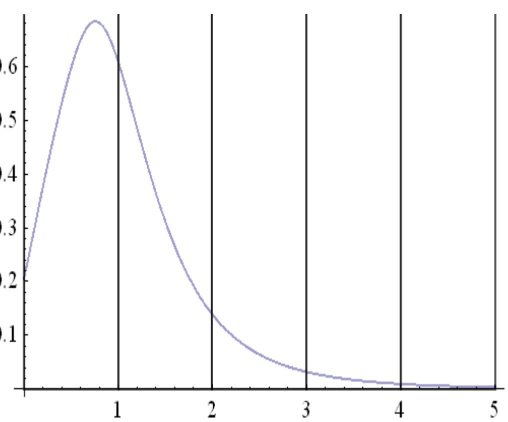
The comparison was applicable for the second and third real data only. The p-values show that DIW, DMIR and TDL do not fit the first real data which ensures that DIKum distribution is the best to model the first real data.



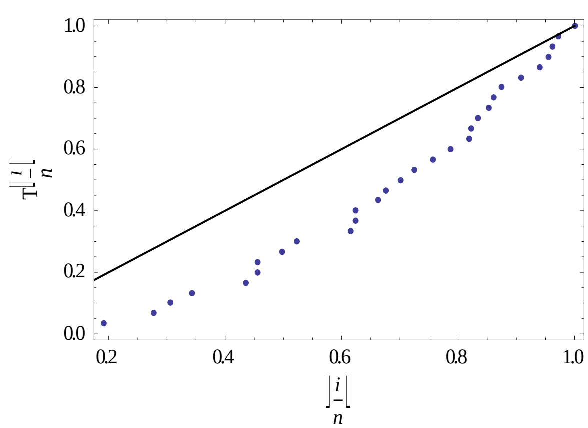
PP-plot for the first data set



QQ-plot for the first data set



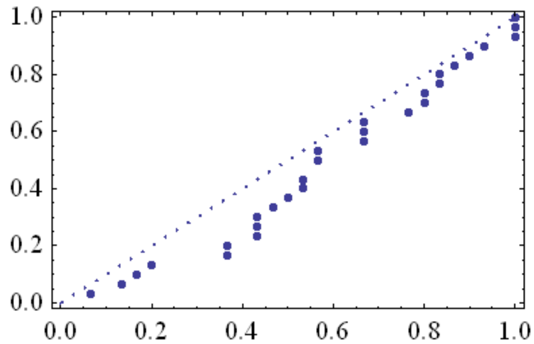
Plot of the fitted PMF



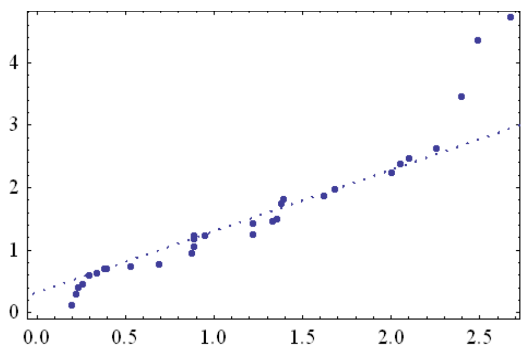
TTT-plot for the first data set

Figure 4: The PP-plot, QQ-plot, fitted PMF and TTT-plot of DIKum distribution for the first data set

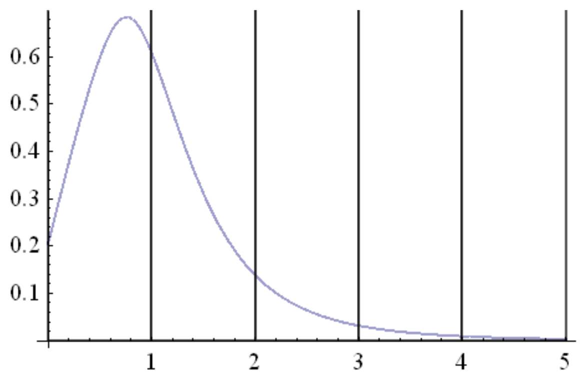
 DOI: http://dx.doi.org/10.18187/pjsor.v18i1.3634



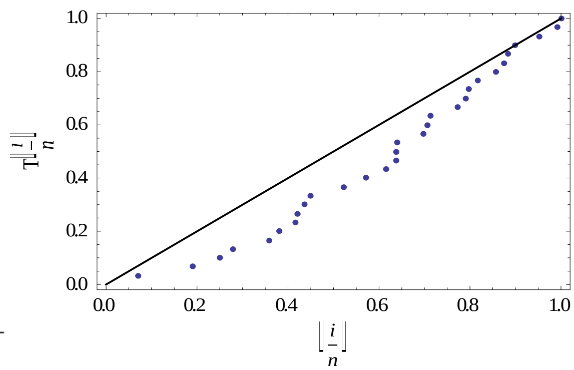
PP-plot for the second data set



QQ-plot for the second data set

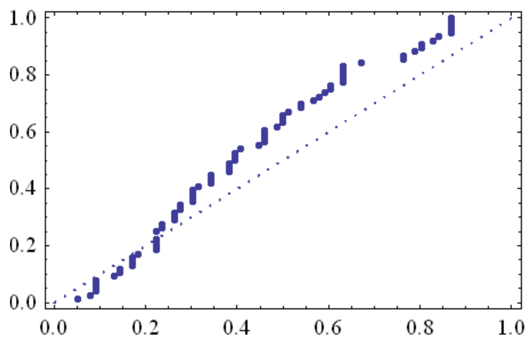


Plot of the fitted PMF

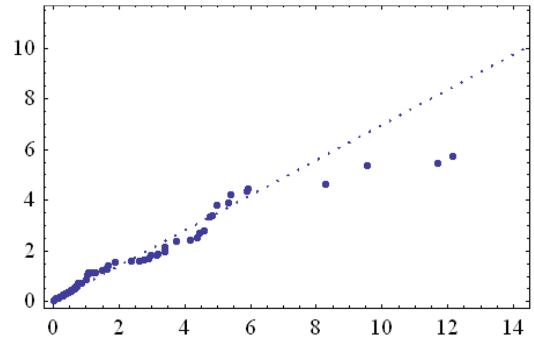


TTT-plot for the second data set

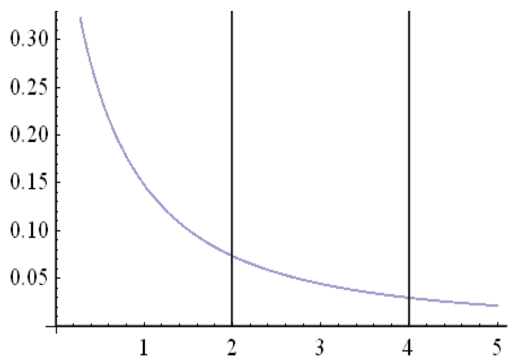
Figure 5: The PP-plot, QQ-plot, fitted PMF and TTT-plot of DIKum distribution for the second data set



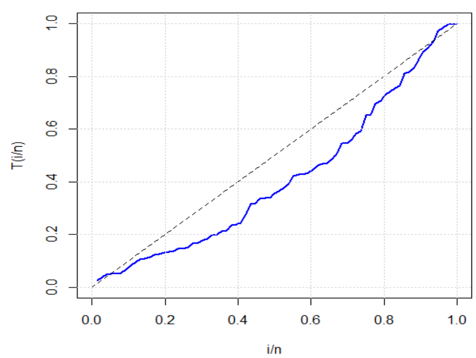
PP-plot for the third data set



QQ-plot for the third data set



Plot of the fitted PMF



TTT-plot for the third data set

Figure 6: The PP-plot, QQ-plot, fitted PMF and TTT-plot of DIKum distribution for the third data set

4.7.3 Concluding remarks

I. From Tables 2, 3 and 7 the RABs, variances and REs of the ML averages of the estimates for the parameters α and β decrease when the sample size n increases. Also, it is observed that as the level of censoring decreases the RABs, variances and REs of the ML estimates of the parameters, SF, HRF and the AHRF estimates decrease. The lengths of the confidence intervals become narrower as the sample size increases.

II. It is noticed, from Tables 4-6, 8 and 9 that the RABs, REs for the estimates of the parameters, SF, HRF, AHRF and the credible interval lengths of the parameters, SF, HRF and AHRF under LINEX loss function have less values than the corresponding RABs, REs and the credible interval lengths under the SE loss function.

III. It is observed that less RABs and REs, obtained for complete sample sizes, are less than the corresponding results for censored samples. Also, the results perform better when n and r get larger.

IV. The Bayes intervals include the estimates (between the LL and UL).

V. Table 9 and 10 conclude the ML estimates and corresponding *standard error* (SE), K-S statistic with its corresponding p-value, -2LL, AIC, BIC and CAIC. The results in these tables indicate DIKum distribution has the smallest values of -2LL, AIC, BIC, CAIC, *K–S* and highest p-value. That means that DIKum distribution is better fit for this data compared with other distributions used here.

VI. The *total time test* (TTT) plot can be used to get information about the shape of the HRF of a given data set, which helps in selecting a particular model to fit a provided data set. Figures 4-6 show the TTT plots of the three real data sets which ensure that the HRF is decreasing. Moreover, the fitted PMF, PP and QQ plots indicate that DIKum distribution fit for the three real data sets.

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Competing Interests
Authors have declared that no competing interests exist.

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Table 2: ML averages relative absolute biases, relative errors, variances of the ML estimates, 95%confidence intervals of the parameters, survival, hazard rate and alternative hazard rate functions from DIKum distribution for different sample sizes n, censoring size r and NR= 5000 ($\alpha = 3$, $\beta = 5$)

n	*r*	Parameters	Averages	RABs	REs	Variance	UL	LL	Length
30	18	α	2.4058	0.1980	0.2090	0.0401	2.7984	2.0132	0.7852
		β	5.3932	0.0849	0.2755	0.7954	7.1413	3.6451	3.4962
		$S(t_0)$	0.6675	0.3703	0.4109	0.0075	0.8374	0.4976	0.3398
		$h(t_0)$	0.5129	0.2072	0.2315	0.0045	0.6437	0.3821	0.2617
		$ah(t_0)$	0.7275	0.3012	0.3241	0.0155	0.9714	0.4837	0.4877
	24	α	2.4147	0.1951	0.2049	0.0351	2.7819	2.0475	0.7343
		β	5.1902	0.0786	0.1949	0.7182	6.6346	3.7459	2.8886
		$S(t_0)$	0.6575	0.3500	0.3859	0.0062	0.8128	0.5023	0.3105
		$h(t_0)$	0.5158	0.2026	0.2276	0.0045	0.6473	0.3924	0.2531
		$ah(t_0)$	0.7334	0.2955	0.3180	0.0149	0.9731	0.4937	0.4793
	30	α	2.5302	0.1589	0.1593	0.0020	2.6198	2.4406	0.1791
		β	5.4246	0.0380	0.1522	0.5430	6.9938	4.8553	2.1384
		$S(t_0)$	0.6337	0.3011	0.3300	0.0043	0.7626	0.5049	0.2576
		$h(t_0)$	0.5446	0.1580	0.1607	0.0003	0.5815	0.5078	0.0736
		$ah(t_0)$	0.7875	0.2435	0.2466	0.0016	0.8666	0.7084	0.1581
60	36	α	2.4268	0.1911	0.1983	0.0252	2.7380	2.1156	0.6223
		β	5.2371	0.0474	0.1952	0.483	6.2950	4.1792	2.1158
		$S(t_0)$	0.6561	0.347	0.3771	0.0052	0.7970	0.5152	0.2818
		$h(t_0)$	0.5205	0.1955	0.2150	0.0019	0.6339	0.4070	0.2269
		$ah(t_0)$	0.7411	0.2882	0.3057	0.0113	0.9491	0.5330	0.4162
	48	α	2.4386	0.1871	0.1920	0.0166	2.6915	2.1857	0.5057
		β	5.0661	0.0247	0.1179	0.2913	5.7275	4.4048	1.3227
		$S(t_0)$	0.6425	0.3191	0.3381	0.0029	0.7492	0.5358	0.2134
		$h(t_0)$	0.5286	0.1829	0.1955	0.0009	0.6162	0.44105	0.1751
		$ah(t_0)$	0.7558	0.2741	0.2852	0.0067	0.9164	0.5951	0.3212
	60	α	2.5232	0.1602	0.1604	0.0010	2.5879	2.4583	0.1296
		β	5.0658	0.0131	0.0688	0.1138	5.9745	5.1571	0.8173
		$S(t_0)$	0.6147	0.2620	0.2809	0.0024	0.7114	0.5180	0.1934
		$h(t_0)$	0.5506	0.1489	0.1505	0.0002	0.5788	0.5223	0.0564
		$ah(t_0)$	0.8004	0.2312	0.2331	0.0009	0.8610	0.7396	0.1214
120	72	α	2.4542	0.1819	0.1852	0.0107	2.6566	2.2517	0.4049
		β	5.1233	0.0322	0.1202	0.0915	5.7162	4.5304	1.1858
		$S(t_0)$	0.6426	0.3193	0.3326	0.0021	0.7316	0.5537	0.1779
		$h(t_0)$	0.5315	0.1784	0.1872	0.0013	0.6033	0.4597	0.1436
		$ah(t_0)$	0.7608	0.2693	0.2770	0.0046	0.8933	0.6283	0.2651
	96	α	2.4543	0.1819	0.1832	0.0041	2.5799	2.3287	0.2512
		β	5.0272	0.0132	0.0653	0.0335	5.3436	4.7107	0.6330
		$S(t_0)$	0.6365	0.3068	0.3122	0.0008	0.6919	0.5751	0.1168
		$h(t_0)$	0.5346	0.1736	0.1766	0.0004	0.5755	0.4937	0.0818
		$ah(t_0)$	0.7658	0.2645	0.2674	0.0016	0.8453	0.6862	0.1590
	120	α	2.5193	0.1565	0.1573	0.0005	2.5625	2.4761	0.0865
		β	5.8389	0.0054	0.0327	0.0261	5.9742	5.7036	0.2706
		$S(t_0)$	0.6025	0.2369	0.2447	0.0008	0.6609	0.5502	0.1107
		$h(t_0)$	0.5545	0.1429	0.1436	8.32×10^{-5}	0.5724	0.5366	0.0358
		$ah(t_0)$	0.8088	0.2232	0.2240	0.0004	0.8474	0.7701	0.0773

Table 3: ML averages relative absolute biases, relative errors, variances of ML estimates, 95%confidence intervals of the parameters, survival, hazard rate and the alternative hazard rate functions from DIKum distribution for different sample sizes n, censoring size r and NR= 5000 ($\alpha = 5,\ \beta = 50$)

n	*r*	Parameters	Averages	RABs	REs	Variance	UL	LL	Length
30	18	α	4.3601	0.1320	0.1329	0.0100	4.5564	4.1638	0.3925
		β	51.5789	0.0316	0.0210	0.9232	52.4621	49.6957	2.7664
		$S(t_0)$	0.9228	0.1613	0.1617	0.0001	0.9461	0.8994	0.0467
		$h(t_0)$	0.6202	0.1934	0.1966	0.0013	0.6908	0.5497	0.1412
		$ah(t_0)$	0.9717	0.3363	0.3386	0.0060	1.1238	0.8195	0.3044
	24	α	4.3934	0.1280	0.1295	0.0071	4.5586	4.2281	0.3305
		β	51.4992	0.0300	0.0370	0.7023	52.1417	49.8566	2.2852
		$S(t_0)$	0.9181	0.1599	0.1606	0.0001	0.9387	0.8974	0.0414
		$h(t_0)$	0.6300	0.1901	0.1958	0.0009	0.6881	0.5719	0.1163
		$ah(t_0)$	0.9967	0.3306	0.3349	0.0043	1.1246	0.8689	0.2557
	30	α	4.3401	0.1213	0.1225	0.0062	4.4944	4.1858	0.3086
		β	51.0067	0.0201	0.0343	0.0930	51.6046	50.4089	1.1957
		$S(t_0)$	0.9239	0.1540	0.1546	7.21×10^{-5}	0.9405	0.9073	0.0333
		$h(t_0)$	0.6177	0.1773	0.1815	0.0007	0.6705	0.5648	0.1057
		$ah(t_0)$	0.9634	0.3134	0.3166	0.0033	1.0755	0.8512	0.2242
60	36	α	4.3838	0.1278	0.1280	0.0022	4.4751	4.2925	0.1827
		β	51.4433	0.0289	0.0191	0.2451	52.4136	50.4730	1.9406
		$S(t_0)$	0.9198	0.1579	0.1580	4.52×10^{-5}	0.9329	0.9066	0.0263
		$h(t_0)$	0.6282	0.1849	0.1854	0.0002	0.6552	0.6012	0.0541
		$ah(t_0)$	0.9900	0.3255	0.3260	0.0012	1.0587	0.9214	0.1372
	48	α	4.4162	0.1232	0.1236	0.0016	4.4948	4.3375	0.1573
		β	51.3599	0.0272	0.0305	0.0909	51.9508	50.7690	1.1818
		$S(t_0)$	0.9147	0.1561	0.1564	3.82×10^{-5}	0.9269	0.9026	0.0242
		$h(t_0)$	0.6373	0.1797	0.1806	0.0001	0.6593	0.6154	0.0439
		$ah(t_0)$	1.0147	0.3180	0.3189	0.0009	1.0741	0.9553	0.1188
	60	α	4.3611	0.1168	0.1170	0.0012	4.4278	4.2944	0.13336
		β	50.9473	0.0189	0.0279	0.0129	51.1703	50.7243	0.4460
		$S(t_0)$	0.9212	0.1498	0.1500	1.92×10^{-5}	0.9297	0.9126	0.0172
		$h(t_0)$	0.6242	0.1678	0.1684	0.0001	0.6446	0.6038	0.0408
		$ah(t_0)$	0.9791	0.3010	0.3017	0.0006	1.0277	0.9304	0.0973
120	72	α	4.3959	0.1259	0.1259	0.0009	4.4555	4.3364	0.1191
		β	51.4420	0.0288	0.0301	0.1853	52.2857	50.5984	1.6873
		$S(t_0)$	0.9181	0.1561	0.1562	2.07×10^{-5}	0.9270	0.9092	0.0178
		$h(t_0)$	0.6316	0.1812	0.1813	6.94×10^{-5}	0.6480	0.6153	0.0327
		$ah(t_0)$	0.9989	0.3205	0.3206	0.0005	1.0428	0.9550	0.0877
	96	α	4.4303	0.1208	0.1210	0.0008	4.4854	4.3751	0.1103
		β	51.3119	0.0262	0.0265	0.0396	51.7021	50.9218	0.7803
		$S(t_0)$	0.9125	0.1541	0.1542	1.95×10^{-5}	0.9212	0.9039	0.0173
		$h(t_0)$	0.6413	0.1752	0.1755	5.92×10^{-5}	0.6564	0.6262	0.0302
		$ah(t_0)$	1.0255	0.3118	0.3122	0.0004	1.0670	0.9840	0.0830
	120	α	4.371	0.1139	0.1141	0.0003	4.4060	4.3353	0.0707
		β	50.9235	0.0185	0.0185	0.0032	51.0341	50.8128	0.2214
		$S(t_0)$	0.9198	0.1470	0.1471	$6.4220 \times 10^{-}$	0.9247	0.9148	0.0099
		$h(t_0)$	0.6270	0.1626	0.1629	$2.4319 \times 10^{-}$	0.6367	0.6174	0.0193
		$ah(t_0)$	0.9863	0.2935	0.2939	0.0002	1.0120	0.9607	0.0513

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Table 4: Bayes averages, relative absolute biases, relative error of the Bayes estimates and 95% credible intervals of the parameters α and β based on Type II censoring of DIKum ($NR = 10000, t_0 = 10, \alpha = 3, \beta = 5$)

n	r	Par	SE						LINEX($v = 0.1$)					
			Averages	RAB	RE	UL	LL	Length	Averages	RAB	RE	UL	LL	Length
30	60% 18	α	2.7699	0.0767	0.0114	2.9931	2.5890	0.4041	2.9710	0.0097	2.64×10^{-4}	3.0140	2.8723	0.1417
		β	4.7736	0.0452	0.0064	4.9676	4.5504	0.4171	5.0615	0.0156	0.0017	5.2274	4.8453	0.3821
	80% 24	α	3.1271	0.0424	0.0036	3.2451	2.9532	0.2919	2.9842	0.0053	2.54×10^{-4}	3.0483	2.9167	0.1316
		β	5.0582	0.0116	0.0015	5.2373	4.8931	0.3442	4.9219	0.0123	2.35×10^{-4}	5.0022	4.7960	0.2062
	100% 30	α	2.9883	0.0039	2.96×10^{-5}	2.9988	2.9750	0.0238	3.0041	0.0014	4.80×10^{-6}	3.0096	2.9968	0.0127
		β	5.0119	0.0024	2.50×10^{-5}	5.0278	4.9935	0.0343	5.0064	0.0013	4.89×10^{-6}	5.0103	4.9996	0.0107
60	60% 36	α	2.9934	0.0022	1.36×10^{-5}	3.0030	2.9790	0.0240	3.0019	6.65×10^{-4}	3.88×10^{-6}	3.0086	2.9944	0.0141
		β	4.9994	0.0027	3.34×10^{-5}	5.0179	4.9793	0.0386	4.9926	0.0014	7.38×10^{-6}	5.0003	4.9850	0.0153
	80% 48	α	3.0063	0.0021	9.97×10^{-6}	3.0134	2.9955	0.0179	3.0012	3.96×10^{-4}	1.48×10^{-6}	3.0077	2.9964	0.0113
		β	4.9863	3.17×10^{-4}	1.72×10^{-5}	5.0024	4.9678	0.0345	4.9992	1.59×10^{-4}	1.22×10^{-6}	5.0052	4.9926	0.0126
	100% 60	α	3.0015	5.12×10^{-4}	7.63×10^{-7}	3.0037	2.9990	0.0047	3.0008	2.80×10^{-4}	1.56×10^{-7}	3.0015	3.0000	0.0015
		β	4.9986	2.84×10^{-4}	2.37×10^{-7}	4.9997	4.9974	0.0023	4.9997	6.66×10^{-5}	2.16×10^{-8}	5.0002	4.9991	0.0011
120	60% 72	α	2.9989	3.76×10^{-4}	3.46×10^{-7}	2.9998	2.9968	0.0030	2.9994	1.91×10^{-4}	8.40×10^{-8}	3.0001	2.9986	0.0014
		β	5.0011	2.27×10^{-4}	3.46×10^{-7}	5.0031	4.9990	0.0041	4.9989	2.23×10^{-4}	1.57×10^{-7}	4.9998	4.9974	0.0024
	80% 96	α	3.0002	6.12×10^{-5}	4.66×10^{-8}	3.0012	2.9992	0.0020	2.9999	4.90×10^{-5}	1.64×10^{-8}	3.0003	2.9993	0.0010
		β	5.0009	1.84×10^{-4}	1.32×10^{-7}	5.0018	4.9991	0.0027	4.9995	1.04×10^{-4}	4.98×10^{-8}	5.0003	4.9986	0.0017
	100% 120	α	3.0001	4.39×10^{-5}	3.80×10^{-9}	3.0003	3.0000	0.0003	3.0000	2.60×10^{-5}	1.15×10^{-9}	3.0000	2.9999	0.0001
		β	4.9999	2.05×10^{-5}	1.38×10^{-9}	5.0000	4.9997	0.0003	5.0001	1.68×10^{-5}	1.16×10^{-9}	5.0002	5.0000	0.0002

Table 5: Bayes averages, relative absolute biases, relative error of the Bayes estimates and 95% credible intervals of the parameters α and β based on Type II censoring of DIKum ($NR = 10000, t_0 = 10, \alpha = 5, \beta = 50$)

n	*r*	Par	SE						LINEX($v = 0.1$)					
			Averages	RAB	RE	UL	LL	Length	Averages	RAB	RE	UL	LL	Length
30	60%	α	3.7607	0.2478	0.1945	4.7620	2.5532	2.2088	5.5311	0.1062	0.0282	5.9715	4.8043	1.1671
	18	β	45.7949	0.0841	0.2240	49.9432	42.4794	7.4637	48.8031	0.0239	0.01432	49.9723	47.6042	2.3681
	80%	α	4.9690	6.20×10^{-3}	5.48×10^{-4}	5.0567	4.7860	0.2707	5.0077	3.53×10^{-3}	5.93×10^{-6}	5.0742	4.9484	0.1258
	24	β	50.1109	2.21×10^{-3}	1.76×10^{-4}	50.2608	49.9402	0.3206	49.9865	2.70×10^{-4}	1.83×10^{-6}	50.0296	49.9394	0.0902
	100%	α	5.0154	3.08×10^{-3}	2.99×10^{-5}	5.0265	4.9987	0.0278	4.9895	2.09×10^{-3}	1.09×10^{-5}	4.9984	4.9789	0.0195
	30	β	49.9949	1.02×10^{-4}	4.14×10^{-7}	50.0038	49.9856	0.0182	50.0019	3.75×10^{-5}	5.64×10^{-8}	50.0059	49.9972	0.0087
60	60%	α	4.9698	6.03×10^{-3}	3.38×10^{-4}	5.0386	4.8380	0.2006	4.9866	2.66×10^{-3}	1.77×10^{-7}	5.0370	4.9306	0.1064
	36	β	49.9037	1.92×10^{-3}	1.20×10^{-4}	49.9911	49.7770	0.2141	49.9851	2.98×10^{-4}	2.22×10^{-6}	50.0729	49.8602	0.2127
	80%	α	4.9893	2.13×10^{-3}	1.61×10^{-5}	4.9988	4.9751	0.0237	5.0055	1.10×10^{-3}	3.03×10^{-6}	5.0123	4.9988	0.0134
	48	β	49.9838	3.23×10^{-4}	4.97×10^{-6}	50.0080	49.9589	0.0491	49.9974	5.15×10^{-5}	6.66×10^{-8}	50.0021	49.9922	0.0098
	100%	α	4.9995	1.01×10^{-4}	3.84×10^{-8}	5.0001	4.9985	0.0016	5.0003	5.23×10^{-5}	1.50×10^{-8}	5.0007	4.9995	0.0012
	60	β	49.9974	5.20×10^{-5}	7.39×10^{-8}	49.9994	49.9959	0.0035	49.9998	3.44×10^{-6}	3.55×10^{-8}	50.0001	49.9993	0.0007
120	60%	α	4.9978	4.32×10^{-4}	4.86×10^{-6}	5.0081	4.9812	0.0269	4.9993	1.47×10^{-4}	6.28×10^{-8}	5.0021	4.9946	0.0074
	72	β	50.0153	3.06×10^{-4}	3.03×10^{-6}	50.0291	49.9934	0.0357	50.0057	1.13×10^{-4}	3.22×10^{-7}	50.0167	49.9980	0.0187
	80%	α	5.0016	3.27×10^{-4}	4.47×10^{-7}	5.0040	4.9994	0.0046	5.0001	1.57×10^{-5}	8.78×10^{-9}	5.0012	4.9992	0.0019
	96	β	50.0021	4.33×10^{-5}	5.98×10^{-8}	50.0035	49.9997	0.0038	50.0014	2.91×10^{-5}	2.14×10^{-8}	50.0023	49.9999	0.0024
	100%	α	4.9999	1.51×10^{-5}	1.66×10^{-9}	5.0001	4.9997	0.0003	5.0000	5.79×10^{-6}	1.27×10^{-10}	5.0000	4.9999	0.0001
	120	β	49.9999	2.85×10^{-6}	2.48×10^{-8}	50.0000	49.9997	0.0003	49.9999	1.18×10^{-6}	4.53×10^{-9}	50.0000	49.9999	0.0001

Table 6: Bayes averages, relative absolute biases, relative errors of the Bayes estimates and 95% credible intervals of the $S(t_0)$, $h(t_0)$ and $ah(t_0)$ based on Type II censoring of DIKum ($NR = 10000, t_0 = 1, \alpha = 5, \beta = 50$)

n	*r*	Par	SE						LINEX($v = 0.1$)					
			Averages	**RAB**	**RE**	**UL**	**LL**	**Length**	**Averages**	**RAB**	**RE**	**UL**	**LL**	**Length**
30	60% 18	$S(t_0)$	0.6191	0.2216	0.0268	0.7561	0.3716	0.3845	0.7347	0.0764	3.01×10^{-3}	0.7961	0.6694	0.1267
		$h(t_0)$	0.5653	0.7727	0.2752	0.7459	0.3704	0.3755	0.6503	0.1507	0.0109	0.7538	0.5545	0.1992
		$ah(t_0)$	2.3258	1.0747	0.9443	3.0717	1.8260	1.2457	1.8209	0.7269	0.4164	2.3576	1.2653	1.0923
	80% 24	$S(t_0)$	0.7839	0.0145	3.62×10^{-3}	0.8911	0.6289	0.2621	0.7894	7.72×10^{-3}	2.64×10^{-4}	0.8277	0.7424	0.0853
		$h(t_0)$	0.6995	0.0864	4.42×10^{-3}	0.7869	0.6105	0.1763	0.7415	0.0316	6.32×10^{-4}	0.7717	0.6967	0.0749
		$ah(t_0)$	1.5985	0.5736	0.2444	1.7861	1.3879	0.3982	1.5890	0.5671	0.2355	1.6912	1.4473	0.2439
	100% 30	$S(t_0)$	0.7968	3.97×10^{-3}	1.05×10^{-5}	0.8041	0.7884	0.0157	0.7987	1.59×10^{-3}	9.87×10^{-6}	0.8022	0.7931	0.0091
		$h(t_0)$	0.7539	1.54×10^{-2}	1.30×10^{-4}	0.7682	0.7408	0.0274	0.7599	7.63×10^{-3}	2.58×10^{-5}	0.7631	0.7511	0.0120
		$ah(t_0)$	1.4573	0.4764	0.1647	1.4699	1.4463	0.0236	1.4550	0.4749	0.1636	1.4617	1.4441	0.0176
60	60% 36	$S(t_0)$	0.6970	0.1237	8.19×10^{-3}	0.7878	0.5400	0.2477	0.8543	0.0739	2.64×10^{-3}	0.8963	0.7926	0.1037
		$h(t_0)$	0.6758	0.2617	0.0359	0.7459	0.6807	0.0652	0.6352	0.1704	0.0164	0.6678	0.6243	0.0435
		$ah(t_0)$	1.5267	0.5242	0.2005	1.6119	1.3908	0.2210	1.5111	0.5134	0.1925	1.6224	1.4116	0.2108
	80% 48	$S(t_0)$	0.8019	8.89×10^{-3}	1.66×10^{-4}	0.8250	0.7809	0.0441	0.7969	1.77×10^{-3}	6.48×10^{-6}	0.8005	0.7880	0.0125
		$h(t_0)$	0.7628	4.06×10^{-3}	1.93×10^{-5}	0.7719	0.7543	0.0176	0.7614	3.81×10^{-3}	1.71×10^{-5}	0.7667	0.7555	0.0112
		$ah(t_0)$	1.4811	0.4928	0.1763	1.5004	1.4460	0.0544	1.4571	0.4762	0.1646	1.4605	1.4503	0.0102
	100% 60	$S(t_0)$	0.7963	1.03×10^{-3}	5.68×10^{-7}	0.7971	0.7952	0.0019	0.7962	9.37×10^{-4}	4.27×10^{-7}	0.7956	0.7944	0.0012
		$h(t_0)$	0.7653	9.60×10^{-4}	6.71×10^{-7}	0.7664	0.7640	0.0024	0.7668	4.81×10^{-4}	3.95×10^{-7}	0.7669	0.7658	0.0011
		$ah(t_0)$	1.4528	0.4733	0.1626	1.4537	1.4514	0.0023	1.4514	0.4727	0.1622	1.4523	1.4511	0.0012

Table 6: continued

n	r	Par	SE						LINEX($v = 0.1$)					
			Averages	RAB	RE	UL	LL	Length	Averages	RAB	RE	UL	LL	Length
120	60% 72	$S(t_0)$	0.8048	0.0116	1.42×10^{-4}	0.8234	0.7831	0.0403	0.7930	3.13×10^{-3}	6.29×10^{-6}	0.7959	0.7885	0.0073
		$h(t_0)$	0.7756	0.0128	8.77×10^{-5}	0.7869	0.7638	0.0230	0.7669	1.52×10^{-3}	8.04×10^{-6}	0.7729	0.7581	0.0147
		$ah(t_0)$	1.4668	0.4829	0.1693	1.4850	1.4436	0.0414	1.4569	0.4761	0.1645	1.4633	1.4484	0.0148
	80% 96	$S(t_0)$	0.7951	9.01×10^{-4}	4.72×10^{-7}	0.7958	0.7937	0.0021	0.7948	4.67×10^{-4}	2.29×10^{-7}	0.7955	0.7939	0.0016
		$h(t_0)$	0.7671	1.69×10^{-3}	1.75×10^{-6}	0.7689	0.7654	0.0035	0.7662	5.61×10^{-4}	1.55×10^{-7}	0.7666	0.7657	0.0009
		$ah(t_0)$	1.4531	0.4735	0.1627	1.4538	1.4514	0.0024	1.4521	0.4728	0.1622	1.4526	1.4514	0.0012
	100% 120	$S(t_0)$	0.7954	1.87×10^{-4}	1.67×10^{-8}	0.7955	0.7953	0.0002	0.7955	8.16×10^{-6}	2.64×10^{-10}	0.7956	0.7955	0.0000
		$h(t_0)$	0.7657	3.96×10^{-5}	4.83×10^{-9}	0.7658	0.7655	0.0003	0.7658	1.90×10^{-5}	1.21×10^{-9}	0.7659	0.7658	0.0001
		$ah(t_0)$	1.4515	0.4724	0.1620	1.4516	1.4514	0.0002	1.4516	0.4724	0.1619	1.4516	1.4515	0.0001

Table 7: ML estimates of the parameters, survival, hazard rate, the alternative hazard rate functions and standard errors for the first real data set based on Type II censoring

n	*r*	Parameters	Estimates	SE
30	21	α	4.3155	0.4686
		β	51.4347	0.2582
		$S(t_0)$	0.9294	0.0179
		$h(t_0)$	0.6094	0.0245
		$ah(t_0)$	0.940	0.2616
	27	α	4.4383	0.3155
		β	50.4820	0.2323
		$S(t_0)$	0.9078	0.0126
		$h(t_0)$	0.6469	0.0141
		$ah(t_0)$	1.0409	0.1687
	30	α	4.4503	0.3021
		β	50.3975	0.1580
		$S(t_0)$	0.9055	0.0121
		$h(t_0)$	0.6503	0.0133
		$ah(t_0)$	1.0506	0.1608

Table 8: Goodness-of-fit measures for fitted models of second real data set

Model	Estimates MLE(SE)		-2LL	AIC	BIC	CAIC	K-S P-value
	α	β					
DIKum	5.7268(0.3142)	50.0817(1.2764)	107.061	111.061	113.864	111.506	0.2666 0.2391
DIW	0.5610(0.2500)	1.7016(0.198)	144.567	148.567	151.37	149.012	0.3333 0.0692
DMIR	0.4366(0.2567)	0.8817(0.2328)	112.494	116.494	119.297	116.939	0.3000 0.1350
EDIR	0.8994(0.2319)	0.4541(0.2558)	161.666	165.666	168.468	166.11	0.3 0.1324
TDL	0.3288(0.2626)	1.6827(0.1986)	111.143	115.143	117.945	115.587	0.3333 0.0708

Table 9: Goodness-of-fit measures for fitted models of third real data set II

Model	Estimates MLE(SE)		-2LL	AIC	BIC	CAIC	K-S P-value
	α	β					
DIKum	0.8617 (0.2193)	1.3172 (0.2116)	382.41	386.41	391.071	386.574	0.0921 0.9067
DIW	0.6405 (0.2233)	0.5418 (0.2252)	408.41	412.41	417.072	412.575	0.2105 0.0686
DMIR	0.6387 (0.2234)	0.1093 (0.2336)	392.625	396.625	401.286	396.789	0.1973 0.1035
EDIR	0.9053 (0.2185)	0.3914 (0.2281)	442.087	446.087	450.749	446.251	0.1973 0.1035
TDL	0.2653 (0.2305)	3.8661 (0.2009)	446.254	450.254	454.915	450.418	0.2763 0.0582

Table 10: Bayes Estimates and standard errors for the α and β of DIKum based on Type II censoring for the three real data sets

Real Data I	n	r	Par	SEL Estimates	SEL SE	LINEX($v = 0.1$) Estimates	LINEX($v = 0.1$) SE
Application I	30	60% 18	α	5.0107	0.1450	4.9990	0.0109
			β	50.0294	0.1170	50.0041	0.00436
		80% 24	α	5.0019	0.0050	5.0001	1.32×10^{-4}
			β	49.9994	5.64×10^{-5}	50.0005	4.00×10^{-5}
		100% 30	α	4.9999	1.57×10^{-5}	5.0000	9.04×10^{-7}
			β	50.0000	1.20×10^{-6}	50.0000	1.06×10^{-7}
Application II	30	60% 18	α	4.9985	0.0031	4.9995	0.0013
			β	50.0006	1.50×10^{-4}	49.9998	6.46×10^{-5}
		80% 24	α	5.0005	0.0013	5.0002	1.82×10^{-4}
			β	49.9991	5.77×10^{-5}	49.9995	4.50×10^{-5}
		100% 30	α	4.9999	3.06×10^{-4}	5.0000	1.43×10^{-4}
			β	49.9998	2.89×10^{-5}	49.9999	1.77×10^{-5}
Application III	30	60% 18	α	0.8988	0.0550	0.9015	0.0186
			β	0.8024	0.0979	0.8004	0.0449
		80% 24	α	0.9010	0.0077	0.8997	0.0022
			β	0.7977	0.0395	0.7985	0.0176
		100% 30	α	0.8995	0.0031	0.9000	0.0014
			β	0.8017	0.0233	0.7990	0.0050

Table 11: Bayes Estimates and standard errors of the $S(t_0)$, $h(t_0)$ and $ah(t_0)$ of DIKum based on Type II censoring for the three real data sets

Real Data I	*n*	*r*	Par	SEL Estimates	SEL SE	LINEX($v = 0.1$) Estimates	LINEX($v = 0.1$) SE
Application I	30	60% 18	$S(t_0)$	0.4507	0.2370	0.4592	0.0613
			$h(t_0)$	0.2605	0.1300	0.2347	0.3100
			$ah(t_0)$	0.2776	0.3060	0.2739	0.2480
		80% 24	$S(t_0)$	0.4585	0.0049	0.4589	7.11×10^{-4}
			$h(t_0)$	0.2352	0.1160	0.2365	0.0309
			$ah(t_0)$	0.2726	0.2290	0.2713	0.21200
		100% 30	$S(t_0)$	0.4587	1.33×10^{-4}	0.4588	3.51×10^{-6}
			$h(t_0)$	0.2374	2.97×10^{-4}	0.2373	3.98×10^{-5}
			$ah(t_0)$	0.2710	0.0217	0.2710	0.2080
Application II	30	60% 18	$S(t_0)$	0.7975	0.0303	0.7967	0.0129
			$h(t_0)$	0.7629	0.0588	0.7654	0.0054
			$ah(t_0)$	1.4517	0.1621	1.451	0.1620
		80% 24	$S(t_0)$	0.7943	0.0104	0.7951	0.1060
			$h(t_0)$	0.7644	0.0203	0.7660	0.3880
			$ah(t_0)$	1.4515	0.1620	1.4516	0.1620
		100% 30	$S(t_0)$	0.7952	0.0047	0.7953	0.4470
			$h(t_0)$	0.7662	0.0045	0.7659	0.1760
			$ah(t_0)$	1.4514	0.1620	1.4516	0.1619
Application III	76	60% 46	$S(t_0)$	0.4599	0.0201	0.4587	0.0025
			$h(t_0)$	0.3203	0.1120	0.3225	0.0030
			$ah(t_0)$	0.3896	0.0058	0.3896	0.0057
		80% 61	$S(t_0)$	0.4589	0.0019	0.4589	8.61×10^{-4}
			$h(t_0)$	0.3249	0.0893	0.3226	0.0014
			$ah(t_0)$	0.3897	0.0058	0.3895	0.0057
		100% 76	$S(t_0)$	0.4588	0.0017	0.4590	6.46×10^{-4}
			$h(t_0)$	0.3226	0.0079	0.3226	1.65×10^{-4}
			$ah(t_0)$	0.3893	0.0057	0.3896	0.0055