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A new three parameter bathtub and increasing failure rate model with applications to real data

Mustapha Muhammad[1,2*], Isyaku Muhammad[3], Mouna Bouchane[4], Muhammad Aslam[5], Sani Musa[6], Sadiya Ali Rano[2]

*Corresponding author

1. Department of Mathematics, Guangdong University of Petrochemical Technology, Maoming, China, mmmahmoud12@sci.just.edu.jo
2. Department of Mathematical Sciences, Bayero University Kano (BUK), Nigeria, mmuhammad.mth@buk.edu.ng (MM), sadiyarano@yahoo.com (SAR)
3. Department of Mechanical Engineering, School of Technology, Kano State Polytechnic, Nigeria, isyakuedu@yahoo.com
4. Key Laboratory of Augmented Reality, College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, China, mouna.bouchane@gmail.com
5. Institute of Physics and Technology, Ural Fedral University, Mira Str. 19, 620002 Yekaterinburg, Russia, aslam@urfu.ru
6. Department of Mathematics and Computer Science, Sule Lamido University, Kafin-Hausa, Nigeria. musasani1010@gmail.com

Abstract

This article introduces a new three-parameter model with an increasing and bathtub failure rate functions as an extension of the Mustapha type-II distribution (Mu-II). The model can be very useful in statistical studies, reliability, computer sciences and engineering. Various mathematical and statistical properties of the distribution are discussed, such as moments, mean deviations, Bonferroni and Lorenz curves, entropy, order statistic, and extreme value distributions. Moreover, we consider the bivariate extension of the new model. Statistical inferences by the maximum likelihood method are discussed and assess by simulation studies. Applications of the proposed model to two right-skewed data are presented for illustration. The new model provides a better fit than some other existing distribution as measured by some model selection criteria and goodness of fits statistics.

Key Words: Mustapha type II distribution; Moments; Entropy; Maximum likelihood estimation

Mathematical Subject Classification: 62E05, 62F10.

1. Introduction

Over decades, several lifetime distributions are extended (generalized) for added flexibility through various techniques as discussed in the literature; one can see Tahir and Cordeiro (2016); Muhammad et al. (2023). Some distributions on the unit interval were extended to a finite support by additional parameter(s) to accommodate various data sets. For example, the uniform distribution (U) on unit interval $U(0,1)$ to an interval $U(a,b)$, beta distribution on unit interval $B(y;p,q)$, $0<y<1$, extended to generalized beta of first kind $GB1(y;a,b,p,q)$, $0<y^a<b^a$ and b,p,q are positive, among others. One of the techniques that received significant attention by authors for the past years is the exponentiation technique; this involved raising the cumulative distribution function of a model to arbitrary positive power. The technique provides a new family of distributions that extends the well-known distributions and provides

better flexible distributions in modeling real data in practice. For instance, Mudholkar and Srivastava (1993) proposed exponentiated Weibull distribution which generalized the Weibull distribution; Gupta and Kundu (2001) introduced the exponentiated exponential distribution; generalized exponential Poisson by Barreto-Souza and Cribari-Neto (2009); generalized BurrXII Poisson Muhammad (2016a); generalized half logistic Poisson Muhammad (2017b); exponentiated Ailamujia Aafaq A. Rather and Alanzi (2022); exponentiated odd Lomax exponential Dhungana and Kumar (2022); exponentiated cotangent generalized distributions Tashkandy et al. (2023); exponentiated sine-G Muhammad et al. (2021a); extended cosine G Muhammad et al. (2021b); exponentiated additive Weibull Ahmad and Ghazal (2020); exponentiated transformation of Gumbel type-II Sindhu et al. (2021); among others.

Our aim in this work is to extend $Mu - II[0, 1]$ by adding two parameters $\beta, \theta > 0$. First, we extended the support to an interval $[0, \beta]$, $\beta > 0$, then we generalized the model using exponentiation technique with additional power parameter $\theta > 0$. Moreover, to provide some mathematical and statistical properties of the new model and discuss its estimation of parameters by maximum likelihood and give some illustrative examples using real data.

The rest of the paper is presented as follows. In Section 2, we present the proposed distributions and of its properties. In Section, 3 bivariate extension of the GMu-II distribution is briefly considered. In Section 4 maximum likelihood method for the parameter estimation is provided. An application to two real data is given in Section 5. Conclusions in Section 6.

2. Model and Properties

In this section, we introduce three-parameter extension of Mustapha type II distribution called *generalized Mustapha type-II (GMu-II)* by extending the support of the Mu-II distribution from $[0, 1]$ to $[0, \beta]$, then the exponentiation procedure by adding the parameter θ. We also provide some mathematical and statistical properties of the proposed distributions. The cumulative distribution of the Mu-II distribution proposed by Mustapha (2017) with parameter $\alpha > 0$ is given by

$$G(x) = e^{x^{\alpha} \ln 2} - 1 \tag{1}$$

The distribution can also be derived as a special case of alpha-power Uniform $(0, 1)$. Now, we obtain the new three parameter model by extending the cumulative distribution function (CDF) in (1) with an arbitrary positive parameters $\beta > 0$ and $\theta > 0$ as

$$F(x) = (e^{(x/\beta)^{\alpha} \ln 2} - 1)^{\theta}, \qquad 0 \le x \le \beta, \tag{2}$$

the corresponding probability density function is given by

$$f(x) = \alpha\theta \ln 2\, \beta^{-\alpha} x^{\alpha-1} e^{(x/\beta)^{\alpha} \ln 2} (e^{(x/\beta)^{\alpha} \ln 2} - 1)^{\theta-1}, \qquad 0 < x \le \beta. \tag{3}$$

The shape properties of the proposed distribution are described below while Figure 1 and 2 provide the plots of the $f(x)$ for some various values of the parameters, showing that the new model can accommodate increasing and decreasing densities. All the plots and computations are conducted using-R software.

Theorem 2.1. *Let $f(x)$ be the probability density function (PDF) given by (3), then, for all $\beta > 0$, $f(x)$ is: (i) monotone increasing function for $\alpha \ge 1$ and $\theta \ge 1$, (ii) monotone decreasing function for $\alpha < 1$ and $0 < \theta \le \frac{1}{2}$, (iii) bathtub shaped for $\alpha < 1$ and $\theta = 1$, and (iv) bathtub shaped for $\alpha = 1$ and $\frac{1}{2} < \theta < 1$.*

Proof. We start by computing

$$(\log f(x))' = \frac{\alpha - 1}{x} + \alpha\beta^{-\alpha} x^{\alpha-1} \ln 2 + \frac{(\theta - 1)\alpha\beta^{-\alpha} x^{\alpha-1} e^{(x/\beta)^{\alpha} \ln 2} \ln 2}{e^{(x/\beta)^{\alpha} \ln 2} - 1}. \tag{4}$$

1. It is clear from (4) if $\alpha \ge 1$ and $\theta \ge 1$ $(\log f(x))' > 0$, thus $f(x)$ is increasing function.

2. We show that $(\log f(x))' < 0$; it is clear that the first term in (4) i.e. $\frac{\alpha-1}{x} < 0$ for $\alpha < 1$, therefore, we need to show that the sum of the last two terms in (4) is less than or equal to zero; simplifying the sum of the last two

terms in (4) we have

$$\frac{\alpha\beta^{-\alpha}x^{\alpha-1}\,(\theta\, e^{(x/\beta)^{\alpha}\ln 2}-1)\,\ln 2}{e^{(x/\beta)^{\alpha}\ln 2}-1}. \tag{5}$$

The above expression is less than or equal to zero if $(\theta\, e^{(x/\beta)^{\alpha}\ln 2}-1)\leq 0$, but $1\leq e^{(x/\beta)^{\alpha}\ln 2}\leq \max|e^{(x/\beta)^{\alpha}\ln 2}|=2$, thus, $(\theta\, e^{(x/\beta)^{\alpha}\ln 2}-1)\leq 0$ if $\theta\leq \frac{1}{2}$.

3. For $\theta=1$ equation (4) becomes $(\log f(x))'=\frac{(\alpha-1)+\alpha\beta^{\alpha}x^{\alpha}\ln 2}{x}$ with the root $x_0=\left(\frac{(1-\alpha)\beta^{\alpha}}{\alpha\ln 2}\right)^{\frac{1}{\alpha}}$, thus, if $\alpha<1$ we have: for $0<x<x_0$ implies $(\log f(x))'<0$, at $x=x_0$ $(\log f(x_0))'=0$, and $(\log f(x))'>0$ for $x_0<x<\beta$, hence f(x) is bathtub shaped.

4. For $\alpha=1$ equation (4) become $(\log f(x))'=\frac{\beta^{-1}(\theta\, e^{(x/\beta)\ln 2}-1)\,\ln 2}{e^{(x/\beta)\ln 2}-1}$ with root say $x^{*}=\frac{-\beta\log\theta}{\log 2}$. Since $x\leq\beta$, therefore, $\frac{-\log\theta}{\log 2}\leq 1$ for a valid x^{*}, which occurs only for $\frac{1}{2}<\theta<1$; hence, we have that $(\log f(x^{*}))'<0$ for $0<x<x^{*}$, $(\log f(x^{*}))'=0$, and $(\log f(x))'>0$ for $x^{*}<x<\beta$, hence f(x) is bathtub shaped.

□

The limiting behavior of density function given by (3) are: (i) for all $\alpha,\theta,\beta>0$, $\lim_{x\to\beta} f(x)=\frac{\alpha\theta\ln 4}{\beta}$, (ii) for $\alpha>1,\theta>1,\beta>0$, $\lim_{x\to 0} f(x)\to 0$, (iii) for $\alpha<1,\theta<1,\beta>0$, $\lim_{x\to 0} f(x)\to\infty$, and (iv) for $\alpha=1,\theta=1,\beta>0$, $\lim_{x\to 0} f(x)=\frac{\ln 2}{\beta}$.
The survival function $S(x)$, hazard rate function $h(x)$ and the reverse hazard rate function $(r(x))$ of the GMu-II distribution are given respectively as

$$\begin{aligned}
s(x)&=1-(e^{(x/\beta)^{\alpha}\ln 2}-1)^{\theta},\\
h(x)&=\frac{\alpha\theta\beta^{-\alpha}x^{\alpha-1}e^{(x/\beta)^{\alpha}\ln 2}\ln 2\,(e^{(x/\beta)^{\alpha}\ln 2}-1)^{\theta-1}}{1-(e^{(x/\beta)^{\alpha}\ln 2}-1)^{\theta}},\\
r(x)&=\frac{\alpha\theta\beta^{-\alpha}x^{\alpha-1}e^{(x/\beta)^{\alpha}\ln 2}\,\ln 2}{e^{(x/\beta)^{\alpha}\ln 2}-1}.
\end{aligned} \tag{6}$$

Theorem 2.2. *The hazard rate function $h(x)$ given by (6) is monotone increasing function for $\alpha\geq 1$ and $\theta\geq 1$.*

Proof. According to the theorem provided by Glaser (1980), we have that

$$\begin{aligned}
(\log h(x))'=&\frac{(\alpha-1)}{x}+\ln 2\,\alpha\beta^{-\alpha}x^{\alpha-1}+\frac{(\theta-1)\,\ln 2\,\alpha\beta^{-\alpha}x^{\alpha-1}e^{(x/\beta)^{\alpha}\ln 2}}{e^{(x/\beta)^{\alpha}\ln 2}-1}\\
&+\frac{\ln 2\,\alpha\theta\beta^{-\alpha}x^{\alpha-1}e^{(x/\beta)^{\alpha}\ln 2}(e^{(x/\beta)^{\alpha}\ln 2}-1)^{\theta-1}}{1-(e^{(x/\beta)^{\alpha}\ln 2}-1)^{\theta}},
\end{aligned} \tag{7}$$

thus, for $\alpha\geq 1$ and $\theta\geq 1$, $(\log h(x))'>0$, hence, $h(x)$ is increasing function. □

The limiting behavior of the hazard rate function given by (6) are: (i) for all $\alpha,\theta,\beta>0$, $\lim_{x\to\beta} h(x)\to\infty$, (ii) for $\alpha>1,\theta>1,\beta>0$, $\lim_{x\to 0} h(x)\to 0$, (iii) for $\alpha<1,\theta<1,\beta>0$, $\lim_{x\to 0} h(x)\to\infty$, and (iv) for $\alpha=1,\theta=1,\beta>0$, $\lim_{x\to 0} h(x)=\frac{\ln 2}{\beta}$.
Figure 3 provide some plots of the hazard function of the GMu-II distribution for various values of parameters, showing that the hazard function can accommodate bathtub and increasing failure rates.
Let $|z|\leq 1$, and $b>0$ real and non integer, then

$$(1-z)^{b-1}=\sum_{i=0}^{\infty}\frac{(-1)^{i}\,\Gamma(b)}{i!\Gamma(b-i)}z^{i}. \tag{8}$$

Here, we want to express the PDF given by (3) in a series form, and we start by

$$f(x)=\alpha\theta\,\ln 2\,\beta^{-\alpha}x^{\alpha-1}e^{\theta(x/\beta)^{\alpha}\ln 2}\,(1-e^{-(x/\beta)^{\alpha}\ln 2})^{\theta-1},$$

by applying (8) and exponential expansion we have

$$f(x) = \alpha\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i(\theta-i)^j(\ln 2)^{j+1}\Gamma(\theta)}{i!\,j!\,\beta^{\alpha(j+1)}\Gamma(\theta-i)}\, x^{\alpha(j+1)-1}. \tag{9}$$

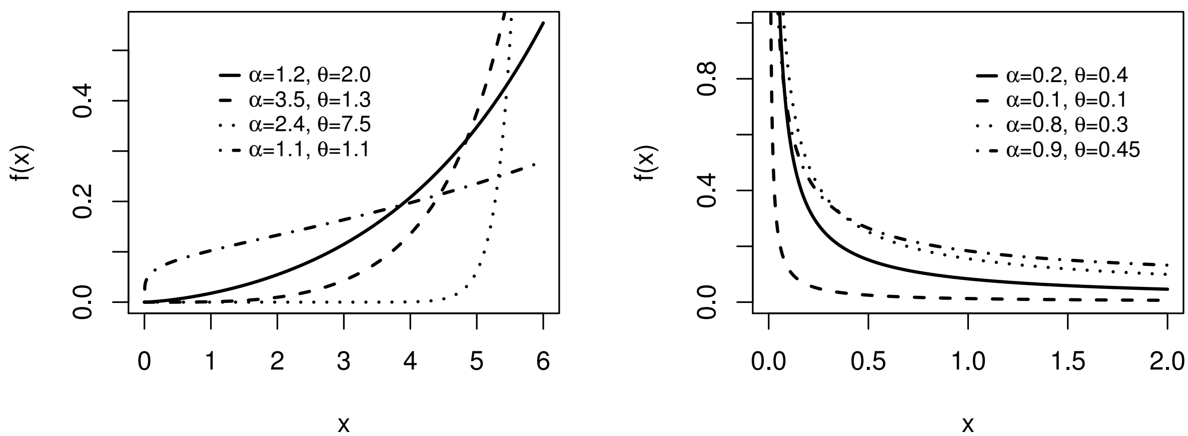

Figure 1: Plots of the GMu-II density function for some values of parameters

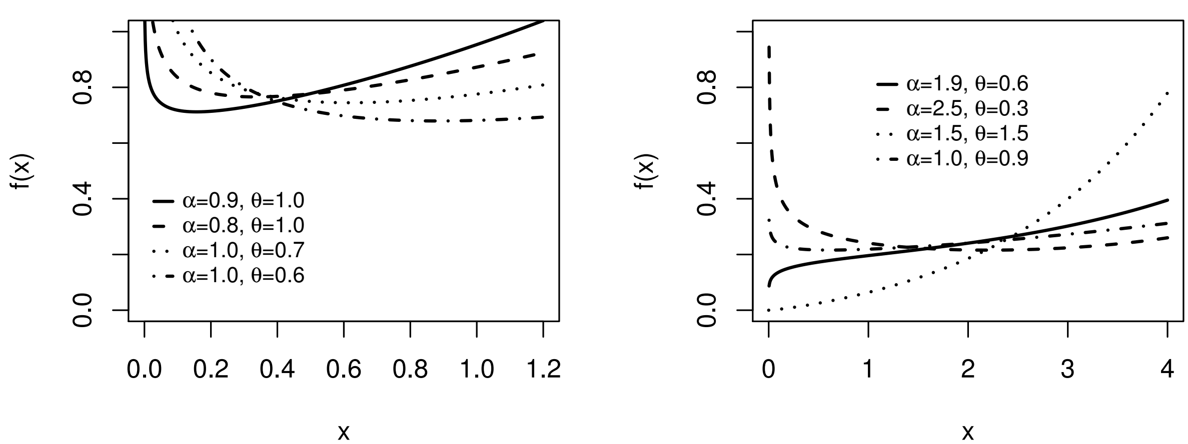

Figure 2: Plots of the GMu-II density function for some values of parameters

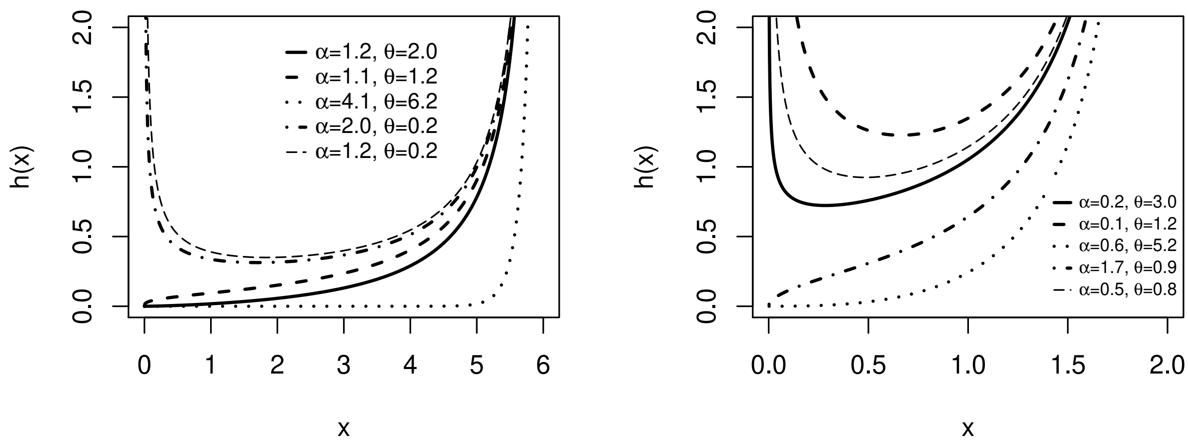

Figure 3: Plots of the GMu-II hazard rate function for some values of parameters

2.1. Moments and quantiles

The quantile function $\zeta(.)$ of the GMu-II can be used for generating random data distributed according to (3) by generating data from uniform distribution and it is given by

$$\zeta(p) = \beta \left(\frac{\log(p^{1/\theta} + 1)}{\log 2} \right)^{\frac{1}{\alpha}}. \tag{10}$$

Therefore, the median (*M*) of the GMu-II can be obtained directly by substituting $p = 1/2$ in (10) as

$$M = \beta \left(\frac{\log((0.5)^{1/\theta} + 1)}{\log 2} \right)^{\frac{1}{\alpha}}. \tag{11}$$

Figure 4 shows that for any value of $\beta > 0$ the median is an increasing function when both α and θ are increasing.

The r^{th} ordinary moments can be computed by $\mu_r = E[X^r] = \int_0^\beta x^r f(x)dx$, thus, we can get the ordinary moments of X in a series form using (9) as

$$E(X^r) = \alpha\theta \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i (\theta - i)^j (\ln 2)^{j+1} \beta^r \, \Gamma(\theta)}{i!\, j!\, (\alpha(j+1) + r)\Gamma(\theta - i)}. \tag{12}$$

We can compute the mean $\mu = E(X)$ and variance $\sigma^2 = E(X^2) - (E(X))^2$ using (12), also, the other higher moments can be obtained by setting $r = 1, 2, 3 \ldots$. Figure 5 below described the behavior of the mean and variance of the GMu-II distribution. The mean is increasing when α and θ are increasing, while the variance is unimodal as α and θ increases.

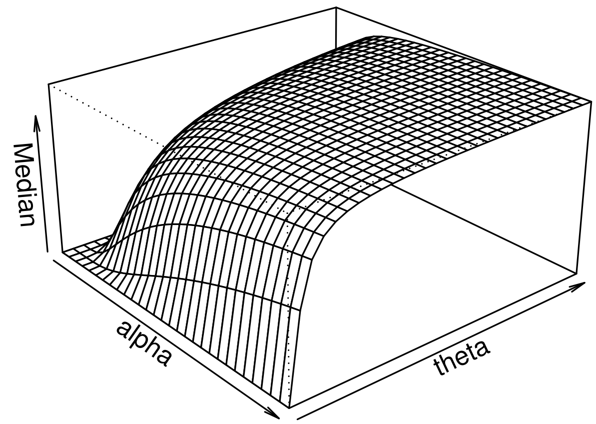

Figure 4: Plots of the median of GMu-II distribution.

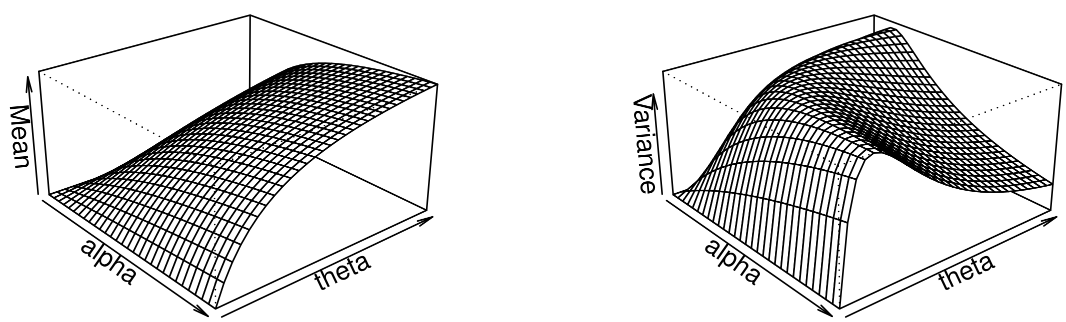

Figure 5: Plots of the mean and variance of GMu-II distribution for $\beta = 5$.

Moreover, the effect of the parameters α and θ on the skewness and kurtosis of GMu-II can also be analyzed using the Bowley skewness (B) and Moores kurtosis (M) which are defined by

$$B = \frac{\zeta(3/4) + \zeta(1/4) - 2\,\zeta(2/4)}{\zeta(3/4) - \zeta(1/4)}, \quad \text{and} \quad M = \frac{\zeta(3/8) - \zeta(1/8) + \zeta(7/8) - \zeta(5/8)}{\zeta(6/8) - \zeta(2/8)},$$

respectively, where $\zeta(.)$ is given by (10), notice that both Bowley skewness and Moores kurtosis are independent of β. Figure 6 below illustrated that the skewness of the GMu-II distribution decreases as both α and θ increases, while the kurtosis is decreasing then increasing as α and θ increases.

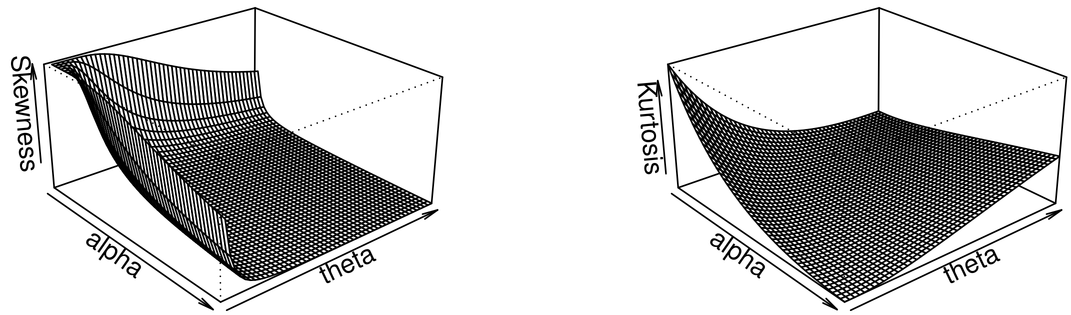

Figure 6: Plots of skewness (B) and kurtosis (M) of GMu-II distribution

2.2. Mean deviations, Bonferroni and Lorenz curves

The mean deviations of a random variable X about the mean (μ_1) and median (M) are defined by $\delta_1(X) = E(|X-\mu_1|)$ and $\delta_2(X) = E(|X - M|)$ which can be expressed as

$$\delta_1(X) = 2\mu_1 F(\mu_1) - 2m_1(\mu_1) \quad \text{and} \quad \delta_2(X) = \mu_1 - 2m_1(M)$$

respectively, where $\mu_1 = E(X)$, $F(\mu_1)$ can be computed from (2), M median of X given by (11) and $m_1(.)$ is the first incomplete moment of X which can be obtain from (12) when $r = 1$ as

$$m_1(t) = \alpha\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i(\theta-i)^j(\ln 2)^{j+1}\, t\,\Gamma(\theta)}{i!\, j!\,(\alpha(j+1)+1)\Gamma(\theta-i)}. \tag{13}$$

Hence, δ_1 and δ_2 can be computed by setting $t = \mu_1$ and $t = M$ in (13) respectively.

Moreover, for a given probability p, the Bonferroni and Lorenz curves are defined by $B(p) = m_1(q)/(p\mu_1)$ and $L(p) = m_1(q)/\mu_1$ respectively, where $q = \beta\left(\log(p^{1/\theta}+1)\right)^{\frac{1}{\alpha}}(\log 2)^{-\frac{1}{\alpha}}$ is the quantile of X at p. These measures have been applied in analyzing many problems in both sciences and social sciences such as economics, insurance, income and poverty, survival analysis and medicine. Therefore, we can get the expressions of the Bonferroni and Lorenz curves of the GMu-II distribution using $t = q$ in (13) respectively as

$$B(p) = \alpha\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i(\theta-i)^j(\ln 2)^{j+1}\, q\,\Gamma(\theta)}{i!\, j!\, p\mu_1(\alpha(j+1)+1)\Gamma(\theta-i)},$$

and

$$B(p) = \alpha\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i(\theta-i)^j(\ln 2)^{j+1}\, q\,\Gamma(\theta)}{i!\, j!\, \mu_1(\alpha(j+1)+1)\Gamma(\theta-i)},$$

respectively.

2.3. Entropy

Entropy is defined as the measure of uncertainty. The two most popularly used entropy measures are the Shannon entropy and Renyi entropy. We first give this important lemma, which is very useful in their computations.

Lemma 2.3. *Let $X \sim f(x)$ in (3), let $a, b \in \mathbb{R}$ and c is a real and non integer, define*

$$\Delta(a,b,c,\alpha,\beta) = \int_0^\beta x^a\, e^{b(x/\beta)^\alpha\,(\ln 2)}(e^{(x/\beta)^\alpha \ln 2} - 1)^c\, dx,$$

then,

$$\Delta(a,b,c,\alpha,\beta) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i \binom{c}{i}(b+c-i)^j (\ln 2)^j\, \beta^{a+1}}{j!(a+\alpha j+1)}. \tag{14}$$

Proof: follow similar to (12).

For a random variable X with pdf given by (3), the Renyi entropy is defined by $I_{R(\rho)} = (1-\rho)^{-1}\log\left[\int_{-\infty}^{\infty} f(x)^\rho dx\right]$, with $\rho > 0$ and $\rho \neq 1$. We first compute $\int_0^\beta f^\rho(x)dx$ by applying the Lemma 2.3 as

$$\begin{aligned}\int_0^\beta f^\rho(x)dx &= \alpha^\rho\theta^\rho\beta^{-\rho\alpha}\int_0^\beta x^{\rho(\alpha-1)} e^{\rho(x/\beta)^\alpha(\ln 2)}(e^{(x/\beta)^\alpha \ln 2} - 1)^{\rho(\theta-1)}dx \\ &= \alpha^\rho\theta^\rho\beta^{-\rho\alpha}(\ln 2)^\rho\,\Delta(\rho(\alpha-1),\rho,\rho(\theta-1),\alpha,\beta).\end{aligned}$$

Thus,

$$I_{R(\rho)} = \frac{1}{1-\rho}\log\left[\alpha^\rho\theta^\rho\beta^{-\rho\alpha}(\ln 2)^\rho\,\Delta(\rho(\alpha-1),\rho,\rho(\theta-1),\alpha,\beta)\right].$$

The Shannon entropy which is defined by $E[-\log f(x)]$ can directly be computed by considering the following the Lemma 2.4.

Lemma 2.4. *Let X∼(3), then,*

$$E[\log(e^{(X/\beta)^\alpha \ln 2} - 1)] = \alpha\theta\beta^{-\alpha}\ln 2\,\frac{\partial}{\partial t}\Delta(\alpha-1,1,\theta+t-1,\alpha,\beta)|_{t=0}, \tag{15}$$

$$E[\log X] = \frac{\partial}{\partial t}E[X^t]|_{t=0}. \tag{16}$$

Proof. follow from Lemma 2.3 □

Therefore, from (15) and (16) we get,

$$\begin{aligned}E[-logf(X)] &= -\log\left(\frac{\alpha\theta\ln 2}{\beta^\alpha}\right) - (\alpha-1)E[\log X] \\ &- (\theta-1)E[\log(2^{(X/\beta)^\alpha} - 1)] - \beta^{-\alpha}(\log 2)\,E[X^\alpha] \\ &= \log\left(\frac{\beta^\alpha}{\alpha\theta\ln 2}\right) - (\alpha-1)\frac{\partial}{\partial t}E[X^t]|_{t=0} \\ &- (\theta-1)\alpha\theta\beta^{-\alpha}(\ln 2)\,\frac{\partial}{\partial t}\Delta(\alpha-1,1,\theta+t-1,\alpha,\beta)|_{t=0} - \beta^{-\alpha}(\log 2)\,E[X^\alpha].\end{aligned}$$

2.4. Order statistics

Let, $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, $j = 1,2,3,\cdots,n$, be a random sample of independent observation of size n from the GMu-II distribution, then the density of the j^{th} order statistic $f_{X_{j:n}}(x)$ can be presented in a series of the form

$$f_{X_{j:n}}(x)=\sum_{l=0}^{n-j}\frac{n!\,(-1)^l}{(j-1)!(n-j-l)!\,l!}f(x)(F(x))^{j+l-1},$$

where $f(x)$ and $F(x)$ are given by (3) and (2) respectively. By expanding $(F(x))^{j+l-1}$ and exponential expansion we have

$$f_{X_{j:n}}(x)=\sum_{l=0}^{n-j}\frac{n!\,(-1)^l\,\ln 2\,\alpha\beta^{-\alpha}\theta}{(j-1)!\,(n-j-l)!\,l!}x^{\alpha-1}e^{(x/\beta)^{\alpha}\,\ln 2}\left(e^{(x/\beta)^{\alpha}\,\ln 2}-1\right)^{\theta+j+l-2}.$$

thus, its the series of GMuII with parameters α,β and $\theta+j+l-1$ as

$$f_{X_{j:n}}(x)=\sum_{l=0}^{n-j}\frac{n!\,(-1)^l\theta f(x;\alpha,\beta,\theta+j+l-1)}{(j-1)!\,(n-j-l)!\,l!\,(\theta+j+l-1)}.$$

The asymptotic distributions for the extreme order statistics $X_{1:n}$ and $X_{n:n}$ from $X_1,X_2,X_3,\cdots,X_n$ with GMu-II are derived. for more detailed see Arnold et al. (1992); Leadbetter et al. (2012). Let W be a random variable with CDF G, then the CDF F is in the domain of maximal attraction of G is the same as $(X_{n,n}-a_n)/b_n\xrightarrow{d}W$, provided there exist a sequence $\{a_n\}$ and $\{b_n>0\}$. Suppose that W^* be a random variable with CDF G^*, then the CDF F is in the domain of minimal attraction of G^* is the same as $(X_{1,n}-a_n^*)/b_n^*\xrightarrow{d}W^*$, provided there exist a sequence $\{a_n^*\}$ and $\{b_n^*>0\}$.

Theorem 2.5. *Let $X_1,X_2,X_3,\cdots,X_n$ be a random sample following GMu-II, let $W_n=(X_{n:n}-a_n)/b_n$, then, $W_n\xrightarrow{d}W$ implies that*

$$\lim_{n\to\infty}P(W_n\le x)=G(x)=e^x,$$

for every point $x\in\mathbb{R}$ of G(x) for which G(x) is continuous, where the normalizing constant can be obtain from (10) by the Theorem 8.3.4 of Arnold et al. (1992), thus, $a_n=\beta$ and $b_n=\beta\left(1-\left(\frac{\log((1-\frac{1}{n})^{1/\theta}+1)}{\log 2}\right)^{1/\alpha}\right)$.

Proof. According to the *Theorem* 8.3.2 of Arnold et al. (1992), we consider

$$\lim_{\epsilon\to0^+}\frac{1-F(F^{-1}(1)-\epsilon x)}{1-F(F^{-1}(1)-\epsilon)}=\lim_{\epsilon\to0^+}\frac{1-\left(e^{(\frac{\beta-\epsilon x}{\beta})^{\alpha}\,\ln 2}-1\right)^{\theta}}{1-\left(e^{(\frac{\beta-\epsilon}{\beta})^{\alpha}\,\ln 2}-1\right)^{\theta}}=x,$$

hence the proof. □

Theorem 2.6. *Let $X_1,X_2,X_3,\cdots,X_n$ be a random sample following GMu-II, let $W_n^*=(X_{1:n}-a_n^*)/b_n^*$, then, $W_n^*\xrightarrow{d}W^*$ is equivalent to*

$$\lim_{n\to\infty}P(W_n^*\le x)=G^*(x;\alpha\theta)=1-e^{-x^{\alpha\theta}},$$

for every point $x\in\mathbb{R}^+$ of $G^*(x;\alpha\theta)$ for which $G^*(x;\alpha\theta)$ is continuous, where the normalizing constant can be computed from (10) by the Theorem 8.3.6 of Arnold et al. (1992), thus, $a_n^*=0$ and $b_n^*=\beta\left(\frac{\log((\frac{1}{n})^{1/\theta}+1)}{\log 2}\right)^{1/\alpha}$.

Proof. According to *Theorem* 8.3.6 of Arnold et al. (1992) we can first consider the asymptotic of $F(x)$ as follows. Since,
$\lim_{x\to0}\left(e^{(x/\beta)^{\alpha}\,\ln 2}-1\right)\sim(x/\beta)^{\alpha}\,\ln 2$, therefore, $\lim_{x\to0}F(x)\sim(x/\beta)^{\alpha\theta}\,(\ln 2)^{\theta}$. Hence,

$$\lim_{\epsilon\to0^+}\frac{F(F^{-1}(0)+\epsilon x)}{F(F^{-1}(0)+\epsilon)}\sim\frac{(\epsilon x/\beta)^{\alpha\theta}\,(\ln 2)^{\theta}}{(\epsilon/\beta)^{\alpha\theta}\,(\ln 2)^{\theta}}=x^{\alpha\theta}.$$

□

3. Bivariate GMuII

In this section, we discuss the bivariate extension for the GMu-II distribution when the shape parameter θ are different. Let, U_1, U_2 and U_3 be mutually independent such that, $U_1 \sim$ GMuII$(\alpha, \beta, \theta_1)$, $U_2 \sim$ GMuII$(\alpha, \beta, \theta_2)$ and $U_3 \sim$ GMuII$(\alpha, \beta, \theta_3)$. Let, $X_1 = \max\{U_1, U_3\}$ and $X_2 = \max\{U_2, U_3\}$, then, the vector (X_1, X_2) has the bivariate GMu-II distribution with parameters $\alpha, \beta, \theta_1, \theta_2$ and θ_3, denoted by BVGMuII$(\alpha, \beta, \theta_1, \theta_2, \theta_3)$. We now briefly discuss on the CDF and the density function of the BVGMu-II.
If $(X_1, X_2) \sim BVGMuII(\alpha, \beta, \theta_1, \theta_2, \theta_3)$, then, the joint CDF of (X_1, X_2) for $x_1 \le \beta$, $x_2 \le \beta$, is given by

$$F_{X_1,X_2}(x_1, x_2) = \left(e^{(x_1/\beta)^\alpha \ \ln 2} - 1\right)^{\theta_1} \left(e^{(x_2/\beta)^\alpha \ \ln 2} - 1\right)^{\theta_2} \left(e^{(z/\beta)^\alpha \ \ln 2} - 1\right)^{\theta_3},$$

where $z = \min\{x_1, x_2\}$.
If $(X_1, X_2) \sim BVGMuII(\alpha, \beta, \theta_1, \theta_2, \theta_3)$, then, the joint CDF of (X_1, X_2) for $x_1 \le \beta$, $x_2 \le \beta$, can be presented as

$$\begin{aligned} F_{X_1,X_2}(x_1, x_2) &= F_{BVGMuII}(x_1, \alpha, \beta, \theta_1) F_{BVGMuII}(x_2, \alpha, \beta, \theta_2) F_{BVGMuII}(z, \alpha, \beta, \theta_3) \\ &= F_{BVGMuII}(x_1, \alpha, \beta, \theta_1 + \theta_3) F_{BVGMuII}(x_2, \alpha, \beta, \theta_2) \qquad if \ \ x_1 < x_2 \\ &= F_{BVGMuII}(x_1, \alpha, \beta, \theta_1) F_{BVGMuII}(x_2, \alpha, \beta, \theta_2 + \theta_3) \qquad if \ \ x_2 < x_1 \\ &= F_{BVGMuII}(x, \alpha, \beta, \theta_1 + \theta_2 + \theta_3) \qquad if \ \ x_1 = x_2 = x. \end{aligned}$$

Proposition 3.1. *If $(X_1, X_2) \sim BVEMu(\alpha, \beta, \theta_1, \theta_2, \theta_3)$, then, the joint PDF of (X_1, X_2) for $x_1 \le \beta$, $x_2 \le \beta$, is given by*

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} f_1(x_1, x_2) & if \ \ x_1 < x_2 \\ f_2(x_1, x_2) & if \ \ x_2 < x_1 \\ f_0(x_1, x_2) & \ x_1 = x_2 = x. \end{cases}$$

where

$$\begin{aligned} f_1(x_1, x_2) &= f_{BVGMuII}(x_1, \alpha, \beta, \theta_1 + \theta_3) f_{BVGMuII}(x_2, \alpha, \beta, \theta_2) \\ &= \alpha^2 \theta_2 (\theta_1 + \theta_3) \beta^{-2\alpha} x_1^{\alpha-1} e^{(x_1/\beta)^\alpha \ \ln 2} (\ln 2)^2 \left(e^{(x_1/\beta)^\alpha \ \ln 2} - 1\right)^{\theta_1+\theta_3-1} \\ &\times x_2^{\alpha-1} e^{(x_2/\beta)^\alpha \ \ln 2} \left(e^{(x_2/\beta)^\alpha \ \ln 2} - 1\right)^{\theta_2-1} \\ f_2(x_1, x_2) &= f_{BVGMuII}(x_1, \alpha, \beta, \theta_1) f_{BVGMuII}(x_2, \alpha, \beta, \theta_2 + \theta_3) \\ &= \alpha^2 \theta_1 (\theta_2 + \theta_3) \beta^{-2\alpha} x_2^{\alpha-1} e^{(x_2/\beta)^\alpha \ \ln 2} (\ln 2)^2 \left(e^{(x_2/\beta)^\alpha \ \ln 2} - 1\right)^{\theta_2+\theta_3-1} \\ &\times x_1^{\alpha-1} e^{(x_1/\beta)^\alpha \ \ln 2} \left(e^{(x_1/\beta)^\alpha \ \ln 2} - 1\right)^{\theta_1-1} \\ f_0(x_1, x_2) &= \frac{\theta_3}{\theta_1 + \theta_2 + \theta_3} f(x, \alpha, \beta, \theta_1 + \theta_2 + \theta_3) \\ &= \alpha \theta_3 \beta^{-\alpha} x^{\alpha-1} e^{(x/\beta)^\alpha \ \ln 2} \ln 2 \left(e^{(x/\beta)^\alpha \ \ln 2} - 1\right)^{\theta_1+\theta_2+\theta_3-1} \end{aligned}$$

Proof. $f_i(x_1, x_2)$ for $i = 1, 2$ can be determined from $\frac{\partial^2 F_{X_1,X_2}(x_1,x_2)}{\partial x_1 \partial x_2}$. For the $f_0(x_1, x_2)$ we follow the fact that,

$$\int_0^\beta \int_0^{x_2} f_1(x_1, x_2) dx_1 dx_2 + \int_0^\beta \int_0^{x_1} f_2(x_1, x_2) dx_2 dx_1 + \int_0^\beta f_0(x) dx = 1,$$

thus,

$$\begin{aligned} \int_0^\beta \int_0^{x_2} f_1(x_1, x_2) dx_1 dx_2 &= \int_0^\beta \alpha \theta_2 \beta^{-\alpha} x_2^{\alpha-1} e^{(x_2/\beta)^\alpha \ \ln 2} \ln 2 \left(e^{(x_2/\beta)^\alpha \ \ln 2} - 1\right)^{\theta_1+\theta_2+\theta_3-1} dx_2, \\ \int_0^\beta \int_0^{x_1} f_2(x_1, x_2) dx_2 dx_1 &= \int_0^\beta \alpha \theta_1 \beta^{-\alpha} x_1^{\alpha-1} e^{(x_1/\beta)^\alpha \ \ln 2} \ln 2 \left(e^{(x_1/\beta)^\alpha \ \ln 2} - 1\right)^{\theta_1+\theta_2+\theta_3-1} dx_1, \end{aligned}$$

hence,

$$\int_0^\beta f_0(x)dx = \int_0^\beta \alpha\theta_3\beta^{-\alpha}x^{\alpha-1}e^{(x/\beta)^\alpha\ \ln 2}\ln 2\,(e^{(x/\beta)^\alpha\ \ln 2}-1)^{\theta_1+\theta_2+\theta_3-1}dx.$$

□

4. Maximum likelihood estimation

In this section, we estimate the unknown parameters of the GMu-II by the method of maximum likelihood and examine by simulation studies. Let, $X_1,\ X_2,\ \cdots, X_n$ be a random sample of size n obtained from the GMu-II distribution, then, the log-likelihood function ($\ell(\Theta)$) where $\Theta = (\alpha, \beta, \theta)$ is given by

$$\ell(\Theta) = n\log\alpha + n\log\theta - n\alpha\log\beta + n\log(\ln 2) + (\alpha-1)\sum_{i=1}^n \log x_i + \log 2\sum_{i=1}^n\left(\frac{x_i}{\beta}\right)^\alpha + (\theta-1)\sum_{i=1}^n \log(e^{(x_i/\beta)^\alpha\ \ln 2}-1).$$

Since $\max\{X_1, X_2, \cdots, X_n\} \leq \beta$, therefore, the MLE of β is $X_{n:n}$ and its a consistent estimator because

$$\lim_{n\to\infty} P(|X_{n:n}-\beta| > \epsilon) = \lim_{n\to\infty}\left(e^{\left(\frac{\beta-\epsilon}{\beta}\right)^\alpha\ \ln 2}-1\right)^{n\theta} \to 0,$$

hence, we can obtain the MLE of α and θ, i.e $\hat\alpha$ and $\hat\theta$ by the solution of the non-linear system $\frac{\partial\ell}{\partial\alpha} = \frac{\partial\ell}{\partial\theta} = 0$, and setting $\hat\beta = X_{n:n} + \vartheta$, $\vartheta \in \mathbb{R}^+$ (or using constrain MLE) where

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - n\log\beta + \sum_{i=1}^n \log x_i + \log 2\sum_{i=1}^n\left(\frac{x_i}{\beta}\right)^\alpha\log\left(\frac{x_i}{\beta}\right) + (\theta-1)\ln 2\sum_{i=1}^n \frac{e^{(x_i/\beta)^\alpha \ln 2}\left(\frac{x_i}{\beta}\right)^\alpha\log\left(\frac{x_i}{\beta}\right)}{(e^{(x_i/\beta)^\alpha\ \ln 2}-1)}, \quad (17)$$

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^n \log(e^{(x_i/\beta)^\alpha\ \ln 2}-1). \quad (18)$$

For the asymptotic interval estimation and hypothesis tests of the parameters α and θ, we need 2×2 Fisher information matrix denoted by $(J(\Theta))$, under the usual condition that are fulfilled for the parameters α and θ in the interior of the parameter space but not on the boundary. The asymptotic distribution of $\sqrt{n}(\hat\Theta-\Theta)$ is $N_2(0, I^{-1}(\Theta))$, which is a Normal 2−variate with zero mean and variance covariance $I(\Theta)$. This condition is also applicable if $I(\Theta)$ is substitute by the information matrix evaluated at $\hat\Theta$, that is $J(\hat\Theta)$. The Normal 2−variate distribution $N_2(0, J^{-1}(\Theta))$ can be used to establish an approximate confidence interval and region for the model parameters α and θ. $J(\Theta) = -[\partial^2\ell/\partial\Theta\partial\Theta^T]$, and the element of $J(\Theta)$ are given in by

$$\frac{\partial^2\ell}{\partial\theta^2} = -\frac{n}{\theta^2}, \quad (19)$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\theta} = \sum_{i=1}^n \frac{\ln 2\, e^{(x_i/\beta)^\alpha\ \ln 2}(x_i/\beta)^\alpha\ \ln(x_i/\beta)}{e^{(x_i/\beta)^\alpha\ \ln 2}-1}, \quad (20)$$

$$\frac{\partial^2\ell}{\partial\alpha^2} = -\frac{n}{\alpha^2} + \ln 2\sum_{i=1}^{n}(x_i/\beta)^{\alpha}\ (\ln(x_i/\beta))^2 + (\theta-1)\ln 2\sum_{i=1}^{n}\frac{e^{(x_i/\beta)^{\alpha}\ \ln 2}(x_i/\beta)^{\alpha}\ (\ln(x_i/\beta))^2}{e^{(x_i/\beta)^{\alpha}\ \ln 2}-1}$$
$$+ (\theta-1)(\ln 2)^2\sum_{i=1}^{n}\frac{e^{(x_i/\beta)^{\alpha}\ \ln 2}(x_i/\beta)^{2\alpha}\ (\ln(x_i/\beta))^2}{e^{(x_i/\beta)^{\alpha}\ \ln 2}-1} - (\theta-1)(\ln 2)^2\sum_{i=1}^{n}\frac{e^{2(x_i/\beta)^{\alpha}\ \ln 2}(x_i/\beta)^{2\alpha}\ (\ln(x_i/\beta))^2}{\left(e^{(x_i/\beta)^{\alpha}\ \ln 2}-1\right)^2}. \tag{21}$$

Moreover, we investigate the existence and uniqueness of the MLEs of α and θ under some possible conditions in similar way to Jafari and Tahmasebi (2016); Muhammad (2017a), among others.

Theorem 4.1. *Let $z_\theta(\theta;\alpha,\beta;x_i)$ be the right hand side of (18), where α and β are true values of parameters, then, $z_\theta(\theta;\alpha,\beta;x_i)=0$ has a unique real root.*

Proof. $\lim_{\alpha\to 0} z_\theta(\theta;\alpha,\beta;x_i)=\infty$ and $\lim_{\alpha\to\infty} z_\theta(\theta;\alpha,\beta;x_i)=\sum_{i=1}^{n}\log(2^{(x_i/\beta)^{\alpha}}-1)<0$, this show that z_θ is a monotone function from positive to negative, thus, $z_\theta=0$ has at least one root. To prove the uniqueness we show that $z_\theta^{'}<0$ and $z_\theta^{'}=-\frac{n}{\theta^2}$ from (19). □

Theorem 4.2. *Let $z_\alpha(\alpha;\beta,\theta,x_i)$ be the right hand side of (17), where θ and β are true values of parameters, then, $z_\alpha(\alpha;\beta,\theta,x_i)=0$ has at least one real root.*

Proof. Now, $\lim_{\alpha\to 0} z_\alpha(\alpha;\beta,\theta,x_i)=\infty$, the we need to show that $\lim_{\alpha\to\infty} z_\alpha$ is negative. From the last term of z_α, since $\lim_{\alpha\to\infty}\frac{(x_i/\beta)^{\alpha}}{(e^{(x_i/\beta)^{\alpha}\ \ln 2}-1)}=1/\ln 2$. Thus, $\lim_{\alpha\to\infty} z_\alpha=\theta\sum_{i=1}^{n}\log(x_i/\beta)<0$, hence the proof. □

4.1. Simulation

In this subsection, simulation studies are conducted to examine the proposed estimation method for the GMu-II. We generate 1000 samples from GMu-II model, each of sample sizes $(30,60,90,\ldots,180)$ for some parameter values and assess based on the average values of the estimators and their standard deviations; the proportion of the convergence for the iterations are considered. The simulation result is given in the Table 1, the performance of the maximum likelihood shows consistency and the standard deviation is decreasing as the sample size increases, also, the proportion of the convergence is approaching more than 95% in most of cases.

5. Illustration

In this section, the GMu-II distribution was fitted to a real data in order to demonstrate its important and applicability in data studies. We compare the fit of the GMu-II with that some existing models such as: Weibull (W), beta (B), exponential uniform (ExU) Javanshiri et al. (2013), logistic-uniform (LU) Torabi and Montazeri (2014), generalized exponential (GE) Gupta and Kundu (2001), Chen (Ch) Chen (2000), exponentiated Nadarajah-Haghighi (ENH) Abdul-Moniem (2015); Lemonte (2013), generalized linear exponential (GLE) Mahmoud and Alam (2010), half logistic Poisson (HLP) Muhammad and Yahaya (2017), Topp-Leone (TL) Topp and Leone (1955), odd-exponential uniform is the limiting distribution of Poisson odd generalized exponential uniform Muhammad (2016c), odd generalized exponential power function Hassan et al. (2019), Mustapha type I (Mu-I) Muhammad (2016b), tau distribution (Tau) Bakouch et al. (2023), and Kumaraswamy (Kw) Kumaraswamy (1980).

The MLEs of the parameters of each model are computed, the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and Kolmogorov-Smirnov (KS) test are used to compare the new distribution and the other models. The model with the smallest value of these measures provides better fits than the other models.

Table 1: **Selected values, average estimates (AE) with standard deviation (Sd) in parenthesis, and proportion of convergence (PC) of the simulated data from GMu-II distribution.**

Sample size	Selected values			Average Estimate(Sd)			Prop. of Converg.
n	α	θ	β	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\beta}$	PC
30	0.1	1.0	1.0	0.4284	1.9869	0.8092	0.92
				(0.3678)	(4.8215)	(0.1598)	
60	0.1	1.0	1.0	0.3936	0.8762	0.8929	0.98
				(0.2846)	(4.1615)	(0.0947)	
90	0.1	1.0	1.0	0.3278	1.0827	0.9211	0.98
				(0.2411)	(2.5222)	(0.0708)	
120	0.1	1.0	1.0	0.3018	1.6796	0.9426	0.97
				(0.2292)	(2.4691)	(0.0522)	
150	0.1	1.0	1.0	0.2695	1.5013	0.9538	0.98
				(0.2146)	(2.0314)	(0.0443)	
180	0.1	1.0	1.0	0.2376	1.4490	0.9595	0.98
				(0.1900)	(1.0006)	(0.0381)	
30	0.2	1.0	1.5	0.8848	1.5360	1.3336	0.93
				(0.6912)	(3.9731)	(0.1499)	
60	0.2	1.0	1.5	1.0615	0.0565	1.4169	0.98
				(0.4649)	(3.5490)	(0.0796)	
90	0.2	1.0	1.5	0.5737	2.3729	1.4405	0.92
				(0.4323)	(2.1762)	(0.0553)	
120	0.2	1.0	1.5	0.4732	2.3455	1.4543	0.94
				(0.4241)	(2.0786)	(0.0453)	
150	0.2	1.0	1.5	0.5244	1.7423	1.4654	0.96
				(0.4213)	(2.0162)	(0.0352)	
180	0.2	1.0	1.5	0.5717	1.1497	1.4699	0.98
				(0.3984)	(1.9861)	(0.0307)	
30	0.2	0.4	1.0	0.3628	1.2241	0.7667	0.96
				(0.2615)	(3.5978)	(0.1816)	
60	0.2	0.4	1.0	0.3361	0.4998	0.8727	0.98
				(0.1950)	(2.074)	(0.1175)	
90	0.2	0.4	1.0	0.2898	0.8509	0.9066	0.99
				(0.1739)	(2.0485)	(0.0889)	
120	0.2	0.4	1.0	0.2705	1.0111	0.9331	0.99
				(0.1712)	(2.0014)	(0.0631)	
150	0.2	0.4	1.0	0.2447	1.1057	0.9420	0.99
				(0.1534)	(1.8234)	(0.0530)	
180	0.2	0.4	1.0	0.2685	0.6271	0.9505	0.98
				(0.1349)	(1.5460)	(0.0473)	
30	0.6	0.9	0.6	2.3881	1.2596	0.5735	0.95
				(1.8085)	(3.2827)	(0.0256)	
60	0.6	0.9	0.6	1.0611	1.2384	0.5871	0.87
				(1.1053)	(3.1316)	(0.0123)	
90	0.6	0.9	0.6	1.2543	2.3964	0.5907	0.89
				(1.1006)	(2.2379)	(0.0086)	
120	0.6	0.9	0.6	1.3963	1.7333	0.5936	0.95
				(1.0016)	(2.1979)	(0.0063)	
150	0.6	0.9	0.6	1.4226	1.2543	0.5949	0.98
				(1.0005)	(2.0025)	(0.0049)	
180	0.6	0.9	0.6	1.4623	1.0342	0.8958	0.99
				(1.0004)	(2.0012)	(0.0042)	

5.1. Data I

The following first data is given by Abouammoh et al. (1994) its the ordered lifetimes (in days) of 43 blood cancer patients from one of the ministry of Health Hospitals in Saudi Arabia, also, studied by Mahmoud and Alam (2010):

115, 181, 255 ,418, 441, 461, 516, 739, 743, 789, 807 ,865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852.

To know some information about the shape of the failure rate of the data, we consider the total time on test (TTT). In Figure 7 it shows that the data possess an increasing failure rate; thus, we can conclude that GMu-II can be an appropriate model for fitting the data sets.

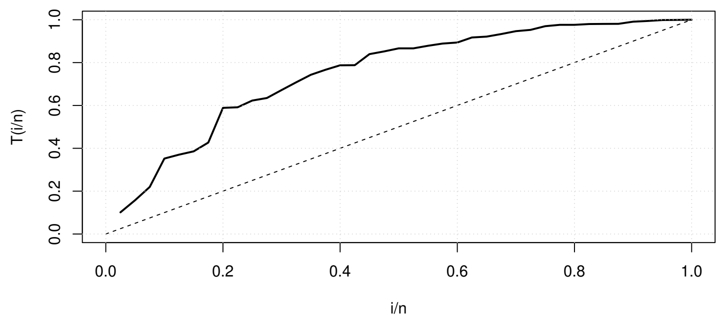

Figure 7: Plots of the total time on test (TTT) for the data I

It can be seen from the Table 2 the GMu-II represent the data set better than the other models as measured by these criteria. For more illustrations, Figure 8 shows the histogram (left) and empirical CDF (right) with the fitted GMu-II distribution for the data I, while Figure 9 give the empirical hazard with the fitted GMu-II (left), and the quantile-quantile plot (right) for the data I. Moreover, the plot of the profile log-likelihood function of the GMu-II for the data I is illustrated in Figure 10.

Table 2: MLEs, ℓ, AIC, BIC, CAIC, K-S and P-values for the data I.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$	$\hat{c}$	ℓ	AIC	BIC	CAIC	K-S	P-value
GMu-II	0.0610	1852.0	18.8704	-	-	-	-	-297.37	600.73	605.79	594.89	0.0585	0.9979
LU	-	0.4159	107.0	-	-	-	4.1375	-317.16	640.32	645.39	634.49	0.1618	0.2206
ExU	115.01	-	-	2.107^{-8}	899.99	-	-	-298.39	602.79	607.86	603.31	1.238	1.0×10^{-10}
ENH	168.0	2.3006	-	4.743×10^{-6}	-	-	-	-304.26	614.51	619.58	608.68	0.1434	0.3489
GLE	2.102×10^{-4}	-	1.5528	-	-	1.389×10^{-6}	-	-305.34	616.68	621.74	610.84	0.1435	0.3488
Ch	0.3116	8.613×10^{-5}	-	-	-	-	-	-302.06	608.13	611.50	604.23	0.0882	0.8879
GE	0.0017	3.6502	-	-	-	-	-	-310.16	624.31	627.69	620.42	0.1655	0.1996
W	9.935×10^{-9}	2.5775	-	-	-	-	-	-304.42	612.83	616.21	608.94	0.1237	0.5323
HLP	0.0004	-	-	4.3496	-	-	-	-322.32	648.65	652.02	648.97	0.2950	0.0014

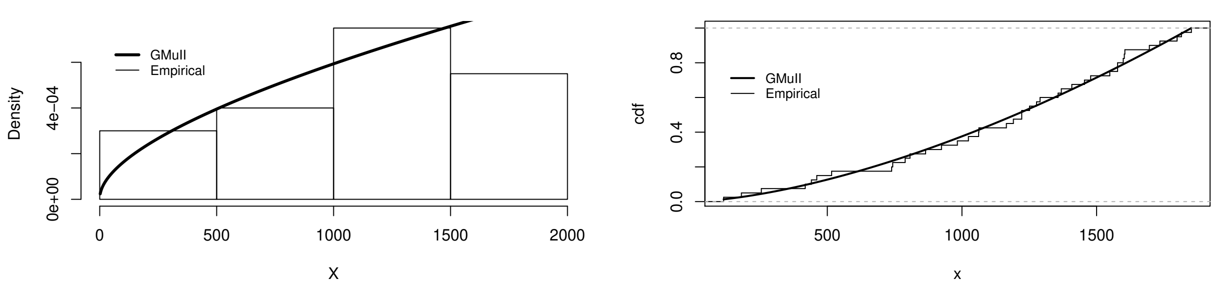

Figure 8: Plots of the histogram (left) and empirical CDF (right) with the fitted GMu-II distribution for the data I

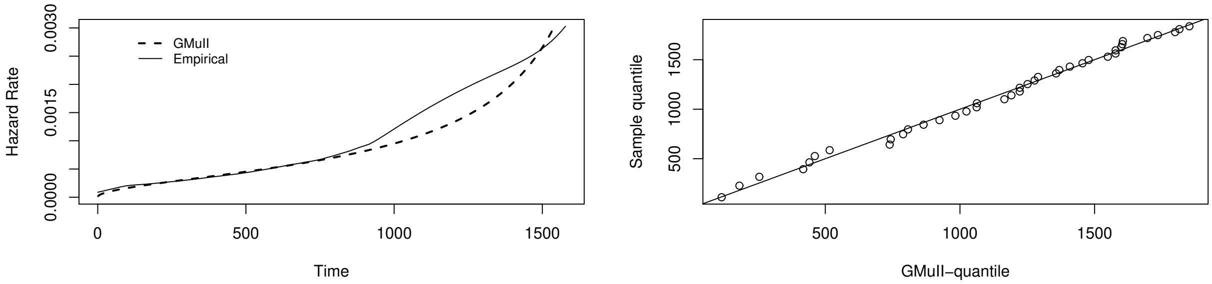

Figure 9: Plots of the empirical hazard with the fitted GMu-II (left), and the quantile-quantile plot (right) for the data I

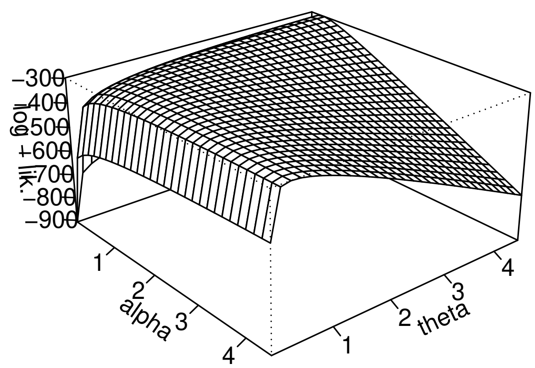

Figure 10: Plots of the profile log- likelihood function of the GMu-II for the data I

5.2. Data II

The following second data is the monthly water capacity of the Shasta reservoir in the time range of August and December from 1975 to 2016 given by Kohansal (2019), also, studied by Tu and Gui (2020):

0.667157, 0.287785, 0.126977, 0.768563, 0.703119, 0.729986, 0.767135, 0.811159, 0.829569, 0.726164, 0.423813, 0.715158, 0.640395, 0.363359, 0.463726, 0.371904, 0.291172, 0.414087, 0.650691, 0.538082, 0.744881, 0.722613, 0.561238, 0.813964, 0.709025, 0.668612, 0.524947, 0.605979, 0.715850, 0.529518, 0.824860, 0.742025, 0.468782, 0.345075, 0.425334, 0.767070, 0.679829, 0.613911, 0.461618, 0.294834, 0.392917, 0.688100

It is clear from the Table 3 the GMu-II provides a better fit to the data better than the other models based on these criteria. Figure 11 give the histogram (left) and empirical CDF (right) with the fitted GMu-II for the data II. Figure 12 the quantile-quantile plot of the GMu-II for the data II. Further, the plot of the profile log-likelihood function of the GMu-II for the data II is given by Figure 13.

Table 3: MLEs, ℓ, AIC, BIC, CAIC, K-S and P-values for the data II.

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$	$\hat{c}$	ℓ	AIC	BIC	CAIC	K-S	P-value
GMu-II	11.1810	0.8296	0.2046	-	-	-	-	20.30	-34.59	-29.38	-40.44	0.1013	0.7445
LU	-	0.4159	0.1260	-	-	-	3.9436	2.71	0.58	5.79	-5.27	0.6818	2.22×10^{-16}
ExU	-	-	-	9.85^{-9}	0.1370	4.4900	-	16.04	-26.08	-20.86	-25.41	0.8409	2.22×10^{-16}
ENH	1.32×10^3	4.5030	-	1.43×10^{-3}	-	-	-	11.83	-17.67	-12.45	-23.51	0.1644	0.1847
GLE	0.2791	-	8.5879	-	-	2.8780	-	10.46	-14.91	-9.70	-20.75	0.1636	0.1889
Ch	3.612	4.1746	-	-	-	-	-	14.99	-25.96	-22.49	-29.86	0.1396	0.3530
GE	4.9943	10.6875	-	-	-	-	-	6.16	-8.33	-4.85	-12.33	0.1643	0.1851
W	5.5277	3.9407	-	-	-	-	-	13.68	-23.35	-19.87	-27.25	0.1505	0.2695
Tau	0.6316	-	3.7681	-	-	-	-	17.95	-31.89	-28.41	-31.58	0.4163	4.16×10^{-7}
TL	3.8782	-	-	-	-	-	-	13.98	-25.95	-24.21	-25.21	0.1951	0.0710
B	-	-	-	-	4.1581	3.0015	-	15.02	-26.05	-22.5	-25.72	0.1403	0.3475
OEU	0.5267	-	-	-	-	1.0000	-	12.70	-21.39	-17.92	-21.07	0.1672	0.1702
Mu-I	-	-	-	-	389.83	0.8196	-	8.31	-12.62	-9.15	-12.30	0.9974	2.22×10^{-16}
Kw	3.4355	3.7681	-	-	-	-	-	15.63	-27.26	-23.79	-26.94	0.1331	0.4104

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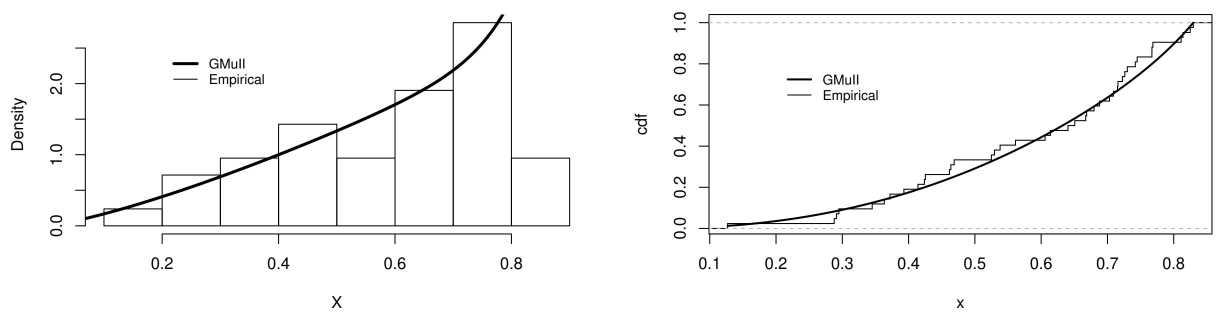

Figure 11: Plots of the histogram (left) and empirical CDF (right) with the fitted GMu-II distribution for the data II

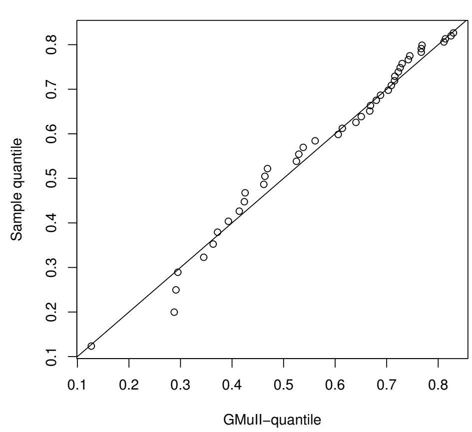

Figure 12: Quantile-quantile plot of GMu-II for the data II

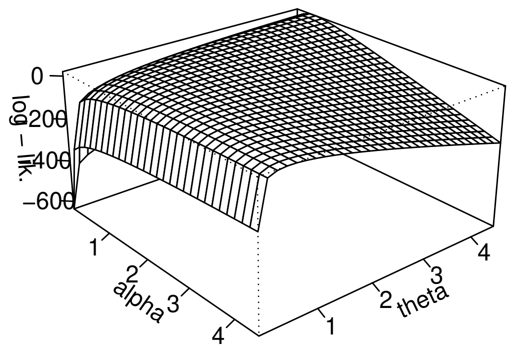

Figure 13: Plots of the profile log- likelihood function of the GMu-II for the data II

6. Conclusions

In this article, we proposed three parameter model with increasing and bathtub failure rate functions called *generalized Mustapha type II distribution* (GMu-II). Some mathematical and statistical properties of the GMu-II are discussed, such as moments, entropy, order statistic, and their extreme value distributions; mean deviations, Bonferroni and Lorenz curves. Further, we consider the bivariate extension of the model. Parameter estimation of the model was conducted by the maximum likelihood method and discussed by simulation studies; the standard deviations of the average estimators decreases as the sample size decreases. Applications of the GMu-II model to two data are provided for illustration. The GMu-II provides a better fit than some other existing distribution as discussed by the AIC, BIC, CAIC, and KS test. The model's performance indicated its capability to study failure data that arise in various fields of studies, especially those with increasing or bathtub failure rates. We hoped that the new model would be useful in statistics, computer sciences, engineering, social sciences, and related fields.

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