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The Double Burr Type XII Model: Censored and Uncensored Validation Using a New Nikulin-Rao-Robson Goodness-of-Fit Test with Bayesian and Non-Bayesian Estimation Methods

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Abstract
After studying the mathematical properties of the Double Burr XII model, we present Bayesian and non-Bayesian estimation for its unknown parameters. Also, we constructed a new statistical test for goodness-of-fit in case of complete and censored samples. The modified test is developed based on the Nikulin-Rao-Robson statistic for validation. Simulations are performed for assessing the new test along with nine applications on real data.

Key Words: Barzilai-Borwein Algorithm; Bayesian Estimation; Validation Nikulin-Rao-Robson; Metropolis Hastings Algorithm; MCMC; Censoring Data; Rényi Entropy; Shannon Entropy.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

Introduction

The Double Burr XII (DBrXII) distribution was originally proposed by Cordeiro et al. (2018) by defining its cumulative distribution function (CDF), probability density function (PDF) and hazard rate function (HRF) as a special case from the Burr XII -G (BrXII -G) family. In this paper, we study the DBrXII distribution with details by studying some of its properties, introducing four applications to illustrate its importance, estimating its unknown parameters via Bayesian and classical methods along with a Markov chain Monte Carlo (MCMC) simulation. Finally, a censored and uncensored validation for it using a developed Nikulin-Rao-Robson (NRR) goodness-of-fit (GOF) test is proposed with simulations and other applications to real data sets.

The DBrXII distribution was constructed based on the well-known BrXII model (Burr (1942)). The CDF of the two-parameter BrXII model is given by

$$G_{\underline{\psi}}(w)|_{(w\geq 0)} = 1-(1+w^{\alpha_2})^{-\beta_2}, \tag{1}$$

where $\underline{\psi} = (\alpha_2, \beta_2)$. The corresponding PDF of (1) is given by

$$g_{\underline{\psi}}(w)|_{(w\geq 0)} = \alpha_2\beta_2 w^{\alpha_1-1}(1+w^{\alpha_2})^{-\beta_2-1}, \tag{2}$$

where both $\alpha_2 > 0$ and $\beta_2 > 0$ are shape parameters. In equation (1), when $\alpha_2 = 1$ the BrXII model reduces to the Lomax (Lx) or Pareto type **II** (Pa**II**) model, when $\beta_2 = 1$ the BrXII model reduces to the log-logistic (LL) model. Details and other properties about the BrXII model can be found in Burr (1942, 1968 and 1973), Burr and Cislak (1968), Rodriguez (1977) and Tadikamalla (1980).

Recently, many authors considered the extension of the BrXII model such as Shao (2004) who discussed the maximum likelihood estimation for the well-known three parameter (3PBrXII) model. Shao et al. (2004) studied models for extremes using the extended 3PBrXII model with application to flood frequency analysis. According to Soliman (2005) the BrXII distribution covers the curve shape characteristics for many other distributions. Silva et al. (2008) proposed a location-scale (L-S) regression model based on the BrXII distribution, Silva et al. (2010), Silva et al. (2011) defined a residual analysis for the log-Burr XII regression model whose empirical model is close to normality. Paranaiba et al. (2011) proposed and studied the beta BrXII (B BrXII) model. Paranaiba et al. (2013) proposed and studied the Kumaraswamy BrXII (KumBrXII) model. Al-Saiari et al. (2014) studied the Marshall-Olkin extended BrXII (MOE BrXII) model. Yousof et al. (2018) derived a new family of Burr-Hatke G (BH-G) models and developed regression model based on it. Cordeiro et al. (2018) proposed and studied the BrXII G (BrXII -G) family for first time ever, the new BrXII -G family was flexible enough to define many other important BrXII model with many useful special cases. Altun et al. (2018a) proposed a new useful BrXII log-location regression model with influence diagnostics, residual analysis, and different real data application. Altun et al. (2018b) studied the Zografos-Balakrishnan BrXII (ZB BrXII) distribution, developed its corresponding regression model for prediction and presented many real data applications with the new model. Nasir et al. (2018) presented a new Weibull BrXII (W BrXII) distribution. Korkmaz et al. (2018) studied the odd Lindley BrXII (OL BrXII) model along with Bayesian analysis, classical inference and some new useful characterizations. Yousof et al. (2019a) introduced a new compound version of BrXII called the zero-truncated Poisson Topp-Leone Burr XII distribution and presented some new useful characterizations and applications for the new model. Yousof et al. (2019b) proposed new BrXII lifetime model based on the Topp-Leone family with regression models, characterizations, and applications. Gad et al. (2019) investigated the Burr XII-Burr XII (BrXII -BrXII) distribution and characterized it along with an application and some statistical properties. Elsayed and Yousof (2019) extended the BrXII model and derived the Poisson generalized Burr XII (PG BrXII) distribution with four applications.

Cordeiro et al. (2018) defined the CDF of the Burr XII-G (BrXII -G) family as

$$F_{\alpha_1,\beta_1,\underline{\psi}}(w) = 1-\left\{\left[\frac{1}{G_{\underline{\psi}}(w)}-1\right]^{-\alpha_1}+1\right\}^{-\beta_1}, \tag{3}$$

where $G_{\underline{\psi}}(w)$ is the baseline CDF. The PDF corresponding to (3) is given by

$$f_{\alpha_1,\beta_1,\underline{\psi}}(w) = \alpha_1\beta_1\left\{1+\left[\frac{1}{G_{\underline{\psi}}(w)}-1\right]^{-\alpha_1}\right\}^{-\beta_1-1}\frac{g_{\underline{\psi}}(w)G_{\underline{\psi}}(w)^{\alpha_1-1}}{\left[1-G_{\underline{\psi}}(w)\right]^{\alpha_1+1}}, \tag{4}$$

where $g_{\underline{\psi}}(w) = dG_{\underline{\psi}}(w)/dx$ is the baseline PDF. Cordeiro et al. (2018) defined also the Double BrXII (DBrXII) with CDF given as

$$F_{(\underline{\Omega})}(w) = 1-\left\{\left[(1+w^{\alpha_2})^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}, \tag{5}$$

where $\underline{\Omega} = (\alpha_1,\beta_1,\alpha_2,\beta_2)$. The PDF corresponding to (5) is given by

$$f_{(\underline{\Omega})}(w) = \alpha_1\beta_1\alpha_2\beta_2 w^{\alpha_1-1}(1+w^{\alpha_2})^{\alpha_1\beta_2-1}\frac{\left[1-(1+w^{\alpha_2})^{-\beta_2}\right]^{\alpha_1-1}}{\left\{1+\left[\frac{1}{1-(1+w^{\alpha_2})^{-\beta_2}}-1\right]^{-\alpha_1}\right\}^{\beta_1+1}}. \tag{6}$$

Equation (5) contains as sub-models several generated distributions. For $\beta_1 = 1$ ($\beta_2 = 1$), we can change the first (second) name of the model by log-logistic (LL). Clearly, the log-logistic-log-logistic (LL-LL) model follows when $\beta_1 = \beta_2 = 1$. For $\alpha_1 = 1$ ($\alpha_2 = 1$), we can change the first (second) name by Pareto type **II** (Pa **II**). So, for $\alpha_1 = 1$ and $\beta_2 = 1$, we obtain the Pareto type **II**-log-logistic (Pa**II**-LL) model. If $\beta_1 \to \infty$ (or $\beta_2 \to \infty$), the first (second) name can be changed by Weibull. If we combine these conditions, we can generate 40 special cases of (5). The DBrXII density and HRF plots for selected parameter values are displayed in Figure 1. From Figure 1(a), we note that the DBrXII PDF can be right skewed ($\alpha_1 = 1, \beta_1 = 5, \alpha_2 = 2, \beta_2 = 10$), ($\alpha_1 = 0.5, \beta_1 = 5, \alpha_2 = 2, \beta_2 = 2$) and ($\alpha_1 = 1, \beta_1 = 5, \alpha_2 = 2, \beta_2 = 2$), left skewed ($\alpha_1 = 10, \beta_1 = 5, \alpha_2 = 1, \beta_2 = 1.2$) and symmetric ($\alpha_1 = 2, \beta_1 = 5, \alpha_2 = 2, \beta_2 = 2$). From Figure 1(b), we note that the DBrXII HRF can be **J**-shape ($\alpha_1 = 5, \beta_1 = 2, \alpha_2 = 2, \beta_2 = 1$), decreasing ($\alpha_1 = 2, \beta_1 = 5, \alpha_2 = 0.5, \beta_2 = 1$) and upside down ($\alpha_1 = 3, \beta_1 = 1, \alpha_2 = 0.4, \beta_2 = 1.25$).

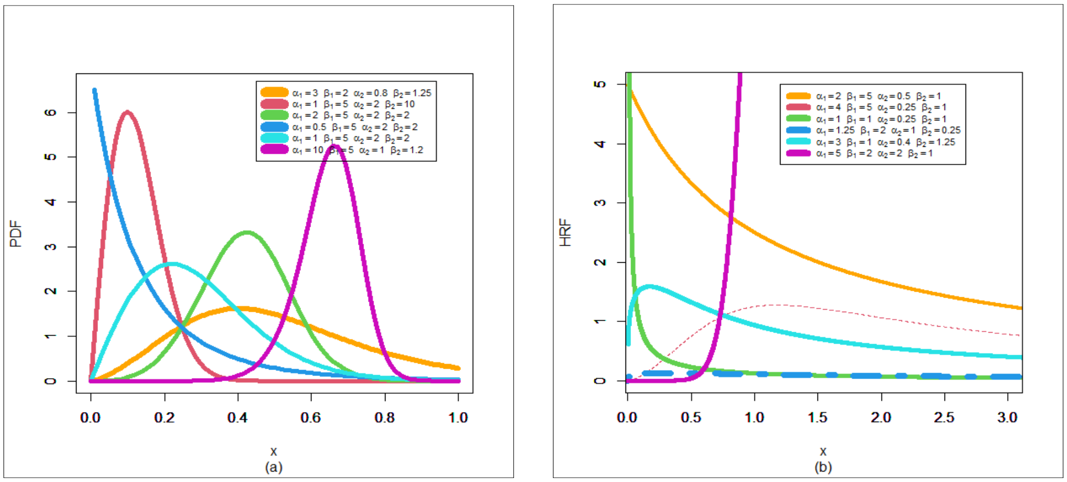

Figure 1: Plots for the DBrXII PDF and HRF.

2. Properties

In this Section, we will present many important mathematical and statistical properties of the new distribution, and some of these features have been presented with numerical analyses and with some useful comments. It is worth noting that there are lot of algebraic derivations that we have neglected to make room for numerical and applied results. We shall employ approaches that provide numerical answers due to the theoretical complexity and the fact that the quantile function is not known in a specific closed form. To make numerical processes easier, pre-made programs like "R" and "MATHCAD" will be used. Numerous factors have contributed to the recent rise in popularity of numerical methods. The presence of several mathematically sophisticated distributions and models, as well as the availability of ready-made statistical programs, are the two most significant. The complexity of models is no longer the main issue facing researchers in the fields of statistical analysis and mathematical modelling, as statistical programs and packages have significantly helped to simplify these complexities by offering numerical solutions. This is a fact that has come to be identified and cannot be ignored.

2.1 Simple linear representation

Let $W \sim \text{DBrXII}(\alpha_1, \beta_1, \alpha_2, \beta_2)$ as shown in (5) and (6). The CDF in (5) can be expressed as

$$F_{(\underline{\Omega})}(w) = 1 - \underbrace{\left\{1 + \left[\frac{1-(1+w^{\alpha_2})^{-\beta_2}}{(1+w^{\alpha_2})^{-\beta_2}}\right]^{\alpha_1}\right\}^{-\beta_1}}_{A(W)}. \tag{7}$$

Consider the power series

$$\left(1 + \frac{q_1}{q_2}\right)^{-\alpha_2} = \sum_{i_1=0}^{\infty} \left(\frac{q_1}{q_2} - 1\right)^{i_1} \left(\frac{1}{2}\right)^{\alpha_2+i_1} \binom{-\alpha_2}{i_1}, \tag{8}$$

$$\left(1 - \frac{q_1}{q_2}\right)^{-\alpha_2} = \sum_{i_2=0}^{\infty} \frac{\Gamma(\alpha_2+i_2)}{\Gamma(1+i_2)\Gamma(\alpha_2)} \left(\frac{q_1}{q_2}\right)^{i_2} \mathrm{I}_{\left(\left|\frac{q_1}{q_2}\right|<1,\ \alpha_2>0\right)}, \tag{9}$$

and

$$\left(1 - \frac{q_1}{q_2}\right)^{\alpha_1-1} = \sum_{i_3=0}^{\infty} \left(\frac{q_1}{q_2}\right)^{i_3} \frac{(-1)^{i_3}\Gamma(\alpha_1)}{\Gamma(1+i_3)\Gamma(\alpha_1-i_3)} \mathrm{I}_{\left(\left|\frac{q_1}{q_2}\right|<1\text{and } \alpha_1>0 \text{ realnon-integer}\right)}. \tag{10}$$

Applying (8) for $A(W)$ in (7), we obtain

$$F_{(\underline{\Omega})}(w)=1-\sum_{i_3=0}^{\infty}\left\{\left[\frac{1-(1+w^{\alpha_2})^{-\beta_2}}{(1+w^{\alpha_2})^{-\beta_2}}\right]^{\alpha_1}-1\right\}^{i_3}\left(\frac{1}{2}\right)^{\alpha_2+i_3}\binom{-\beta_1}{i_3}.$$

First, using the binomial expansion, the last equation can be expressed as

$$F_{(\underline{\Omega})}(w)=1-\sum_{i_3=0}^{\infty}\sum_{i_1=0}^{i_3}\frac{(-1)^{i_1}\left(\frac{1}{2}\right)^{\alpha_2+i_3}}{[1-(1+w^{\alpha_2})^{-\beta_2}]^{-\alpha_1(i_3-i_1)}}\binom{i_3}{i_1}\binom{-\beta_1}{i_3}\underbrace{\left\{1-\left[1-(1+w^{\alpha_2})^{-\beta_2}\right]\right\}^{-\alpha_1(i_3-i_1)}}_{B(W)}.$$

Second, applying (9) for $B(W)$ in the last equation, we can write

$$F_{(\underline{\Omega})}(w)=1-\sum_{i_2,i_3=0}^{\infty}\sum_{i_1=0}^{i_3}\tau_{(i_1,i_2,i_3)}\ \Pi_{\vartheta_{(i,\alpha_1)}}(w), \tag{11}$$

where

$$\Pi_{\vartheta_{(i,\alpha_1)}}(w)=\left[1-(1+w^{\alpha_2})^{-\beta_2}\right]^{\vartheta_{(i,\alpha_1)}}$$

is the CDF of the exp-BrXII model with power parameter $\vartheta_{(i,\alpha_1)}=(-i_1+i_3)\alpha_1+i_2$ and

$$\tau_{(i_1,i_2,i_3)}=(-1)^{i_1}\frac{\Gamma\left(\vartheta_{(i,\alpha_1)}\right)}{i_2!\,\Gamma\left(\alpha_1(i_3-i_1)\right)}\left(\frac{1}{2}\right)^{\alpha_2+i_3}\binom{i_3}{i_1}\binom{-\beta_1}{i_3}.$$

By differentiating (11) and applying (10), we obtain

$$f_{(\underline{\Omega})}(w)=\sum_{\zeta=0}^{\infty}\Omega_{(\zeta)}\,g_{[\alpha_2,\beta_2(1+\zeta)]}(w), \tag{12}$$

and $g_{[\alpha_2,\beta_2(1+\zeta)]}(w)$ is the BrXII PDF with parameters $[\alpha_2,\beta_2(1+\zeta)]$ and

$$\Omega_{(\zeta)}=\frac{(-1)^{\zeta+1}}{\zeta!\,(1+\zeta)\Gamma\left(\vartheta_{(i,\alpha_1)}-\zeta\right)}\sum_{i_2,i_3=0}^{\infty}\sum_{i_1=0}^{i_3}\tau_{(i_1,i_2,i_3)}\left(\vartheta_{(i,\alpha_1)}\right)\Gamma\left(\vartheta_{(i,\alpha_1)}\right)|_{(i_2+i_3\geq 1)}.$$

Equation (12) reveals that the DBrXII PDF is a linear combination of BrXII PDF.

2.2 Ordinary moment

The $m^{(\mathrm{th})}$ ordinary moment of w is given by

$$\mu'_m=E(W^m)=\sum_{\zeta=0}^{\infty}\Omega_{(\zeta)}\int_0^{\infty}w^m\ g_{[\alpha_2,\beta_2(1+\zeta)]}(w)dw.$$

Then, we obtain

$$\mu'_m|_{(m<\alpha_2\beta_2(1+\zeta))}=E(W^m)=\sum_{\zeta=0}^{\infty}\Omega_{(\zeta)}\,\beta_2(1+\zeta)B\left(\beta_2(1+\zeta)-\frac{m}{\alpha_2},\frac{m}{\alpha_2}+1\right). \tag{13}$$

Setting $m=1$ in (13), we have the mean of W.

2.3 Moment generating function

The moment generating function (M.G.F) of W, say $M_W(t)=E[exp(tW)]$, can be obtained from (12) as

$$M_W(t)=\sum_{\zeta=0}^{\infty}\Omega_{(\zeta)}\,M_{[\alpha_2,\beta_2(1+\zeta)]}(t).$$

Next, we require the Meijer **G**-function defined by

$$G_{(p,q)}^{(m,n)}\left(w\middle|\begin{matrix}(\alpha_2)_1,\ldots,(\alpha_2)_p\\(\beta_2)_1,\ldots,(\beta_2)_q\end{matrix}\right)=\frac{1}{2\sqrt{-1}\pi}\int_{(\mathbf{L})}\frac{\begin{bmatrix}\prod_{j=1}^{n}\quad\Gamma\left((\beta_2)_j+t\right)\\\prod_{j=1}^{n}\quad\Gamma\left(1-(\alpha_2)_j-t\right)\end{bmatrix}}{\begin{bmatrix}\prod_{j=n+1}^{p}\quad\Gamma\left((\alpha_2)_j+t\right)\\\prod_{i_2=m+1}^{p}\quad\Gamma\left(1-(\beta_2)_j-t\right)\end{bmatrix}}w^{-t}dt,$$

where $\sqrt{-1}$ is the complex unit and $\mathbf{L}$ denotes the path of the integration. The Meijer **G**-function (MjGF) contains as particular cases many integrals with elementary and special functions. We now assume that $\alpha_2\ =m/\beta_2$, where m and β_2 are positive integers. This condition is not restrictive since every positive real number can be approximated by a rational number. We have the following result, which holds for positive integers m and β_2, $\mu>-1$ and $p>0$,

$$I\left(p,\mu,\frac{m}{\beta_2},C\right)|_0^{\infty}=\int_0^{\infty}exp(-pw)\,w^{\mu}\left(1+w^{\frac{m}{\beta_2}}\right)^{v}dw=CG_{(\beta_2+m,\beta_2)}^{(\beta_2,\beta_2+m)}\left(\frac{m^m}{p^m}\middle|\begin{matrix}\Delta(m,-\mu),\Delta(\beta_2,v+1)\\\Delta(\beta_2,0)\end{matrix}\right),$$

where

$$C = \frac{\beta_2^{-v} m^{\mu+\frac{1}{2}}}{(2\pi)^{\frac{m-1}{2}} \Gamma(-v) p^{\mu+1}}$$

and

$$\Delta(\alpha_2, \beta_2) = \frac{\alpha_2}{\beta_2}, \frac{\alpha_2+1}{\beta_2}, \frac{\alpha_2+2}{\beta_2}, \dots, \frac{\alpha_2+\beta_2}{\beta_2}.$$

We can write (for $t < 0$)

$$M_W(t) = mI\left(-t, \frac{m}{\beta_2} - 1, \frac{m}{\beta_2}, -\beta_2 - 1\right).$$

Hence, the M.G.F of W can be expressed in terms of the MjGF as

$$M_W(t) = m \sum_{\zeta=0}^{\infty} \Omega_{(\zeta)}\, I\left(-t, \frac{m}{\beta_2(1+\zeta)} - 1, \frac{m}{\beta_2(1+\zeta)}, -[1+\beta_2(1+\zeta)]\right).$$

2.4 Incomplete moments

The $s^{(\mathrm{th})}$ incomplete moment, say $I_{(s,W)}(t)$, of the DBrXII distribution is given by

$$I_{(s,W)}(t) = \int_0^t w^s f(w) dw.$$

We can write from equation (12)

$$I_{(s,W)}(q) = \sum_{\zeta=0}^{\infty} \Omega_{(\zeta)} \int_0^q w^s\, g_{[\alpha_2, \beta_2(1+\zeta)]}(w) dw,$$

and then using the lower incomplete gamma function, we obtain

$$I_{(s,W)}(t)|_{(s<\alpha_2\beta_2(1+\zeta))} = \sum_{\zeta=0}^{\infty} \Omega_{(\zeta)}\, \beta_2(1+\zeta)\left[B_{[t^{\alpha_1}]}\left(\beta_2(1+\zeta) \ - \frac{s}{\alpha_2}, \frac{s}{\alpha_2} + 1\right)\right].$$

The first incomplete moment of W, denoted by $I_{(1,W)}(t)$, is simply determined from the above equation by setting $s = 1$. The first incomplete moment has important applications related to the mean residual life, Bonferroni and Lorenz curves.

2.5 Residual and reversed residual life functions

The $m^{(\mathrm{th})}$ moment of the residual life (RL), is given by

$$m_m(t) = E[(W-t)^m]|_{(W>t,\ m=1,2,\dots)}.$$

The $m^{(\mathrm{th})}$ moment of the residual life of W is given by

$$m_m(t) = \frac{1}{1 - F_{(\underline{\Omega})}(t)} \int_t^{\infty} (w-t)^m\, dF_{(\underline{\Omega})}(w).$$

Then, we can write

$$m_m(t)|_{(m<\alpha_2\beta_2(1+\zeta))} = \frac{1}{1-F(t)} \sum_{i_1=0}^{m} \sum_{\zeta=0}^{\infty} \Omega_{(\zeta)} \frac{(-1)^{m-i_1} m!\, t^{m-i_1}}{i_1!\, \Gamma(m-i_1+1)} \beta_2(1+\zeta) B_{[t^{\alpha_1}]}\left(\beta_2(1+\zeta) - \frac{m}{\alpha_2}, \frac{m}{\alpha_2} + 1\right).$$

The $m^{(\mathrm{th})}$ moment RRL, say

$$M_m(t) = E[(t-W)^m]|_{(W \le t, t>0 \text{ and } m=1,2,\dots,)}$$

Then, $M_m(t)$ is defined by

$$M_m(t) = \frac{1}{F_{(\underline{\Omega})}(t)} \int_0^t (t-w)^m\, dF_{(\underline{\Omega})}(t).$$

The $m^{(\mathrm{th})}$ moment of the RRL of W

$$M_m(t)|_{(m<\alpha_2\beta_2(1+\zeta))} = \frac{1}{F(t)} \sum_{i_1=0}^{m} \sum_{\zeta=0}^{\infty} \frac{(-1)^{i_1} m!}{i_1!\,(m-i_1)!} \Omega_{(\zeta)} \beta_2(1+\zeta) B_{[t^{\alpha_1}]}\left(\beta_2(1+\zeta) - \frac{m}{\alpha_2}, \frac{m}{\alpha_2} + 1\right).$$

3. Entropies and numerical analysis

Entropy is a measurable physical characteristic and a scientific notion that is frequently connected to a condition of disorder, unpredictability, or uncertainty. From classical thermodynamics, where it was originally recognized, through the microscopic description of nature in statistical physics, to the fundamentals of information theory, the phrase and concept are utilized in a variety of disciplines. It has numerous applications in physics and chemistry, biological systems and how they relate to life, cosmology, economics, sociology, weather science, and information systems, especially the exchange of information. In this Section, we will present all the algebraic derivations of the three types of entropy. We will also present the numerical analysis needed to highlight the importance of the three types with some useful comments.

3.1 Rényi entropy

The Rényi entropy of a continuous random variable (RV) W represents a measure of variation of the uncertainty. Unlike the discrete case, Rényi entropy can be negative for continuous RVs (see Table 1), and so Rényi entropy is typically only used for discrete RVs. For a continuous RV, the Rényi entropy is defined by

$$I_\theta(W) = \frac{1}{1-\theta} log \int_{-\infty}^{\infty} f(w)^\theta \, dw|_{(\theta>0 \text{ and } \theta\neq1)}.$$

For $\theta = 0$ we have the Max-entropy (Hartley entropy). By using Equation (6), we can write

$$I_\theta(X) = \frac{1}{(1-\theta)} log\left[\sum_{\substack{i_1,i_2=0\\ i_1+i_2\geq1}}^{\infty} \sum_{i_3=0}^{i_2} a_{i_3,i_1,i_2}\, \mathbf{I}_0^\infty(\theta)\right],$$

where

$$\mathbf{I}_0^\infty(\mathbf{P}) = \int_{-\infty}^{\infty} \left(\alpha_2\beta_2 w^{\alpha_1-1}(1+w^{\alpha_2})^{-\beta_2-1}\right)^{\mathbf{p}} \left[1-(1+w^{\alpha_2})^{-\beta_2}\right]^{\alpha_1(i_2-i_3+\theta)+i_1-\mathbf{P}} dx.$$

The integration $\mathbf{I}_0^\infty(\mathbf{P})$ can be easily simplified and expressed as the $m^{(\text{th})}$ ordinary moment. Table 1 gives some values of the Rényi entropy for some parameter values. From Table 1 we note that the Rényi entropy can have a wide range in the interval $(-17.75737, 1.09352)$.

The Rényi entropy reaches its maximum value when $\theta = 0.5, \alpha_1 = 2, \beta_1 = 0.5, \alpha_2 = 1.2$ and $\beta_2 = 1.5$. The Rényi entropy reaches its minimum value when $\theta = 0.5, \alpha_1 = 1, \beta_1 = 1, \alpha_2 = 5$ and $\beta_2 = 5$. The Rényi entropy decreases as θ increases.

3.2 δ -entropy

The δ-entropy, say $H_\delta(W)$, is defined by

$$\Lambda_\delta(W) = \frac{1}{\delta-1} log\left[1-\int_{-\infty}^{\infty} f(w)^\delta \, dx\right]|_{(\delta>0 \text{ and } \delta\neq1)},$$

and then we have

$$\Lambda_\delta(W) = \frac{1}{\delta-1} log\left(1-\int_{-\infty}^{\infty} \sum_{\substack{i_1,i_2=0\\ i_1+i_2\geq1}}^{\infty} \sum_{i_3=0}^{i_2} a_{i_3,i_1,i_2}\, \mathbf{I}_0^\infty(\delta)\right)|_{(\delta>0 \text{ and } \delta\neq1)},$$

Table 2 gives some values of δ-entropy for some parameter values. From Table 2 we note that the δ-entropy is always positive. The δ-entropy reaches ∞ as θ increases.

3.3 Shannon entropy

The Shannon entropy, say $H_1(W)$, of a RV W is defined by

$$H_1(X) = \lim_{\theta\to\infty} [I_\theta(W)] = -E[log\, f(W)].$$

Table 3 gives some values of Shannon entropy for some parameter values. From Table 3 we note that the Shannon entropy is always positive. The Shannon entropy can have a wide range in the interval (0.019911, 1.723628). The Shannon entropy reaches its minimum value when $\alpha_1 = 2, \beta_1 = 0.5, \alpha_2 = 0.2$ and $\beta_2 = 0.2$. The Shannon entropy reaches its maximum value when $\alpha_1 = 3, \beta_1 = 1, \alpha_2 = 1$ and $\beta_2 = 1$.

Table 1: Numerical analysis for the Rényi entropy.

θ	$I_\theta(X)\|\underline{\Omega} = (2,0.5,1.2,1.5)$
0.5	1.09352
15	0.00843
25	0.00317

40	0.00129
50	0.00084
100	0.00022
θ	$I_{\theta}(X)\vert\underline{\Omega} = (10,10,0.25,2)$
0.5	-0.73722
5	-0.01031
30	-0.00019
50	-0.00007
150	-0.00001
θ	$I_{\theta}(X)\vert\underline{\Omega} = (1,1,5,5)$
0.1	1.646730
0.5	-17.75737
5	-0.065810
30	-0.001030
50	-0.000350
800	-0.000001
θ	$I_{\theta}(X)\vert\underline{\Omega} = (1,1,5,5)$
0.1	-1.161670
0.5	-2.057420
5	-0.021210
30	-0.000380
50	-0.000130
300	-0.000003

Table 2: Numerical analysis for the δ-entropy.

δ	$\Lambda_{\delta}(X)\vert\underline{\Omega} = (2,0.5,1.2,1.5)$
1.5	1.5000
10	4.9500
30	1052.5
50	∞
δ	$\Lambda_{\delta}(X)\vert\underline{\Omega} = (1.6,0.4,1.5,2)$
1.5	1.444327
10	3.712108
30	411.2866
50	∞
δ	$I_{\theta}(X)\vert\underline{\Omega} = (0.5,0.3,5,5)$
1.5	1.040901
10	0.524755
30	1.042385
50	2.721963
100	36.965389
δ	$I_{\theta}(X)\vert\underline{\Omega} = (0.2,0.1,3,3)$
1.5	14.50
10	5541
30	656.9
50	11376
100	∞

Table 3: Numerical analysis for Shannon entropy.

α_1	β_1	α_2	β_2	$H_1(X)$
2.0	0.5	0.20	0.20	0.019911
1.5	1.5	0.25	0.10	0.025604
3.0	1.0	1.00	1.00	1.723628
2.5	0.5	0.90	0.60	0.251462

0.9	0.9	1.10	0.70	0.446482
2.8	0.8	0.20	0.40	0.073508
1.9	1.2	0.50	0.75	0.522898

4. Uncensored Non-Bayesian estimation methods

We will consider the following methods: the maximum likelihood estimation (MLE) method, method of ordinary least square estimation (OLSE) and the Cramer-Von-Mises estimation (CVME) method. Many authors used different estimation methods in their work and performed useful simulations and applications such as Ibrahim and Yousof (2020) (under the transmuted Topp-Leone Weibull lifetime distribution), Ibrahim et al. (2020b) (under the Burr XII exponentiated exponential model), Salah et al. (2020) (under an expanded Fréchet model), Yousof et al. (2020) (under the two-parameter xgamma Fréchet distribution), Ibrahim et al. (2021) (under a new three-parameter xgamma Fréchet distribution) and Alizadeh et al. (2022) (under the odd log-logistic transmuted-G family of distributions).

4.1 The MLE

Maximum MLE is a statistical technique for estimating the parameters of a probability distribution that has been assumed given some observed data. This is accomplished by maximizing a likelihood function to make the observed data as probable as possible given the assumed statistical model. The maximum likelihood estimate is in the location in the parameter space where the likelihood function is maximized. Maximum likelihood is a popular approach for making statistical inferences since its rationale is clear and adaptable. The derivative test for locating maxima can be used if the likelihood function is differentiable. When all observed outcomes are assumed to have Normal distributions with the same variance, the ordinary least squares estimator for a linear regression model maximizes the likelihood. In some circumstances, the first-order requirements of the likelihood function can be solved analytically. The MLE is typically equal to maximum a posteriori (MAP) estimation with uniform prior distributions from the standpoint of Bayesian inference (or a normal prior distribution with a standard deviation of infinity). The MLE is a specific example of an extremum estimator in frequentist inference, with likelihood as the objective function. As part of the MLE technique, we represent a collection of observations as a random sample drawn from an unknowable joint probability distribution that is specified in terms of a number of parameters. Finding the parameters for which the observed data have the highest joint probability is the aim of maximum likelihood estimation. The log likelihood function ($log\, L$) for the new model is

$$log\, L = m\, log\, \alpha_1 + m\, log\, \beta_1 + m\, log\, \alpha_2 + m\, log\, \beta_2 + (\alpha_1 - 1)\sum_{i=1}^{m} log(w_{i:m}) + (\beta_2\alpha_1 - 1)\sum_{i=1}^{m} log(\xi_i)$$
$$+(\alpha_1 - 1)\sum_{i=1}^{m} log\left(-\xi_i^{-\beta_2} + 1\right) - (\beta_1 + 1)\sum_{i=1}^{m} log\left[\left(\xi_i^{\beta_2} - 1\right)^{\alpha_1} + 1\right],$$

where $w_{i:m}^{\alpha_2} + 1 = \xi_i$ and

$$\frac{\partial}{\partial\alpha_1} log\, L = \frac{m}{\alpha_1} + \sum_{i=1}^{m} log(w_{i:m}) + \beta_2\sum_{i=1}^{m} log\, \xi_i + \sum_{i=1}^{m} log\left(-\xi_i^{-\beta_2} + 1\right)$$
$$-(\beta_1 + 1)\sum_{i=1}^{m} \frac{\left(\xi_i^{\beta_2} - 1\right)^{\alpha_1} log\left(\xi_i^{\beta_2} - 1\right)}{\left(\xi_i^{\beta_2} - 1\right)^{\alpha_1} + 1},$$

$$\frac{\partial}{\partial\beta_1} log\, L = \frac{m}{\beta_1} - \sum_{i=1}^{m} log\left[\left(\xi_i^{\beta_2} - 1\right)^{\alpha_1} + 1\right],$$

$$\frac{\partial}{\partial\alpha_2} log\, L = \frac{m}{\alpha_2} - m\, log\, c + (\beta_2\alpha_1 - 1)\sum_{i=1}^{m} \frac{w_i^{\alpha_2}\, log\, w_{i:m}}{\xi_i} + (\alpha_1 - 1)\sum_{i=1}^{m} \frac{\beta_2\xi_i^{-\beta_2 - 1} w_{i:m}^{\alpha_2}\, log\, w_{i:m}}{-\xi_i^{-\beta_2} + 1}$$
$$-(\beta_1 + 1)\sum_{i=1}^{m} \frac{\alpha_1\beta_2\xi_i^{\beta_2 - 1}\left(\xi_i^{\beta_2} - 1\right)^{\alpha_1 - 1} w_{i:m}^{\alpha_2}\, log\, w_{i:m}}{\left(\xi_i^{\beta_2} - 1\right)^{\alpha_1} + 1},$$

and

$$\frac{\partial}{\partial\beta_2}log\,L=\frac{m}{\beta_2}+\alpha_1\sum_{i=1}^{m}log\,\xi_i+(\alpha_1-1)\sum_{i=1}^{m}\frac{\xi_i^{-\beta_2}\,log(\xi_i)}{-\xi_i^{-\beta_2}+1}-(\beta_1+1)\sum_{i=1}^{m}\frac{\alpha_1\xi_i^{\beta_2}\,log(\xi_i)\left(\xi_i^{\beta_2}-1\right)^{\alpha_1-1}}{\left(\xi_i^{\beta_2}-1\right)^{\alpha_1}+1}.$$

Numerical methods can be used in maximizing the above equations.

4.2 The OLSE

Let $F_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(w_{i:m})$ denote the CDF of DBrXII version and let $w_1<w_2<\cdots<w_m$ be the m ordered random sample. The OLSE are obtained by minimizing

$$OLS(\underline{\Omega})=\sum_{i=1}^{m}\left[F_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(w_m)-c\right]^2,$$

where $c=c_{(m,i)}=\frac{i}{m+1}$. Now using CDF of DBrXII we get

$$OLS(\alpha_1,\beta_1,\alpha_2,\beta_2)=\sum_{i=1}^{m}\left(1-\left\{\left[\left(1+w_{i:m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}-c\right)^2.$$

The OLSEs of $\alpha_1,\beta_1,\alpha_2,\beta_2$ can derived by solving

$$\sum_{i=1}^{m}\left(1-\left\{\left[\left(1+w_{i:m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}-c\right)\Upsilon_{\alpha_1}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=0,$$

$$\sum_{i=1}^{m}\left(1-\left\{\left[\left(1+w_{i:m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}-c\right)\Upsilon_{\beta_1}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=0,$$

$$\sum_{i=1}^{m}\left(1-\left\{\left[\left(1+w_{i:m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}-c\right)\Upsilon_{\alpha_2}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=0,$$

and

$$\sum_{i=1}^{m}\left(1-\left\{\left[\left(1+w_{i:m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}-c\right)\Upsilon_{\beta_2}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=0,$$

where $\Upsilon_{\alpha_1}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=\partial F_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(w_m)/\partial\alpha_1$, $\Upsilon_{\beta_1}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=\partial F_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(w_m)/\partial\beta_1$, $\Upsilon_{\alpha_2}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=\partial F_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(w_m)/\partial\alpha_2$ and $\Upsilon_{\beta_2}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=\partial F_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(w_m)/\partial\beta_2$ are the first partial derivatives of the CDF of DBrXII distribution with respect to $\alpha_1,\beta_1,\alpha_2,\beta_2$ respectively.

4.3 The CVME

The CVME are obtained by minimizing

$$\ell_{\mathrm{CVM}}(\underline{\Omega})=\frac{1}{12m}+\sum_{i=1}^{m}\left[F_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(w_{i:m})-v\right]^2,$$

with respect to the parameter $\alpha_1,\beta_1,\alpha_2$ and β_2 respectively, where $v=v_{(2m,2i)}=\frac{2i-1}{2m}$, then

$$\ell_{\mathrm{CVM}}(\underline{\Omega})=\sum_{i=1}^{m}\left(1-\left\{\left[\left(1+w_{i:m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}-v\right)^2.$$

The CVMEs are obtained by solving

$$\sum_{i=1}^{m}\left(1-\left\{\left[\left(1+w_{i:m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}-v\right)\Upsilon_{\alpha_1}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=0,$$

$$\sum_{i=1}^{m}\left(1-\left\{\left[\left(1+w_{i:m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}-v\right)\Upsilon_{\beta_1}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=0,$$

$$\sum_{i=1}^{m}\left(1-\left\{\left[\left(1+w_{i:m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}-v\right)\Upsilon_{\alpha_2}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=0,$$

and

$$\sum_{i=1}^{m}\left(1-\left\{\left[\left(1+w_{i:m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{-\beta_1}-v\right)\Upsilon_{\beta_2}(w_{i:m},\alpha_1,\beta_1,\alpha_2,\beta_2)=0.$$

5. Uncensored Bayesian estimation

Assume the gamma priors of the parameters $\alpha_1,\beta_1,\alpha_2,\beta_2$ are of the following forms

$$\pi_1^{(\phi_1,\psi_1)}(\alpha_1)\sim \text{Gamma}(\phi_1,\psi_1),\pi_2^{(\phi_2,\psi_2)}(\beta_1)\sim \text{Gamma}(\phi_2,\psi_2),$$
$$\pi_3^{(\phi_3,\psi_3)}(\alpha_2)\sim \text{Gamma}(\phi_3,\psi_3) \text{ and } \pi_4^{(\phi_4,\psi_4)}(\beta_2)\sim \text{Gamma}(\phi_4,\psi_4),$$

where, Gamma $(\phi_i,\psi_i)|_{(i=1,2,3,4)}$ stands for standard gamma model with shape parameter ϕ_i and scale parameter ψ_i. We assume that the parameters are independently distributed. The joint prior distribution $\pi(\underline{\Omega})$ is given by

$$\pi_{(\phi_i,\psi_i)}(\alpha_1,\beta_1,\alpha_2,\beta_2)=\frac{\psi_1^{\phi_1}}{\Gamma(\phi_1)}\frac{\psi_2^{\phi_2}}{\Gamma(\phi_2)}\frac{\psi_3^{\phi_3}}{\Gamma(\phi_3)}\frac{\psi_4^{\phi_4}}{\Gamma(\phi_4)}\frac{\alpha_1^{\phi_1-1}\,\beta_1^{\phi_2-1}\,\alpha_2^{\phi_3-1}\,\beta_2^{\phi_4-1}}{exp[(\alpha_1\psi_1+\beta_1\psi_2+\alpha_2\psi_3+\beta_2\psi_4)]}.$$

The posterior distribution $\pi(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{x})$ of the parameters is defined as

$$\pi(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{x})\propto \text{likelihood}(\alpha_1,\beta_1,\alpha_2,\beta_2|\underline{x})\times\pi_{(\phi_i,\psi_i)}(\alpha_1,\beta_1,\alpha_2,\beta_2).$$

Under squared error loss, the Bayesian estimators of α_1, β_1, α_2 and β_2 are the means of their marginal posteriors. It is not possible to get the Bayesian estimates through the above formulae. So, the numerical approximation is needed. We propose the use of MCMC techniques namely Gibbs sampler and Metropolis Hastings (M-H) algorithm. Since the conditional posteriors $\pi_i(\cdot\mid\cdot,\cdot,\cdot,\underline{x})$ of the parameters α_1, β_1, α_2 and β_2 cannot be obtained in any standard forms, therefore, using a hybrid MCMC for drawing samples from the joint posterior of the parameters is suggested. the full conditional posteriors of $\alpha_1,\beta_1,\alpha_2$ and β_2 are given by

$$\pi_1(\alpha_1|_{(\beta_1,\alpha_2,\beta_2)},\underline{x})\propto\frac{\psi_1^{\phi_1}}{\Gamma(\phi_1)}\frac{\psi_2^{\phi_2}}{\Gamma(\phi_2)}\frac{\psi_3^{\phi_3}}{\Gamma(\phi_3)}\frac{\psi_4^{\phi_4}}{\Gamma(\phi_4)}\alpha_1^{m+\phi_1-1}\prod_{i=1}^{m}K_i\;exp[-(\alpha_1\psi_1+\beta_1\psi_2+\alpha_2\psi_3+\beta_2\psi_4)]$$

$$\pi_2(\beta_1|_{(\alpha_1,\alpha_2,\beta_2)},\underline{x})\propto\frac{\psi_1^{\phi_1}}{\Gamma(\phi_1)}\frac{\psi_2^{\phi_2}}{\Gamma(\phi_2)}\frac{\psi_3^{\phi_3}}{\Gamma(\phi_3)}\frac{\psi_4^{\phi_4}}{\Gamma(\phi_4)}\beta_1^{m+\phi_2-1}\prod_{i=1}^{m}K_i\;exp[-(\alpha_1\psi_1+\beta_1\psi_2+\alpha_2\psi_3+\beta_2\psi_4)],$$

$$\pi_3(\alpha_2|_{(\alpha_1,\beta_1,\beta_2)},\underline{x})\propto\frac{\psi_1^{\phi_1}}{\Gamma(\phi_1)}\frac{\psi_2^{\phi_2}}{\Gamma(\phi_2)}\frac{\psi_3^{\phi_3}}{\Gamma(\phi_3)}\frac{\psi_4^{\phi_4}}{\Gamma(\phi_4)}\alpha_2^{m+\phi_3-1}\prod_{i=1}^{m}K_i\;exp[-(\alpha_1\psi_1+\beta_1\psi_2+\alpha_2\psi_3+\beta_2\psi_4)],$$

and

$$\pi_4(\beta_2|_{(\alpha_1,\beta_1,\alpha_2)},\underline{x})\propto\frac{\psi_1^{\phi_1}}{\Gamma(\phi_1)}\frac{\psi_2^{\phi_2}}{\Gamma(\phi_2)}\frac{\psi_3^{\phi_3}}{\Gamma(\phi_3)}\frac{\psi_4^{\phi_4}}{\Gamma(\phi_4)}\beta_2^{m+\phi_4-1}\prod_{i=1}^{m}K_i\;exp[-(\alpha_1\psi_1+\beta_1\psi_2+\alpha_2\psi_3+\beta_2\psi_4)],$$

where

$$K_i=w_i^{\alpha_1-1}\left(1+w_i^{\alpha_2}\right)^{\beta_2\alpha_1-1}\frac{\left[-\left(1+w_i^{\alpha_2}\right)^{-\beta_2}+1\right]^{\alpha_1-1}}{\left\{\left[\left(1+w_i^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1\right\}^{\beta_1+1}}.$$

The simulation algorithm is given as:

- **I.** Provide the initial values, say $\alpha_1,\beta_1,\alpha_2$ and β_2 then at $i^{(\text{th})}$ stage,
- **II.** Using M-H algorithm, generate $[\alpha_1]_{(i)}\sim\pi_1\left([\alpha_1]_{(i)}|_{\{[\beta_1]_{(i)},[\alpha_2]_{(i)},[\beta_2]_{(i)}\}},\underline{x}\right)$,
- **III.** Using M-H algorithm, generate $[\beta_1]_{(i)}\sim\pi_2\left([\beta_1]_{(i)}|_{\{[\alpha_1]_{(i-1)},[\alpha_2]_{(i)},[\beta_2]_{(i)}\}},\underline{x}\right)$,
- **IV.** Using M-H algorithm, generate $[\alpha_2]_{(i)}\sim\pi_3\left([\alpha_2]_{(i)}|_{\{[\alpha_1]_{(i-1)},[\beta_1]_{(i-1)},[\beta_2]_{(i)}\}},\underline{x}\right)$,
- **V.** Using M-H algorithm, generate $[\beta_2]_{(i)}\sim\pi_4\left([\beta_2]_{(i)}|_{\{[\alpha_1]_{(i-1)},[\beta_1]_{(i-1)},[\alpha_2]_{(i-1)}\}},\underline{x}\right)$,
- **VI.** Repeat steps $2-5$, $M=100000$ times to get the samples of size M from the corresponding posteriors of

interest. Obtain the Bayesian estimates of $\alpha_1, \beta_1, \alpha_2$ and β_2 using the following formulae

$$\widehat{\alpha_1}_{\text{Bayesian}} = \frac{1}{M - M_0} \sum_{j=1+M_0}^{M} \alpha_1^{[j]}, \widehat{\beta_1}_{\text{Bayesian}} = \frac{1}{M - M_0} \sum_{j=1+M_0}^{M} \beta_1^{[j]},$$

$$\widehat{\alpha_2}_{\text{Bayesian}} = \frac{1}{M-M_0} \sum_{j=1+M_0}^{M} \alpha_2^{[j]} \text{ and } \widehat{\beta_2}_{\text{Bayesian}} = \frac{1}{M-M_0} \sum_{j=1+M_0}^{M} \beta_1^{[j]},$$

respectively, where $M_0(\approx 50000)$ is the burn-in period of the generated MCMC.

6. Simulations for comparing the uncensored Bayesian non-Bayesian estimation methods

The MCMC simulation studies are performed for assessing and comparing the performance of the estimators. This assessment is performed using the average values (AVs) of estimates and the mean squared errors (MSEs). First, we generated 1000 samples of the DBrXII distribution, where $n = 50,\ 100,\ 200$ and 500 choosing

	α_1	β_1	α_2	β_2
I	2	0.5	1.2	1.5
II	1.2	1.5	0.9	2

The AVs and MSEs are obtained and listed in Tables 4, 5, 6 and 7. Based on Tables 4, 5, 6 and 7 we note that all methods performed well but the Bayesian method is the best for all sample sizes.

Table 4: AVs and MSEs for m=50.

	Bayesian	MLE	OLS	CVM
α_1=2	1.99209	2.00325	1.95615	2.16442
	(0.04776)	(0.07030)	(0.18154)	(0.63099)
β_1=0.5	0.50419	0.50408	0.50421	0.50390
	(0.00490)	(0.00461)	(0.00735)	(0.00591)
α_2=1.2	1.21097	1.20653	1.13861	1.14755
	(0.01889)	(0.03018)	(0.23325)	(0.06981)
β_2=1.5	1.49463	1.50338	1.49828	1.50752
	(0.02081)	(0.02120)	(0.02383)	(0.02337)
α_1=1.2	1.20149	1.22260	1.20758	1.22193
	(0.01798)	(0.02358)	(0.06857)	(0.07330)
β_1=1.5	1.58934	1.50721	1.54003	1.51822
	(0.06366)	(0.07226)	(0.08517)	(0.06940)
α_2=0.9	0.88050	0.91284	0.90818	0.90799
	(0.00709)	(0.00797)	(0.01174)	(0.00984)
β_2=2	2.08309	1.98324	2.03585	2.00733
	(0.07041)	(0.09195)	(0.09730)	(0.09185)

Table 5: AVs and MSEs for m=100.

	Bayesian	MLE	OLS	CVM
α_1=2	1.94481	2.00817	1.97742	1.98372
	(0.03092)	(0.04178)	(0.09006)	(0.09382)
β_1=0.5	0.49713	0.50365	0.49976	0.50239
	(0.00250)	(0.00261)	(0.00294)	(0.00302)
α_2=1.2	1.17579	1.20646	1.14954	1.18303
	(0.00984)	(0.01726)	(0.08246)	(0.04404)
β_2=1.5	1.50445	1.50483	1.50092	1.50442
	(0.01336)	(0.01204)	(0.01167)	(0.01131)
α_1=1.2	1.17109	1.20035	1.19442	1.21642
	(0.01024)	(0.01111)	(0.03250)	(0.03860)
β_1=1.5	1.39801	1.50832	1.52894	1.50547
	(0.03050)	(0.03436)	(0.03868)	(0.03474)
α_2=0.9	0.90986	0.90048	0.90066	0.90511
	(0.00398)	(0.00390)	(0.00580)	(0.00469)
β_2=2	1.88030	1.99896	2.02740	2.00009
	(0.04235)	(0.04469)	(0.04573)	(0.04318)

Table 6: AVs and MSEs for m=200.

	Bayesian	MLE	OLS	CVM
α_1=2	1.92878	2.00270	1.99917	1.99324
	(0.01848)	(0.02013)	(0.05335)	(0.05156)
β_1=0.5	0.47360	0.50099	0.50142	0.50070
	(0.00186)	(0.00124)	(0.00151)	(0.00152)

α_2=1.2	1.16534	1.20404	1.17707	1.18015
	(0.00623)	(0.00828)	(0.01158)	(0.01123)
β_2=1.5	1.45561	1.50034	1.50348	1.50164
	(0.00527)	(0.00586)	(0.00582)	(0.00578)
α_1=1.2	1.19933	1.20343	1.20715	1.21240
	(0.00476)	(0.00568)	(0.01796)	(0.01757)
β_1=1.5	1.48944	1.50379	1.50644	1.49742
	(0.01050)	(0.01578)	(0.01912)	(0.01611)
α_2=0.9	0.87792	0.90186)	0.90370	0.90408
	(0.00179)	(0.00197)	(0.00303)	(0.00229)
β_2=2	2.01909	1.99830	2.00498	1.99354
	(0.01594)	(0.02158)	(0.02294)	(0.02024)

Table 7: AVs and MSEs for m=500.

	Bayesian	MLE	OLS	CVM
α_1=2	2.02876	2.00289	1.99057	1.999986
	(0.00736)	(0.00856)	(0.02392)	(0.02538)
β_1=0.5	0.49749	0.50093	0.49914	0.50009
	(0.00052)	(0.00052)	(0.00062)	(0.00064)
α_2=1.2	1.23210	1.20253	1.18356	1.18828
	(0.00327)	(0.00359)	(0.00516)	(0.00491)
β_2=1.5	1.48453	1.50133	1.49756	1.50024
	(0.00224)	(0.00245)	(0.00218)	(0.00231)
α_1=1.2	1.22057	1.19975	1.20634	1.20424
	(0.00236)	(0.00209)	(0.00743)	(0.00813)
β_1=1.5	1.52407	1.50313	1.49908	1.50289
	(0.00486)	(0.00632)	(0.00666)	(0.00737)
α_2=0.9	0.90584	0.89985	0.90261	0.90134
	(0.00074)	(0.00074)	(0.00109)	(0.00115)
β_2=2	2.02922	2.00135	1.99823	2.00326
	(0.00720)	(0.00867)	(0.00807)	(0.00861)

7. Four applications for comparing uncensored Bayesian non-Bayesian methods

Four examples with real data sets are introduced for comparing Bayesian and classical estimators. We consider the Cramér-Von Mises (W*) and the Anderson-Darling (A*) statistic. The 1st data is the breaking stress data (see [29]). The 2nd data presents survival times of guinea pigs see [30]. The 3rd data are taxes revenue data see [20-21]. The 4th data called leukemia data see [28]. For data sets **I, III** and **IV**, all methods perform well (see Table 8, Table 9, Table 10 and Table 11). For data set **II**, the CVM is the best method with W*= 0.12782 and A*= 0.75961, however, all other methods preform well (see Table 9).

Table 8: Estimators, W* and A* for the 1st data set.

Method	α_1	β_1	α_2	β_2	W*	A*
ML	7.90802	11.08759	0.44455	0.57891	0.06830	0.39622
Bayesian	9.18110	17.49517	0.37689	0.60070	0.06415	0.39325
OLS	8.80655	16.73587	0.39450	0.59011	0.06464	0.39325
CVM	8.27016	19.51729	0.43308	0.55734	0.06363	0.39346

Table 9: Estimators, W* and A* for the 2nd data set.

Method	α_1	β_1	α_2	β_2	W*	A*
ML	18.12813	0.6688842	0.10887	0.79877	0.15900	0.93678
Bayesian	3.63440	50.69499	0.71415	0.30328	0.12794	0.76187
OLS	8.91991	13.11140	0.25657	0.72190	0.15064	0.88425
CVM	3.26713	51.00056	0.86068	0.26390	0.12782	0.75961

Table 10: Estimators, W* and A* for the 3rd data set.

Method	α_1	β_1	α_2	β_2	W*	A*
ML	6.64754	0.71808	7.54324	0.04047	0.05501	0.33235
Bayesian	6.55043	0.59917	6.96374	0.04552	0.04680	0.29956
OLS	7.44767	0.44835	7.86524	0.04194	0.04392	0.31066
CVM	7.70330	0.44299	0.75780	0.04347	0.04469	0.31757

Table 11: Estimators, W* and A* for the 4th data set.

Method	α_1	β_1	α_2	β_2	W*	A*
ML	3.91956	127.23786	0.31499	0.18368	0.09489	0.63921
Bayesian	6.06638	244.81409	0.16265	0.33376	0.09423	0.64148
OLS	8.74562	7.10599	0.08498	0.68958	0.09788	0.65752
CVM	5.49470	244.95433	0.16053	0.30677	0.09459	0.63855

8. Four uncensored applications for comparing the competitive models

For all data sets, we compare the DBrXII distribution, with BrXII distributions as listed below. We consider the well-known GOF statistics: the Akaike Information Criterion (C_{AI}), Bayesian Criterion (C_{Bayes}), Hannan-Quinn Criterion (C_{HQ}), Consistent Akaike Criterion (C_{CA}). Tables 12, 13, 14 and 15 give the MLEs, standard errors (SEs), confidence interval (CL) with C_{AI}, C_{Bayes}, C_{HQ} and C_{CA} values for the four data set respectively. Based on the values in Tables 12, 13, 14 and 15 DBrXII model has the best fits as compared to BrXII other models in the four applications with small values for C_{AI}, C_{Bayes}, C_{HQ} and C_{CA}. Other useful data sets are given in Aryal and Yousof (2017), Aryal et al. (2017), Ali et al. (2021a,b), Alizadeh et al. (2018, 2020a,b), Almazah et al. (2021), Chesneau and Yousof (2021), Elgohari and Yousof (2020a,b,c), Elgohari et al. (2021), Hamedani et al. (2017, 2018,2019,2022), Karamikabir et al. (2019), Korkmaz et al. (2018a,2018b,2020), Merovci et al. (2017,2020), Nascimento et al. (2019). Figures 2-5 gives the plots for data set **I**, **II**, **III** and **IV** respectively.

Table 12: MLEs, SEs and CL with C_{AI}, C_{Bayes}, C_{HQ} and C_{CA} for the data set I.

Model	$\hat{\lambda},\hat{\theta},\hat{\alpha},\hat{\beta},\hat{\gamma}$	C_{AI}, C_{Bayes}, C_{HQ} and C_{CA}
B XII	--,--, 5.941, 0.187,--	382.94, 388.15, 383.06, 385.05
	--,--, (1.279) ,(0.044),--	
	--,--, (3.43,8.45),(0.10,0.27),--	
MOB XII	--,--, 1.192,4.834,838.73	305.78, 313.61, 306.03, 308.96
	--,--, (0.952),(4.896),(229.34)	
	--,--, 0, 3.06),(0,14.43),(389.22,1288.24)	
TLB XII	--,--, 1.350,1.061,13.728	323.52, 331.35, 323.77, 326.70
	--,--, 0.378) ,(0.384) ,(8.400)	
	--,--, (0.61, 2.09), (0.31,1.81) ,(0, 30.19)	
KwB XII	48.103 ,79.516 ,0.351 ,2.730,--	303.76, 314.20, 304.18, 308.00
	(19.348) ,(58.186) ,(0.098) ,(1.077) ,--	
	(10.18,86.03) ,(0,193.56) ,(0.16,0.54), (0.62,4.84),--	
BBXII	359.683 ,260.097 ,0.175 ,1.123 ,--	305.64, 316.06, 306.06, 309.85
	(57.941) ,(132.213),(0.013),(0.243),--	
	(246.1,473.2), (0.96,519.2), (0.14,0.20), (0.65,1.6),--	
BE BXII	0.381, 11.949, 0.937, 33.402, 1.705	305.82, 318.84, 306.46, 311.09
	(0.078), (4.635), (0.267), (6.287),(0.478)	
	(0.23,0.53) ,(2.86,21), (0.41,1.5), (21,45), (0.8,2.6)	
FKw BXII	0.542,4.223, 5.313, 0.411, 4.152	305.50, 318.55, 306.14, 310.80
	(0.137), (1.882), (2.318), (0.497), (1.995)	
	(0.3, 0.8), (0.53,7.9), (0.9,9), (0, 1.7), (0.2,8)	
ZB BXII	123.101,--,0.368, 139.247,--	302.96, 310.78, 303.21, 306.13
	(243.011), --, (0.343), (318.546),--	
	(0, 599.40), --,(0, 1.04), (0, 763.59),--	
DBrXII	7.91, 11.09, 0.445, 0.58, --	290.55, 300.97, 290.97, 294.77
	(15.8), (30.64), (1.04), (0.79), --	
	(0, 39.9), (0, 71), (0, 3.3), (0, 2.16), --	

Table 13: MLEs, SEs and CL with C_{AI}, C_{Bayes}, C_{HQ} and $C_{\mathrm{CA}}$for the data set II.

Model	$\hat{\lambda},\hat{\theta},\hat{\alpha},\hat{\beta},\hat{\gamma}$	C_{AI}, C_{Bayes}, C_{HQ} and C_{CA}
B XII	--,--, 3.102, 0.465, --	209.60, 214.15, 209.77, 211.40
	--,--, (0.538), (0.077),--	
	--,--, (2.05,4.16), (0.31,0.62),--	
MO BXII	--,--, 2.259,1.533, 6.760	209.74, 216.56, 210.09, 212.44
	--,--, (0.864), (0.907), (4.587)	
	--,--, (0.57,3.95), (0,3.31), (0, 15.75)	
TL BXII	--,--, 2.393,0.458,1.796	211.80, 218.63, 212.15, 214.52
	--,--, (0.907), (0.244),(0.915)	
	--,--, (0.62,4.17),(0, 0.94),(0.002,3.59)	
TL BXII	--,--, 2.393,0.458,1.796	211.80, 218.63, 212.15, 214.52
	--,--, (0.907), (0.244),(0.915)	
	--,--, (0.62,4.17),(0, 0.94),(0.002,3.59)	

Kw BXII	14.105,7.424, 0.525, 2.274,-- (10.805), (11.850), (0.279),(0.990),-- (0, 35.28), (0.30.65), (0, 1.07),(0.33, 4.21),--	208.76, 217.86, 209.36, 212.38
FBB XII	0.621, 0.549,3.838, 1.381, 1.665 (0.541), (1.011), (2.785), (2.312), (0.436) (0, 1.7), (0, 2.5), (0, 9.3), (0, 5.9), (0.8, 4.5)	206.80, 218.20, 207.71, 211.30
FKwB XII	0.558,0.308, 3.999, 2.131, 1.475 (0.442), (0.314), (2.082), (1.833), (0.361) (0, 1.4), (0, 0.9), (0, 3.1), (0, 5.7), (0.76, 2.2)	206.50, 217.90, 207.41, 211.00
DBrXII	3.329, 1.465, 0.872, 0.688, -- (2.83), (1.05), (0.8), (0.3), -- (0, 8.9), (0, 3.6), (0, 2.47), (1.3), --	205.44, 214.55, 206.04, 209.07

Table 14: MLEs, SEs and CL with C_{AI} , C_{Bayes} , C_{HQ} and C_{CA} for the data set III.

Model	$\hat{\lambda},\hat{\theta},\hat{\alpha},\hat{\beta},\hat{\gamma}$	C_{AI} , C_{Bayes} , C_{HQ} and C_{CA}
B XII	--,--, 5.615, 0.072,-- --,--, (15.048), (0.194),-- --,--, (0, 35.11), (0, 0.45),--	518.46, 522.62, 518.67, 520.08
MOB XII	--,--, 8.017, 0.419, 70.359 --,--, (22.083), (0.312), (63.831) --,--, (0, 51.29), (0, 1.03), (0, 195.47)	387.22, 389.38, 387.66, 389.68
TLB XII	--,--, 91.320, 0.012, 141.073 --,--, (15.071), (0.002), (70.028) --,--, (61.78,120.86) (0.008, 0.02) (3.82,278.33)	385.94, 392.18, 386.38, 388.40
KwB XII	18.130, 6.857, 10.694, 0.081,-- (3.689), (1.035), (1.166), (0.012),-- (10.89,25.36), (4.83,8.89), (8.41,12.98), (0.06,0.10),--	385.58, 393.90, 386.32, 388.86
BB XII	26.725, 9.756, 27.364, 0.020,-- (9.465), (2.781), (12.351), (0.007),-- (8.17,45.27), (4.31,15.21), (3.16,51.57), (0.006,0.03),--	385.56, 394.10, 386.30, 389.10
BEB XII	2.924, 2.911, 3.270, 12.486, 0.371 (0.564), (0.549), (1.251), (6.938), (0.788) (1.82,4.03), (1.83,3.99), (0.82,5.72), (0, 26.08), (0, 1.92)	387.04, 397.42, 388.17, 391.09
FBB XII	30.441, 0.584, 1.089, 5.166, 7.862 (91.745), (1.064), (1.021), (8.268), (15.036) (0, 210.26), (0, 2.67), (0, 3.09), (0, 21.37), (0, 37.33)	386.74, 397.14, 387.87, 390.84
FKwB XII	12.878, 1.225, 1.665, 1.411, 3.732 (3.442), (0.131), (0.034), (0.088), (1.172) (6.1,19.6), (0.9,1.48), (1.56,1.73), (1.24,1.58), (1.4,6.03)	386.96, 397.36, 388.09, 391.06
DBrXII	6.65, 0.72, 7.5, 0.04,-- (1.74), (0.44), (4.95), (0.03)-- (3.2, 10), (0, 1.6), (9, 17.3), (0, 0.1), --	386.57, 394.88, 387.31, 389.81

Table 15: MLEs, SEs and CL with C_{AI} , C_{Bayes} , C_{HQ} and C_{CA} for the data set IV.

Model	$\hat{\lambda},\hat{\theta},\hat{\alpha},\hat{\beta},\hat{\gamma}$	C_{AI} , C_{Bayes} , C_{HQ} and C_{CA}
B XII	--,--, 58.711,0.006,-- --,--, (42.382), (0.004),-- --,--, (0, 141.78), (0, 0.01),--	328.20, 331.19, 328.60, 329.19
MOB XII	--,--, 11.838, 0.078, 12.251 --,--, (4.368), (0.013), (7.770) --,--, (0, 141.78), (0, 0.01), (0, 27.48)	315.54, 320.01, 316.37, 317.04
TLB XII	--,--,0.281, 1.882 ,50.215 --,--, (0.288), (2.402), (176.50) --,--, (0, 0.85), (0, 6.59), (0, 396.16)	316.26, 320.73, 317.09, 317.76
KwB XII	9.201, 36.428, 0.242,0.941,-- (10.060), (35.650), (0.167), (1.045),-- (0, 28.912), (0, 106.30), (0, 0.57), (0, 2.99),--	317.36, 323.30, 318.79, 319.34
BB XII	96.104, 52.121, 0.104, 1.227,-- (41.201), (33.490), (0.023), (0.326),-- (15.4,176.8),(0, 117.8), (0.6, 0.15), (0.59,1.9),--	316.46, 322.45, 317.89, 318.47
BEB XII	0.087, 5.007, 1.561, 31.270, 0.318 (0.077), (3.851), (0.012), (12.940), (0.034) (0, 0.3), (0, 12.6), (1.5, 1.6), (5.9, 56.6), (0.3,0.4)	317.58, 325.06, 319.80, 320.09
FBB XII	15.194, 32.048, 0.233, 0.581, 21.855 (11.58), (9.867), (0.091), (0.067), (35.548) (0, 37.8), (12.7,51.4), (0.05,0.4), (0.45,0.7), (0, 91.5)	317.86, 325.34, 320.08, 320.36
FKwB XII	14.732, 15.285, 0.293, 0.839, 0.034 (12.390), (18.868), (0.215), (0.854), (0.075) (0, 39.02), (0, 52.27), (0, 0.71), (0, 2.51), (0, 0.18)	317.76, 325.21, 319.98, 320.26

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ZBB XII	41.973, --,0.157, 44.263, -- (38.787), --, (0.082), (47.648), -- (0, 117.99), --, (0, 0.32,) (0, 137.65),--	313.86, 318.35, 314.39, 315.36
DBrXII	2.94, 56.347, 0.514, 0.117, -- 1.19, 78.45, 0.39, 0.12, -- (0.5, 5,3), (0, 212), (0, 1.3), (0, 0.36), --	315.13, 321.12, 316.56, 317.15

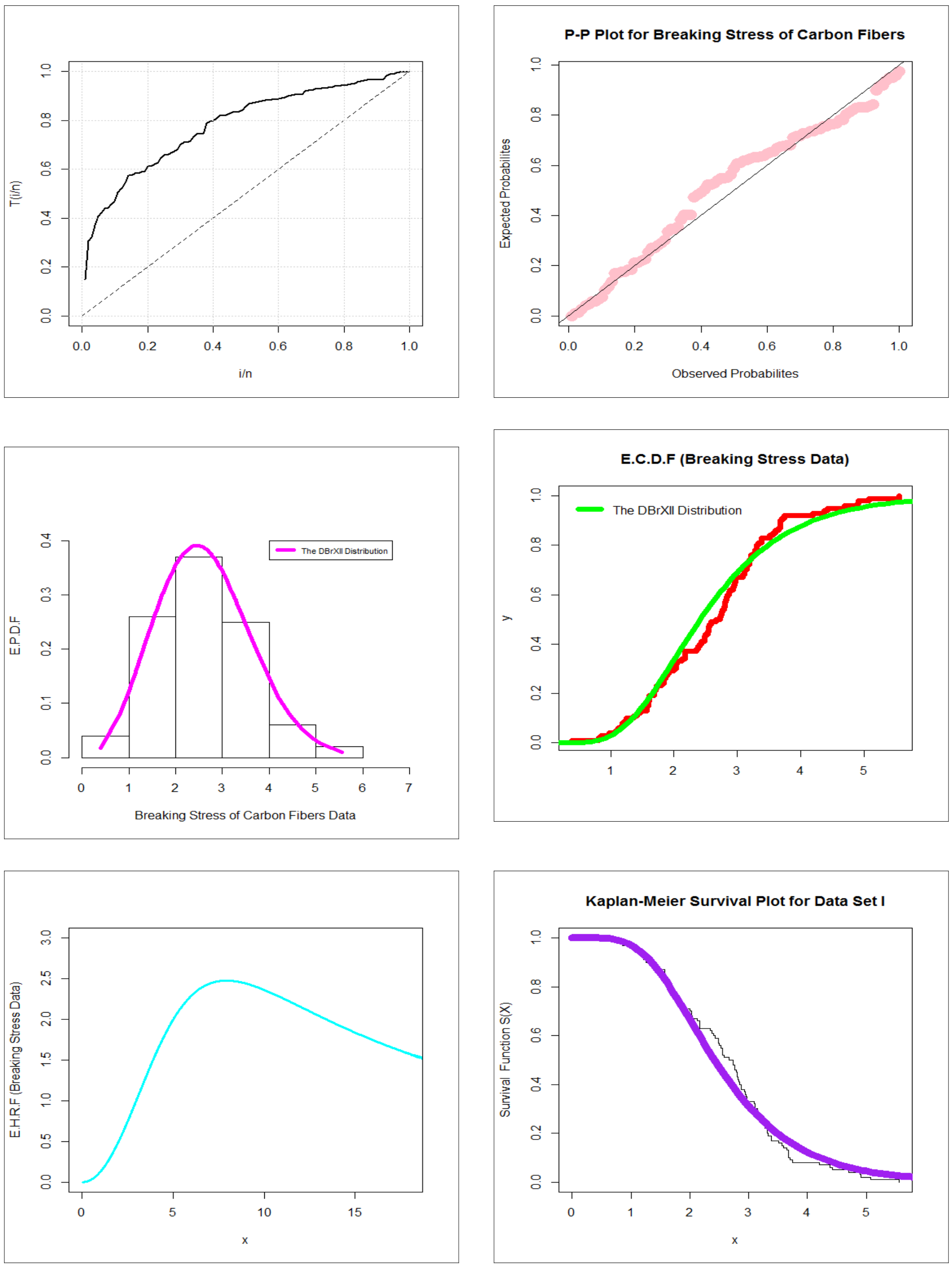

Figure 2: Plots for data set **I**.

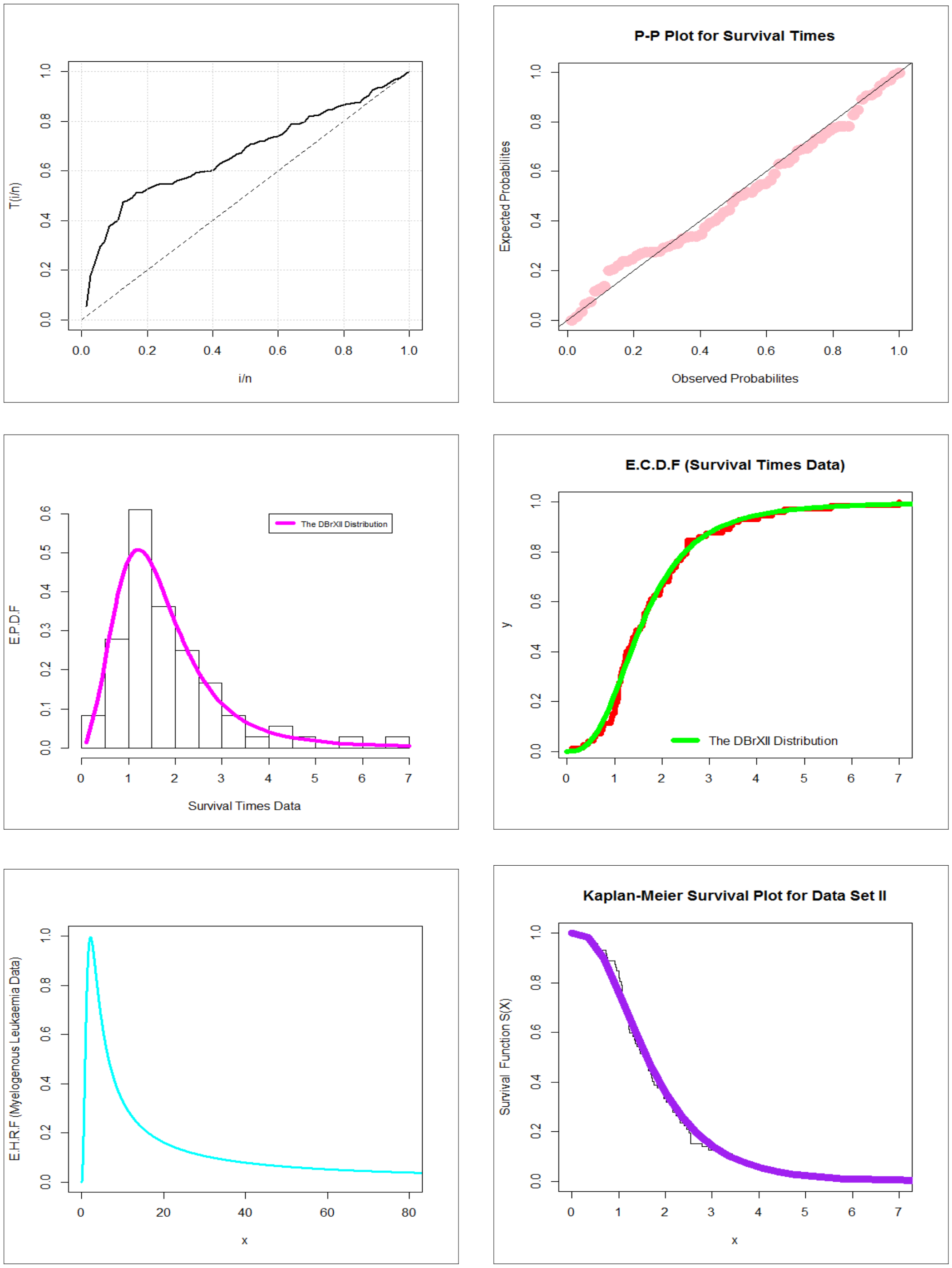

Figure 3: Plots for data set **II**.

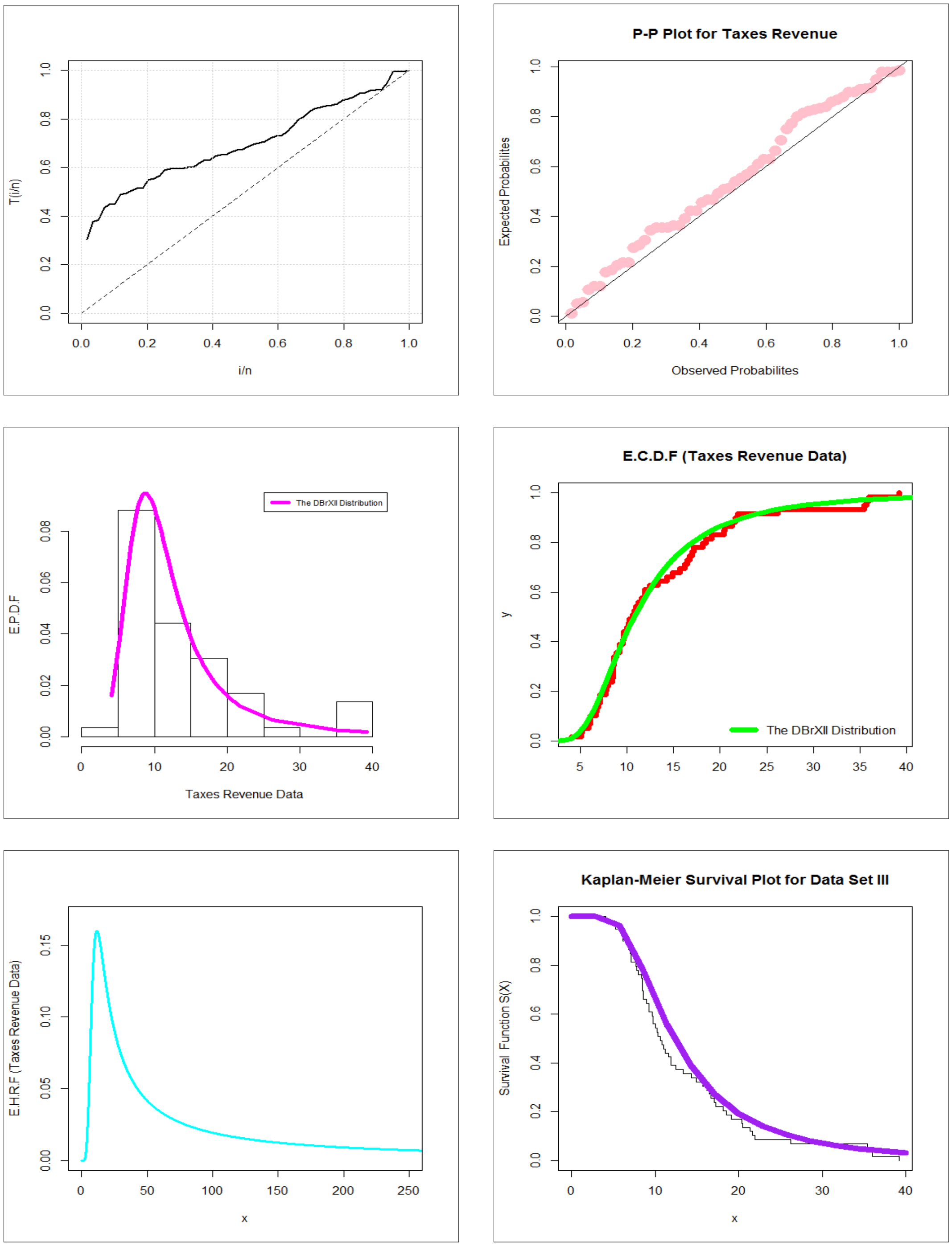

Figure 4: Plots for data set **III**.

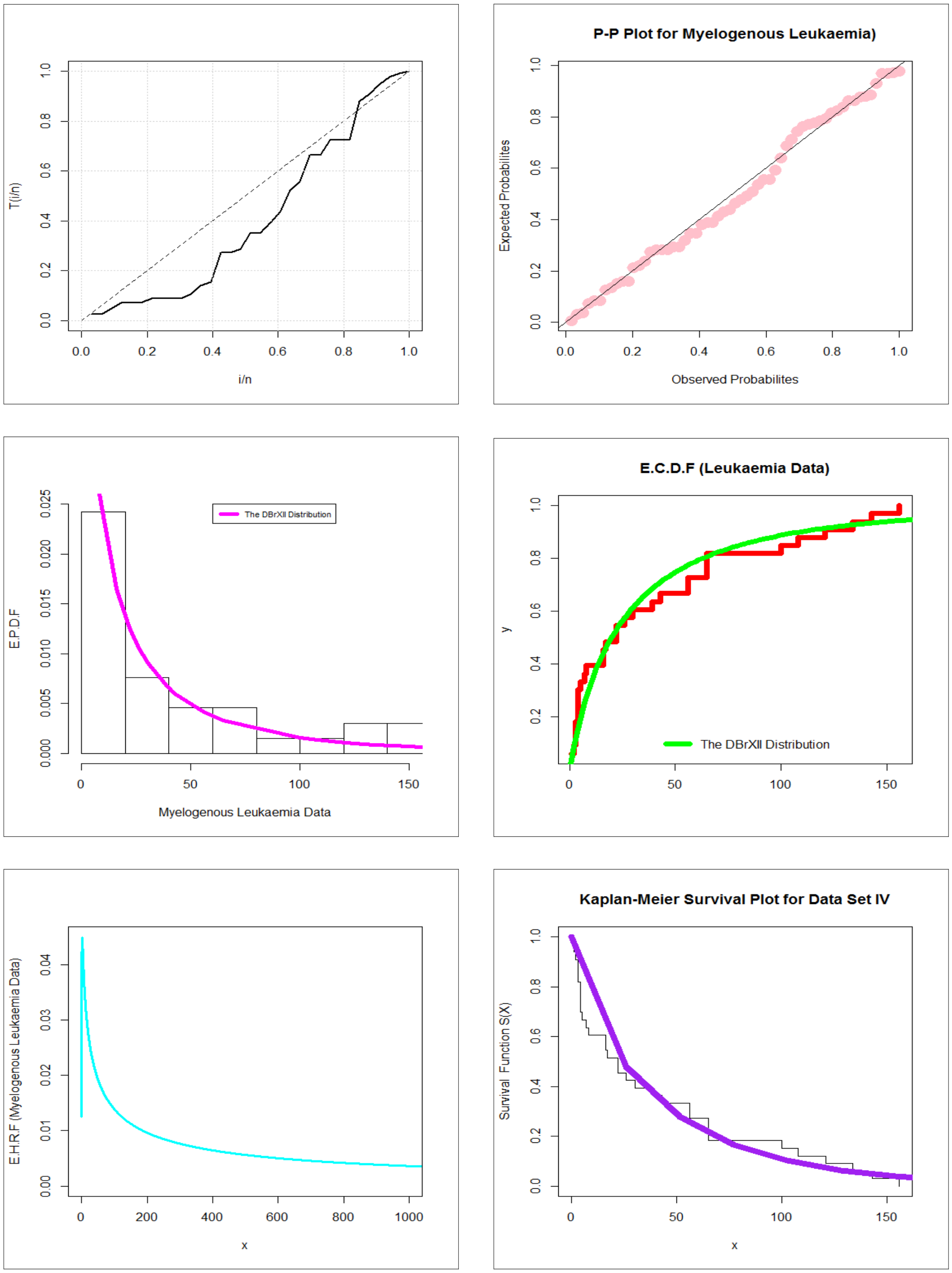

Figure 5: Plots for data set **IV**.

8. Censored Validation

Generally, there are several criteria that may be applied to determine if a statistical model is legitimate. For the uncensored data, the most popular tests are those based on the empirical functions, such as the likelihood ratio test, Akaike information criteria, Bayesian information criteria, or chi-square tests. These tests include Kolmogorov-Smirnov, Anderson-Darling, and other statistics. The NRR statistic, based on the MLEs on initial non-grouped data, is of particular importance among these goodness-of-fit evaluations. This Nikulin (1973a,b,c) and Rao and Robson (1974) statistic restores information lost during data grouping and has a chi-square distribution. However, the existence of censorship renders all the conventional goodness-of-fit tests invalid and leads to several practical issues. As a result, several researchers offered various revisions of current goodness-of-fit tests.

A modified NRR statistic was created by Bagdonavicius and Nikulin (2011) for statistical distributions with unknown parameters and right censoring. This version of the NRR statistic may be used to fit data from domains like survival analysis, dependability, and others where data is often censored since it recovers all the information lost during data regrouping. In this study, we will provide modified NRR chi-square goodness-of-fit test statistics for fitting full and right-censored data to the suggested model, following Nikulin (1973a,b,c) and Rao and Robson (1974). The NRR statistic is a well-known variant of the traditional chi-squared tests in the situation of full data. It is based on differences between two estimators of the probability for falling into grouping intervals. One estimate is based on the empirical distribution function, and the other on maximum likelihood estimates of the tested model's unobserved parameters using ungrouped initial data, (see Nikulin (1973a,b,c), and Rao and Robson (1974) for more details and see Goual and Yousof (2020a), Goual et al. (2019), Goual and Yousof (2020b), Yousof et al. (2021a,b) for more relevant applications under uncensored schemes).

8.1 Maximum likelihood estimation

Let T ~DBrXII $(\alpha_1,\beta_1,\alpha_2,\beta_2)$, for a certain individual (i), lifetime $(T_{[i]})$ and censorship time $(\nabla_{[i]})$ where $T_{[i]}$ and $\nabla_{[i]}$ are independent R.V.s. and the data consists of m independent observations, where

$$t_i|_{(i=1,2\dots,m)} = min(T_{[i]},\nabla_{[i]}).$$

Let the model of $\nabla_{[i]}$ does not depend on $T_{[i]}$ which have the unknown parameters (case of non-informative censorship), the likelihood function $(L(t,\alpha_1,\beta_1,\alpha_2,\beta_2))$ is given as

$$L(t,\alpha_1,\beta_1,\alpha_2,\beta_2) = \prod_{i=1}^{m} \lambda^{[\delta_i]}\,(t_i,\underline{\Omega}) S_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(t_i)|_{\left([\delta_i]=1_{\{T_{[i]}\le\nabla_{[i]}\}}\right)}.$$

Then, the loglikelihood function is

$$\ell = \ell(t,\alpha_1,\beta_1,\alpha_2,\beta_2) = -\beta_1\sum_{i=1}^{m} ln\left\{1+\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}\right\}$$

$$+\sum_{i=1}^{m}\left[[\delta_i]\begin{pmatrix} ln(\alpha_1)+ln(\beta_1)+ln(\alpha_2)+ln(\beta_2)+(\alpha_1-1)\,ln(t_{i,m})+(\beta_2\alpha_1-1)\,ln\left(1+t_{i.m}^{\alpha_2}\right) \\ +(\alpha_1-1)\,ln\left[1-\left(1+t_{i.m}^{\alpha_2}\right)^{-\beta_2}\right]-ln\left\{1+\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}\right\}\end{pmatrix}\right],$$

or

$$\ell = m[ln(\alpha_1)+ln(\beta_1)+ln(\alpha_2)+ln(\beta_2)]+(\alpha_1-1)\sum_{[i\in\mathbf{F}]} ln(t_{i,m})$$
$$+(\beta_2\alpha_1-1)\sum_{[i\in\mathbf{F}]} ln\left(1+t_{i.m}^{\alpha_2}\right)+(\alpha_1-1)\sum_{[i\in\mathbf{F}]} ln\left[1-\left(1+t_{i.m}^{\alpha_2}\right)^{-\beta_2}\right]$$
$$-\sum_{[i\in\mathbf{F}]} ln\left\{1+\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}\right\}-\beta_1\sum_{[i\in C]} ln\left\{1+\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}\right\},$$

where C is the set of censored observations, r is the number of failures. Then the score functions can be derived as

$$\frac{\partial\ell}{\partial\alpha_1} = \frac{r}{\alpha_1}-\sum_{[i\in\mathbf{F}]} ln(t_{i,m})+\beta_2\sum_{[i\in\mathbf{F}]} ln\left(1+t_{i.m}^{\alpha_2}\right)+\sum_{[i\in\mathbf{F}]} ln\left[1-\left(1+t_{i.m}^{\alpha_2}\right)^{-\beta_2}\right]$$
$$-\sum_{[i\in\mathbf{F}]}\frac{\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1} ln\left\{1+\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]\right\}}{\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1}$$
$$-\sum_{[i\in C]}\frac{\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1} ln\left\{1+\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]\right\}}{\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1},$$

$$\frac{\partial\ell}{\partial\beta_1} = \frac{\rho}{\beta_1}-\sum_{[i\in C]} ln\left\{1+\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}\right\},$$

$$\frac{\partial \ell}{\partial \alpha_2} = \frac{\rho}{\alpha_2} + (\beta_2\alpha_1 - 1)\sum_{[i\in \mathbf{F}]} \frac{t_{i.m}^{\alpha_2}\, ln(t_i)}{t_{i.m}^{\alpha_2}+1} + (\alpha_1 - 1)\sum_{[i\in \mathbf{F}]} \frac{\beta_2 t_{i.m}^{\alpha_2}\, ln(t_i)\left(t_{i.m}^{\alpha_2}+1\right)^{-\beta_2-1}}{1-\left(t_{i.m}^{\alpha_2}+1\right)^{-\beta_2}}$$
$$+\sum_{[i\in \mathbf{F}]} \frac{\beta_2\alpha_1 t_{i.m}^{\alpha_2}\, ln(t_i)\left(t_{i.m}^{\alpha_2}+1\right)^{\beta_2-1}\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1-1}}{\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1}$$
$$-\beta_1\sum_{[i\in C]} \frac{\beta_2\alpha_1 t_{i.m}^{\alpha_2}\, ln(t_i)\left(t_{i.m}^{\alpha_2}+1\right)^{\beta_2-1}\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1-1}}{\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1},$$

$$\frac{\partial \ell}{\partial \beta_2} = \frac{\rho}{\beta_2} + \alpha_1\sum_{[i\in ?]} ln\left(1+t_{i.m}^{\alpha_2}\right) + (\alpha_1 - 1)\sum_{[i\in \mathbf{F}]} \frac{\left(1+t_{i.m}^{\alpha_2}\right)^{-\beta_2} ln\left(1+t_{i.m}^{\alpha_2}\right)}{1-\left(1+t_{i.m}^{\alpha_2}\right)^{-\beta_2}}$$
$$+\sum_{[i\in \mathbf{F}]} \frac{\alpha_1\left(t_{i.m}^{\alpha_2}+1\right)^{\beta_2} ln\left(t_{i.m}^{\alpha_2}+1\right)\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1-1}}{\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1}$$
$$-\beta_1\sum_{[i\in C]} \frac{\alpha_1\left(t_{i.m}^{\alpha_2}+1\right)^{\beta_2} ln\left(t_{i.m}^{\alpha_2}+1\right)\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1-1}}{\left[\left(1+t_{i.m}^{\alpha_2}\right)^{\beta_2}-1\right]^{\alpha_1}+1},$$

8.2 Simulation

We can simulate the DBrXII model where $M = 10000$, $m = 30{,}100{,}250{,}500$ and $\alpha_1 = 2.5, \beta_1 = 0.9, \alpha_2 = 1.5, \beta_2 = 2.0$. The AVs and their mean MSEs are presented in Table 16 from which one can note that the MLEs are convergent.

Table 16: AVs and their MSEs (censored data).

M=10000	m=30	100	250	500
α_1	2.5438	2.5374	2.5163	2.5048
MSE	0.0477	0.0433	0.0298	0.0021
β_1	0.9446	0.9307	0.9183	0.9051
MSE	0.0428	0.0278	0.0164	0.0030
α_2	1.5481	1.533	1.5127	1.5049
MSE	0.0439	0.0324	0.0209	0.0027
β_2	2.1847	2.0943	2.0477	2.0115
MSE	0.0449	0.0267	0.0195	0.0017

9. The developed Nikulin-Rao-Robson GOF test for censored validation

9.1 Nikulin-Rao-Robson statistic NRR GOF test (Y^2)

To test the hypothesis $H_{[0]}$ (see Nikulin (1973a,b,c), Rao and Robson (1974), Goual and Yousof (2020), Goual et al. (2012,2020)) we have

$$H_{[0]} \;:\; Pr\{T_i \le t\} = F_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(t)|_{[t\in R,\,(\Theta_1,\Theta_2,\cdots,\Theta_s)^T]}.$$

For defining the NRR test, consider that $T_{[1]}, T_{[2]}, \cdots, T_{[m]}$ are grouped in ρ subintervals which are mutually disjoint

$$I_j =]a_{(j)-1}; a_{(j)}], j = 1,2,\ldots,\rho.$$

The limits $a_{(j)}$ of the intervals I_j are obtained such that

$$p_j(\alpha_1,\beta_1,\alpha_2,\beta_2)|_{(\,j=1,2,\cdots,\rho)} = \int_{a_{(j-1)}}^{a_{(j)}} f_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(t)\,dt,$$

where $a_{(j)}|_{(j=1,\cdots,\rho-1)} = F^{-1}\left(\frac{j}{\rho}\right)$. Let $\nu_j = (\nu_1,\nu_2,\cdots,\nu_\rho)^T$ be a frequencies vector which obtained via the grouping the data in I_j are intervals $\nu_j|_{(j=1,2,\ldots,\rho)} = \sum_{i=1}^{m} 1_{\{t_i\in I_j\}}$. The NRR statistic $\left(Y^2\left(\underline{\widehat{\Omega}}_m\right)\right)$ can be derived as

$$Y^2\left(\underline{\widehat{\Omega}}_m\right) = \chi_m^2\left(\underline{\widehat{\Omega}}_m\right) + m^{-1}\ell^T\left(\underline{\widehat{\Omega}}_m\right)\left(\mathbf{I}\left(\underline{\widehat{\Omega}}_m\right) - \mathbf{J}\left(\underline{\widehat{\Omega}}_m\right)\right)^{-1}\ell\left(\underline{\widehat{\Omega}}_m\right),$$

where $\mathbf{I}(\underline{\widehat{\Omega}}_m)$ is the estimated Fisher information matrix (FIM), where

$$\chi_m^2(\underline{\Omega}) = \left(\frac{u_1 - mp_1(\alpha_1,\beta_1,\alpha_2,\beta_2)}{\sqrt{mp_1(\alpha_1,\beta_1,\alpha_2,\beta_2)}}, \frac{u_2 - mp_2(\alpha_1,\beta_1,\alpha_2,\beta_2)}{\sqrt{mp_2(\alpha_1,\beta_1,\alpha_2,\beta_2)}}, \dots, \frac{u_r - mp_r(\alpha_1,\beta_1,\alpha_2,\beta_2)}{\sqrt{mp_r(\alpha_1,\beta_1,\alpha_2,\beta_2)}}\right)^T,$$

and $\mathbf{J}(\underline{\Omega})$ is the FIM but for the grouped data, where $\mathbf{J}(\underline{\Omega}) = B(\underline{\Omega})^T B(\underline{\Omega})$, and

$$B(\alpha_1,\beta_1,\alpha_2,\beta_2)|_{(h=1,2,\cdots,r\ )} = \left[\frac{1}{\sqrt{p}_h}\frac{\partial p_h(\alpha_1,\beta_1,\alpha_2,\beta_2)}{\partial\mu}\right]_{r\times s},$$

then

$$l(\alpha_1,\beta_1,\alpha_2,\beta_2) = \begin{pmatrix} l_1(\alpha_1,\beta_1,\alpha_2,\beta_2), \\ l_2(\alpha_1,\beta_1,\alpha_2,\beta_2), \\ \dots, \\ l_s(\alpha_1,\beta_1,\alpha_2,\beta_2) \end{pmatrix}^T,$$

with

$$B(\alpha_1,\beta_1,\alpha_2,\beta_2)|_{(i=1,2,\cdots,\rho \text{ and } k=1,\cdots,s)} = \left[\frac{1}{\sqrt{p_{(i)}}}\frac{\partial}{\partial\mu}p_{(i)}(\alpha_1,\beta_1,\alpha_2,\beta_2)\right]_{\rho\times s}.$$

where

$$L(\alpha_1,\beta_1,\alpha_2,\beta_2) = \begin{pmatrix} L_1(\alpha_1,\beta_1,\alpha_2,\beta_2), \\ L_1(\alpha_1,\beta_1,\alpha_2,\beta_2), \\ \dots, \\ L_s(\alpha_1,\beta_1,\alpha_2,\beta_2) \end{pmatrix}^T$$

and

$$L_k(\alpha_1,\beta_1,\alpha_2,\beta_2) = \sum_{i=1}^{\rho} \frac{\nu_i}{p_{(i)}}\frac{\partial}{\partial\underline{\Omega}_k}p_{(i)}(\alpha_1,\beta_1,\alpha_2,\beta_2).$$

The Y^2 statistic follows a chi-square $\chi 2$ distribution with $(\rho-1)$ degrees of freedom.

9.2 NRR. statistic for the DBrXII model

For testing $H_{[0]}$ that a certain R.S. belongs to the DBrXII model, the Y^2 (NRR. statistic) is calculated for $M = 10000$ simulated samples with sizes $m = 30,50,100,250,500$ and $\varepsilon = 0.02,0.05,0.01,0.1$ then, the average number of the non-rejection of $H_{[0]}$ is calculated, when $Y^2 \leq \ \chi_\varepsilon^2(\rho-1)$. Table 17 show that the values of the empirical levels calculated are very close to those of their corresponding theoretical levels.

Table 17: Empirical levels and corresponding theoretical levels.

M=10000	ε=0.02	0.0500	0.0100	0.100
m_1 =30	0.9848	0.9536	0.9930	0.903
m_2 =50	0.9840	0.9533	0.9925	0.903
m_3 =100	0.9819	0.9521	0.9918	0.902
m_4 =250	0.9808	0.9504	0.9905	0.901
m_5 =500	0.9802	0.9502	0.9903	0.900

So, we conclude that the modified test provides a good fit to the DBrXII model.

9.3 Real data modeling (Taxes revenue data)

Using the BB algorithm, we compute the MLEs and since $[Y^2 = 12.51847] < [\chi_{0.05}^2(7-1) = 12.59159]$, we say that taxes revenue data arise appropriately from the DBrXII model.

10. Validation of DBrXII model in case of censored data via $Y_{(m)}^2$

Consider the modified NRR statistic $(Y_{(m)}^2)$ presented above. Suppose $H_{[0]}$: $T_{[i]} \sim$ DBrXII, where survival function is $S_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(t) = 1 - F_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(t)$. Then $\mathrm{Q}(\tau,\alpha_1,\beta_1,\alpha_2,\beta_2) = -lnS_{(\alpha_1,\beta_1,\alpha_2,\beta_2)}(\tau)$ where

$$E_k = \sum_{(h:z_h > a_{(K)})} \left[\mathrm{Q}\left(a_{(K)} \wedge \tau_h, \underline{\widehat{\Omega}}\right) - \mathrm{Q}\left(a_{(K-1)}, \underline{\widehat{\Omega}}\right)\right].$$

Under such choice of intervals, we have a constant value of

$$e_\kappa = \frac{E_k}{k}\ \forall\ k.$$

10.1 Simulation study

We calculate NRR statistic $\left(Y_{(m)}^2\right)$ for $M = 10000,\ m = 30,\ 100,\ 250,\ 500$ and $\varepsilon = 0.02,\ 0.05,\ 0.01,\ 0.1$, and after calculating the mean of the number of no rejections of $H_{[0]}$when $Y_{(m)}^2 \leq \chi_\varepsilon^2(\rho)$ (see Table 18) , we find that the ε_i levels of the $Y_{(m)}^2$ synchronize with those corresponding to the theoretical ε_i levels of the chi-square model at degrees of freedom $= \rho$.

Table 18: Empirical and theoretical levels.

M=10000	ε=0.02	0.05	0.01	0.1
m_1 =30	0.9836	0.9525	0.9921	0.9027
m_2 =100	0.9829	0.9514	0.9915	0.9020
m_3 =250	0.9818	0.9507	0.9907	0.9006
m_4 =500	0.9805	0.9501	0.9902	0.9003

From Table 19 we can say that the empirical ε_i of the $Y_{(m)}^2$ synchronize with those corresponding theoretical ε_i which means that the proposed GOF test can properly fit censored data drawn from the DBrXII model.

10.2 Application to real censored data (times of failure data)

Considered the data of [37]. The data are "0.4680, 0.7250, 0.8380, 0.8530, 0.9650, 1.1390, 1.1420, 1.3040, 1.3170, 1.4270, 1.5540, 1.6580, 1.764, 1.7760, 1.990, 2.010, 2.2240, 2.2790*, 2.2440*, 2.2860*" (* censored). Then, the value of the test statistic $Y_{(m)}^2 = 9.31854$, on the other hand, the critical value is $\left[Y_{(m)}^2 = 9.31854\right] < \left[\chi_{0.05}^2(4) = 9.48773\right]$. Hence, the times of failure data is compatible with the DBrXII model.

13. Conclusions

In this work, we studied the four-parameter DBrXII model. First, we examined its PDF and HRF. It is noted that the DBrXII PDF can be right skewed, left skewed and symmetric and its HRF can be **J**-shape, decreasing and upside down. Some of its mathematical properties are presented. However other works may be allocated to characterize this model. We presented the Bayesian and non-Bayesian estimation for its unknown parameters along with MCMC simulations. Further future works may be allocated for using other estimation methods. We constructed a modified test for goodness-of-fit in case of completeness and censorship. The new test is formed based on the Nikulin-Rao-Robson statistic for validation. Simulations are performed for assessing the new test in case of completeness and censorship. Nine applications on real data are presented, four of them for comparing methods, four for comparing models however the final one is employed for NRR validation. For data set **I (**breaking stress data**), III (**taxes revenue data**)** and **IV (**leukemia data**)**, all methods perform well. For data set **II (**survival times**)**, the Cramer-Von-Mises method is the best method however all other methods perform well. Below some important results and findings of this work:

1. Depending of (5), for $\beta_1 = 1$ (β_2 =1), we can replace the 1st (2nd) name of the model by log-logistic (Log-L). The log-logistic-log-logistic (LogL-LogL) model obtained when $\beta_1 = \beta_2 = 1$. For $\alpha_1 = 1$ (α_2=1), we can change the 1st (2nd) name by Pareto type **II** (Pa **II**). So, for $\alpha_1 = 1$ and β_2 =1, we obtain the Pareto type **II**-log-logistic (Pa**II**-LogL) model. If $\beta_1 \to \infty$ (or $\beta_2 \to \infty$), the 1st (2nd) name can be changed by Weibull. If we combine these conditions, we can get fourteen special cases of (5).
2. The PDF of the DBrXII PDF is a linear combination of BrXII PDF (see Subsection 2.1).
3. The DBrXII PDF can be right skewed ($\alpha_1 = 1, \beta_1 = 5, \alpha_2 = 2, \beta_2 = 10$), ($\alpha_1 = 0.5, \beta_1 = 5, \alpha_2 = 2, \beta_2 = 2$) and ($\alpha_1 = 1, \beta_1 = 5, \alpha_2 = 2, \beta_2 = 2$), left skewed ($\alpha_1 = 10, \beta_1 = 5, \alpha_2 = 1, \beta_2 = 1.2$) and symmetric ($\alpha_1 = 2, \beta_1 = 5, \alpha_2 = 2, \beta_2 = 2$) (see Figure 1 the left panel).
   The DBrXII HRF can be can be "decreasing ($\alpha_1 = 2, \beta_1 = 5, \alpha_2 = 0.5, \beta_2 = 1$)", "reversed **J**-shape ($\alpha_1 = 1, \beta_1 = 1, \alpha_2 = 0.25, \beta_2 = 1$)", "constant ($\alpha_1 = 1.25, \beta_1 = 2, \alpha_2 = 1, \beta_2 = 0.25$)", "upside down ($\alpha_1 = 3, \beta_1 = 1, \alpha_2 = 0.4, \beta_2 = 1.25$)" and "**J**-shape ($\alpha_1 = 5, \beta_1 = 2, \alpha_2 = 2, \beta_2 = 1$)" (see Figure 1 the right panel).
4. The Rényi entropy can have a wide range in the interval $(-17.75737, 1.09352)$ and reaches its maximum value when $\theta = 0.5, \alpha_1 = 2, \beta_1 = 0.5, \alpha_2 = 1.2,\ \beta_2 = 1.5$ (see Subsection 3.1). The Rényi entropy reaches its minimum value when $\theta = 0.5, \alpha_1 = 1, \beta_1 = 1, \alpha_2 = 5,\ \beta_2 = 5$ and decreases as θ increases (see Subsection 3.1).
5. The δ-entropy is always positive and reaches ∞ as θ increases (see Subsection 3.2).
6. The Shannon entropy is always positive and can range in the interval (0.019911, 1.723628) (see Subsection 3.2). The Shannon entropy reaches its minimum value when $\alpha_1 = 2, \beta_1 = 0.5, \alpha_2 = 0.2,\ \beta_2 = 0.2$ and reaches its maximum value when $\alpha_1 = 3, \beta_1 = 1, \alpha_2 = 1,\ \beta_2 = 1$ (see Subsection 3.3).
7. Based on the simulation studies we note that all estimation methods perform well but the Bayesian method is the best for all sample sizes (see Tables 4, 5, 6 and 7).

8. For data sets **I, III** and **IV**, all methods perform well (see Table 8, Table 9, Table 10 and Table 11). For data set **II**, the CVM is the best method with W*= 0.12782 and A*= 0.75961, however, all other methods preform well (see Table 9).
9. Based on the values in Tables 12, 13, 14 and 15, the DBrXII model has the best fits as compared to BrXII other models in the four applications with small values for C_{AI}, C_{Bayes}, C_{HQ} and C_{CA}.
10. The MLEs (uncensored case) for the DBrXII model is convergent using the Barzilai-Borwein (BB) algorithm (see Table 16).
11. The MLEs (censored case) for the DBrXII model is convergent (see Table 17).
12. The values of the empirical levels calculated are very close to those of their corresponding theoretical levels. So, we conclude that the modified GOF test provides a good fit to the DBrXII distribution (see Table 18).
13. The empirical significance levels of the $Y^2_{(m)}$ synchronize with those corresponding theoretical ones which means that the proposed GOF test can properly fit censored data drawn from the DBrXII model. The same result is proofed via an application to real data set (see Subsection 12.2).

To model count real-life data, it is suggested that a novel discrete DBrXII model be presented; for more details, see Aboraya et al. (2020,2022), Chesneau et al. (2022), Yousof et al. (2021c), and Ibrahim et al. (2022b). Additionally, using the Bagdonaviius-Nikulin, see, for example, Ibrahim et al. (2019, 2020a), Goual et al. (2019, 2020), Yadav et al. (2020 and 2022), Goual and Yousof (2020), Ibrahim et al. (2022a), Aidi et al. (2021) and Yousof et al. (2022)). Following Altun et al. (2018a,b) and Yousof et al. (2019) and under the DBrXII distribution, some new developments of certain new regression models for modelling censored data sets. The generalized stress-strength parameter under the DBrXII model could be inferred using Bayesian and classical methods due to Saber and Yousof (2022) as well as reliability estimation for the remained stress-strength model under the DBrXII distribution due to Saber et al. (2022a,b). A single acceptance sampling strategy with its associated application in quality and risk decisions might be given in the manner of Ahmed and Yousof (2022) and Ahmed et al. (2022a,b). Finally, one might follow Mohamed et al. (2022a,b,c) and Salem et al. (2022) to find applications for more insurance studies under some time series models such as Autoregressive models, moving average models, autoregressive moving average modes and autoregressive integrated moving average models. Many bivariate versions can be derived following Al-babtain et al. (2020a,b) and Mansour et al. (2020a-f), Shehata and Yousof (2022), Shehata et al. (2021,2022), Yousof et al. (2020) and Ali et al (2022b). In fact, the new distribution can be generalized using many new families, and these generalizations will have their importance in statistical and mathematical modeling processes and applications in many applied fields (see Ali et al. (2022a), Altun et al. (2020) and Chesneau and Yousof (2022)).

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