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A New Lifetime Model: Copulas, Properties and Real Lifetime Data Applications

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Abstract

A new four parameter lifetime model called the Weibull generalized Lomax is proposed and studied. The new density function can be "right skewed", "symmetric" and "left skewed" and its corresponding failure rate function can be "monotonically decreasing", " monotonically increasing" and "constant". The skewness of the new distribution can negative and positive. The maximum likelihood method is employed and used for estimating the model parameters. Using the "biases" and "mean squared errors", we performed simulation experiments for assessing the finite sample behavior of the maximum likelihood estimators. The new model deserved to be chosen as the best model among many well-known Lomax extension such as exponentiated Lomax, gamma Lomax, Kumaraswamy Lomax, odd log-logistic Lomax, Macdonald Lomax, beta Lomax, reduced odd log-logistic Lomax, reduced Burr-Hatke Lomax, special generalized mixture Lomax and the standard Lomax distributions in modeling the "failure times" and the "service times" data sets.

Key Words: Lomax model; Simulations; Renyi's entropy copula; Farlie Gumbel Morgenstern copula; Simulation; Estimation; Modeling real data.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1.Introduction

The Lomax or Pareto type II distribution (Lomax (1954)), is a heavy-tail probabilistic model used in modeling business, actuarial science, biological sciences, engineering, economics, income and wealth inequality, queueing theory, size of cities and Internet traffic data sets. Harris (1968) and Atkinson and Harrison (1978) employed the Lomax (Lx) distribution in modeling data obtained from income and wealth. Corbellini et al. (2007) used the Lx distribution firm size data modeling. For applications in reliability and life testing experiments see Hassan Al-Ghamdi (2009). The Lx model is known as a special distribution form of Pearson system (type VI) and has also considered as a mixture of standard exponential (Exp) and standard gamma (Gam) distributions. The Lx model belongs to the family of "monotonically decreasing" hazard rate function (HRF) and considered as a limiting model of residual lifetimes at great age (see Balkema and de Hann (1974) and Chahkandi and Ganjali (2009)). The Lx distribution has been suggested as heavy tailed alternative model to the standard Exp, standard Weibull (W) and standard Gam distributions (see Bryson (1074)). For details about relation between the Lx model and the Burr XII and Compound Gamma (CGam) models see Tadikamalla (1980) and Durbey (1970).

The main aim of this work is to provide a flexible extension of the Lx distribution using the Weibull generalized-G (WG-G) family defined by Yousof et al. (2018). A random variable (RV) Y has the Lomax (Lx) distribution with two parameters α and σ if it has cumulative distribution function (CDF) (for $y>0$) given by

$$G_{\alpha,\sigma}(y)=1-\left(1+\frac{y}{\sigma}\right)^{-\alpha}, \tag{1}$$

where $\alpha > 0$ and $\sigma > 0$ are the shape and scale parameters, respectively. Then the corresponding probability density function (PDF) of (1) is

$$g_{\lambda,\sigma}(y) = \frac{\alpha}{\sigma}\left(1 + \frac{y}{\sigma}\right)^{-(\alpha+1)}. \tag{2}$$

Due to Yousof et al. (2018), the CDF of the WG-G family is given by

$$F_{v,\gamma,\underline{\zeta}}(y) = 1 - exp\left\{-\left[\overline{G}_{\underline{\zeta}}(y)^{-\gamma} - 1\right]^{v}\right\}|_{x\in R, v>0 \text{ and } \gamma>0}, \tag{3}$$

where $\overline{G}_{\underline{\zeta}}(y) = 1 - G_{\underline{\zeta}}(y)$. The PDF corresponding to (3) is given by

$$f_{v,\gamma,\underline{\zeta}}(y) = v\gamma g_{\underline{\zeta}}(y)\frac{\left[1 - \overline{G}_{\underline{\zeta}}(y)^{\gamma}\right]^{v-1}}{\overline{G}_{\underline{\zeta}}(y)^{v\gamma+1}} exp\left\{-\left[\overline{G}_{\underline{\zeta}}(y)^{-\gamma} - 1\right]^{v}\right\}|_{x\in R, v>0 \text{ and } \gamma>0}. \tag{4}$$

The Weibull generalized Lomax (WG-Lx) survival function (SF) is given by

$$S_{\underline{\Psi}}(y) = exp\left\{-\left[\left(1 + \frac{y}{\sigma}\right)^{\gamma\alpha} - 1\right]^{v}\right\}|_{(x\in R^{+} \text{ and } \underline{\Psi}=v,\gamma,\alpha,\sigma\in R^{+})}, \tag{5}$$

where $S_{\underline{\Psi}}(y) = 1 - F_{\underline{\Psi}}(y)|_{(\underline{\Psi}=v,\gamma,\alpha,\sigma)}$. The PDF corresponding to (5) is given by

$$f_{\underline{\Psi}}(y) = v\gamma\frac{\alpha}{\sigma}\frac{\left[1 - \left(1 + \frac{y}{\sigma}\right)^{-\gamma\alpha}\right]^{v-1}}{\left(1 + \frac{y}{\sigma}\right)^{(\gamma+1)\alpha+2}} exp\left\{-\left[\left(1 + \frac{y}{\sigma}\right)^{\gamma\alpha} - 1\right]^{v}\right\}|_{(x\in R^{+} \text{ and } v,\gamma,\alpha,\sigma\in R^{+})}. \tag{6}$$

For simulation of the WG-Lx model, we obtain the QF of Y by inverting (5), say $y_u = F^{-1}(u)$, as

$$y_u = \sigma\left\{1 + [-ln(1 - u)]^{\frac{1}{v}}\right\}^{\frac{1}{\alpha\gamma}} - 1, \tag{6}$$

Equation (7) is used for simulating the new model. The HRF for the new model can be derived from $f_{\underline{\Psi}}(y)/k_{\underline{\Psi}}(y)$. Many useful Lx extensions can be found in Tahir et al. (2015) (the Weibull Lomax distribution), Cordeiro et al. (2018) (the one parameter Lomax system of densities), Altun et al. (2018a) (the odd log-logistic Lomax), Altun et al. (2018a) (Zografos-Balakrishnan Lomax distribution), Elbiely and Yousof (2018) (the Weibull generalized Lomax, Rayleigh generalized Lomax and exponential generalized Lomax distributions), Yousof et al. (2019) (the Topp Leone Poisson Lomax distribution), Goual and Yousof (2019) (the Lomax inverse Rayleigh), Gad et al. (2019) (the Burr type XII Lomax, the Lomax Burr type XII and the Lomax Lomax distributions), Yadav et al. (2020) (the Topp Leone Lomax distribution) and Ibrahim and Yousof (2020) (the Poisson Burr X generalized Lomax and Poisson Rayleigh generalized Lomax distributions). In this paper, we derive some new bivariate WG-Lx (BvWG-Lx) via Farlie Gumbel Morgenstern (FGM) Copula, modified Farlie Gumbel Morgenstern (FGM) Copula, Renyi's entropy and Clayton Copula. The Multivariate WG-Lx (MvWG-Lx) type is also presented using the Clayton Copula. However, future works could be allocated to study these new models. The concept of copulas is recently used by many authors such as Ibrahim et al. (2020) (Topp-Leone Lindley distribution), Al-babtain et al. (2020) (Marshall–Olkin binomial-exponential distribution), Elgohari and Yousof (2020b) (Marshall-Olkin generalized-Weibull distribution), Salah et al. (2020) (odd burr Fréchet distribution), Yousof et al. (2021) (Xgamma reciprocal Rayleigh distribution), El-Morshedy et al. (2021) (the Poisson generalized exponential G family) and Ali et al. (2021a, b) (odd Burr generalized Rayleigh distribution and Marshall-Olkin Lehmann Burr type X distribution).

To illustrate the flexibility of the new PDF and its corresponding HRF we present Figure 1. Figure 1(a) gives some PDF plots for some selected parameters value. Figure 2(b) gives some HRF plots for some selected parameters value. Based on Figure 1(a) the WG-Lx density can be "right skewed", "symmetric", "left skewed" and "uniformed-PDF". Based on Figure 2(b) the WG-Lx HRF can be "monotonically decreasing" ($v = \gamma = 0.5, \alpha = 1000, \sigma = 10000$), "monotonically increasing" ($v = 2.5, \gamma = 2.5, \alpha = 215, \sigma = 1000$) & ($v = 1.25, \gamma = 3000, \alpha = 0.65, \sigma = 8000$) and "constant" ($v = 1, \gamma = 1000, \alpha = 0.5, \sigma = 6000$).

The WG-Lx model could be useful in statistical modeling in following cases:

1- The real-life data sets with " monotonically increasing HRF (asymmetric increasing HRF)".
2- The real-life data sets which have no extreme observations.
3- The real-life data sets which their nonparametric Kernel density estimation is bimodal and symmetric with right and left simple tail.

The WG-Lx model proved its applicability against many well-known Lx extensions in following cases:

1- In statistical modeling of the failure times of the aircraft windshield observations, the WG-Lx model is better than many well-known Lx extension such as the McDonald Lx extension, the special generalized mixture Lx extension, the odd log-logistic Lx extension, the Gamma Lx extension, the Burr-Hatke Lx extension, the transmuted Topp-Leone Lx extension, the exponentiated Lx extension, the proportional reversed hazard rate Lx extension and the Kumaraswamy Lx extension under the Akaike-Information-Criteria, Consistent-Information-Criteria, Bayesian-Information-Criteria and Hannan-Quinn-Information-Criteria.

2- In modeling the service times of the aircraft windshield, the WG-Lx model is better than many well-known Lx extension such as the McDonald Lx extension, the special generalized mixture Lx extension, the odd log-logistic Lx extension, the Gamma Lx extension, the Burr-Hatke Lx extension, the transmuted Topp-Leone Lx extension, the exponentiated Lx extension, the proportional reversed hazard rate Lx extension and the Kumaraswamy Lx extension under the Akaike-Information-Criteria, Consistent-Information-Criteria, Bayesian-Information-Criteria and Hannan-Quinn-Information-Criteria.

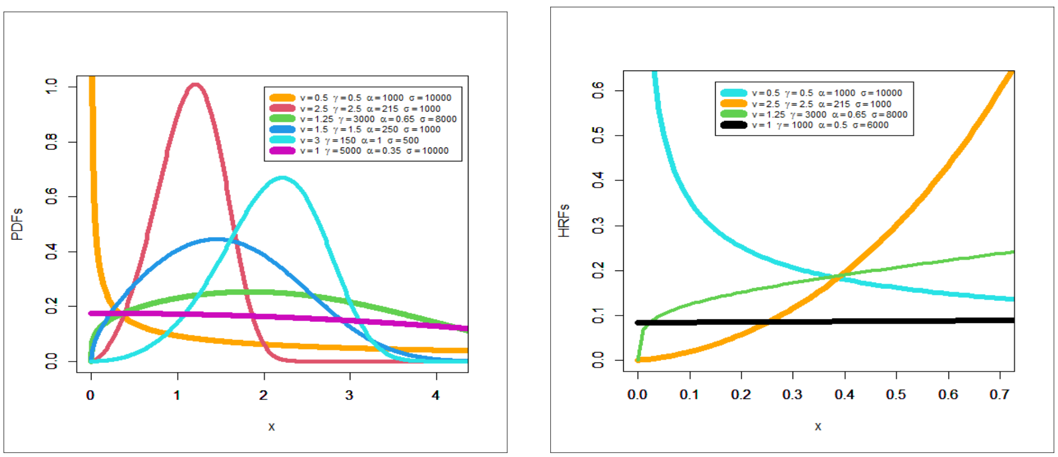

Figure 1: PDF and HRF plots for some selected parameters value.

2.Copulas

BvWG-Lx type via FGM Copula

Consider the joint CDF of the FGM family then

$$C_{\Upsilon\in(-1,1)}(s,w)|_{\Upsilon\in(-1,1)} = sw(1+\Upsilon sw),$$

where the marginal function $S = F_1(y_1)$, $W = F_2(y_2)$ is a dependence parameter and for every $S, W \in (0,1)^2$, $C_\Upsilon(S,0) = C_\Upsilon(0,W) = 0$ which is "grounded minimum" and $C_\Upsilon(S,1) = S$ and $C_\Upsilon(1,W) = W$ which is "grounded maximum". Then for $\varepsilon_{(y_l,\sigma_l)} = \left(1+\frac{y_l}{\sigma_l}\right)$, setting

$$\overline{s} = \overline{s}_{\underline{\Psi}_1} = exp\left[-\left(d_{y_1;\sigma_1}^{\gamma_1\alpha_1}-1\right)^{v_1}\right]|_{d_{y_1;\sigma_1}=\left(1+\frac{y_1}{\sigma_1}\right)}$$

and

$$\overline{w} = \overline{w}_{\underline{\Psi}_2} = exp\left[-\left(d_{y_2;\sigma_2}^{\gamma_2\alpha_2}-1\right)^{v_2}\right]|_{d_{y_{21};\sigma_2}=\left(1+\frac{y_2}{\sigma_2}\right)}$$

then we have

$$F_\Upsilon(y_1,y_2) = C_\Upsilon(F_{\underline{\Psi}_1}(y_1), F_{\underline{\Psi}_2}(y_2)) = \left\{1-exp\left[-\left(d_{y_1;\sigma_1}^{\gamma_1\alpha_1}-1\right)^{v_1}\right]\right\}$$
$$\times\left\{1-exp\left[-\left(d_{y_2;\sigma_2}^{\gamma_2\alpha_2}-1\right)^{v_2}\right]\right\}\left(1+\Upsilon\left\{exp\left[-\left(d_{y_1;\sigma_1}^{\gamma_1\alpha_1}-1\right)^{v_1}-\left(d_{y_2;\sigma_2}^{\gamma_2\alpha_2}-1\right)^{v_2}\right]\right\}\right).$$

The joint PDF can then derived from $c_\Upsilon(s,w) = 1+\Upsilon s^* w^*|_{(s^*=1-2s \text{ and } w^*=1-2w)}$.

BvOBGR type via modified FGM Copula

Consider the following modified version of the bivariate FGM copula defined as (see Rodriguez-Lallena and Ubeda-Flores (2004)):

$$C_\Upsilon(s,w)|_{\Upsilon\in[-1,1]} = sw[1+\Upsilon\varphi(s)\psi(w)] = sw+\Upsilon A(s)B(w),$$

where $A(s) = s\varphi(s)$, and $B(w) = w\psi(w)$. Where $\varphi(s)$ and $\psi(w)$ are two absolutely continuous functions on $(0,1)$ with the following conditions:
1-The boundary condition: $\varphi(0) = \varphi(1) = \psi(0) = \psi(1) = 0$.
2-Let

$$\tau_1 = lnf\left\{\frac{\partial}{\partial s}A(s)|T_1(s)\right\} < 0,$$

$$\tau_2 = sup\left\{\frac{\partial}{\partial s}A(s)|T_1(s)\right\} < 0,$$

$$\pi_1 = lnf\left\{\frac{\partial}{\partial w}B(w)|T_2(w)\right\} > 0,$$

and

$$\pi_2 = sup\left\{\frac{\partial}{\partial w}B(w)|T_2(w)\right\} > 0,$$

Then, $mln(\tau_1\tau_2, \pi_1\pi_2) \geq 1$ where

$$\frac{\partial}{\partial s}A(s) = \varphi(s) + s\frac{\partial}{\partial s}\varphi(s),$$

$$T_1(s) = \left\{s: s \in (0,1)|\frac{\partial}{\partial s}A(s) \quad \text{exists}\right\} \text{ and } T_2(w) = \left\{w: w \in (0,1)|\frac{\partial}{\partial w}B(w) \quad \text{exists}\right\}.$$

BvWG-Lx-FGM (Type I) model

Here, we consider the following functional form for both $A(s)$ and $B(w)$ as

$$C_\Upsilon(s,w) = \left\{1 - exp\left[-\left(d_{s;\sigma_1}^{\gamma_1\alpha_1} - 1\right)^{v_1}\right]\right\}\left\{1 - exp\left[-\left(d_{w;\sigma_2}^{\gamma_2\alpha_2} - 1\right)^{v_2}\right]\right\} + \Upsilon A(s)B(w) ,$$

where $A(s) = s\, exp\left[-\left(d_{s;\sigma_1}^{\gamma_1\alpha_1} - 1\right)^{v_1}\right]$ and $B(w) = w\, exp\left[-\left(d_{w;\sigma_2}^{\gamma_2\alpha_2} - 1\right)^{v_2}\right]$.

BvWG-Lx-FGM (Type II) model

Due to Mansour et al. (2020a) the CDF of the BvWG-Lx-FGM (Type II) model can be derived from

$$C_\Upsilon(s,w) = sF^{-1}(w) + wF^{-1}(s) - F^{-1}(s)F^{-1}(w),$$

where

$$1 + F^{-1}(s) = \sigma_1\left\{1 + [-\,ln(1-s)]^{\frac{1}{v_1}}\right\}^{\frac{1}{\alpha_1\gamma_1}}$$

and

$$1 + F^{-1}(w) = \sigma_2\left\{1 + [-\,ln(1-w)]^{\frac{1}{v_2}}\right\}^{\frac{1}{\alpha_2\gamma_2}}.$$

BvWG-Lx-FGM (Type III) model

Consider the following functional form for both $\varphi(s)$ and $\psi(w)$ which satisfy all the conditions stated earlier where

$$\varphi(s)|_{(\Upsilon_1>0)} = S^{\Upsilon_1}(1-s)^{1-\Upsilon_1} \text{ and } \psi(w)|_{(\Upsilon_2>0)} = W^{\Upsilon_2}(1-w)^{1-\Upsilon_2}.$$

The corresponding bivariate copula (henceforth, BvWG-Lx-FGM (Type III) copula) can be derived from

$$C_{\Upsilon,\Upsilon_1,\Upsilon_2}(s,w) = sw[1 + \Upsilon s^{\Upsilon_1}w^{\Upsilon_2}(1-s)^{1-\Upsilon_1}(1-w)^{1-\Upsilon_2}].$$

BvWG-Lx-FGM (Type IV) model

Consider the following functional form for both $C(s)$ and $D(w)$ which satisfy all the conditions stated earlier where

$$C(s) = s\, log(1+\overline{s}) \text{ and } D(w) = w\, log(1+\overline{w}).$$

In this case, one can also derive a closed form expression for the associated CDF of the BvWG-Lx-FGM (Type IV) as

$$C_\Upsilon(S,W) = SW\big(1 + \Upsilon C(s)D(w)\big).$$

BvWG-Lx type via Renyi's entropy

Consider theorem of Pougaza and Djafari (2011) where

$$C(s,w) = y_2 s + y_1 w - y_1 y_2,$$

where S and W are two absolutely continuous functions on $(0,1)$. Then, the associated BvWG-Lx will be

$$C(y_1,y_2) = -y_1y_2 + y_2\left\{1 - exp\left[-\left(d_{y_1;\sigma_1}^{\gamma_1\alpha_1} - 1\right)^{v_1}\right]\right\} + y_1\left\{1 - exp\left[-\left(d_{y_2;\sigma_2}^{\gamma_2\alpha_2} - 1\right)^{v_2}\right]\right\}.$$

BvWG-Lx type via Clayton Copula

The Clayton Copula can be considered as

$$C_\Upsilon(w_1, w_2) = (w_1^{-\Upsilon} + w_2^{-\Upsilon} - 1)^{-\frac{1}{\Upsilon}}|_{\Upsilon\in[0,\infty]}.$$

Let us assume that $Y \sim$ WG-Lx $(\underline{\Psi_1})$ and $Z \sim$ WG-Lx $(\underline{\Psi_2})$. Then, setting

$$w_1 = w(y|\underline{\Psi_1}) = 1 - exp\left[-\left(d_{y;\sigma_1}^{\gamma_1\alpha_1} - 1\right)^{v_1}\right],$$

and

$$w_2 = w(z|\underline{\Psi_2}) = 1 - exp\left[-\left(d_{z;\sigma_2}^{\gamma_2\alpha_2} - 1\right)^{v_2}\right].$$

Then, the BvWG-Lx type distribution can be derived as

$$F_\Upsilon(y,z) = \left[\left\{1 - exp\left[-\left(d_{y;\sigma_1}^{\gamma_1\alpha_1} - 1\right)^{v_1}\right]\right\}^{-\Upsilon} + \left\{1 - exp\left[-\left(d_{z;\sigma_2}^{\gamma_2\alpha_2} - 1\right)^{v_2}\right]\right\}^{-\Upsilon} - 1\right]^{-\frac{1}{\Upsilon}}.$$

MvWG-Lx extension via Clayton Copula

A straightforward n -dimensional extension from the above will be

$$C(w_l) = \left[\sum_{l=1}^{n} w_l^{-\Upsilon} + 1 - n\right]^{-\frac{1}{\Upsilon}}.$$

Then, the MvWG-Lx extension can be expressed as

$$C(\underline{Z}) = \left(\sum_{l=1}^{n}\left\{1 - exp\left[-\left(d_{z;\sigma_2}^{\gamma_2\alpha_2} - 1\right)^{v_2}\right]\right\}^{-\Upsilon} + 1 - n\right)^{-\frac{1}{\Upsilon}},$$

where $\underline{Z} = z_1, z_2, \cdots, z_n$.

3.Mathematical properties

Useful representations

Due to Yousof et al. (2018), the PDF in (6) can be expressed as

$$f(y) = \sum_{k=0}^{\infty} \Delta_k\, g_{(1+k),\alpha,\sigma}(y), \tag{8}$$

where

$$\Delta_k = \sum_{l,j=0}^{\infty} \frac{(-1)^{1+l+j+k}}{\Gamma(1+l)}\binom{lv}{j}\binom{\gamma(j-lv)}{k}$$

and $g_{(1+k),\alpha,\sigma}(y)$ is the PDF of the Lx model with power parameter $1+k$. By integrating Equation (8), the CDF of Y becomes

$$F(y) = \sum_{k=0}^{\infty} \Delta_k\, G_{(1+k),\alpha,\sigma}(y), \tag{9}$$

where $G_{(1+k),\alpha,\sigma}(y)$ is the CDF of the Lx distribution with power parameter $1+k$.

Moments and incomplete moments

The r^{th} ordinary moment of Y is given by $\mu'_{r,Y} = E(y^r) = \int_{-\infty}^{\infty} y^r f(y)dy$, then we obtain

$$\mu'_{r,Y} = \sum_{k=0}^{\infty}\sum_{d=0}^{r} \Delta_{k,d}^{(1+k,r)}\, B\left(1+k, 1+\frac{d-r}{\alpha}\right)|_{(\alpha>r)}, \tag{10}$$

where

$$\Delta_{k,d}^{(1+k,r)} = \Delta_k(1+k)\sigma^r(-1)^d\binom{r}{d}$$

and

$$B(\nabla_1, \nabla_2) = \int_0^1 s^{\nabla_1-1}(1-s)^{\nabla_2-1}ds.$$

Setting $r = 1,2,3$ and 4 in (10), we have

$$E(Y) = \sum_{k=0}^{\infty}\sum_{d=0}^{1} \Delta_{k,d}^{(1+k,1)}\, B\left(1+k, \frac{d-1}{\alpha}+1\right)|_{(\alpha>1)},$$

$$E(Y^2) = \sum_{k=0}^{\infty}\sum_{d=0}^{2} \Delta_{k,d}^{(1+k,2)}\, B\left(1+k, \frac{d-2}{\alpha}+1\right)|_{(\alpha>2)},$$

$$E(Y^3) = \sum_{k=0}^{\infty}\sum_{d=0}^{3} \Delta_{k,d}^{(1+k,3)}\, B\left(1+k, \frac{d-3}{\alpha}+1\right)|_{(\alpha>3)},$$

and

$$E(Y^4) = \sum_{k=0}^{\infty}\sum_{d=0}^{4} \Delta_{k,d}^{(1+k,4)}\, B\left(1+k, \frac{d-4}{\alpha}+1\right)|_{(\alpha>4)},$$

where $E(Y) = \mu'_{1,Y}$ is the mean of Y. The r^{th} incomplete moment, say $I_{r,Y}(t)$, of Y can be expressed, from (9) as

$$I_{r,Y}(t) = \int_{-\infty}^{t} y^r f(y)dy = \sum_{k=0}^{\infty} \Delta_k \int_{-\infty}^{t} y^r\, g_{(1+k),\alpha,\sigma}(y)dy$$

then

$$I_{r,Y}(t) = \sum_{k=0}^{\infty}\sum_{d=0}^{r} \Delta_{k,d}^{(1+k,r)}\, B_t\left(1+k, 1+\frac{d-r}{\alpha}\right)|_{(\alpha>r)},$$

where

$$B_u(\nabla_1, \nabla_2) = \int_0^u w^{\nabla_1-1}(1-w)^{\nabla_2-1}dw.$$

The first incomplete moment given by (11) with $r=1$ as

$$I_{1,y}(t) = \sum_{k=0}^{\infty}\sum_{d=0}^{1} \Delta_{k,d}^{(1+k,1)}\, B_t\left(1+k, 1+\frac{d-1}{\alpha}\right)|_{(\alpha>1)}.$$

The index of dispersion IxDis or the variance (μ_2) to mean ratio can derived as IxDis $(Y) = \mu_2/\mu'_1$. It is a measure used to quantify whether a set of observed occurrences are clustered or dispersed compared to a standard statistical model.

Numerical analysis

By analyzing the μ'_1, μ_2, skewness (β_1), kurtosis (β_2) and IxDis (Y) numerically in Table 1, it is noted that, the β_1 of the WG-Lx distribution can be negative and also positive. The spread for the β_2 of the WG-Lx model is ranging from -1129.85 to 311.698. The IxDis (Y) for the WG-Lx model can be in $(0,1)$ and also >1 so it may be used as an "under-dispersed" and "over-dispersed" model.

Table 1: μ_1', μ_2, β_1, β_2 and IxDis(Y) of the WG-Lx model.

v	γ	α	σ	μ_1'	μ_2	β_1	β_2	IxDis(Y)
0.5	2	5	15	1.115318	1.701461	1.582538	5.420254	1.5255390
1				0.935914	0.461627	0.813496	3.230566	0.4932369
5				1.000615	0.032425	−0.5441214	3.28545	0.0324055
10				1.034504	0.009234	−0.8243438	4.090645	0.0089257
5	0.5	5	3	0.8871517	0.030056	−0.4445040	3.114689	0.0338792
	1			0.4140281	0.005868	−0.5105485	3.224178	0.0141719
	2			0.2001230	0.001297	−0.5441214	3.285450	0.0064811
	3			0.1319302	0.000553	−0.5553979	3.306883	0.0041941
	4			0.0983970	0.000305	−0.5610533	3.317809	0.0030991
	5			0.0784549	0.000193	−0.5644505	−716.0823	0.0024573
1.5	1.5	0.5	0.5	0.7048251	0.281039	1.3835870	5.710077	0.3987369
		1		0.2595549	0.025489	0.7938293	3.500659	0.0982025
		2		0.1130104	0.003996	0.5420847	2.916349	0.0353562
		3		0.0720898	0.001529	0.4630558	2.774388	0.0212056
		4		0.0529074	0.000799	0.4243545	3.268787	0.0150956
		5		0.0417835	0.000489	0.4013848	1.682906	0.0117068
4	3	2	0.25	0.028029	4×10^{-5}	96.54233	−1129.85	0.0014246

0.5	0.056057	0.000159	−0.3924604	311.698	0.0028493
1	0.112115	0.000639	−0.3925147	2.99214	0.0056985
5	0.560573	0.015972	−0.3924368	2.99139	0.0284925
10	1.121145	0.063889	−0.3924372	2.99139	0.0569851
50	5.605726	1.597213	−0.3924372	2.99130	0.2849253
100	11.21145	6.388852	−0.3924372	2.99139	0.5698506
500	56.05726	159.7213	−0.3924372	2.99139	2.8492530
1000	112.1145	638.8852	−0.3924372	2.99139	5.6985060

Some generating functions (GF)

The moment generating function (MGF) can be derived using (8) as

$$M_Y(t)=\sum_{k=0}^{\infty}\varDelta_k\, M_{(1+k),\alpha,\sigma}(t),$$

where $M_{(1+k),\alpha,\sigma}(t)$ is the MGF of the ExpLx model, then

$$M_Y(t)=\sum_{k,r=0}^{\infty}\sum_{d=0}^{r}\varDelta_{k,d,r}^{(1+k,r)}\,B\left(1+k,1+\frac{d-r}{\alpha}\right)|_{(\alpha>r)},$$

where

$$\varDelta_{k,d,r}^{(1+k,r)}=\tfrac{t^r}{r!}\varDelta_{k,d}^{(1+k,r)}\ .$$

The first r derivatives of $M_Y(t)$, with respect to t at $t=0$, yield the first r moments about the origin, i.e., $\mu'_{r,Y}=E(Y^r)=\frac{d^r}{dt^r}M_Y(t)|_{(t=0\text{ and }r=1,2,3,\ldots)}$. The generating function GF (CGF) is the logarithm of the MGF. Thus, r^{th} cumulant, say κ_r, can be obtained from

$$\kappa_r=\frac{d^r}{dt^r}log\left[\sum_{k,r=0}^{\infty}\sum_{d=0}^{r}\varDelta_{k,d,r}^{(1+k,r)}\,B\left(1+k,1+\frac{d-r}{\alpha}\right)\right]|_{(t=0,\text{ and }r=1,2,3,\ldots)}.$$

The 1st cumulant is the mean ($\kappa_1=\mu'_1$), the 2nd cumulant is the variance, and the 3rd cumulant is the same as the 3rd central moment $\kappa_3=\mu_3$. But 4th and higher order cumulants are not equal to central moments. In some cases, theoretical treatments of problems in terms of cumulants are simpler than those using moments. When two or more RVs are statistically independent, the r^{th} order cumulant of their sum is equal to the sum of their r^{th} order cumulants. Moreover, the cumulants can be also obtained from

$$\kappa_r|_{r\geq 1}=\mu'_r-\sum_{m=0}^{r-1}\binom{r-1}{m-1}\mu'_{r-m}\kappa_m.$$

Moment of the reversed residual

The n^{th} moment of the reversed residual life, say $V_n(t)=E\left[(t-y)^n\ |_{y\leq t,t>0\text{ and }n=1,2,\ldots}\right]$. We obtain

$$V_n(t)=\frac{1}{F(t)}\int_0^t(t-y)^n\,dF(y).$$

Then, the n^{th} moment of the reversed residual life of Y becomes

$$V_n(t)=\frac{1}{F(t)}\sum_{k=0}^{\infty}\sum_{d=0}^{r}\varDelta_{k,d}^{(1+k,n)*}\,B_t\left(1+k,1+\frac{d-r}{\alpha}\right)|_{(\alpha>r)},$$

where

$$\varDelta_{k,d}^{(1+k,n)*}=\varDelta_k(1+k)\sigma^r(-1)^d\binom{r}{d}\sum_{d=0}^{n}\ \ (-1)^d\binom{n}{d}t^{n-d}.$$

4.Graphical assessment

Graphically and using the biases and mean squared errors (MSEs), we can perform the simulation experiments to assess the finite sample behavior of the maximum likelihood estimations (MLEs). The assessment was based on N =1000 replication for all $n|_{(n=50,100,\ldots,500)}$. The following algorithm is considered:

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I. Generate N =1000 samples of size $n|_{(n=50,100,\ldots,500)}$ from the WG-Lx distribution using (7);
II. Compute the MLEs for the N =1000 samples;
III. Compute the SEs of the MLEs for the 1000 samples. The standard errors (SEs) were computed by inverting the observed information matrix.
IV. Compute the biases and mean squared errors given for $\underline{\Psi} = v, \gamma, \alpha, \sigma$. We repeated these steps for $n|_{(n=50,100,\ldots,500)}$ with $v = 1,2,..100$, $\gamma = 1,2,..100$, $\alpha = 1,2,..100$, $\sigma = 1,2,..100$, so computing biases $\left(\mathrm{Bias}_{\underline{\Psi}}(n)\right)$, mean squared errors (MSEs) $\left(\mathrm{MSE}_{h}(n)\right)$ for $\underline{\Psi} = v, \gamma, \alpha, \sigma$ and $n|_{(n=50,100,\ldots,500)}$.
Figures 2, 3, 4 and 5 gives the biases (left panels) and MSEs (right panels) for the parameters v, γ, α and σ respectively.

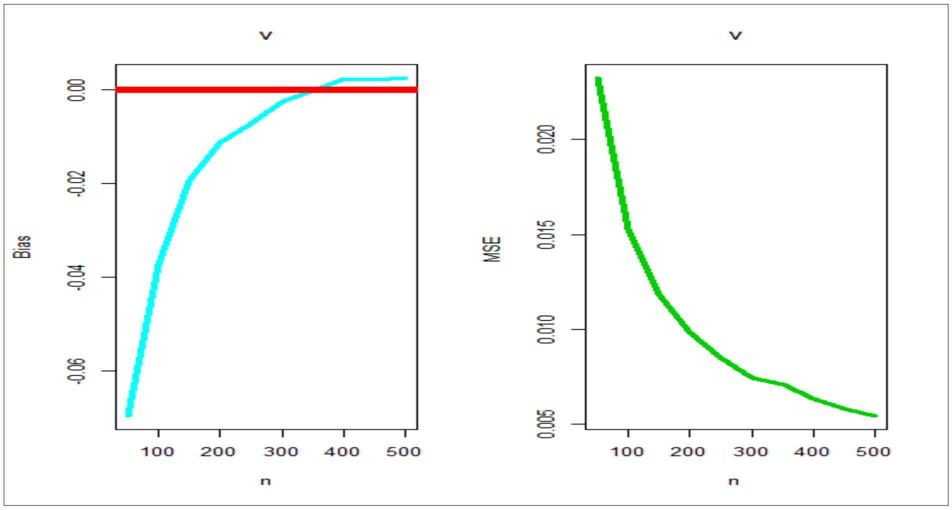

Figure 2: biases and mean squared errors for the parameter v .

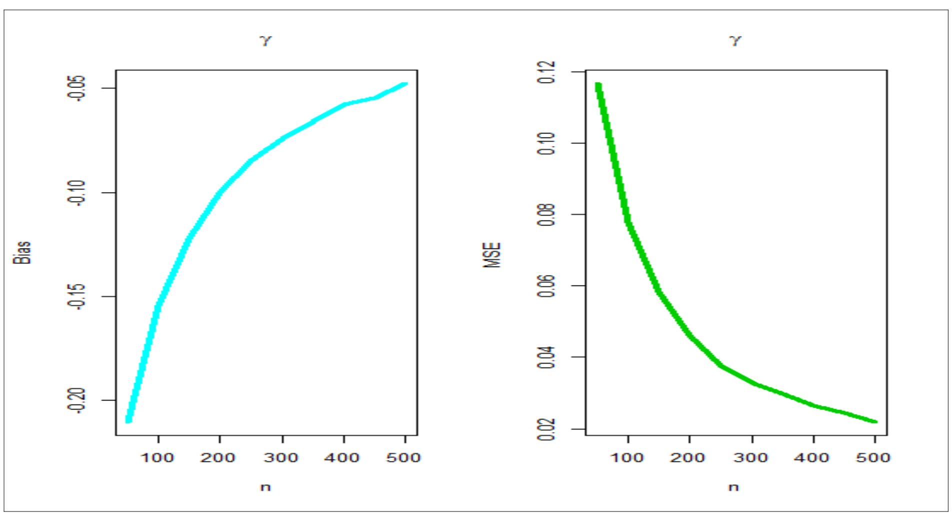

Figure 3: biases and mean squared errors for the parameter γ .

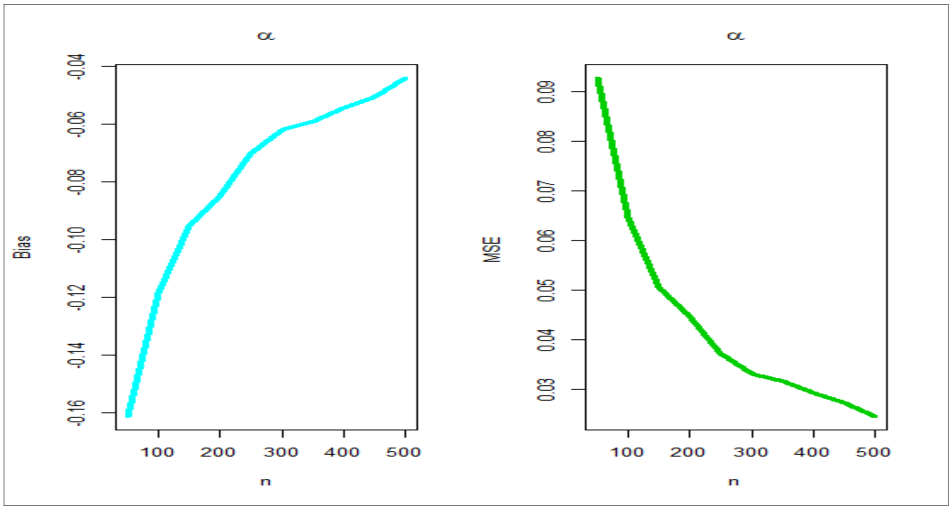

Figure 4: biases and mean squared errors for the parameter α .

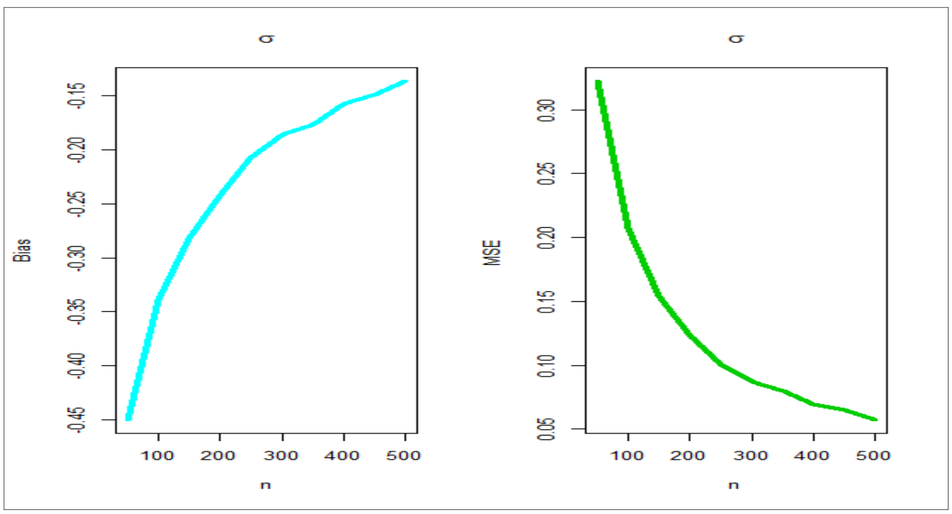

Figure 5: biases and mean squared errors for the parameter σ .

The left panels from show how the four biases vary with respect to n . The right panels show how the four MSEs vary with respect to n . The broken line in red in Figure 2 corresponds to the biases being 0 . From Figures 2-5 (left panels), the biases for each parameter are generally negative and tends to zero as $n \rightarrow \infty$. From Figures 2-5 (right panels), the MSEs for each parameter decrease to zero as $n \rightarrow \infty$.

5.Applications

In this section, we provide two real life applications to two real data sets to illustrate the importance and flexibility of the WG-Lx model. We compare the fit of the WG-Lx with some well-known competitive models (see Table 2)

Table 2: Competitive models.

N.	Model	Abbreviation	Author
1	Lomax	Lx	Lomax (1954)
2	Exponentiated Lx	ExpLx	Gupta et al. (1998)
3	Kumaraswamy Lx	KumLx	Lemonte and Cordeiro (2013)
4	Macdonald Lx	McLx	Lemonte and Cordeiro (2013)

5	Beta Lx	BLx	Lemonte and Cordeiro (2013)
6	Gamma Lx	GamLx	Cordeiro et al. (2015)
7	Transmuted Topp-Leone Lx	TTLLx	Yousof et al. (2017)
8	Reduced TTLLx	RTTLLx	Yousof et al. (2017)
9	Odd log-logistic Lx	OLLLx	Altun et al. (2018a)
10	Reduced OLLLx	ROLLLx	Altun et al. (2018a)
11	Reduced Burr-Hatke Lx	RBHLx	Yousof et al. (2018)
12	Special generalized mixture Lomax	SGMLx	Chesneau and Yousof (2021)
13	Reduced WG-Lx	RWG-Lx model	New
14	Proportional reversed hazard rate Lx	PRHRLx	New

First data set: Failure times of 84 Aircraft Windshield: The first real data set (dataset I) represents the data on failure times of 84 aircraft windshield, this data is recently analyzed by El-Morshedy et al. (2021). The data are: 0.0400, 1.866, 2.3850, 3.443, 0.3010, 1.876, 2.4810, 3.4670, 0.3090, 1.8990, 2.610, 3.4780, 0.557, 1.9110, 2.625, 3.5780, 0.943, 1.9120, 2.632, 3.5950, 1.0700, 1.914, 2.6460, 3.6990, 1.1240, 1.9810, 2.661, 3.7790,1.248, 2.0100, 2.688, 3.9240, 1.2810, 2.0380, 2.820, 3, 4.035, 1.281, 2.0850, 2.890, 4.121, 1.3030, 2.089, 2.9020, 4.167, 1.4320, 2.097, 2.934, 4.2400, 1.480, 2.1350, 2.962, 4.2550, 1.505, 2.1540, 2.9640, 4.2780, 1.506, 2.190, 3.000, 4.3050, 1.568, 2.1940, 3.103, 4.376, 1.615, 2.2230, 3.114, 4.449, 1.6190, 2.2240, 3.1170, 4.4850, 1.652, 2.2290, 3.166, 4.570, 1.652, 2.3000, 3.344, 4.602, 1.7570, 2.324, 3.3760, 4.6630. Second data set: Service times of 63 Aircraft Windshield: The second real data set (dataset II) represents the data on service times of 63 aircraft windshield, this data is recently analyzed by El-Morshedy et al. (2021). The data are: 0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.7190, 2.7170, 0.2800, 1.794, 2.819, 0.3130, 1.915, 2.820, 0.389, 1.9200, 2.878, 0.4870, 1.9630, 2.950, 0.622, 1.978, 3.0030, 0.9000, 2.053, 3.1020, 0.9520, 2.0650, 3.3040, 0.9960, 2.1170, 3.483, 1.0030, 2.1370, 3.500, 1.0100, 2.141, 3.6220, 1.085, 2.1630, 3.6650, 1.092, 2.1830, 3.695, 1.1520, 2.2400, 4.015, 1.1830, 2.3410, 4.6280, 1.2440, 2.4350, 4.8060, 1.2490, 2.4640, 4.8810, 1.2620, 2.5430, 5.1400. Many other useful real life data sets can be found in Aryal et al. (2017), Yousof et al. (2018), Elbiely and Yousof (2018), Ibrahim and Yousof (2020), Yadav et al. (2020), Mansour et al. (2020e), Goual et al. (2020), Elgohari and Yousof (2020a) and Ibrahim et al. (2020). For exploring the extreme values, the box plot is plotted (see Figure 6). Based on Figure 6, we note that no extreme values were found in the two real life data sets. For checking the normality, the Quantile-Quantile (QQ) plot is sketched (see Figure 7). Based on Figures 7, we note that the normality is nearly exists. For exploring the HRF for real data, the total time test (TTT) plot is provided (see Figure 8). Based on Figure 8, we note that the HRF is "monotonically increasing" for the two real life data sets. For exploring the initial shape of real data nonparametrically, kernel density estimation (KDE) is provided (see Figure 9). Figure 9 show nonparametric KDE for exploring the data. Figures 10 and 11 give, Probability-Probability (P-P) plot (top left), estimated PDF (EPDF) (top right), estimated CDF (ECDF) (bottom left) and estimated HRF (EHRF) (bottom right) for data set I and II respectively.

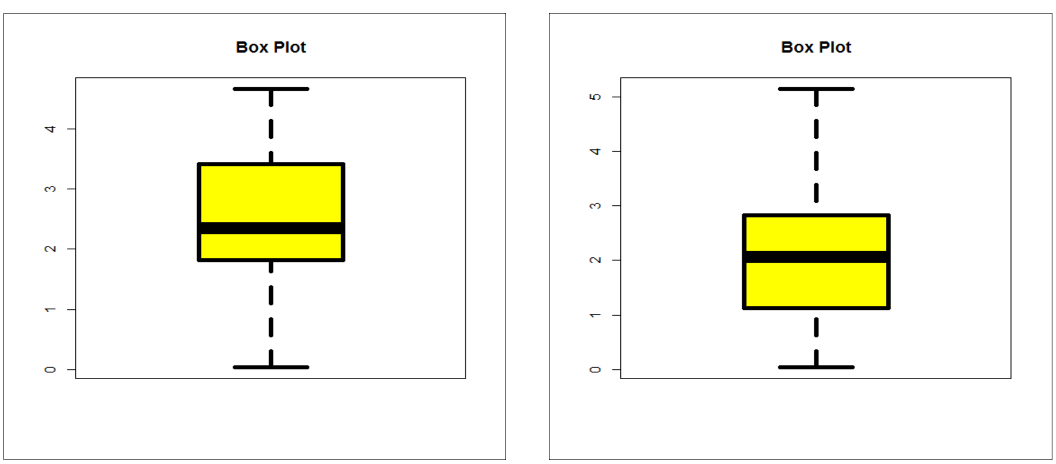

Figure 6: Box plots.

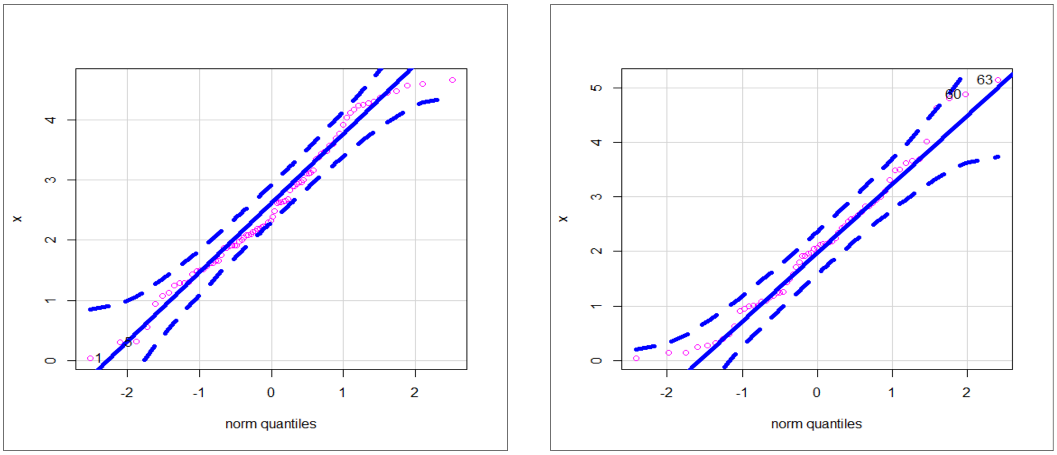

Figure 7: Normal QQ plots.

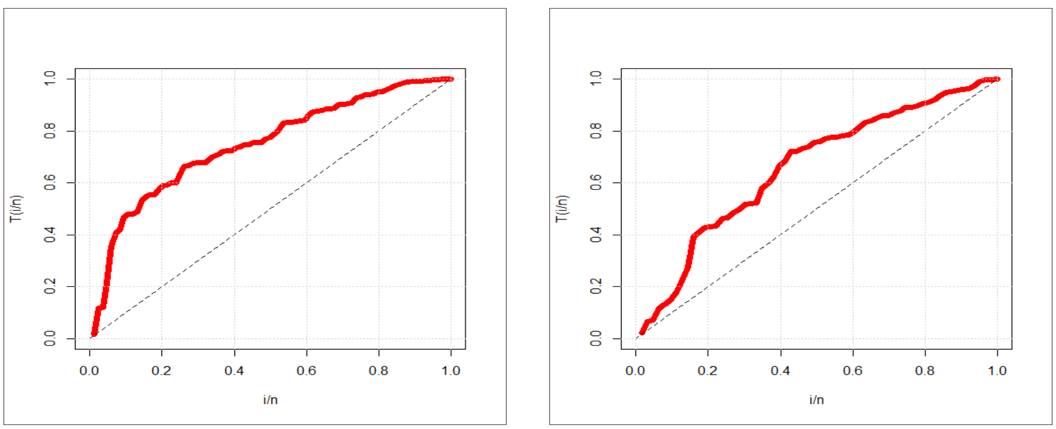

Figure 8: TTT plots.

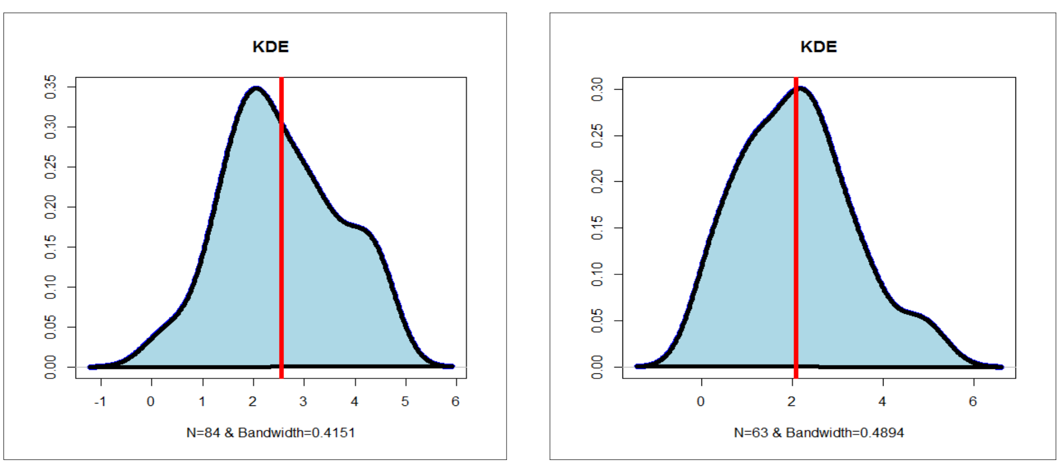

Figure 9: Nonparametric KDE.

We estimate the unknown parameters of each model by maximum likelihood using "L-BFGS-B" method and the goodness-of-fit statistics Akaike information criterion (AIC), Bayesian IC (BIC), Consistent AIC (CAIC), Hannan-Quinn IC (HQIC), Anderson-Darling (A^*) and Cramér-von Mises (C^*) are used to compare the five models. For failure times data: the analysis results of are listed in Tables 3 and 4. Table 3 gives the MLEs and standard errors (SEs) for failure times data. Table 4 gives the $-\hat{\ell}$ and goodness-of-fits statistics for failure times data. For service times data: the analysis results of are listed in Tables 5 and 6. Table 5 gives the MLEs and SEs for service times data. Table 6 give the $-\hat{\ell}$ and goodness-of-fits statistics for the service times data. Based on Tables 4 and 6, we note that the WG-Lx model gives the lowest values for the AIC, CAIC, BIC, HQIC, A^* and C^* among all fitted models. Hence, it could be chosen as the best model under these criteria.

Table 3: MLEs and SEs for failure times data.

Model	Estimates				
WG-Lx(v,γ,α,σ)	**1.790561**	**0.55679**	**402.4722**	**947.204**	
	(0.16317)	**(0.23738)**	**(17.5762)**	**(11.871)**	
McLx(v,γ,c,α,σ)	2.1875	119.1751	12.4171	19.9243	75.6606
	(0.5211)	(140.297)	(20.845)	(38.9601)	(147.24)
TTLLx(v,γ,α,σ)	-0.80750	2.47663	(15608.2)	(38628.3)	
	(0.13960)	(0.54176)	(1602.37)	(123.936)	
KumLx(v,γ,α,σ)	2.6150	100.2756	5.27710	78.6774	
	(0.3842)	(120.486)	(9.8116)	(186.005)	
BLx(v,γ,α,σ)	3.60360	33.63870	4.830700	118.8374	
	(0.6187)	(63.7145)	(9.23820)	(428.927)	
PRHRLx(γ,α,σ)	3.73×10^6	4.707×10^{-1}	4.49×10^6		
	1.01×10^6	(0.00001)	37.14684		
RTTLLx(v,γ,α)	-0.84732	5.520572	1.156782		
	(0.10010)	(1.18479)	(0.09588)		
SGMLx(v,α,σ)	-1.04×10^{-1}	9.83×10^6	1.18×10^7		
	(0.1233)	(4843.3)	(501.04)		
RWG-Lx(v,γ,α)	3.00116	0.667532	0.77532		
	(0.27521)	(0.00321)	(0.0093)		
OLLLx(v,α,σ)	2.32636	(7.17×10^5)	2.34×10^6)		
	(2.14×10^{-1})	(1.19×10^4)	(2.61×10^1)		
GamLx(v,α,σ)	3.587600	52001.49	37029.66		
	(0.50334)	(7955.00)	(81.1644)		
ExpLx(v,α,σ)	3.626101	20074.51	26257.68		
	(0.6236)	(2041.83)	(99.7417)		
ROLLLx(v,α)	3.890564	0.573161			
	(0.36524)	(0.01946)			
RBHLx(α,σ)	10801754	51367189			
	(983309)	(232312)			
Lx(α,σ)	51425.353	131789.8			
	(5933.492)	(296.119)			

Table 4: ℓ and goodness-of-fits statistics for failure times data.

Model	ℓ	AIC	CAIC	BIC	HQIC	A^*	C^*
WG-Lx	**-127.8950**	**263.7899**	**264.2962**	**273.5132**	**267.6986**	**0.5599**	**0.0696**
McLx	-129.8023	269.6045	270.3640	281.8178	274.5170	0.6672	0.0858
RWG-Lx	-132.1993	270.3987	270.6987	277.6911	273.3302	0.7593	0.0772
OLLLx	-134.4235	274.8470	275.1470	282.1394	277.7785	0.9407	0.1009
TTLLx	-135.5710	279.1410	279.6464	288.8633	283.0487	1.1257	0.1270
GamLx	-138.4042	282.8083	283.1046	290.1363	285.7559	1.3666	0.1618
BLx	-138.7177	285.4354	285.9354	295.2062	289.3654	1.4084	0.1682
ExpLx	-141.3997	288.7994	289.0957	296.1273	291.7469	1.7435	0.2194
ROLLLx	-142.8452	289.6904	289.8385	294.5520	291.6447	1.9566	0.2554
SGMLx	-143.0874	292.1747	292.4747	299.4672	295.1062	1.3467	0.1578

RTTLLx	-153.9810	313.9618	314.2618	321.2542	316.8933	3.7527	0.5592
PRHRLx	-162.8771	331.7543	332.0542	339.0464	334.6855	1.3672	0.1609
Lx	-164.9884	333.9767	334.1230	338.8620	335.9417	1.3976	0.1666
RBHLx	-168.6041	341.2082	341.3562	346.0697	343.1624	1.6711	0.2069

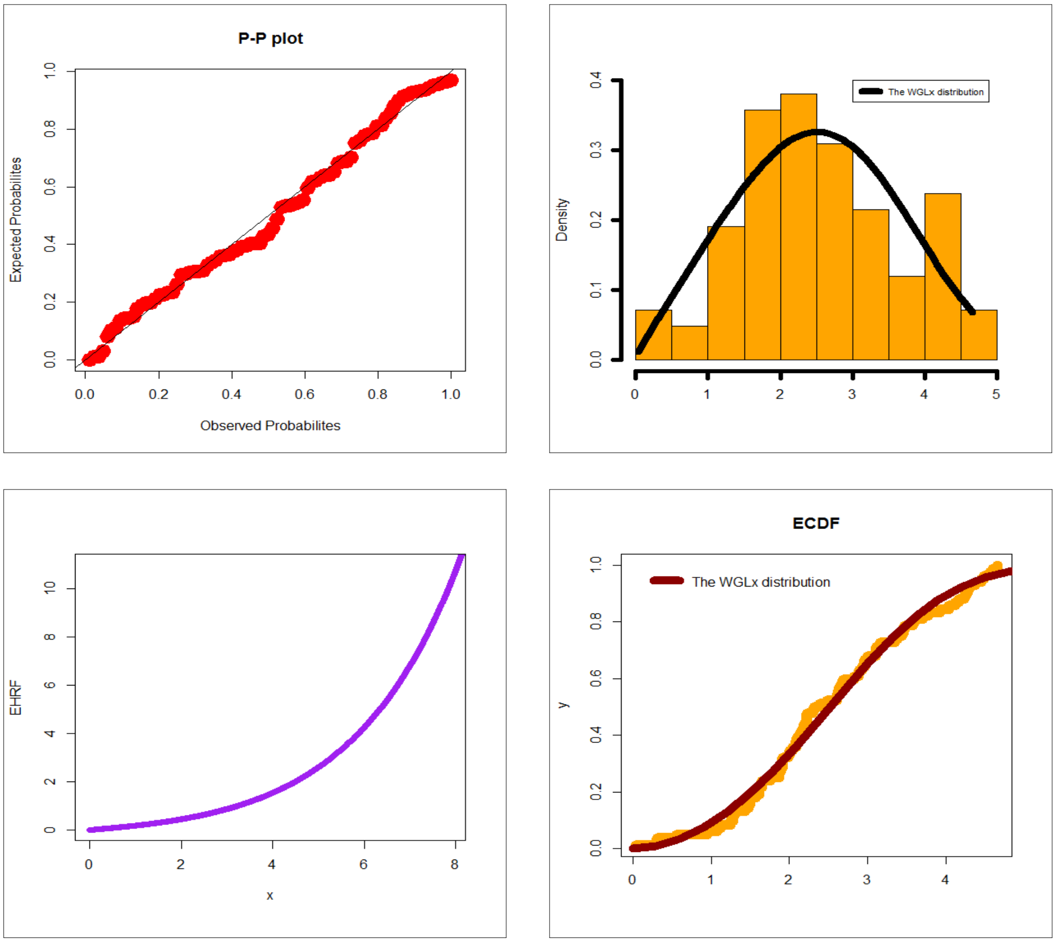

Figure 10: P-P plot, EPDF, ECDF and EHRF for data set **I**.

Table 5: MLEs and SEs for service times data.

Model	Estimates			
WG-Lx(ν,γ,α,σ)	**1.22070**	**3642.784**	**0.53098**	**6732.61**
	(0.13189)	**(32.1769)**	**(0.3201)**	**(8.1764)**
KumLx(ν,γ,α,σ)	1.669132	60.5673	2.56490	65.06400
	(0.25701)	(86.0131)	(4.7589)	(177.592)
BLx(ν,γ,α,σ)	1.9218	31.25943	4.968432	169.5719
	(0.3184)	(316.841)	(50.528)	(339.207)
TTLLx(ν,γ,α,σ)	(-0.6070)	1.785780	2123.391	4822.789
	(0.21371)	(0.41522)	(163.915)	(200.009)
PRHRLx(γ,α,σ)	1.59×10^{6}	3.93×10^{-1}	1.30×10^{6}	
	2.01×10^{3}	0.0004×10^{-1}	0.95×10^{6}	
RTTLLx(ν,γ,α)	-0.671451	2.74496	1.012381	
	(0.18746)	(0.6696)	(0.11405)	
SGMLx(ν,α,σ)	-1.04×10^{-1}	6.451×10^{6}	6.33×10^{6}	

	(4.1×10^{-10})	(3.21×10^{6})	(3.8573)
RWG-Lx(v,γ,α)	1.92707	1.349822	0.43660
	(0.21096)	(12.6473)	(4.09087)
OLLLx(v,α,σ)	1.66419	6.340×10^{5}	2.01×10^{6}
	(1.79×10^{-1})	(1.68×10^{4})	7.22×10^{6}
GamLx(v,α,σ)	1.907334	35842.433	39197.57
	(0.3213)	(6945.074)	(151.653)
ExpLx(v,α,σ)	1.91454	22971.154	32881.99
	(0.34822)	(3209.533)	(162.230)
ROLLLx(v,α)	2.37233	0.6910933	
	(0.26825)	(0.04488)	
RBHLx(α,σ)	14055522	53203423	
	(422.005)	(28.52323)	
Lx(α,σ)	99269.78	207019.37	
	(11863.5)	(301.2366)	

Table 6: ℓ and goodness-of-fits statistics for the service times data.

Model	ℓ	AIC	CAIC	BIC	HQIC	A^*	C^*
WG-Lx	**-98.40022**	**204.8004**	**205.4901**	**213.3730**	**208.1721**	**0.3270**	**0.0522**
KumLx	-100.8676	209.7353	210.4249	218.3078	213.1069	0.7391	0.1219
RWG-Lx	-101.8349	209.6698	210.0766	216.0992	212.1985	0.8836	0.1459
TTLLx	-102.4498	212.8996	213.5892	221.4723	216.2713	0.9431	0.1555
GamLx	-102.8332	211.6663	212.0731	218.0958	214.1951	1.1120	0.1836
SGMLx	-102.8939	211.7881	212.1949	218.2175	214.3168	1.1135	0.1839
BLx	-102.9612	213.9224	214.6119	222.4948	217.2939	1.1336	0.1873
ExpLx	-103.5498	213.0995	213.5063	219.5289	215.6282	1.2332	0.2037
OLLLx	-104.9042	215.8082	216.2151	222.2376	218.3369	0.9425	0.1545
PRHRLx	-109.2985	224.5973	225.0044	231.0267	227.1265	1.1263	0.1862
Lx	-109.2989	222.5976	222.7976	226.8839	224.2834	1.1265	0.1862
ROLLLx	-110.7288	225.4573	225.6573	229.7436	227.1431	2.3473	0.3908
RTTLLx	-112.1855	230.3711	230.7777	236.8004	232.8997	2.6876	0.4532
RBHLx	-112.6010	229.2010	229.4010	233.4873	230.8869	1.3985	0.2318

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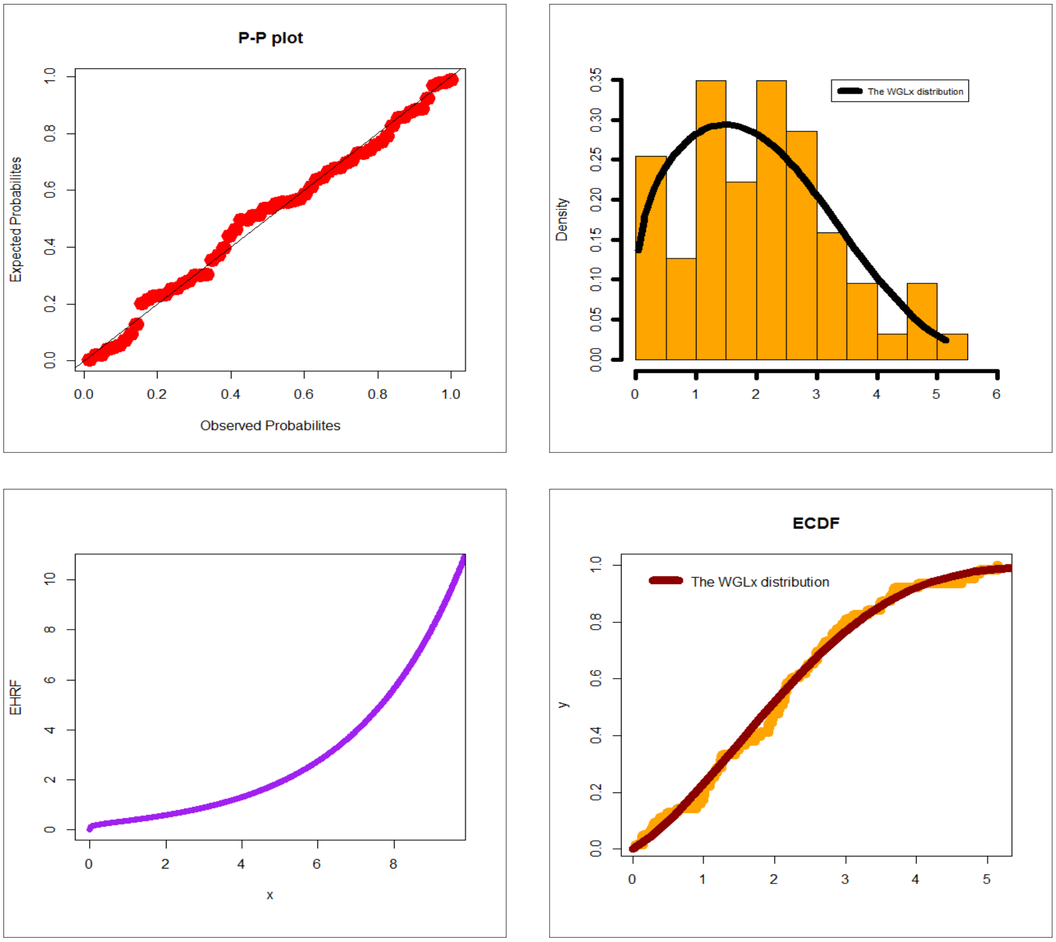

Figure 11: P-P plot, EPDF, ECDF and EHRF for data set II.

6.Conclusions

A new four parameter lifetime model called the Weibull generalized Lomax (WG-Lx) is proposed and studied. The WG-Lx density function can be "right skewed", "symmetric", "left skewed" and "uniformed density". The WG-Lx failure rate function can be "monotonically decreasing", " monotonically increasing" and "constant". The new WG-Lx density can be expressed as a mixture of the exponentiated Lomax model. The skewness of the WG-Lx distribution can negative and positive. The spread for the kurtosis of the WG-Lx model is ranging from $-$ 1129.85 to 311.698. The index of dispersion for the WG-Lx model can be in $(0,1)$ and also >1 so it may be used as an "under-dispersed" and "over-dispersed" model. The maximum likelihood method is used to estimate the WG-Lx parameters. Using the "biases" and "mean squared errors", we performed simulation experiments for assessing the finite sample behavior of the maximum likelihood estimators. It is noted that, the biases for all parameters are generally negative and tends to 0 as $n \rightarrow \infty$ and the mean squared errors for all parameter decrease to 0 as $n \rightarrow \infty$. The WG-Lx model deserved to be chosen as the best model among many well-known Lomax extension such as exponentiated Lomax, gamma Lomax, Kumaraswamy Lomax, odd log-logistic Lomax, Macdonald Lomax, beta Lomax, reduced odd log-logistic Lomax, reduced Burr-Hatke Lomax, reduced WG-Lx, special generalized mixture Lomax and the standard Lomax distributions in modeling the "failure times" and the "service times" data sets. As a future potential work, the WG-Lx model can be validated using many new useful goodness-of-fit statistic tests in case of the right censored data sets such as the goodness-of-fit test of Nikulin-Rao-Robson (NRR) for right censored data, the modified NRR goodness-of-fit test for right censored data, the goodness-of-fit test of Bagdonavičius-Nikulin (BgN) for right censored data, modified BgN goodness-of-fit test for right censored data as recently performed by Ibrahim et al. (2019), Alizadeh et al. (2020), Goual et al. (2019, 2020), Mansour et al. (2020a-f), Yadav et al. (2020) and Goual and Yousof (2020), among others.

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