Pakistan Journal of Statistics and Operation Research

The Type I Quasi Lambert Family: Properties, Characterizations and Different Estimation Methods

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Abstract

A new G family of probability distributions called the type I quasi Lambert family is defined and applied for modeling real lifetime data. Some new bivariate type G families using "Farlie-Gumbel-Morgenstern copula", "modified Farlie-Gumbel-Morgenstern copula", "Clayton copula" and "Renyi's entropy copula" are derived. Three characterizations of the new family are presented. Some of its statistical properties are derived and studied. The maximum likelihood estimation, maximum product spacing estimation, least squares estimation, Anderson-Darling estimation and Cramer-von Mises estimation methods are used for estimating the unknown parameters. Graphical assessments under the five different estimation methods are introduced. Based on these assessments, all estimation methods perform well. Finally, an application to illustrate the importance and flexibility of the new family is proposed.

Keywords: Characterizations; Copula; Maximum Product Spacing; Maximum Likelihood; Anderson-Darling Estimation.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction

In mathematics, the "Lambert function", also called the "omega function" or "product logarithm", is a multivalued function, namely the branches of the inverse relation of the function $f(W) = W\exp(W)$ where W is any complex number and $exp(W)$ is the exponential function. In this paper, we define and study a new G family called the type **I** quasi Lambert (TIQL) family. The cumulative distribution function (CDF) of the TIQL family can be expressed as

$$F_{\underline{\boldsymbol{\Phi}}}(x) = W_{\underline{\boldsymbol{\Phi}}}(x)\exp\left[\overline{W}_{\underline{\boldsymbol{\Phi}}}(x)\right]|_{x\in R}, \quad (1)$$

where $\underline{\boldsymbol{\Phi}} = (\alpha, \underline{\boldsymbol{\Psi}})|_{\alpha>0}$ refers to the parameter vector of the TIQL family. The argument $W_{\underline{\boldsymbol{\Phi}}}(x)$ is defined as $W_{\underline{\boldsymbol{\Phi}}}(x) = \left(\frac{2-G_{\underline{\boldsymbol{\Psi}}}(x)}{G_{\underline{\boldsymbol{\Psi}}}(x)}\right)^{\alpha}$, whereas the argument $\overline{W}_{\underline{\boldsymbol{\Phi}}}(x)$ is defined as $\overline{W}_{\underline{\boldsymbol{\Phi}}}(x) = 1 - W_{\underline{\boldsymbol{\Phi}}}(x)$. The fnction $G_{\underline{\boldsymbol{\Psi}}}(x)$ is the CDF of any baseline model and $\underline{\boldsymbol{\Psi}}$ refers to the parameter vector. For $\alpha = 1$, the TIQL family reduces to the reduced TIQL (RTIQL) family. The corresponding probability density function (PDF) can be expressed as

$$f_{\underline{\boldsymbol{\Phi}}}(x) = 2\alpha\frac{g_{\underline{\boldsymbol{\Psi}}}(x)}{G_{\underline{\boldsymbol{\Psi}}}(x)^{\alpha+1}}\left[2 - G_{\underline{\boldsymbol{\Psi}}}(x)\right]^{\alpha-1}\left[W_{\underline{\boldsymbol{\Phi}}}(x) - 1\right]exp\left[\overline{W}_{\underline{\boldsymbol{\Phi}}}(x)\right]|_{x\in R}, \quad (2)$$

where $g_{\underline{\boldsymbol{\Psi}}}(x) = dG_{\underline{\boldsymbol{\Psi}}}(x)/dx$ refers to the PDF of the baseline model. Many well-known generators can be cited such as beta-G (Eugene et al. (2002)), transmuted exponentiated generalized-G (Yousof et al. (2015)), generalized odd generalized exponential family by Alizadeh et al. (2017), exponentiated generalized-G Poisson family (Aryal and Yousof (2017)), transmuted Topp-Leone G family (Yousof et al. (2017a)), beta Weibull-G family (Yousof et al. (2017b)), Topp-Leone odd log-logistic family (Brito et al. (2017)), Burr XII system of densities (Cordeiro et al. (2018)), transmuted Weibull-G family (Alizadeh et al. (2018)), generalized odd Weibull generated family (Korkmaz et al. (2018a)), exponential Lindley odd log-logistic G family (Korkmaz et al. (2018b)), Marshall-Olkin generalized-G Poisson family (Korkmaz et al. (2018c)) and The Odd Power Lindley Generator (Korkmaz et al. (2019)) and odd Nadarajah-Haghighi family (Nascimento et al. (2019)), generalized transmuted Poisson-G family (Yousof et al. (2018a)), Marshall-Olkin generalized-G family (Yousof et al. (2018b)), Burr-Hatke G family (Yousof et al. (2018c)), Type I general exponential class of distributions (Hamedani et al. (2017)), new extended G family (Hamedani et al. (2018)), Type II general exponential class of distributions (Hamedani et al. (2019)), exponential Lindley odd log-logistic-G family (Korkmaz et al. (2018b)), dd power Lindley generator of probability distributions (Korkmaz et al. (2019)), Weibull generalized G family (Yousof et al. (2018d)), Weibull-G Poisson family (Yousof et al. (2020)) and Weibull Topp-Leone generated family (Karamikabir et al. (2020)). Using the power series, the CDF in (1) can be written as

$$F_{\underline{\boldsymbol{\Phi}}}(x) = W_{\underline{\boldsymbol{\Phi}}}(x)\sum_{i=0}^{\infty}\frac{1}{i!}\overline{W}_{\underline{\boldsymbol{\Phi}}}(x)^{i} \tag{3}$$

If $\left|\frac{s_1}{s_2}\right| < 1$ and $s_3 > 0$ is a real non-integer, the following power series holds

$$\left(1-\frac{s_1}{s_2}\right)^{s_3-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(s_3)}{j!\,\Gamma(s_3-j)}\left(\frac{s_1}{s_2}\right)^{j}. \tag{4}$$

Applying (4) to (3) we have

$$F_{\underline{\boldsymbol{\Phi}}}(x) = \sum_{i,j=0}^{\infty}(-1)^j\frac{2^{\alpha(1+j)}\Gamma(1+i)}{i!\,j!\,\Gamma(1+i+j)}\frac{\left[1-\frac{1}{2}G_{\underline{\boldsymbol{\Psi}}}(x)\right]^{\alpha(1+j)}}{G_{\underline{\boldsymbol{\Psi}}}(x)^{\alpha(1+j)}}. \tag{5}$$

Applying (4) again to the term $\left[1-\frac{1}{2}G_{\underline{\boldsymbol{\Psi}}}(x)\right]^{\alpha(1+j)}$, Equation (5) becomes

$$F_{\underline{\boldsymbol{\Phi}}}(x) = \sum_{j,\kappa=0}^{\infty}c_{j,\kappa}\ \Pi_{\kappa^*}\left(x;\underline{\boldsymbol{\Psi}}\right)|_{\kappa^*=\kappa-\alpha(1+j)}, \tag{7}$$

where

$$c_{j,\kappa} = \sum_{i=0}^{\infty}2^{\alpha(1+j)-\kappa}\frac{(-1)^{j+\kappa}\Gamma(1+i)\Gamma\left(1+\alpha(1+j)\right)}{i!\,j!\,\kappa!\,\Gamma(1+i+j)\Gamma(1+\alpha(1+j)-\kappa)}$$

and $\Pi_{\kappa^*}\left(x;\underline{\boldsymbol{\Psi}}\right)$ is the CDF of the exp-G family with power parameter $\kappa^* > 0$. Similarly, the PDF of the TIQL family can also be expressed as a mixture of exp-G PDFs as

$$f_{\underline{\boldsymbol{\Phi}}}(x) = \sum_{j,\kappa=0}^{\infty}c_{j,\kappa}\ \pi_{\kappa^*}\left(x;\underline{\boldsymbol{\Psi}}\right), \tag{7}$$

where $\pi_{\kappa^*}\left(x;\underline{\boldsymbol{\Psi}}\right) = d\Pi_{\kappa^*}\left(x;\underline{\boldsymbol{\Psi}}\right)/dx$ s the PDF of the exp-G family with power parameter $\kappa^* > 0$.

2. Properties

2.1 Moments

Let Y_{κ^*} be a r.v. having density $\pi_{\kappa^*}(x;\underline{\boldsymbol{\Psi}})$. The r th ordinary moment of X , say $\mu'_{r,X}$, follows from (7) as

$$\mu'_{r,X} = \mathrm{E}(X^r) = \sum_{j,\kappa=0}^{\infty}c_{j,\kappa}\ \mathrm{E}(Y_{\kappa^*}^{r}), \tag{8}$$

where

$$\mathrm{E}(Y_{\varsigma}^{r}) = \varsigma\int_{-\infty}^{\infty}x^r\ g_{\underline{\boldsymbol{\Psi}}}(x)G_{\underline{\boldsymbol{\Psi}}}(x)^{\varsigma-1}\ dx,$$

can be evaluated numerically in terms of the baseline qf $Q_G(u) = G^{-1}(u)$ as

$$\mathrm{E}(Y_{\varsigma}^{r}) = \varsigma\int_{0}^{1}u^{\varsigma-1}\quad[Q_G(u)]^r du.$$

Setting $r = 1$ in (8) gives the mean of X.

2.2 Incomplete moments

The r th incomplete moment of X is defined by $m_{r,X}(y) = \int_{-\infty}^{y}x^r\ f_{\underline{\boldsymbol{\Phi}}}(x)dx$. We can write from (7)

$$m_{r,X}(y) = \sum_{j,\kappa=0}^{\infty} c_{j,\kappa}\ m_{r,\kappa^*}(y), \quad (9)$$

where

$$m_{r,\alpha}(y) = \int_0^{G(y)} u^{\alpha-1}\ [Q_G(u)]^r du. \quad (10)$$

The integral $m_{r,\alpha}(y)$ can be determined analytically for special models with closed-form expressions for $Q_G(u)$ or computed at least numerically for most baseline distributions. Two important applications of the first incomplete moment are related to the mean deviations about the mean and median and to the Bonferroni and Lorenz curves.

2.3 Moment generating functions

The moment generating function (mgf) of X, say $M(t) = E\ (\exp(tX))$, is obtained from (7) as

$$M_X(t) = \sum_{j,\kappa=0}^{\infty} c_{j,\kappa}\ M_{\kappa^*}(t), \quad (11)$$

where $M_\varsigma(t)$ is the generating function of Y_ς given by

$$M_\varsigma(t) = \varsigma \int_{-\infty}^{\infty} exp(tx)\, g_{\underline{\Psi}}(x)\left[G_{\underline{\Psi}}(x)\right]^{\varsigma-1} dx = \varsigma \int_0^1 u^{\varsigma-1}\, exp[\, tQ_G(u;\alpha)]du. \quad (12)$$

The last two integrals can be computed numerically for most parent distributions.

3. Copula

In probability theory, a copula is a multivariate CDF for which the marginal probability distribution of each variable is uniform on the interval $[0,1]$. copulas are used to describe the dependence between random variables. In this Section, we derive some new bivariate TIQL (B-TIQL) type distributions using Farlie Gumbel Morgenstern (FGM) copula (see Morgenstern (1956), Gumbel (1958), Gumbel (1960), Johnson and Kotz (1975) and Johnson and Kotz (1977)), modified FGM copula (see Rodriguez-Lallena and Ubeda-Flores (2004)), Clayton copula and Renyi's entropy (Pougaza and Djafari (2011)). The Multivariate TIQL (M-TIQL) type is also presented. However, future works may be allocated to the study of these new models. First, we consider the joint CDF of the FGM family, where

$$C_\varsigma(\tau,u) = \tau u(1+\varsigma\tau^{\cdot}u^{\cdot})|_{\tau^{\cdot}=1-\tau,u^{\cdot}=1-u},$$

and the marginal function $\tau = F_1$, $u = F_2$, $\varsigma \in (-1,1)$ is a dependence parameter and for every $\tau,u \in (0,1)$, $C(\tau,0) = C(0,u) = 0$ which is "grounded minimum" and $C(\tau,1) = \tau$ and $C(1,u) = u$ which is "grounded maximum", $C(\tau_1,u_1) + C(\tau_2,u_2) - C(\tau_1,u_2) - C(\tau_2,u_1) \geq 0$.

3.1 Via FGM family

A copula is continuous in τ and u; actually, it satisfies the stronger Lipschitz condition, where

$$|C(\tau_2,u_2) - C(\tau_1,u_1)| \leq |\tau_2-\tau_1| + |u_2-u_1|.$$

For $0 \leq \tau_1 \leq \tau_2 \leq 1$ and $0 \leq u_1 \leq u_2 \leq 1$, we have

$$Pr(\tau_1 \leq \tau \leq \tau_2, u_1 \leq u \leq u_2) = C(\tau_1,u_1) + C(\tau_2,u_2) - C(\tau_1,u_2) - C(\tau_2,u_1) \geq 0.$$

Then, setting $\tau^{\cdot} = 1 - F_{\underline{\Phi}_1}(x_1)|_{[\tau^{\cdot}=(1-\tau)\in(0,1)]}$ and $u^{\cdot} = 1 - F_{\underline{\Phi}_2}(x_2)|_{[u^{\cdot}=(1-u)\in(0,1)]}$, we can esaily get the get the joint CDF of the TIQL using the FGM family

$$C_\varsigma(\tau,u) = W_{\underline{\Phi}_1}(\tau)W_{\underline{\Phi}_2}(u)\, exp\left[\overline{W}_{\underline{\Phi}_1}(\tau) + \overline{W}_{\underline{\Phi}_2}(u)\right]$$
$$\times\left[1+\varsigma\begin{pmatrix}\left\{1-W_{\underline{\Phi}_1}(\tau)\, exp\left[\overline{W}_{\underline{\Phi}_1}(\tau)\right]\right\}\\ \left\{1-W_{\underline{\Phi}_2}(u)\, exp\left[\overline{W}_{\underline{\Phi}_2}(u)\right]\right\}\end{pmatrix}\right].$$

The joint PDF can then be derived from $c_\varsigma(\tau,u) = 1 + \ \varsigma\ \tau^{\cdot}u^{\cdot}|_{(\tau^{\cdot}=1-2\tau \text{ and } u^{\cdot}=1-2u)}$ or from $c_\varsigma(\tau,u) = f(x_1,x_2) = C(F_1,F_2)f_1f_2$.

3.2 Via modified FGM family

The modified FGM copula is defined as $C_\varsigma(\tau,u) = \tau u[1+\varsigma B(\tau)A(u)]|_{\varsigma\in(-1,1)}$ or $C_\varsigma(\tau,u) = \tau u + \varsigma\boldsymbol{Q}(\tau)\boldsymbol{Q}(u)|_{\varsigma\in(-1,1)}$, where $\boldsymbol{Q}(\tau) = \tau B(\tau)$, and $\boldsymbol{Q}(u) = uA(u)$ and $B(\tau)$ and $A(u)$ are two continuous functions on $(0,1)$ with $B(0) = B(1) = A(0) = A(1) = 0$.

Type I: Consider the following functional form for both $\boldsymbol{B(\tau)}$ and $\boldsymbol{A(u)}$. Then, the B-TIQL-FGM (Type **I**) can be derived from

$$C_\varsigma(\tau,u) = W_{\underline{\Phi}_1}(\tau)W_{\underline{\Phi}_2}(u)\,exp\left[\overline{W}_{\underline{\Phi}_1}(\tau)+\overline{W}_{\underline{\Phi}_2}(u)\right] +\varsigma\begin{pmatrix} W_{\underline{\Phi}_1}(\tau)\,exp\left[\overline{W}_{\underline{\Phi}_1}(\tau)\right]\left\{1-W_{\underline{\Phi}_1}(\tau)\,exp\left[\overline{W}_{\underline{\Phi}_1}(\tau)\right]\right\} \\ W_{\underline{\Phi}_2}(u)\,exp\left[\overline{W}_{\underline{\Phi}_2}(u)\right]\left\{1-W_{\underline{\Phi}_2}(u)\,exp\left[\overline{W}_{\underline{\Phi}_2}(u)\right]\right\}\end{pmatrix}|_{\varsigma\in(-1,1)}.$$

Type II: Let $\boldsymbol{B(\tau)}$ and $\boldsymbol{A(u)}$ be two functional form satisfying all the conditions stated earlier where $\boldsymbol{B(\tau)}^{\cdot}|_{(\varsigma_1>0)} = \boldsymbol{\tau^{\varsigma_1}(1-\tau)^{1-\varsigma_1}}$ and $\boldsymbol{A(u)}^{\cdot}|_{(\varsigma_2>0)} = \boldsymbol{u^{\varsigma_2}(1-u)^{1-\varsigma_2}}$. Then, the corresponding B-TIQL-FGM **(**Type **II)** can be derived from $\boldsymbol{C_{\varsigma,\varsigma_1,\varsigma_2}(\tau,u) = \tau u[1+\varsigma B(\tau)^{\cdot}\,A(u)^{\cdot}]}$. Thus

$$C_{\varsigma,\varsigma_1,\varsigma_2}(\tau,u) = W_{\underline{\Phi}_1}(\tau)W_{\underline{\Phi}_2}(u)\,exp\left[\overline{W}_{\underline{\Phi}_1}(\tau)+\overline{W}_{\underline{\Phi}_2}(u)\right] \times\left[1+\varsigma\begin{pmatrix}\left\{W_{\underline{\Phi}_1}(\tau)\,exp\left[\overline{W}_{\underline{\Phi}_1}(\tau)\right]\right\}^{\varsigma_1} \\ \left\{W_{\underline{\Phi}_2}(u)\,exp\left[\overline{W}_{\underline{\Phi}_2}(u)\right]\right\}^{\varsigma_2} \\ \left(1-W_{\underline{\Phi}_1}(\tau)\,exp\left[\overline{W}_{\underline{\Phi}_1}(\tau)\right]\right)^{1-\varsigma_1} \\ \left(1-W_{\underline{\Phi}_2}(\tau)\,exp\left[\overline{W}_{\underline{\Phi}_2}(u)\right]\right)^{1-\varsigma_2}\end{pmatrix}\right]$$

Type III: Let $\boldsymbol{\mathcal{W}(\tau)}^{\cdot} = \boldsymbol{\tau\log(1+\tau^{\cdot})}$ and $\boldsymbol{\mathcal{W}(u)}^{\cdot} = \boldsymbol{u\log(1+u^{\cdot})}$ for all $\boldsymbol{B(\tau)}$ and $\boldsymbol{A(u)}$ which satisfies all the conditions stated earlier. In this case, one can also derive a closed form expression for the associated CDF of the B-TIQL-FGM (Type **III**) from $\boldsymbol{C_\varsigma(\tau,u) = \tau u[1+\varsigma\mathcal{W}(\tau)^{\cdot}\mathcal{W}(u)^{\cdot}]}$. Then

$$C_\varsigma(\tau,u) = W_{\underline{\Phi}_1}(\tau)W_{\underline{\Phi}_2}(u)\,exp\left[\overline{W}_{\underline{\Phi}_1}(\tau)+\overline{W}_{\underline{\Phi}_2}(u)\right] \times\left[1+\varsigma\begin{pmatrix} W_{\underline{\Phi}_1}(\tau)\,exp\left[\overline{W}_{\underline{\Phi}_1}(\tau)\right] \\ W_{\underline{\Phi}_2}(u)\,exp\left[\overline{W}_{\underline{\Phi}_2}(u)\right] \\ \left[\log\left(2-W_{\underline{\Phi}_1}(\tau)\,exp\left[\overline{W}_{\underline{\Phi}_1}(\tau)\right]\right)\right] \\ \left[\log\left(2-W_{\underline{\Phi}_2}(u)\,exp\left[\overline{W}_{\underline{\Phi}_2}(u)\right]\right)\right]\end{pmatrix}\right].$$

3.3 Via Clayton copula

The Clayton copula can be considered as $C(u_1,u_2) = [(1/u_1)^\varsigma + (1/u_2)^\varsigma - 1]^{-\varsigma^{-1}}|_{\varsigma\in(0,\infty)}$. Setting $u_1 = F_{\underline{\Phi}_1}(\tau)$ and $u_2 = F_{\underline{\Phi}_2}(x)$, the B-TIQL type can be derived from $C(u_1,u_2) = C(F_{\underline{\Phi}_1}(u_1), F_{\underline{\Phi}_1}(u_2))$. Then

$$C(u_1,u_2) = \left\{W_{\underline{\Phi}_1}(u_1)^{-\varsigma}\,exp\left[-\varsigma\overline{W}_{\underline{\Phi}_1}(u_1)\right] + W_{\underline{\Phi}_2}(u)^{-\varsigma}\,exp\left[-\varsigma\overline{W}_{\underline{\Phi}_2}(u)\right] - 1\right\}^{-\varsigma^{-1}}|_{\varsigma\in(0,\infty)}$$

Similarly, the M-TIQL can be derived from

$$C(u_i) = \left(\sum_{i=1}^{d} u_i^{-\varsigma} + 1 - d\right)^{-\varsigma^{-1}}.$$

3.4 Via Renyi's entropy

Using the theorem of Pougaza and Djafari (2011) where $C(\tau,u) = x_2\tau + x_1u - x_1x_2$, the associated B-TIQL can be derived from

$$C(\tau,u) = x_2W_{\underline{\Phi}_1}(x_1)\,exp\left[\overline{W}_{\underline{\Phi}_1}(x_1)\right] + x_1W_{\underline{\Phi}_2}(x_2)\,exp\left[\overline{W}_{\underline{\Phi}_2}(x_2)\right] - x_1x_2.$$

4. Characterizations of the TIQL Distribution

To understand the behavior of the data obtained through a given process, we need to be able to describe this behavior via its approximate probability law. This, however, requires establishing conditions which govern the required probability law. In other words, we need to have certain conditions under which we may be able to recover the probability law of the data. So, characterization of a distribution is important in applied sciences, where an investigator is vitally interested to find out if their model follows the selected distribution. Therefore, the investigator relies on conditions under which their model would follow a specified distribution. A probability distribution can be characterized in different directions one of which is based on the truncated moments. This type of characterization initiated by Galambos and Kotz (1978) and followed by other authors such as Kotz and Shanbhag (1980), Glänzel et al. (1984), Glänzel (1987), Glänzel and Hamedani (2001) and Kim and Jeon (2013), to name a few. For example, Kim and Jeon (2013) proposed a credibility theory based on the truncation of the loss data to estimate conditional mean loss for a given risk function. It should also be mentioned that characterization results are mathematically challenging and elegant. In this section, we present three characterizations of the TIQL distribution based on: (i) conditional expectation (truncated moment) of certain function of a random variable; (ii) the reversed hazard function and (iii) in terms of the conditional expectation of a function of a random variable.

4.1 Characterizations based on two truncated moments

This subsection deals with the characterizations of TIQL distribution in terms of a simple relationship between two truncated moments. We will employ Theorem 1 of Glänzel (1987) given in the Appendix A. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 4.1.1. Let X is be a continuous random variable and let

$$\boldsymbol{Q}_1(x) = exp\left[W_{\underline{\boldsymbol{\Phi}}}(x) - 1\right]|_{x\in R},$$

and

$$\boldsymbol{Q}_2(x) = \boldsymbol{Q}_1(x)\left[W_{\underline{\boldsymbol{\Phi}}}(x) - 1\right]|_{x\in R},$$

Then X has PDF (2) if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x) = \frac{2}{3}\left[W_{\underline{\boldsymbol{\Phi}}}(x) - 1\right]|_{x\in R}$$

Proof. If X has PDF (2), then

$$\left[1 - F_{\underline{\boldsymbol{\Phi}}}(x)\right]E[\boldsymbol{Q}_1(X) \mid X \geq x] = \frac{1}{2}\left[W_{\underline{\boldsymbol{\Phi}}}(x) - 1\right]^2|_{x\in R},$$

and

$$\left[1 - F_{\underline{\boldsymbol{\Phi}}}(x)\right]E[\boldsymbol{Q}_2(X) \mid X \geq x] = \frac{1}{3}\left[W_{\underline{\boldsymbol{\Phi}}}(x) - 1\right]^3|_{x\in R},$$

and hence

$$\xi(x) = \frac{2}{3}\left[W_{\underline{\boldsymbol{\Phi}}}(x) - 1\right]|_{x\in R}.$$

We also have

$$\xi(x)\boldsymbol{Q}_1(x) - \boldsymbol{Q}_2(x) = \frac{1}{3}\boldsymbol{Q}_1(x)\overline{W}_{\underline{\boldsymbol{\Phi}}}(x) < 0|_{x\in R}.$$

Conversely, if $\xi(x)$ is of the above form, then

$$s'(x) = \frac{\xi'(x)\boldsymbol{Q}_1(x)}{\xi(x)\boldsymbol{Q}_1(x) - \boldsymbol{Q}_2(x)} = -4\alpha\frac{g_{\underline{\Psi}}(x)\left(2 - G_{\underline{\Psi}}(x)\right)^{\alpha-1}}{G_{\underline{\Psi}}(x)^{\alpha+1}\overline{W}_{\underline{\boldsymbol{\Phi}}}(x)}|_{x\in R},$$

and

$$s(x) = -2\log\overline{W}_{\underline{\boldsymbol{\Phi}}}(x)|_{x\in R}.$$

Now, according to Theorem 1, X has density (2) .

Corollary 4.1.1. Suppose X is a continuous random variable. Let $\boldsymbol{Q}_1(x)$ be as in Proposition 4.1.1. Then X has density (2) if and only if there exist functions $\boldsymbol{Q}_2(x)$ and ξ defined in Theorem 1 for which the following first order differential equation holds

$$\frac{\xi'(x)\boldsymbol{Q}_1(x)}{\xi(x)\boldsymbol{Q}_1(x) - \boldsymbol{Q}_2(x)} = -4\alpha\frac{g_{\underline{\Psi}}(x)\left[2 - G_{\underline{\Psi}}(x)\right]^{\alpha-1}}{G_{\underline{\Psi}}(x)^{\alpha+1}\overline{W}_{\underline{\boldsymbol{\Phi}}}(x)}|_{x\in R}.$$

Corollary 4.1.2. The differential equation in Corollary 4.1.1 has the following general solution

$$\xi(x) = \overline{W}_{\underline{\boldsymbol{\Phi}}}(x)^{-1}\left[\int \frac{4\alpha g_{\underline{\Psi}}(x)\left(2 - G_{\underline{\Psi}}(x)\right)^{\alpha-1}}{G_{\underline{\Psi}}(x)^{\alpha+1}}[\boldsymbol{Q}_1(x)]^{-1}\boldsymbol{Q}_2(x) + D\right],$$

where D is a constant. A set of functions satisfying the above differential equation is given in Proposition 4.1.1 with $D = 0$. Clearly, there are other triplets $\left(\boldsymbol{Q}_1(x), \boldsymbol{Q}_2(x), \xi(x)\right)$ satisfying the conditions of Theorem 1.

4.2 Characterization based on reverse hazard function

The reverse hazard function, $r_{F_{\underline{\boldsymbol{\Phi}}}}$, of a twice differentiable distribution function, F , is defined as

$$r_{F_{\underline{\Phi}}}(x)=\frac{f_{\underline{\Phi}}(x)}{F_{\underline{\Phi}}(x)},\quad x\in \text{support of } F.$$

In this subsection we present a characterizations of the TIQL which is not of the above trivial form.

Proposition 4.2.1. Suppose X is a continuous random variable. Then, X has density (2) if and only if its hazard function $r_F(x)$ satisfies the following first order differential equation

$$r'_{F_{\underline{\Phi}}}(x)-\frac{g'_{\underline{\Psi}}(x)}{g_{\underline{\Psi}}(x)}r_{F_{\underline{\Phi}}}(x)=2\alpha g_{\underline{\Psi}}(x)\frac{d}{dx}\left\{\frac{\left[2-G_{\underline{\Psi}}(x)\right]^{\alpha}-G_{\underline{\Psi}}(x)^{\alpha}}{G_{\underline{\Psi}}(x)^{\alpha+1}\left[2-G_{\underline{\Psi}}(x)\right]}\right\}|_{x\in R}.$$

Proof. Is straightforward and hence omitted.

4.3 Characterizations based on the Conditional Expectation of a Function of the Random Variable

Hamedani (2013) established the following proposition which can be used to characterize the TIQL distribution.

Proposition 4.3.1. Suppose $X:\Omega\to(a,b)$ is a continuous random variable with CDF F. If $\psi(x)$ is a differentiable function on (a,b) with $lim_{x\to b^-}\psi(x)=1$. Then, for $\delta\neq 1$,

$$E[\psi(X)\mid X\leq x]=\delta\psi(x)|_{x\in(a,b)},$$

implies that

$$\psi(x)=\left[F_{\underline{\Phi}}(x)\right]^{\frac{1}{\delta}-1}|_{x\in(a,b)}.$$

Remark 4.3.1. Let $(a,b)=R,\psi(x)=\left(\frac{2-G_{\underline{\Psi}}(x)}{G_{\underline{\Psi}}(x)}\right)exp\left[\frac{1}{\alpha}-\frac{1}{\alpha}W_{\underline{\Phi}}(x)\right]$ and $\delta=\frac{\alpha}{\alpha+1}$, then Proposition 4.3.1 presents a characterization of TIQL distribution. Clearly, there are other suitable functions than the one we employed for simplicity.

5. Different methods of estimation

In this section, five different estimation methods have been derived to estimate the parameters of the TIQL distribution. The details are given below.

5.1 Maximum likelihood estimation

In this subsection, we derive estimations of the parameters α and $\underline{\boldsymbol{\Phi}}$ via method of the maximum likelihood (ML) estimation. Let $X_1,X_2,\ldots,X_n$ be a random sample from the TIQL distribution with observed values $x_1,x_2,\ldots,x_n$. Then, the log-likelihood function is given by

$$\ell(\underline{\boldsymbol{\Phi}})=n\log 2+n\log\alpha+\sum_{i=1}^{n}\log g_{\underline{\boldsymbol{\Psi}}}(x_i)-(\alpha+1)\sum_{i=1}^{n}\log G_{\underline{\boldsymbol{\Psi}}}(x_i)$$
$$+(\alpha-1)\sum_{i=1}^{n}\log\left[2-G_{\underline{\boldsymbol{\Psi}}}(x_i)\right]+\sum_{i=1}^{n}\log\left[W_{\underline{\boldsymbol{\Phi}}}(x)-1\right]+n-\sum_{i=1}^{n}W_{\underline{\boldsymbol{\Phi}}}(x_i).$$

Then, the ML estimates (MLEs) of α and $\underline{\boldsymbol{\Psi}}$, say $\hat{\alpha}$ and $\underline{\boldsymbol{\Psi}}$, are obtained by maximizing $\ell(\boldsymbol{\Phi})$ with respect to $\underline{\boldsymbol{\Phi}}$. Mathematically, this is equivalent to solve the following non-linear equation with respect to the parameters:

$$\frac{\partial}{\partial\alpha}\ell(\underline{\boldsymbol{\Phi}})=0 \text{ and } \frac{\partial}{\partial\underline{\boldsymbol{\Psi}}}\ell(\underline{\boldsymbol{\Phi}})=0.$$

Hence, the numerical methods are needed to obtain the MLEs. Under mild regularity conditions, one can use the multivariate normal distribution with mean $\mu=(\alpha,\underline{\boldsymbol{\Psi}})$ and covariance matrix I^{-1}, where I denotes the following $(p+1)\times(p+1)$ observed information matrix of real numbers to construct confidence intervals or likelihood ratio test on the parameters. The components of I can be requested from the authors when it is needed.

5.2 Maximum product spacing estimation

The maximum product spacing (MPS) method has been introduced by Cheng and Amin (1979). It is based on the idea that differences (spacings) between the values of the CDF at consecutive data points should be identically distributed. Let $X_{(1)},X_{(2)},\ldots,X_{(n)}$ be the ordered statistics from the TIQL distribution with sample size n, and $x_{(1)},x_{(2)},\ldots,x_{(n)}$ be the ordered observed values. Then, we define the MPS function by

$$MPS(\underline{\boldsymbol{\Phi}}) = \frac{1}{n+1}\sum_{i=1}^{n+1} \log\left[F(x_{(i)},\underline{\boldsymbol{\Phi}}) - F(x_{(i-1)},\underline{\boldsymbol{\Phi}})\right],$$

where $F(x,\underline{\boldsymbol{\Phi}}) = F_{\underline{\boldsymbol{\Phi}}}(x)$. The MPS estimates (MPSEs), say $\hat{\alpha}_{MPS}$ and $\underline{\boldsymbol{\Psi}}_{MPS}$, can be obtained by minimizing $MPS(\underline{\boldsymbol{\Phi}})$ with respect to $\underline{\boldsymbol{\Phi}}$. They are also given as the simultaneous solution of the following non-linear equations:

$$\frac{\partial MPS(\underline{\boldsymbol{\Phi}})}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1} \left[\frac{F'_{\alpha}(x_{(i)},\underline{\boldsymbol{\Phi}}) - F'_{\alpha}(x_{(i-1)},\underline{\boldsymbol{\Phi}})}{F(x_{(i)},\underline{\boldsymbol{\Phi}}) - F(x_{(i-1)},\underline{\boldsymbol{\Phi}})}\right] = 0$$

and

$$\frac{\partial MPS(\underline{\boldsymbol{\Phi}})}{\partial \underline{\boldsymbol{\Psi}}} = \frac{1}{n+1}\sum_{i=1}^{n+1} \left[\frac{F'_{\underline{\boldsymbol{\Psi}}}(x_{(i)},\underline{\boldsymbol{\Phi}}) - F'_{\underline{\boldsymbol{\Psi}}}(x_{(i-1)},\underline{\boldsymbol{\Phi}})}{F(x_{(i)},\underline{\boldsymbol{\Phi}}) - F(x_{(i-1)},\underline{\boldsymbol{\Phi}})}\right] = 0,$$

where $F'_{\alpha}(x,\underline{\boldsymbol{\Phi}}) = \frac{\partial}{\partial \alpha}F(x,\underline{\boldsymbol{\Phi}})$ and $F'_{\underline{\boldsymbol{\Psi}}}(x,\underline{\boldsymbol{\Phi}}) = \frac{\partial}{\partial \underline{\boldsymbol{\Psi}}}F(x,\underline{\boldsymbol{\Phi}})$.

5.3 Least squares estimation

The least square estimates (LSEs) $\hat{\alpha}_{LSE}$ and $\underline{\boldsymbol{\Psi}}_{LSE}$ of α and $\underline{\boldsymbol{\Psi}}$, respectively, are obtained by minimizing the following function:

$$LSE(\underline{\boldsymbol{\Phi}}) = \sum_{i=1}^{n} \left(F(x_{(i)},\underline{\boldsymbol{\Phi}}) - E\left[F(X_{(i)},\underline{\boldsymbol{\Phi}})\right]\right)^2,$$

with respect to $\underline{\boldsymbol{\Phi}}$, where $E\left[F(X_{(i)},\underline{\boldsymbol{\Phi}})\right] = i/(n+1)$ for $i = 1,2,\ldots,n$. Then, $\hat{\alpha}_{LSE}$ and $\underline{\boldsymbol{\Psi}}_{LSE}$ are solutions of the following equations:

$$\frac{\partial LSE(\underline{\boldsymbol{\Phi}})}{\partial \alpha} = 2\sum_{i=1}^{n} F'_{\alpha}(x_{(i)},\underline{\boldsymbol{\Phi}})\left(F(x_{(i)},\underline{\boldsymbol{\Phi}}) - \frac{i}{n+1}\right) = 0,$$

and

$$\frac{\partial LSE(\underline{\boldsymbol{\Phi}})}{\partial \underline{\boldsymbol{\Psi}}} = 2\sum_{i=1}^{n} F'_{\underline{\boldsymbol{\Psi}}}(x_{(i)},\underline{\boldsymbol{\Phi}})\left(F\left(x_{(i)},\underline{\boldsymbol{\Phi}}\right) - \frac{i}{n+1}\right) = 0,$$

respectively, where $F'_{\alpha}(x_{(i)},\underline{\boldsymbol{\Phi}})$ and $F'_{\underline{\boldsymbol{\Psi}}}(x_{(i)},\underline{\boldsymbol{\Phi}})$ are mentioned before.

5.4 Anderson-Darling estimation

The Anderson-Darling minimum distance estimates (ADEs) $\hat{\alpha}_{AD}$ and $\underline{\boldsymbol{\Psi}}_{AD}$ of α and $\underline{\boldsymbol{\Psi}}$, respectively, are obtained by minimizing the following function:

$$AD\left(\underline{\boldsymbol{\Phi}}\right) = -n - \sum_{i=1}^{n} \frac{2i-1}{n}\left[\log F\left(x_{(i)},\underline{\boldsymbol{\Phi}}\right) + \log\left\{1 - F(x_{(n+1-i)},\underline{\boldsymbol{\Phi}})\right\}\right],$$

with respect to $\underline{\boldsymbol{\Phi}}$. Therefore, $\hat{\alpha}_{AD}$ and $\hat{\beta}_{AD}$ can be obtained as the solutions of the following system of equations:

$$\frac{\partial AD\left(\underline{\boldsymbol{\Phi}}\right)}{\partial \alpha} = -\sum_{i=1}^{n} \frac{2i-1}{n}\left[\frac{F'_{\alpha}\left(x_{(i)},\underline{\boldsymbol{\Phi}}\right)}{F\left(x_{(i)}\underline{\boldsymbol{\Phi}}\right)} - \frac{F'_{\alpha}\left(x_{(n+1-i)},\underline{\boldsymbol{\Phi}}\right)}{1 - F\left(x_{(n+1-i)},\underline{\boldsymbol{\Phi}}\right)}\right] = 0$$

and

$$\frac{\partial AD\left(\underline{\boldsymbol{\Phi}}\right)}{\partial \underline{\boldsymbol{\Psi}}} = -\sum_{i=1}^{n} \frac{2i-1}{n}\left[\frac{F'_{\underline{\boldsymbol{\Psi}}}(x_{(i)},\underline{\boldsymbol{\Phi}})}{F(x_{(i)},\underline{\boldsymbol{\Phi}})} - \frac{F'_{\underline{\boldsymbol{\Psi}}}(x_{(n+1-i)},\underline{\boldsymbol{\Phi}})}{1 - F(x_{(n+1-i)},\alpha,\beta)}\right] = 0.$$

5.5 The Cramer-von Mises estimation

The Cramer-von Mises minimum distance estimates (CVMEs) $\hat{\alpha}_{CVM}$ and $\underline{\boldsymbol{\Psi}}_{CVM}$ of α and β , respectively, are obtained by minimizing the following function:

$$CVM\left(\underline{\boldsymbol{\Phi}}\right) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x_{(i)},\underline{\boldsymbol{\Phi}}) - \frac{2i-1}{2n}\right]^2,$$

with respect to $\underline{\boldsymbol{\Phi}}$. Therefore, the estimates $\hat{\alpha}_{CVM}$ and $\underline{\boldsymbol{\Psi}}_{CVM}$ can be obtained as the solution of the following system of equations:

$$\frac{\partial CVM(\underline{\boldsymbol{\Phi}})}{\partial \alpha} = 2\sum_{i=1}^{n} \left(F(x_{(i)},\underline{\boldsymbol{\Phi}}) - \frac{2i-1}{2n}\right) F'_{\alpha}(x_{(i)},\underline{\boldsymbol{\Phi}}) = 0$$

and

$$\frac{\partial CVM(\underline{\boldsymbol{\Phi}})}{\partial \underline{\boldsymbol{\Psi}}} = 2\sum_{i=1}^{n} \left(F\left(x_{(i)},\underline{\boldsymbol{\Phi}}\right) - \frac{2i-1}{2n}\right) F'_{\underline{\boldsymbol{\Psi}}}\left(x_{(i)},\underline{\boldsymbol{\Phi}}\right) = 0.$$

We note that it may be seen \cite{chen1995general} for the information about the AD and CVM goodness-of-fits statistics. Since all estimating equations except those of the MLE method contain non-linear functions, it is not possible to obtain explicit forms of all estimators directly. Therefore, they have to be solved by using numerical methods such as the Newton-Raphson and quasi-Newton algorithms. In addition, the Equations (13), (14), (15) and (16) can be also optimized directly by using the software such as R (constrOptim, optim and maxLik functions), S-Plus and Mathematica to numerically optimize $\ell(\underline{\boldsymbol{\Phi}})$ and $MPS(\underline{\boldsymbol{\Phi}})$, LSE $(\underline{\boldsymbol{\Phi}})$, $AD(\underline{\boldsymbol{\Phi}})$ and $CVM(\underline{\boldsymbol{\Phi}})$ functions.

6. Simulations for comparing methods

In this section, we perform a graphical simulation study to see the performance of the above estimates of the special member of the new family with respect to varying sample size n. We take Topp-Leone (TL) distribution (Topp and Leone (1955)) as baseline model. Hence, the CDF of the extended TL distribution, called by TIQLTL distribution, is given by

$$F_{\alpha,\beta}(x) = \left[\frac{2 - x^{\beta}(2-x)^{\beta}}{x^{\beta}(2-x)^{\beta}}\right]^{\alpha} exp\left\{1 - \left[\frac{2 - x^{\beta}(2-x)^{\beta}}{x^{\beta}(2-x)^{\beta}}\right]^{\alpha}\right\},$$

where, $0 < x < 1$, $\alpha, \beta > 0$. We generate $N = 1000$ samples of size $n = 20,30,\ldots,500$ from the TIQLTL distribution based on the actual parameter values. We take them as the $\alpha = 0.1$, $\beta = 0.25$ for simulation study. The random numbers generation is obtained by the solution of the its CDF via uniroot function in R software as well as all the estimations based on the estimation methods have been obtained by using the optim function in the same software. Further, we calculate the empirical mean, bias and mean square error (MSE) of the estimations for comparisons between estimation methods. For $\varepsilon = \alpha$ and β, the bias and MSE are calculated by

$$Bias_{\varepsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\varepsilon_i - \hat{\varepsilon}_i), MSE_{\varepsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\varepsilon_i - \hat{\varepsilon}_i)^2,$$

respectively. We expect that the empirical means are close to true values when the MSEs and biases are near zero. The results of this simulation study are shown in the Figure 1

Figure 1 shows that all estimates are consistent since the MSE and biasedness decrease to zero with increasing sample size as expected. All estimates are asymptotic unbiased also. According to the simulation study, the empirical biases and MSEs are closing each other on the increasing sample size. Generally, the performances of all estimates are close. Therefore, all methods can be chosen as more reliable than another estimate of the newly defined model. The similar results can be also obtained for different parameter settings.

7. Modeling data for comparing competitive models

In this section, a real data set is analyzed to prove the empirical importance and modeling ability of a special member of the I-QL family. The used data set consist of the times between successive failures (in thousands of hours) in events of secondary reactor pumps studied by Salman et al. (1999), Bebbington et al. (2007) and Lucena et al. (2015).

The data are: 2.160, 0.746, 0.402, 0.954, 0.491, 6.560, 4.992, 0.347,0.150, 0.358, 0.101, 1.359, 3.465, 1.060, 0.614, 1.921,4.082 ,0.199 ,0.605 ,0.273 ,0.070, 0.062, 5.320. Using Weibull (W) baseline model, we will explore the data modeling ability of the TIQLW distribution on above data set. Corresponding pdf of the TIQLW distribution is given by

$$f_{\alpha,\beta,\theta}(x) = 2\alpha\beta\theta^{\beta} \frac{x^{\beta-1}\, exp\left[-(\theta x)^{\beta}\right]\left\{1 + exp\left[-(\theta x)^{\beta}\right]\right\}^{\alpha-1}}{\{1 - exp[-(\theta x)^{\beta}]\}^{\alpha+1}}$$
$$\times\left(\left\{\frac{1 + exp\left[-(\theta x)^{\beta}\right]}{1 - exp\left[-(\theta x)^{\beta}\right]}\right\}^{\alpha} - 1\right) exp\left(1 - \left\{\frac{1 + exp\left[-(\theta x)^{\beta}\right]}{1 - exp\left[-(\theta x)^{\beta}\right]}\right\}^{\alpha}\right),$$

where, $0 < x$ and $\alpha, \beta, \theta > 0$. We compare performance of the real data fitting of the the TIQLW distribution under the MLE method with well know unit distribution in the literature. These competitor distributions are:

- Modified Weibull (MW) distribution (Lai et al. (2003)):

$$f_{\alpha,\beta,\theta}(x) = \theta(\alpha + \beta x)x^{\alpha-1} exp\left(\beta x - \theta x^{\alpha} e^{\beta x}\right),$$

where, $0 < x$ and $\alpha, \beta, \theta > 0$.

- Beta Weibull (BW) distribution (Famoye et al. (2005)):

$$f_{\alpha,\beta,\theta,\gamma}(x) = \beta\theta^{\beta} \frac{x^{\beta-1} exp\left[-(\theta x)^{\beta}\right]}{B(\alpha,\gamma)} \left\{1 - exp\left[-(\theta x)^{\beta}\right]\right\}^{\alpha-1} exp\left[-\gamma(\theta x)^{\beta}\right],$$

where, $0 < x$ and $\alpha, \beta, \theta, \gamma > 0$ and $B(\alpha, \gamma)$ is the beta function.

- Odd log-logistic Weibull (OLLW) distribution (Gleaton and Lynch (2006)):

$$f_{\alpha,\beta,\theta}(x) = \beta\theta^{\beta} \frac{x^{\beta-1} exp\left[-\alpha(\theta x)^{\beta}\right] \left\{1 - exp\left[-(\theta x)^{\beta}\right]\right\}^{\alpha-1}}{(\{1 - exp[-(\theta x)^{\beta}]\}^{\alpha} + exp[-(\theta x)^{\beta}])^{2}},$$

where, $0 < x$ and $\alpha, \beta, \theta > 0$.

- Kumaraswamy Weibull (KwW) distribution (Cordeiro et al. (2010)):

$$f_{\alpha,\beta,\theta,\gamma}(x) = \alpha\gamma\beta\theta^{\beta} x^{\beta-1} exp\left[-(\theta x)^{\beta}\right] \left\{1 - exp\left[-(\theta x)^{\beta}\right]\right\}^{\alpha-1} \left(1 - \left\{1 - exp\left[-(\theta x)^{\beta}\right]\right\}^{\alpha}\right)^{\gamma-1}.$$

where, $0 < x$ and $\alpha, \beta, \theta, \gamma > 0$. The $\hat{\ell}$ values, Akaike Information Criteria (AIC), Bayesian information criterion (BIC), Kolmogorov-Smirnov (KS), Cramer-von-Mises, (W^*) and Anderson-Darling (A^*) goodness of-fit statistics have been obtained based on all distribution models to determine the optimum model In general, it can be chosen as the optimum model the one which has the smaller the values of the AIC, BIC, KS, W^* and A^* statistics and the larger the values of $\hat{\ell}$ and p-value of the goodness-of-statistics.

Firstly, we fit the Weibull distribution, which has the CDF $F_{\beta,\theta}(x) = 1 - exp\left[-(\theta x)^{\beta}\right]$ for $0 < x$ and $\beta, \theta > 0$,to this data set. For this model, we obtained the $\hat{\ell}$ value and KS statistics as -32.5139 and 0.1184 (with p-value=0.8667) respectively. We give the data analysis results belong to other competitor models in Table 1. Table 1 shows that the TIQLW distribution has the has the biggest $\hat{\ell}$ value as well as it has the lowest values of the AIC, BIC, A^* and W^* statistic among application models. The BW distribution has the lowest A^* and KS statistics with the biggest p-value. However, the TIQLW model has fever parameters than the BW model. This is the advantage of the TIQLW model. It implies that the TIQLW model will be the best choice for the modeled data set.

Figure 2 displays the fitted pdfs and CDFs for all models. It is clear that proposed TIQLW model captures shapes of the data set graphically and its CDF fitting is sufficient. Figure 3 shows that the plotted lines of the probability-probability (PP) is very closer the diagonal line which indicates that the performance of the TIQLW distribution is acceptable for the modeled data.

Table 1: MLEs, standard errors of the estimates (in parentheses), $\hat{\ell}$ and goodness-of-fits statistics for the data set (p-value is given in [.])

Model	Estimates				Goodness-of-fits statistics					
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\gamma}$	$\hat{\ell}$	AIC	BIC	A^*	W^*	KS [p-value]
TIQLW	0.1645	2.0450	0.1043		-30.984	67.9681	71.3746	0.2470	0.0275	0.1023
	(0.2505)	(3.1421)	(0.0621)							[0.9494]
MW	0.7922	0.0093	0.7517		-32.508	71.0165	74.4230	0.4160	0.0643	0.1199
	(0.1925)	(0.0850)	(0.2199)							[[0.8565]
OLLW	1.3160	0.6362	0.6855		-32.475	70.9504	74.3569	0.3412	0.0462	0.1028
	(1.5414)	(0.6725)	(0.3050)							[0.9479]
BW	30.2595	0.0027	0.1353	55.8427	-31.8016	71.6033	76.1453	0.2406	0.0286	0.0981
	(1.4132)	(0.0026)	(0.0202)	(1.4365)						[0.9641]
Kw-W	5.4565	0.6823	25.8738	0.1077	-31.487	70.9740	75.5160	0.4428	0.0626	0.1111
	(0.0001)	(0.0003)	(0.0006)	(0.0224)						[[0.9096]

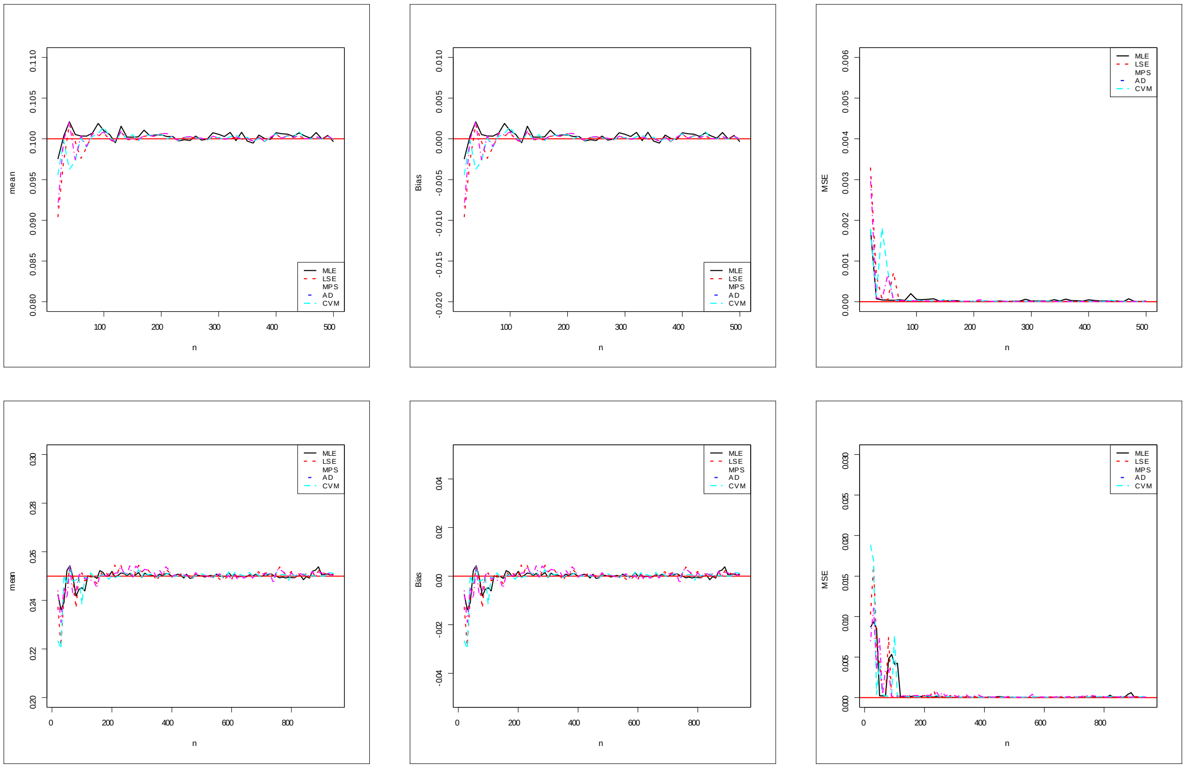

Figure 1: The results of on the parameters α (top) and β (bottom) for the simulation study.

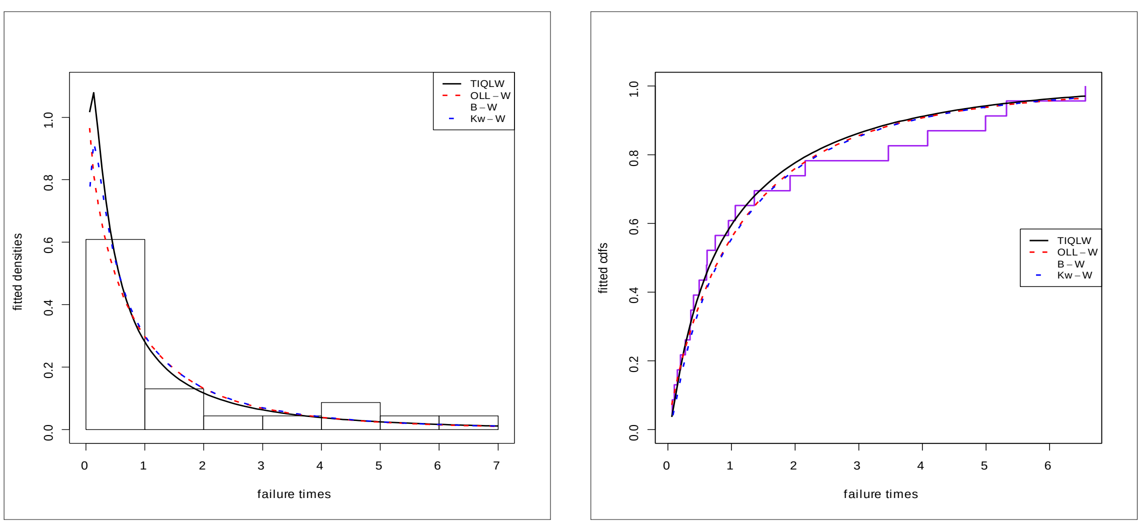

Figure 2: PP plots for the fitted models based on the data set.

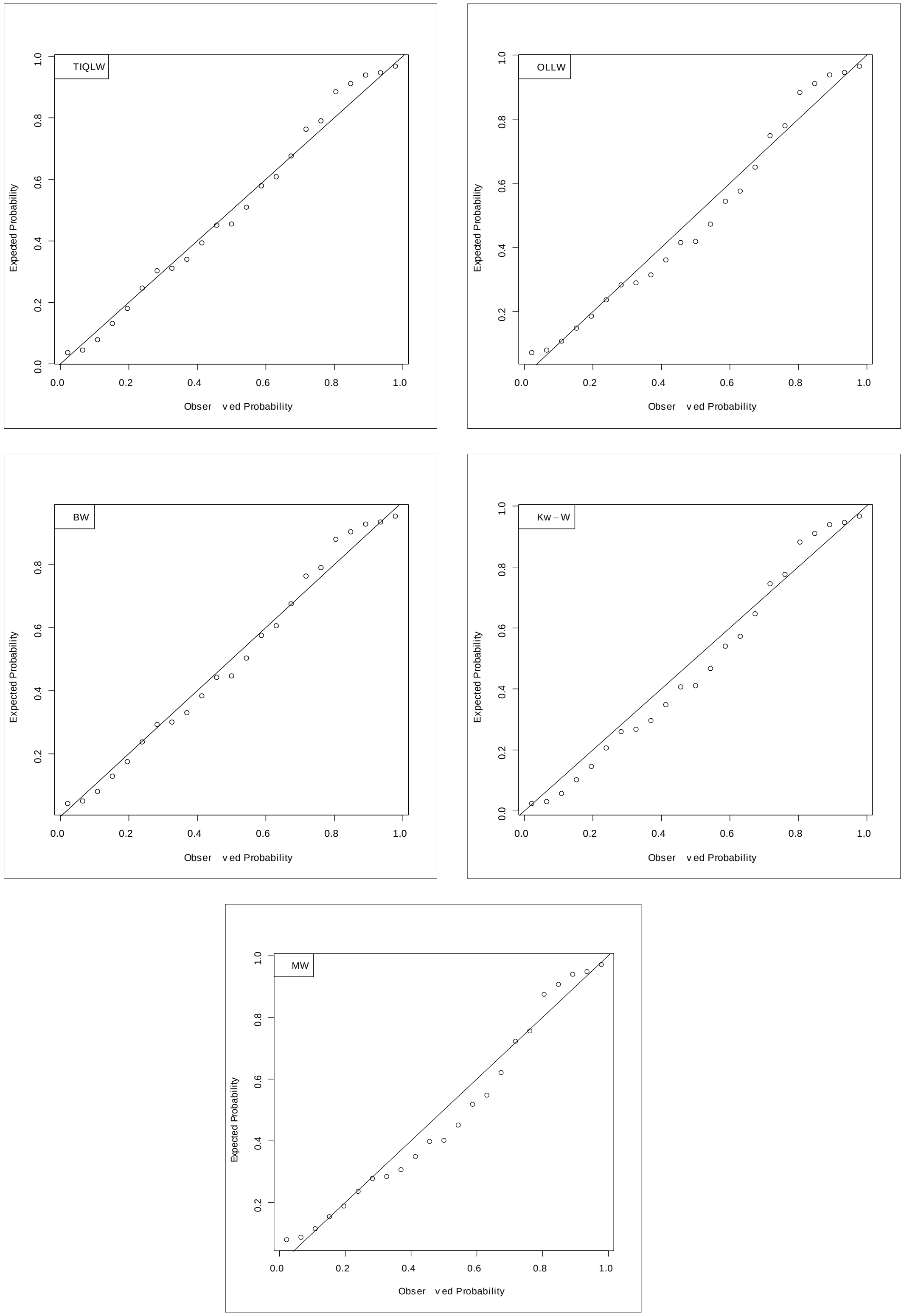

Figure 3: PP plots for the fitted models based on the data set.

8. Concluding remarks

A new G family of continuous probability distributions called the type I quasi Lambert family is defined and studied. Some new bivariate type G families using "copula of Farlie-Gumbel-Morgenstern", "modified Farlie-Gumbel-Morgenstern copula", "Clayton copula" and "Renyi's entropy copula" are derived. Three characterizations based on conditional expectation (truncated moment) of certain function of a random variable; the reversed hazard function and in terms of the conditional expectation of a function of a random variable are presented. Some of its statistical properties including moments, incomplete moments and moment generating functions are derived and studied. The maximum likelihood estimation, maximum product spacing estimation, least squares estimation, Anderson-Darling estimation and Cramer-von Mises estimation methods are used for estimating the unknown parameters. A graphical assessment under the five different estimation methods is introduced. Graphical assessments under five different estimation methods are introduced. Based on these assessments, all estimation methods perform well. Finally, an application to illustrate the importance and flexibility of the new family is proposed.

References

1. Alizadeh, M., Ghosh, I., Yousof, H. M., Rasekhi, M., & Hamedani, G. G. (2017). The generalized odd generalized exponential family of distributions: properties, characterizations and applications. Journal of Data Science, 15(3), 443-465.
2. Alizadeh, M., Rasekhi, M., Yousof, H. M. and Hamedani G. G. (2018). The transmuted Weibull G family of distributions. Hacettepe Journal of Mathematics and Statistics, 47, 1-20.
3. Aryal, G. R., & Yousof, H. M. (2017). The exponentiated generalized-G Poisson family of distributions. Stochastics and Quality Control, 32(1), 7-23.
4. Bebbington, M., Lai, C. D. and Zitikis, R. (2007). A flexible Weibull extension. Reliability Engineering & System Safety, 92(6), 719-726.
5. Brito, E., Cordeiro, G. M., Yousof, H. M., Alizadeh, M. & Silva, G. O. (2017). Topp-Leone odd log-logistic family of distributions, Journal of Statistical Computation and Simulation, 87(15), 3040-3058.
6. Cheng, R. C. H. and Amin, N. A. K. (1979). Maximum product of spacings estimation with application to the lognormal distribution. Math Report, 791.
7. Cordeiro, G. M., Ortega, E. M.and Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. Journal of the Franklin Institute, 347(8), 1399-1429.
8. Cordeiro, G. M., Yousof, H. M., Ramires, T. G. & Ortega, E. M. M. (2017b). The Burr XII system of densities: properties, regression model and applications. Journal of Statistical Computation and Simulation, 88(3), 432-456.
9. Famoye, F., Lee, C. and Olumolade, O. (2005). The beta-Weibull distribution. Journal of Statistical Theory and Applications, 4(2), 121-136.
10. Farlie, D. J. G. (1960) The performance of some correlation coefficients for a general bivariate distribution. Biometrika, 47, 307-323.
11. Fréchet, M. (1927). Sur la loi de probabilité de lécart maximum. Ann. de la Soc. polonaisede Math, 6, 93-116.
12. Galambos, J. and Kotz, S. Characterizations of probability distributions. A unified approach with emphasis on exponential and related models, Lecture Notes in Mathematics, p.675. Springer, Berlin (1978).
13. Glänzel, W., A characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, (1987), 75-84.
14. Glänzel, W., Some consequences of a characterization theorem based on truncated moments, *Statistics: A* Journal of Theoretical and Applied Statistics, 21(4), (1990), 613-618.
15. Glänzel, W, Telcs, A, Schubert, A. Characterization by truncated moments and its application to Pearson-type distributions, Z. Wahrsch. Verw. Gebiete 66, 173-182 (1984).
16. Glänzel, W. and Hamedani, G.G. Characterizations of the univariate distributions, Studia Scien. Math. hung., 37, 83-118 (2001).
17. Gleaton, J. U. and Lynch, J. D. (2006). Properties of generalized loglogistic families of lifetime distributions. Journal of Probability and Statistical Science, 4(1), 51-64.
18. Gumbel, E. J. (1961). Bivariate logistic distributions. Journal of the American Statistical Association, 56(294), 335-349.
19. Gumbel, E. J. (1960) Bivariate exponential distributions. Journ. Amer. Statist. Assoc., 55, 698-707.
20. Gupta, R. C., Gupta, P. L. and Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. Communications in Statistics-Theory and methods, 27(4), 887-904.
21. Hamedani, G.G., On certain generalized gamma convolution distributions II, Technical Report, No. 484,

MSCS, Marquette University, 2013.

22. Hamedani, G. G., Altun, E., Korkmaz, M. C., Yousof, H. M., & Butt, N. S. (2018). A new extended G family of continuous distributions with mathematical properties, characterizations and regression modeling. Pakistan Journal of Statistics and Operation Research, 737-758.
23. Hamedani, G. G., Rasekhi, M., Najibi, S., Yousof, H. M., & Alizadeh, M. (2019). Type II general exponential class of distributions. Pakistan Journal of Statistics and Operation Research, 503-523.
24. Hamedani G. G. Yousof, H. M., Rasekhi, M., Alizadeh, M. & Najibi, S. M. (2017). Type I general exponential class of distributions. Pak. J. Stat. Oper. Res., XIV(1), 39-55.
25. Johnson, N. L. and Kotz, S. (1975) On some generalized Farlie-Gumbel-Morgenstern distributions. Commun. Stat. Theory, 4, 415-427.
26. Johnson, N. L. and Kotz, S. (1977) On some generalized Farlie-Gumbel-Morgenstern distributions- II: Regression, correlation and further generalizations. Commun. Stat.Theory, 6, 485-496.
27. Kim, J.H. and Jeon, Y. Credibility theory based on trimming, Insur. Math. Econ. 53(1), 36-47 (2013).
28. Karamikabir, H., Afshari, M., Yousof, H. M., Alizadeh, M. and Hamedani, G. (2020). The Weibull Topp-Leone generated family of gistributions: statistical properties and applications. Journal of The Iranian Statistical Society, 19(1), 121-161.
29. Korkmaz, M. C., Alizadeh, M., Yousof, H. M. and Butt, N. S. (2018a). The generalized odd Weibull generated family of distributions: statistical properties and applications. Pak. J. Stat. Oper. Res., 14(3), 541-556.
30. Korkmaz, M. C., Altun, E., Yousof, H. M. and Hamedani G. G. (2019). The odd power Lindley generator of probability distributions: properties, characterizations and regression modeling, International Journal of Statistics and Probability, 8(2), 70-89.
31. Korkmaz, M. C., Yousof, H. M. and Hamedani, G. G. (2018b). The exponential Lindley odd log-logistic-G family: properties, characterizations and applications. Journal of Statistical Theory and Applications, 17(3), 554-571.
32. Korkmaz, M. C., Yousof, H. M., Hamedani G. G. and Ali, M. M. (2018c). The Marshall-Olkin generalized-G Poisson family of distributions, Pakistan Journal of Statistics, 34(3), 251-267.
33. Lai, C. D., Xie, M. and Murthy, D. N. P. (2003). A modified Weibull distribution. IEEE Transactions on reliability, 52(1), 33-37.
34. Lee, C., Famoye, F. and Olumolade, O. (2007). Beta-Weibull distribution: some properties and applications to censored data. Journal of Modern Applied Statistical Methods, 6, 17.
35. Lucena, S. E., Silva, A. H. A. and Cordeiro, G. M. (2015). The transmuted generalized gamma distribution: Properties and application. Journal of Data Science, 13(1), 187-206.
36. Merovci, F., Alizadeh, M., Yousof, H. M. and Hamedani G. G. (2017). The exponentiated transmuted-G family of distributions: theory and applications, Communications in Statistics-Theory and Methods, 46 (21), 10800-10822.
37. Morgenstern, D. (1956). Einfache beispiele zweidimensionaler verteilungen. Mitteilingsblatt fur Mathematische Statistik, 8, 234-235.
38. Nascimento, A. D. C., Silva, K. F., Cordeiro, G. M., Alizadeh, M. and Yousof, H. M. (2019). The odd Nadarajah-Haghighi family of distributions: properties and applications. Studia Scientiarum Mathematicarum Hungarica, 56(2), 1-26.
39. Pougaza, D. B. and Djafari, M. A. (2011). Maximum entropies copulas. Proceedings of the 30th international workshop on Bayesian inference and maximum Entropy methods in Science and Engineering, 329-336.
40. Rezaei, S., Nadarajah, S., and Tahghighnia, N. A (2013). New three-parameter lifetime distribution, Statistics, 47, 835-860.
41. Rodriguez-Lallena, J. A. and Ubeda-Flores, M. (2004). A new class of bivariate copulas. Statistics and Probability Letters, 66, 315-25. Salman S., M. and Prayoto, S. (1999). Total time on test plot analysis for mechanical components of the RSG-GAS reactor. Atom Indones, 25(2), 155-61.
42. Topp, C. W. and Leone, F. C. (1955). A family of J-shaped frequency functions. Journal of the American Statistical Association, 50(269), 209-219.
43. Yousof, H. M., Afify, A. Z., Alizadeh, M., Butt, N. S., Hamedani, G. G. and Ali, M. M. (2015). The transmuted exponentiated generalized-G family of distributions. Pak.j.stat.oper.res., 11, 441-464.
44. Yousof, H. M., Afify, A. Z., Alizadeh, M., Hamedani G. G., Jahanshahi, S. M. A. and Ghosh, I. (2018a). The generalized transmuted Poisson-G family of Distributions. Pak. J. Stat. Oper. Res., 14 (4), 759-779.
45. Yousof, H. M., Afify, A. Z., Alizadeh, M., Nadarajah, S., Aryal, G. R. and Hamedani, G. G. (2018b). The Marshall-Olkin generalized-G family of distributions with Applications, STATISTICA, 78(3), 273- 295.
46. Yousof, H. M., Alizadeh, M., Jahanshahi and, S. M. A., Ramires, T. G., Ghosh, I. and Hamedani G. G. (2017a).

The transmuted Topp-Leone G family of distributions: theory, characterizations and applications, Journal of Data Science. 15, 723-740.

47. Yousof, H. M., Altun, E., Ramires, T. G., Alizadeh, M. and Rasekhi, M. (2018c). A new family of distributions with properties, regression models and applications, Journal of Statistics and Management Systems, 21(1), 163-188.
48. Yousof, H. M., Rasekhi, M., Afify, A. Z., Alizadeh, M., Ghosh, I. and Hamedani G. G. (2017b). The beta Weibull-G family of distributions: theory, characterizations and applications, Pakistan Journal of Statistics, 33, 95-116.
49. Yousof, H. M., Majumder, M., Jahanshahi, S. M. A., Ali, M. M. and Hamedani G. G. (2018d). A new Weibull class of distributions: theory, characterizations and applications, Journal of Statistical Research of Iran, 15, 45-83.
50. Yousof, H., Mansoor, M., Alizadeh, M., Afify, A. and Ghosh, I. (2020). The Weibull-G Poisson family for analyzing lifetime data. Pakistan Journal of Statistics and Operation Research, 131-148.

Appendix A

Theorem 1. Let (Ω, F, P) be a given probability space and let $H=[a,b]$ be an interval for some $d<b$ $(a=-\infty, b=\infty$ might as well be allowed). Let $X:\Omega\rightarrow H$ be a continuous random variable with the distribution function F and let $\boldsymbol{Q}_1(x)$ and $\boldsymbol{Q}_2(x)$ be two real functions defined on H such that

$$E[\boldsymbol{Q}_2(X) \mid X\geq x]=E[\boldsymbol{Q}_1(X) \mid X\geq x]\xi(x), \quad x\in H,$$

is defined with some real function $\xi(x)$. Assume that $\boldsymbol{Q}_1(x), \boldsymbol{Q}_2(x)\in C^1(H)$, $\xi(x)\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi(x)\boldsymbol{Q}_1(x)=\boldsymbol{Q}_2(x)$ has no real solution in the interior of H. Then F is uniquely determined by the functions $\boldsymbol{Q}_1(x), \boldsymbol{Q}_2(x)$ and $\xi(x)$, particularly

$$F_{\underline{\boldsymbol{\Phi}}}(x)=\int_a^x C\left|\frac{\xi'(u)}{\xi(u)\boldsymbol{Q}_1(u)-\boldsymbol{Q}_2(u)}\right| exp\big(-s(u)\big)\ du\,,$$

where the function s is a solution of the differential equation $s'=\frac{\xi'\,\mathcal{Q}_1}{\xi\mathcal{Q}_1-\mathcal{Q}_2}$ and C is the normalization constant, such that $\int_H \ dF=1$.

Note: The goal is to have the function $\xi(x)$ as simple as possible.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel, 1990), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions $\boldsymbol{Q}_{1n}(x)$, $\boldsymbol{Q}_{2n}(x)$ and $\xi_n(x)$ $(n\in N)$ satisfy the conditions of Theorem 1 and let $\boldsymbol{Q}_{1n}(x)\rightarrow\boldsymbol{Q}_1(x), \boldsymbol{Q}_{2n}(x)\rightarrow\boldsymbol{Q}_2(x)$ for some continuously differentiable real functions $\boldsymbol{Q}_1(x)$ and $\boldsymbol{Q}_2(x)$. Let, finally, X be a random variable with distribution F. Under the condition that $\boldsymbol{Q}_{1n}(X)$ and $\boldsymbol{Q}_{2n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if $\xi_n(x)$ converges to $\xi(x)$, where

$$\xi(x)=\frac{E[\boldsymbol{Q}_2(X) \mid X\geq x]}{E[\boldsymbol{Q}_1(X) \mid X\geq x]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions $\boldsymbol{Q}_1(x)$, $\boldsymbol{Q}_2(x)$ and $\xi(x)$, respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha\rightarrow\infty$. A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions $\boldsymbol{Q}_1(x)$, $\boldsymbol{Q}_2(x)$ and, specially, $\xi(x)$ should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose $\xi(x)$ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics. In some cases, one can take $\boldsymbol{Q}_1(x)\equiv 1$, which reduces the condition of Theorem 1 to $E[\boldsymbol{Q}_2(X) \mid X\geq x]=\xi(x), \quad x\in H$. We, however, believe that employing three functions $\boldsymbol{Q}_1(x)$, $\boldsymbol{Q}_2(x)$ and $\xi(x)$ will enhance the domain of applicability of Theorem 1.