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A Novel Accelerated Failure Time Model: Characterizations, Validation Testing, Different Estimation Methods and Applications in Engineering and Medicine

Haitham M. Yousof[1,*], Hafida Goual[2], Meribout Kaouter Khaoula[2], G.G. Hamedani[3], Abdullah H. Al-Aefaie[4], Mohamed Ibrahim[4,5], Nadeem Shafique Butt[6], Moustafa Salem[7]

* Corresponding Author

[1] Department of Statistics, Mathematics and Insurance, Benha University, Egypt, haitham.yousof@fcom.bu.edu.eg
[2] Laboratory of Probability and Statistics, University of Badji Mokhtar, Annaba, Algeria, hafida.goual@univ-annaba.dz, kaouter.meribout@gmail.com
[3] Department of Mathematical and Statistical Sciences, Marquette University, USA, gholamhoss.hamedani@marquette.edu
[4] Quantitative Methods Department, School of Business, King Faisal University, Saudi Arabia, aalnefaie@kfu.edu.sa
[5] Department of Applied, Mathematical and Actuarial Statistics, Damietta University, Damietta, Egypt, mohamed_ibrahim@du.edu.eg
[6] Department of Family and Community Medicine, King Abdul Aziz University, Jeddah, Kingdom of Saudi Arabia, nshafique@kau.edu.sa
[7] Department of Applied Statistics, Damanhour University, Damanhour, Egypt, moustafasalemstat@com.dmu.edu.eg

Abstract

In this paper we present a new exponential accelerated failure time model. Some of its properties and characterization results are derived. Different estimation methods are considered for assessing the finite sample behavior of the estimators. Simulation studies for comparing the estimation methods are performed. Finally, we present a novel modified chi-square test for the novel exponential accelerated failure time model in both complete and right censored data cases. The validity of the new model is checked by using the theoretical global of the Nikulin-Rao-Robson. The maximum likelihood method is considered for this purpose. Two simulation studies are performed for assessing the exponential accelerated failure time model and the efficiency of the Nikulin-Rao-Robson test statistic, respectively. Three real data sets are considered for illustrating the efficiency of the test statistic in validation.

Key Words: Accelerated Failure Time; Characterization; Nikulin-Rao-Robson; Simulation; Validation.

Mathematical Subject Classification: 62E10, 60K10, 60N05

1 Introduction

An appropriate parametric model is often of interest for analyzing survival data since it provides an overview of the failure times characteristics and the risk functions. However, when failure rates of the products or death or remission of patients or any other diseases can have different causes, simple parametric models cannot measure the influence of each cause. In this case, accelerated failure time (AFT) models were proposed in the statistical literature, where the stresses (explanatory variable, temperature, pressure, dose of medicine...etc) represented by covariates affect directly the functions of interest of the model such as the failure rate and survival functions. The AFT models are primarily fully parametric, in contrast to proportional hazards models, where Cox's semi-parametric proportional hazards model is more frequently used than parametric models. Also, The regression parameter estimates from AFT models are resistant to omitted covariates, unlike proportional hazards models. Additionally, they are less impacted by the probability distribution of choice. Depending on the values affected to the covariables, by increasing or decreasing them, engineers and practitioners can achieve the desired results, this is why the AFT models are widely used in reliability studies and survival analysis. The objective of this theory is to know the influence of the stresses (covariates) on the life duration of the items. Based on classical distributions called baseline, several AFT models are studied such as the exponential, Weibull, log-logistic and log-normal AFT models (Bagdonavicius and Nikulin (2002), Lawless (2003), Bagdonavicius et al. (2010)), the generalized inverse Weibull AFT model (Goual and Seddik-Ameur (2014). Bagdonavicius and Nikulin (2011), Bagdonavicius et al. (2011), gave chi-squared goodness-of-fit tests for regression models such as accelerated failure time, proportional hazards, generalized proportional hazards, frailty models, models with cross-effects of survival functions.

The exponential distribution is likely the statistical model that is used the most frequently across a variety of fields among the parametric distributions. Its significance is due in part to the exponential model's constant failure rate function. Furthermore, this model was the first lifetime model for which extensive statistical tools were created in the literature on life testing. In a random process where events happen at a predetermined pace, the waiting period before the first occurrence is distributed using an exponential function. It is a relatively simple distribution; a random variable having this distribution is necessarily positive, and it is one of the more important distributions among those used for positive random variables. The cumulative distribution function (CDF) of the exponential distribution can be written as $G_{\lambda}(x) = 1 - exp(-\lambda x)$ where $\lambda > 0$ and $x \geq 0$, the moments, the moment generating function (MGF) and several other properties of this distribution can be expressed in terms of the elementary functions. In the last decANDEs, many new distributions are developed by adding one or more parameters to classical distributions in order to and more flexibility to these distributions.

The most popular AFT model is provided by the log-logistic distribution. It can display a non-monotonic hazard function that rises early and falls later, unlike the Weibull distribution. Although it has heavier tails, it has a form that is relatively comparable to the log-normal distribution. When fitting data with censoring, the log-logistic cumulative distribution function's straightforward closed form plays a crucial computational role. The survival function, which is the complement of the cumulative distribution function, is required for the censored observations. It is unique among distribution families that the Weibull distribution (which includes the exponential distribution as a special example) can be parameterized as either an AFT model or a proportional hazards model. There are two ways to interpret the outcomes of fitting a Weibull model. This model's biological application, however, might be constrained by the danger function's monotonicity, that is, its ability to be either decreasing or growing. The log-normal, gamma, and inverse Gaussian distributions are additional distributions appropriate for AFT models; however, they are less common than the log-logistic distribution, in part because their cumulative distribution functions do not have a closed form. The Weibull, log-normal, and gamma distributions are special examples of the generalised gamma distribution, a three-parameter distribution.

In this work, we introduce an exponential model dubbed the Burr-Hatke exponential (BHE) distribution and investigate its mathematical features in the manner of Yousof et al. (2018). The novel model simply has two parameters and can be written as linear combinations of the well-known exponentiated exponential density. Its probability distribution function (PDF) also has a straightforward shape. The asymptotics results can be used to assess how the two parameters affect the BHE distribution's tails. The novel PDF, CDF, and hazard rate function (HRF) asymptotics results are obtained, correspondingly. Using two truncated moments, the HRF, and the conditional expectation of a random variable-based function, various descriptions of the BHE distribution are provided.

The finite sample behaviour of the estimators is evaluated using a variety of estimation techniques, such as the maximum likelihood, Cramer-von-Mises, Anderson Darling, right tail-Anderson Darling, left tail-Anderson Darling, and method of L-moments. Simulated studies are carried out to compare the estimation techniques. Various sample sizes and parameter values are used to accomplish the simulation experiments. The bias, root mean-standard errors, the mean of the absolute difference between the theoretical (MADv) and the estimates and the maximum absolute difference between the true parameters and estimates (MaxADv) are all taken into consideration when comparing the estimates. Then, we propose a new Burr-Hatke exponential accelerated failure time (BHE-AFT) model as a parametric accelerated life model when the baseline survival function belongs to BHE model. The new BHE-AFT model can be used in reliability modeling and life time testing in many applied fields such as electric insulating, medicine and life time studies. For assessing the estimates of the BHE-AFT model and depending on using Barzilai-Borwein (BZB) algorithm, the averages of the simulated values of the maximum likelihood estimators (MXLEs) and their corresponding mean squared errors are reported under different sample sizes.

The BHE-AFT model is tested using a novel modified chi-square test in both the complete and right censored data situations. The theoretical framework of Nikulin-Rao-Robson (NIKRR) statistics is used to assess the viability of the BHE-AFT model (see Nikulin (1973a,b,c) and Rao and Robson (1974)). In several validation procedures, the NIKRR test statistic has recently been enhanced (see, for instance, Goual and Yousof (2019), Goual et al. (2019), Goual et al. (2020), Yadav et al. (2020), and Yadav et al. (2022)). The modified NIKRR test statistic for the BHE-AFT model is evaluated using the maximum likelihood approach at a few empirical levels and equivalent theoretical levels. In order to evaluate the effectiveness of the NIKRR test statistic in validation, three real data sets are also taken into account. For more detials, applications and real-life data, see Salah et al. (2020), Mansour et al. (2020a,b), Ibrahim et al. (2019, 2020, 2021), Aidi et al. (2021), Emam et al. (2023), Khalil et al. (2023), Yousof et al. (2021, 2023a,b).

2 The Burr Hatke exponential model

2.1 Formulation

Based on the Burr-Hatke differential equation, Yousof et al. (2018) presented a new family called the BH-G family. According to Yousof et al. (2018), the CDF of the BHE distribution can be derived as

$$F_{\theta,\lambda}(x) = 1 - \frac{1}{1+\lambda x} exp(-\lambda\theta x). \tag{1}$$

The PDF corresponding to (1) is given by

$$f_{\theta,\lambda}(x) = \lambda(1+\lambda x)^{-2}[\theta(1+\lambda x)+1]exp(-\lambda\theta x)\,. \tag{2}$$

The HRF of the BHE model can be expressed as

$$h_{\theta,\lambda}(x) = \frac{\lambda}{1+\lambda x}[\theta(1+\lambda x)+1]. \tag{3}$$

Mixture representations for Equations (2) and (3) are obtained. Consider the following expansions,

$$\left(1-\frac{\zeta_1}{\zeta_2}\right)^{\zeta_3} = \sum_{\zeta_4=0}^{\infty} (-1)^{\zeta_4}\binom{\zeta_3}{\zeta_4}\left(\frac{\zeta_1}{\zeta_2}\right)^{\zeta_3}, |\frac{\zeta_1}{\zeta_2}| < 1, \tag{4}$$

and

$$\log\left(1-\frac{\zeta_1}{\zeta_2}\right) = -\sum_{\zeta_4=0}^{\infty}\frac{1}{1+\zeta_4}\left(\frac{\zeta_1}{\zeta_2}\right)^{1+\zeta_4}, |\frac{\zeta_1}{\zeta_2}| < 1. \tag{5}$$

Firstly, the CDF (2) can be rewriten as

$$F_{\theta,\lambda}(x) = 1 - \frac{A_{\theta,\lambda}(x)}{B_\lambda(x)},$$

where $A_{\theta,\lambda}(x) = \{1-[1-exp(-\lambda x)]\}^\theta$ and $B_\lambda(x) = 1-\log\{1-[1-exp(-\lambda x)]\}$. Applying (4) to $A_{\theta,\lambda}(x)$. Then,

$$A_{\theta,\lambda}(x) = \sum_{\mathcal{k}=0}^{\infty} a_{\mathcal{k}}[1-exp(-\lambda x)]^{\mathcal{k}},$$

where $a_{\mathcal{k}} = (-1)^{\mathcal{k}}\binom{\theta}{\mathcal{k}}$.

Now, applying (5) to $B_\lambda(x)$, still in Equation (2), we obtain

$$B_\lambda(x) = 1 + \sum_{\imath=0}^{\infty}\frac{1}{\imath+1}[1-exp(-\lambda x)]^{\imath+1}.$$

Then,

$$B_\lambda(x) = \sum_{\mathcal{k}=0}^{\infty} b_{\mathcal{k}}[1-exp(-\lambda x)]^{\mathcal{k}},$$

where $b_0 = 1$, $\mathcal{k} \geq 1$ and $b_{\mathcal{k}} = \frac{-1}{\mathcal{k}}$. Then, Equation (2) can be writen as

$$F(x;\theta,\lambda) = 1 - \frac{\sum_{\mathcal{k}=0}^{\infty} a_{\mathcal{k}}[1-exp(-\lambda x)]^{\mathcal{k}}}{\sum_{\mathcal{k}=0}^{\infty} b_{\mathcal{k}}[1-exp(-\lambda x)]^{\mathcal{k}}},$$

then,

$$F(x;\theta,\lambda)=1-\sum_{k=0}^{\infty} c_k[1-exp(-\lambda x)]^{k}$$

where $c_0=\frac{a_o}{b_0}$ and, for $k\geq 1$, we have

$$c_k=\frac{1}{b_0}\left(a_k-\frac{1}{b_0}\sum_{r=1}^{k} b_r c_{k-r}\right).$$

At the end, the CDF (2) can be writen as

$$F_{\theta,\lambda}(x)=\sum_{k=0}^{\infty} d_k \mathbf{\Pi}_{1+k}(x;\lambda), \tag{6}$$

where $d_0=1-c_k$, for $k\geq 1$ we have $d_0=-c_k$ and $\mathbf{\Pi}_{1+k}(x;\lambda)=[1-exp(-\lambda x)]^{1+k}$ is the CDF of the exponentiated exponential model with power parameter $1+k$. By differentiating (6), we obtain the same mixture representation

$$f_{\theta,\lambda}(x)=\sum_{k=0}^{\infty} d_k \boldsymbol{\pi}_{1+k}(x;\lambda), \tag{7}$$

where $\boldsymbol{\pi}_{\varsigma}(x)=(1+k)\lambda exp(-\lambda x)[1-exp(-\lambda x)]^{k}$ is the PDF of the exponentiated exponential with power parameter $(\boldsymbol{\varsigma})$. Equation (7) demonstrates that the exponentiated exponential densities are combined linearly to form the BHE density function. As a result, it is possible to derive some structural characteristics of the new model, including the generating function, ordinary and incomplete moments, and exp-E distribution, right away. Many authors have recently explored the exponentiated exponential distribution's properties.

2.2 Properties

Let $a=\inf\{x|F_{\theta,\lambda}(x)>0\}$, the asymptotics of CDF, PDF and HRF as $x\to a$ are given by

$$F_{\theta,\lambda}(x)\sim 1-exp(-\lambda x)|_{x\to a}\,,$$
$$f_{\theta,\lambda}(x)\sim \lambda exp(-\lambda x)|_{x\to a},$$

and

$$h_{\theta,\lambda}(x)\sim \lambda exp(-\lambda x)|_{x\to a}.$$

The asymptotics of CDF, PDF and HRF as $x\to\infty$ are given by

$$1-F_{\theta,\lambda}(x)\sim \frac{1}{\lambda x}exp(-\theta\lambda x)|_{x\to\infty},$$
$$f_{\theta,\lambda}(x)\sim \frac{1}{\lambda x^2}exp(-\theta\lambda x)(1-\theta\,\lambda x)|_{x\to\infty},$$

and

$$h_{\theta,\lambda}(x)\sim \frac{1}{x}(1-\theta\,\lambda x)|_{x\to\infty}.$$

The effect of the parameters on tails of distribution can be evaluated by means of the above equations.

Theorem 2.2.1:

Let T be a random variable with the exponentiated exponential distribution with positive parameters λ and $\boldsymbol{\varsigma}$. Then, for any $r>-1$, the r th ordinary and incomplete moments of T are given by

$$\mu'_{r,T}=\sum_{w=0}^{\infty} C_w^{(r,\varsigma)}\,\Gamma(1+r)$$

and

$$\mathbf{I}_{r,T}(t)=\sum_{w=0}^{\infty} C_w^{(r,\varsigma)}\,\gamma\big(1+r,(\lambda t)\big),$$

respectively, where

$$C_w^{(r,\varsigma)}=\boldsymbol{\varsigma}\lambda^{-r}\frac{(-1)^{w}}{(w+1)^{(1+r)}}\binom{\boldsymbol{\varsigma}-1}{w}$$

and $\gamma(\zeta_1,\zeta_2)$ is the incomplete gamma function which can be expressed as

$$\gamma(\zeta_1,\zeta_2)=\int_0^{\zeta_2} exp(-w)dw=\frac{1}{\zeta_1}\zeta_2^{\zeta_1}\{{}_1\mathbf{F}_1[\zeta_1;\zeta_1+1;-\zeta_2]\}=\sum_{\kappa=0}^{\infty}\frac{(-1)^{\kappa}}{\kappa!\,(\zeta_1+\kappa)}\zeta_2^{\zeta_1+\kappa},$$

and ${}_1\mathbf{F}_1[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function. Based on Theorem 1, the r^{th} ordinary moment of X is given by $\mu'_{r,X}=E(X^r)=\int_{-\infty}^{\infty} x^r f(x)dx$. Then, we obtain

$$\mu'_{r,X}=\sum_{k,w=0}^{\infty} C_{k,w}^{(1+k,r)}\Gamma(1+r)|_{r>-1}, \tag{8}$$

where $C_{k,w}^{(1+k,r)}=d_k C_w^{(r,1+k)}$ and

$$C_w^{(r,1+k)}=(1+k)\frac{(-1)^w}{(w+1)^{(1+r)}}\binom{k}{w}$$

The cumulants, central moment, skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Based on Theorem 1, the r th incomplete moment of X, say $\mathbf{I}_{r,X}(t)=\int_{-\infty}^{t} x^r f(x)dx$, can be determined from (7) and (8) as

$$\mathbf{I}_{r,X}(t)=\int_{-\infty}^{t} x^r f(x)dx=\sum_{k,w=0}^{\infty} C_{k,w}^{(1+k,r)}\gamma\big(1+r,(\lambda t)\big)|_{r>-1}. \tag{9}$$

The MGF of X follows from (7) and (8) as

$$M_X(t)=\sum_{k,w,r=0}^{\infty}\frac{t^r}{r!}C_{k,w}^{(1+k,r)}\Gamma(1+r)|_{r>-1}.$$

3 Characterization results

The characterizations of the BHE distribution are covered in this section in the following ways:
First: based on two truncated moments.
Second: in terms of the hazard function.
Third: based on the conditional expectation of a function of the random variable.

The CDF does not necessarily required to have a closed form for the initial characterization. The next subsections will present the different categorizations. Those characterization theorems are considered with more details in Yousof et al. (2021) and Yousof et al. (2022).

3.1 Characterizations based on two truncated moments

Characterizations based on two truncated moments are often used in statistical analysis to describe probability distributions. In this context, a characterization is a set of properties or relationships that uniquely identify a specific probability distribution. Characterizations based on two truncated moments can provide insights into the properties of a distribution, help identify specific distributions, and guide the selection of appropriate models for statistical analysis. In practice, characterizations based on two truncated moments are particularly useful when dealing with data for which the theoretical distribution is not known or when the distribution is suspected to deviate from common parametric models. By using truncated moments, you can gain insights into the distribution of your data and make informed decisions about the appropriate statistical methods to apply. The characterizations of the BHE distribution based on the relationship between two truncated moments are the focus of this subsection. The first characterization makes use of a Glänzel theorem (1987), Theorem 3.1.1 given below. Clearly, the result holds as well when the interval H is not a closed. This characterization is stable in the sense of weak convergence, please see reference Glänzel (1990).

Theorem 3.1.1.
Let $(\Omega,\mathcal{F},\mathbf{P})$ be a given probability space and let $H=[d,e]$ be an interval for some $d<e$ $(d=-\infty,e=\infty$ mightaswellbeallowed). Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$\mathbf{E}[g(X)|X\geq x]=\mathbf{E}[h(X)|X\geq x]\xi(x),\quad x\in H,$$

is defined with some real function ξ. Assume that $g,h\in C^1(H)$, $\xi\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi h=g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and ξ, particularly

$$F(x) = \int_a^x C \left| \frac{1}{\xi(u)h(u)-g(u)} \xi'(u) \right| exp\big(-s(u)\big)du,$$

where the function s is a solution of the differential equation $s' = \frac{\xi' h}{\xi h - g}$ and C is the normalization constant, such that $\int_H dF = 1$.

Proposition 3.1.1.
The random variable $X: \Omega \to (0, \infty)$ is continuous, and assume

$$h(x) = \frac{1}{\theta(1+\lambda x)+1}(1+\lambda x)^2$$

and $g(x) = h(x)exp(-\lambda\theta x)$ for $x > 0$. Then, the density of X is given in (2) if and only if the function ξ defined in Theorem 3.1.1 is

$$\xi(x) = \frac{1}{2}exp(-\lambda\theta x), x > 0.$$

Proof. If X has PDF (2), then

$$\big(1-F(x)\big)E[h(X)|X \geq x] = \frac{1}{\theta}exp(-\lambda\theta x), x > 0,$$

and

$$\big(1-F(x)\big)E[g(X)|X \geq x] = \frac{1}{2\theta}exp(-2\lambda\theta x), x > 0,$$

and finally

$$\xi(x)h(x) - g(x) = -\frac{1}{2}h(x)exp(-\lambda\theta x) < 0, x > 0.$$

Conversely, if ξ has the above form, then

$$s'(x) = \frac{1}{\xi(x)h(x)-g(x)}\xi'(x)h(x) = \lambda\theta,$$

and hence

$$s(x) = \lambda\theta x, \quad x > 0.$$

In view of Theorem 3.1.1, X has PDF (2).

Corollary 3.1.1.
If $X: \Omega \to (0, \infty)$ is a continuous random variable and $h(x)$ is as in Proposition 3.1.1. Then, X has PDF (2) if and only if there exist functions g and ξ defined in Theorem 3.1.1 satisfying the following first order differential equation

$$\frac{1}{\xi(x)h(x)-g(x)}\xi'(x)h(x) = \lambda\theta.$$

Corollary 3.1.2.
The general solution of the above differential equation is

$$\xi(x) = e^{\lambda\theta x}\left[-\int \lambda\theta e^{-\lambda\theta x}\big(h(x)\big)^{-1}g(x) + D\right],$$

where D is a constant. A set of functions satisfying this differential equation is presented in Proposition 3.1.1 with $D = 0$. Clearly, there are other triplets (h, g, ξ) satisfying the conditions of Theorem 3.1.1.

3.2 Characterization based on hazard function

Characterization based on the hazard function, also known as the hazard rate, plays a fundamental role in survival analysis and reliability theory. The hazard function is a key component in understanding and modeling the behavior of time-to-event data. Here are some statistical applications of characterizations based on the hazard function:

I. The primary application of the hazard function is in survival analysis. Survival analysis involves modeling the time until an event of interest occurs, such as failure, death, or event occurrence. The hazard function provides insights into how the event rate changes over time and is a fundamental component of survival models like the Kaplan-Meier estimator, the Cox proportional hazards model, and parametric survival models (e.g., Weibull, exponential, and log-logistic models).

II. In reliability engineering, the hazard function is used to analyze the reliability of products or systems. The hazard function characterizes the failure rate of a system over time, and reliability engineers use it to assess product durability, make predictions about the lifetime of components, and perform maintenance planning.

III. In clinical trials and medical research, the hazard function is employed to compare the effects of different treatments or interventions. The hazard ratio, which quantifies the relative hazard rates of two groups, is often used to assess the impact of a treatment on survival outcomes.

IV. In epidemiology, the hazard function is utilized to study the risk of disease onset, transmission, or other time-to-event outcomes. It helps epidemiologists understand how the risk of an event varies over time and how different factors influence this risk.

V. In economics and decision analysis, the hazard function can be used to model the expected time to the occurrence of certain events that may have economic consequences. This is valuable in cost-benefit analysis and decision-making processes.

VI. In engineering, the hazard function is employed to study and characterize failure modes of systems or components. Engineers use the hazard function to assess the reliability of critical components and to make design decisions.

VII. The hazard function can be used to predict future event occurrences based on historical data. This is valuable in forecasting failures, disease outbreaks, and other time-dependent events.

VIII. In quality control and manufacturing, the hazard function is used to evaluate the time-to-failure of products and to assess the reliability of manufactured items.

IX. Businesses use the hazard function to study customer churn or attrition. By modeling the hazard of customers leaving over time, companies can develop retention strategies and improve customer loyalty.

X. The hazard function is applied in demography and population studies to analyze life events such as birth, marriage, and mortality. It helps researchers understand patterns of event occurrence and their variations over time.

Characterizations based on the hazard function are essential for understanding time-to-event data and making informed decisions in various fields. These characterizations help researchers and practitioners model the risk and reliability of systems, assess the impact of interventions, and make predictions about event occurrences over time. The hazard function, h_F, of a twice differentiable distribution function, F with density f, satisfies the first following trivial first differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only hazard function-based characterization that is currently available. The BHE distribution is described in detail in the Proposition given below.

Proposition 3.2.1.

Suppose $X: \Omega \to (0, \infty)$ is a continuous random variable. The density of X is (2) if and only if the differential equation holds

$$h_{\theta,\lambda}'(x) + \lambda(1+\lambda x)^{-1} h_{\theta,\lambda}(x) = \lambda^2 \theta (1+\lambda x)^{-1}, \qquad x > 0,$$

with the initial condition $\lim_{x\to 0} h_{\theta,\lambda}(x) = \lambda(\theta + 1)$. Proof. Is straightforward and hence omitted.

3.3 Characterizations based on conditional expectation

Characterizations based on conditional expectations are fundamental in statistics and provide a powerful framework for understanding, analyzing, and modeling various aspects of data. Conditional expectation plays a central role in regression analysis. In simple linear regression, the conditional expectation of the response variable given a predictor (covariate) is used to model the relationship between the two. In multiple regression, it extends to modeling the conditional expectation of the response variable given multiple covariates. Various regression techniques, such as linear regression, logistic regression, and Poisson regression, are based on modeling conditional expectations. Conditional expectation is used for making predictions in statistical models. Given a set of predictor variables, you can estimate the conditional expectation of the response variable, which represents the expected value of the response variable for a given set of predictors. This is the basis for prediction in regression and machine learning models. In survival analysis, conditional expectations are used to estimate quantities such as survival probabilities, hazard rates, and median survival times. Conditional expectations of survival times are key to survival curves and survival models like Kaplan-Meier and Cox proportional hazards models. In Bayesian statistics, the posterior distribution is often characterized by conditional expectations. The conditional expectation of a parameter given data is a key element of Bayesian inference. This expectation is updated as more data becomes available, allowing for iterative and adaptive modeling. In experimental design, conditional expectations are used to optimize the allocation of treatments or resources. For example, in response surface methodology, researchers aim to maximize or minimize the conditional expectation of a response variable while varying experimental factors. Characterizations based on conditional expectations provide a flexible and powerful framework for understanding and analyzing data in a wide range of statistical applications. They are fundamental in building statistical models, making predictions, conducting hypothesis tests, and drawing inferences from data. Hamedani (2013) makes the following claim, thus we will make use of it to describe the BHE distribution.

Proposition 3.3.1.
Suppose the random variable $X:\Omega \to (a,b)$ is continuous with CDF F. If $\psi(x)$ is a differentiable function on (a,b) with $\lim_{x\to 0^+}\psi(x) = 1$, then for $\delta \neq 1$,

$$E[\psi(X)|X \geq x] = \delta\psi(x), \quad x \in (a,b),$$

if and only if

$$\psi(x) = \left(1 - F(x)\right)^{\frac{1}{\delta}-1}, \quad x \in (a,b).$$

Remark 3.3.1. Taking

$$(a,b) = (0,\infty), \psi(x) = \frac{1}{(1+\lambda x)^{1/\theta}} exp(-\lambda x)$$

and

$$\delta = \frac{\theta}{\theta+1},$$

Proposition 3.3.1 presents a characterization of BHE distribution. Clearly, there are other possible function.

4 Different estimation methods

4.1 Maximum likelihood method

Maximum likelihood estimation (MXLE) is a widely used method for estimating the parameters of a statistical model. It has several important statistical properties:

I. The MXLE is asymptotically efficient, which means that as the sample size (n) increases, the MXLE becomes the most efficient estimator in terms of achieving the smallest possible variance among all consistent estimators. In simple terms, for large sample sizes, the MXLE produces parameter estimates that are close to the true values and have low variability.

II. MXLE is a consistent estimator, meaning that as the sample size grows, the MXLE converges in probability to the true population parameter. In other words, as you collect more data, the MXLE provides parameter estimates that approach the true values.

III. When the sample size is large, the distribution of the MXLE approximates a normal distribution. This property allows you to construct asymptotic confidence intervals and conduct hypothesis tests using normal theory.

IV. MXLE is invariant to transformations of the parameter space. This means that if you reparameterize the model (e.g., changing from one set of parameters to another through a one-to-one transformation), the MXLE remains consistent and efficient under the new parameterization.

V. MXLE often results in sufficient statistics, which means that the parameter estimates capture all the information about the parameters contained in the data. In other words, the MXLE utilizes the most relevant information in the data for parameter estimation.

VI. In many cases, the MXLE is asymptotically unbiased, meaning that as the sample size becomes large, the expected value of the MXLE approaches the true parameter value. However, MXLE can be biased for small sample sizes or in certain situations.

VII. MXLE can be sensitive to outliers or deviations from the assumed model. In the presence of outliers, the MXLE may provide parameter estimates that are highly influenced by the outliers. To mitigate this, alternative estimators like robust M-estimators or Bayesian methods can be considered.

VIII. In small samples, the efficiency of the MXLE may be compromised, and other estimators, such as method of moments or Bayesian estimators, may perform better in terms of mean squared error. MXLE efficiency tends to manifest in larger samples.

IX. MXLE may still be consistent under model misspecification, meaning it can provide reasonable estimates even when the assumed model is not entirely correct. However, this depends on the extent of the misspecification.

X. MXLE provides parameter estimates that are straightforward to interpret because they maximize the likelihood of the observed data under the assumed model. In many cases, MXLE has well-defined closed-form solutions, making it computationally efficient and easy to implement.

The choice to use MXLE as an estimation method depends on the specific problem, the availability of data, and the appropriateness of the assumed model. While MXLE has numerous desirable statistical properties, it is not universally suitable for all situations, and alternative estimation methods may be more appropriate depending on the context and data characteristics. Let $x_1, x_2, \dots, x_n$ be a RS from this distribution with parameter vector $(\theta, \lambda)^{\intercal}$. The log-likelihood function for (θ, λ), say $\ell(\theta, \lambda)$, is given by

$$\ell(\theta, \lambda) = n\log\lambda - 2n\sum_{i=0}^{n} \log(1 + \lambda x_{i:n}) + \sum_{i=0}^{n} \log[\theta(1 + \lambda x_{i:n}) + 1] - \lambda\theta\sum_{i=0}^{n} \log x_{i:n}$$

which can be maximized either using the statistical programs or by solving the nonlinear system obtained from $\ell(\theta, \lambda)$ by differentiation. The score vector, $\mathbf{U}(\theta, \lambda) = \left(\frac{\partial}{\partial\theta}\ell(\theta, \lambda), \frac{\partial}{\partial\lambda}\ell(\theta, \lambda)\right)^{\intercal}$, are easily derived.

4.2 Cramér-von-Mises method

The Cramér-von-Mises estimation (CVOME) method is consistent, meaning that as the sample size (n) increases, the estimated parameters converge in probability to the true population parameters. In other words, as you collect more data, the estimates become more accurate. When the sample size is sufficiently large, the distribution of the Cramér-von-Mises statistic converges to a normal distribution. This property allows you to construct asymptotic confidence intervals for the estimated parameters. The Cramér-von-Mises estimator is efficient if it achieves the Cramér-Rao lower bound, which is the smallest possible variance for an unbiased estimator. In some cases, the Cramér-von-Mises estimator may be asymptotically efficient, meaning that it attains the smallest possible variance among all consistent estimators. The Cramér-von-Mises estimator can be robust to outliers, meaning that it may still provide reasonable parameter estimates even when the data contains a small number of extreme values or outliers. However, its robustness properties depend on the specific application and the underlying distribution being estimated. The Cramér-von-Mises method does not assume a specific underlying distribution for the data. It is a non-parametric method, which means it can be applied in a wide range of situations where you do not have a prior distributional assumption. Like all estimators, the Cramér-von-Mises estimator may exhibit a bias-variance trade-off. This means that you might need to balance the bias (systematic error) and the variance (random error) of the estimator depending on the sample size and the properties of the data. The choice to use the Cramér-von-Mises estimation method depends on the specific problem and the assumptions you are willing to make about the underlying distribution. It is a valuable tool for non-parametric estimation and goodness-of-fit testing in statistics. The CVOME of the parameters θ and λ are obtained via minimizing the following expression with respect to the parameters θ and λ respectively, where

$$CVM_{(\theta,\lambda)} = \frac{1}{12}n^{-1} + \sum_{i=1}^{n} \left[F_{\theta,\lambda}(x_{i:n}) - \zeta_{(i,n)}\right]^2,$$

where

$$\zeta_{(i,n)} = \frac{2i-1}{2n}$$

and

$$CVM_{(\theta,\lambda)} = \sum_{i=1}^{n} \left(1 - \frac{1}{1 + \lambda x_{i:n}} exp(-\lambda\theta x_{i:n}) - \zeta_{(i,n)}\right)^2.$$

The, CVOME of the parameters θ and λ are obtained by solving the two following non-linear equations

$$\sum_{i=1}^{n}\left(1-\frac{1}{1+\lambda x_{i:n}}exp(-\lambda\theta x_{i:n})-\zeta_{(i,n)}\right)\boldsymbol{\varsigma}_{(\theta)}(x_{i:n},\theta,\lambda)=0,$$

and

$$\sum_{i=1}^{n}\left(1-\frac{1}{1+\lambda x_{i:n}}exp(-\lambda\theta x_{i:n})-\zeta_{(i,n)}\right)\boldsymbol{\varsigma}_{(\lambda)}(x_{i:n},\theta,\lambda)=0,$$

where $\boldsymbol{\varsigma}_{\theta}(x_{i:n},\theta,\lambda)$ and $\boldsymbol{\varsigma}_{\lambda}(x_{i:n},\theta,\lambda)$ are the first derivatives of the CDF of BHE distribution with respect to θ and λ respectively.

4.3 Method of L-moments

The Method of L-Moments is an alternative approach to moments-based estimators, such as the method of moments and maximum likelihood estimation. L-Moments are linear combinations of ordered statistics (usually sample quantiles) and provide several statistical properties that make them useful in various applications, especially in the analysis of heavy-tailed and skewed distributions. Here are some statistical properties of the Method of L-Moments:

I. L-Moments are less sensitive to outliers and heavy tails compared to traditional moments-based estimators. This makes them particularly useful for analyzing data with extreme values or when the underlying distribution is not well-behaved.

II. Like traditional moments, L-Moments are invariant to linear transformations. This means that they provide consistent estimates of the parameters regardless of the units in which the data is measured.

III. L-Moments can be used to estimate traditional moments (mean, variance, skewness, and kurtosis) as well as other distribution characteristics. They are often used for estimating these moments in situations where traditional methods are not applicable.

IV. L-Moments are valuable for characterizing the shape of a distribution. In particular, L-Moments can help identify the tail behavior and asymmetry of a distribution.

V. L-Moment ratios can be used for goodness-of-fit tests to assess how well a particular distribution fits a dataset. This is useful in distribution selection and model diagnostics.

VI. L-Moments are helpful in selecting an appropriate distribution for a dataset by comparing the L-Moment ratios of different distributions with those estimated from the data.

VII. L-Moment ratios can be used to estimate the parameters of specific probability distributions, such as the Pearson Type III, generalized extreme value (GEV), and generalized logistic distributions, among others. This makes the L-Moment method particularly relevant for hydrology, environmental science, and engineering applications.

VIII. L-Moments may be more efficient in the estimation of distribution parameters in certain situations, especially for heavy-tailed distributions, compared to traditional moment-based methods.

IX. L-Moments extend naturally to higher order moments (beyond the fourth moment) and can provide information about the entire distribution of a dataset.

X. L-Moments are commonly used in hydrology and engineering to estimate return periods and design floods, as they provide information about the tail of the distribution, which is critical for extreme value analysis.

XI. L-Moment estimators often have simpler closed-form expressions, making them computationally efficient and less prone to numerical issues than maximum likelihood estimation in some cases.

The Method of L-Moments has been widely adopted in various fields, particularly in areas where heavy-tailed and skewed distributions are prevalent. It offers valuable alternatives to traditional moment-based and maximum likelihood estimation methods, and its robustness and ease of use make it suitable for a range of practical applications. The L-moments for the population can be obtained from

$$\gamma_r=\frac{1}{r}\sum_{m=0}^{r-1}(-1)^m\binom{r-1}{m}\mathbf{E}(x_{r-m:m})|_{(r\geq 1)}.$$

The first four L-moments are given by

$$\gamma_1(\theta,\lambda)=\mathbf{E}(x_{1:1})=\mu_1'=\mathbb{L}_1,$$

and

$$\gamma_2(\theta,\lambda)=\frac{1}{2}\mathbf{E}(x_{2:2}-x_{1:2})=\frac{1}{2}(\mu_{2:2}'-\mu_{1:2}')=\mathbb{L}_2,$$

where $\mathbb{L}_i|_{(i=1.2.3.4)}$ is the L-moments for the sample. Then, the L-moments estimators $\hat{\theta}_{(\mathrm{LME})}$ and $\hat{\lambda}_{(\mathrm{LME})}$ of the parameters θ and λ can be obtained by solving the following four equations numerically

$$\gamma_1\left(\hat{\theta}_{(\mathrm{LME})},\hat{\lambda}_{(\mathrm{LME})}\right)=\mathbb{L}_1,$$

and

$$\gamma_2\left(\hat{\theta}_{(\mathrm{LME})},\hat{\lambda}_{(\mathrm{LME})}\right)=\mathbb{L}_2.$$

4.4 Anderson Darling method

The Anderson Darling estimation (ANDE) of $\hat{\theta}_{(\mathrm{ANDE})}$ and $\hat{\lambda}_{(\mathrm{ANDE})}$ are obtained by minimizing the function

$$\mathrm{ANDE}(\theta,\lambda)=-n-n^{-1}\sum_{i=1}^{n}(2i-1)\begin{Bmatrix}\log F_{(\theta,\lambda)}(x_{i:n})\\+\log\left[1-F_{(\theta,\lambda)}(x_{[-i+1+n:n]})\right]\end{Bmatrix}.$$

The parameter estimates of $\hat{\theta}_{(\mathrm{ANDE})}$ and $\hat{\lambda}_{(\mathrm{ANDE})}$ follow by solving the nonlinear equations

$$\frac{\partial}{\partial\theta}[\mathrm{ANDE}(\theta,\lambda)]=0,$$

and

$$\frac{\partial}{\partial\lambda}[\mathrm{ANDE}(\theta,\lambda)]=0.$$

4.5 Right Tail-Anderson Darling method

The Tail-Anderson Darling estimation (RT-ANDE) of $\hat{\theta}_{(\mathrm{RT-ANDE})}$ and $\hat{\lambda}_{(\mathrm{RT-ANDE})}$ are obtained by minimizing

$$\mathrm{RT-ANDE}(\theta,\lambda)=\frac{1}{2}n-2\sum_{i=1}^{n}F_{(\theta,\lambda)}(x_{i:n})$$
$$-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log\left[1-F_{(\theta,\lambda)}(x_{[-i+1+n:n]})\right]\right\}.$$

The parameter estimates of $\hat{\theta}_{(\mathrm{RT-ANDE})}$ and $\hat{\lambda}_{(\mathrm{RT-ANDE})}$ follow by solving the nonlinear equations

$$\frac{\partial}{\partial\theta}[RTADE(\theta,\lambda)]=0,$$

and

$$\frac{\partial}{\partial\lambda}[RTADE(\theta,\lambda)]=0.$$

4.6 Left Tail-Anderson Darling method

The left Tail-Anderson Darling estimation (LTANDE) of $\hat{\theta}_{(\mathrm{LTANDE})}$ and $\hat{\lambda}_{(\mathrm{LTANDE})}$ are obtained by minimizing

$$LTADE(\theta,\lambda)=-\frac{3}{2}n+2\sum_{i=1}^{n}F_{(\theta,\lambda)}(x_{i:n})-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log F_{(\theta,\lambda)}(x_{i:n}).$$

The parameter estimates of $\hat{\theta}_{(\mathrm{LTANDE})}$ and $\hat{\lambda}_{(\mathrm{LTANDE})}$ follow by solving the nonlinear equations

$$\frac{\partial}{\partial\theta}[LTADE(\theta,\lambda)]=0,$$

and

$$\frac{\partial}{\partial\lambda}[LTADE(\theta,\lambda)]=0.$$

5 Simulation studies for comparing estimation methods

To compare the traditional estimating techniques, a numerical simulation is run. The BHE distribution's N=1000 generated data sets serve as the foundation for the simulation investigation, where $n=50,100,200$ and 300 and

	θ	λ
Table 1	1.5	2.0
Table 2	0.5	0.8
Table 3	0.7	0.7

The estimates are compared in terms of their

1-Bias $\left(\mathrm{BIAS}_{(\theta,\lambda)}\right)$:

$$\mathrm{BIAS}_{(\theta)} = \frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\theta_\imath} - \theta\right)$$

and

$$\mathrm{BIAS}_{(\lambda)} = \frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\lambda_\imath} - \lambda\right),$$

2-Root mean-standard error $\left(\mathrm{RMSE}_{(\theta,\lambda)}\right)$:

$$\mathrm{RMSE}_{(\theta)} = \sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\theta_\imath} - \theta\right)^2}$$

and

$$\mathrm{RMSE}_{(\lambda)} = \sqrt{\frac{1}{B}\sum_{i=1}^{B}\left(\widehat{\lambda_\imath} - \lambda\right)^2}$$

3-The MADv $\left(D_{(\mathrm{abs})}\right)$:

$$\mathrm{D}_{(\mathrm{abs})} = \frac{1}{nB}\sum_{i=1}^{B}\sum_{j=1}^{n}\left|F_{(\theta,\lambda)}(x_{ij}) - F_{(\hat{\theta},\hat{\lambda})}(t_{ij})\right|,$$

4-The MaxADv $\left(D_{(\max)}\right)$:

$$\mathrm{D}_{(\max)} = \frac{1}{B}\sum_{i=1}^{B}\max_j\left|F_{(\theta,\lambda)}(x_{ij}) - F_{(\hat{\theta},\hat{\lambda})}(w_{ij})\right|.$$

Table 1 gives simulation results for parameters $\theta = 1.5$ and $\lambda = 2$ under the MXLE, CVOM, L-moment, ANDE, RT-ANDE and LE-ANDE methods. Table 2 presents simulation results for parameters $\theta = 0.5$ and $\lambda = 0.8$ under the MXLE, CVOM, L-moment, ANDE, RT-ANDE and LE-ANDE methods. Table 3 shows simulation results for parameters $\theta = 0.7$ and $\lambda = 0.7$ under the MXLE, CVOM, L-moment, ANDE, RT-ANDE and LE-ANDE methods. From From Tables 1, 2 and 3 we note that the BIAS $_{(\theta,\lambda)}$ tend to zero when n increases which means that all estimators are non-biased and the RMSE $_{(\theta,\lambda)}$ tend to zero when n increases which means incidence of consistency property. In other ordes: based on Table 1, Table 2 and Table 3, the $\mathrm{BIAS}_{(\theta)}$ approaches zero as the sample size increases for all estimation methods that were used. Based on Tables 1, 2 and 3, the $\mathrm{RMSE}_{(\theta)}$ decreases and approaches zero as the sample size increases for all estimation methods that were used. Based on Table 1, Table 2 and Table 3, the $\mathrm{BIAS}_{(\lambda)}$ approaches zero as the sample size increases for all estimation methods that were used. Based on Tables 1, 2 and 3, the $\mathrm{RMSE}_{(\lambda)}$ decreases and approaches zero as the sample size increases for all estimation methods that were used. The MXLE is the best method for all sample sizes. However, most methods performed well for $= 50{,}100{,}200$ and 300.

6 The BHE-AFT model

The AFT estimation and reliability analysis are closely related and often used together in engineering, product design, quality control, and other fields to assess and improve the durability and performance of systems and components. AFT models are a class of survival models used to estimate the time-to-failure or survival time of a system or component. These models relate the survival time to covariates or factors, often by assuming a particular probability distribution for the survival times. Reliability analysis is the process of assessing the reliability of a system or component, which refers to its ability to perform a required function without failure for a specified period under given conditions. It includes tasks like designing for reliability, maintaining and improving system reliability, and conducting reliability tests. AFT models help in assessing the reliability of a system or component by estimating the time-to-failure under different conditions. Engineers can use these estimates to make informed decisions about system design and maintenance. AFT models are commonly applied in stress testing to accelerate the failure of components or systems. By applying higher levels of stress (e.g., temperature, load), you can estimate how long a component will last in real-world conditions. This is crucial for reliability analysis.

In this section, we propose a new Burr-Hatke exponential accelerated failure time model. For this, we suppose that n independent failure time variables are observed and we consider that the hypothesis H_0 stating that the survival function given the vector of explanatory variables $z(t) = (z_0(t), z_1(t), \dots, z_m(t))$, $z_0(t) = 1$ (covariates such as temperature, stress,...etc) has the form

$$S(t|z) = S_0(I_0^t[\beta; z(u)]; \zeta),$$

where $\beta = (\beta_0, \beta_1, \dots, \beta_m)^T$ and

$$I_0^t[\beta; z(u)] = \int_0^t e^{-\beta^T z(u)} du$$

is a vector of unknown regression parameters, the function S_0 is a specified functional of time and does not depend on z_i. If explanatory variables are constant over time, the parametric accelerated failure time (AFT) model has the form

$$S(t|z) = S_0[exp(-\beta^T z)t; \zeta].$$

Consider the BHE distribution as baseline distribution where

$$H_0 = F(t) = F_{\mathrm{AFT}}(t, \theta, \lambda, \beta) = F_{\mathrm{AFT}}.$$

So, the CDF of the AFT model can be expressed as

$$F_{\mathrm{AFT}} = 1 - \frac{1}{1 + \lambda texp(-\beta^T \mathrm{z})} exp[-\lambda\theta texp(-\beta^T \mathrm{z})], t > 0; \theta, \lambda > 0,$$

and then, the PDF of the AFT model can be re-expressed as

$$f_{\mathrm{AFT}} = \lambda\theta \frac{1}{1 + \lambda texp(-\beta^T \mathrm{z})} exp(-\beta^T \mathrm{z}) exp[-\lambda\theta texp(-\beta^T \mathrm{z})]$$
$$+\lambda \frac{1}{\left(1 + \lambda texp(-\beta^T \mathrm{z})\right)^2} exp(-\beta^T \mathrm{z}) exp[-\lambda\theta texp(-\beta^T \mathrm{z})].$$

Then,

$$f_{\mathrm{AFT}} = \frac{\lambda}{\left(1 + \lambda texp(-\beta^T \mathrm{z})\right)^2} exp(-\beta^T \mathrm{z}) exp[-\lambda\theta texp(-\beta^T \mathrm{z})]\left[\theta\left(1 + \lambda texp(-\beta^T \mathrm{z})\right) + 1\right].$$

Analogously, the corresponding survival function (SF), HRF and cumulative HRF of the AFT model are given by

$$S_{\mathrm{AFT}} = S_0[texp(-\beta^T \mathrm{z})] = \frac{exp[-\lambda\theta texp(-\beta^T \mathrm{z})]}{1 + \lambda texp(-\beta^T \mathrm{z})}$$
$$h_{\mathrm{AFT}} = \lambda exp(-\beta^T \mathrm{z}) \frac{\{\theta[1 + \lambda texp(-\beta^T \mathrm{z})] + 1\}}{1 + \lambda texp(-\beta^T \mathrm{z})},$$

and

$$H_{\mathrm{AFT}} = -\log\left\{\frac{exp[-\lambda\theta texp(-\beta^T \mathrm{z})]}{1 + \lambda texp(-\beta^T \mathrm{z})}\right\}.$$

Table 1: Simulation results for parameters $\theta = 1.5$ and $\lambda = 2$

	n	$\mathrm{BIAS}_{(\theta)}$	$\mathrm{BIAS}_{(\lambda)}$	$\mathrm{RMSE}_{(\theta)}$	$\mathrm{RMSE}_{(\lambda)}$	D(abs)	D(max)
MXLE	50	0.03481	0.02596	0.33077	0.31835	0.00675	0.00992
CVOM		0.02881	0.02338	0.39524	0.35856	0.00581	0.00854
L-moment		0.06710	0.06305	0.34102	0.33405	0.01447	0.02119
ANDE		0.03710	0.03330	0.35057	0.32448	0.00787	0.01153
RT-ANDE		0.03174	0.02829	0.33610	0.31629	0.00672	0.00984
LE-ANDE		0.06546	0.05515	0.42992	0.38206	0.01339	0.01961
MXLE	100	0.02663	0.02087	0.22994	0.22089	0.00530	0.00778
CVOM		0.01933	0.01578	0.27279	0.24796	0.00393	0.00576
L-moment		0.02624	0.02424	0.23598	0.23241	0.00566	0.00830
ANDE		0.01039	0.00894	0.24543	0.22768	0.00217	0.00318
RT-ANDE		0.00851	0.00704	0.23648	0.22335	0.00174	0.00256
LE-ANDE		0.02188	0.01795	0.29163	0.25996	0.00446	0.00654
MXLE	200	0.01482	0.01290	0.15810	0.15293	0.00310	0.00456
CVOM		0.01685	0.01456	0.19073	0.17375	0.00351	0.00517
L-moment		0.00839	0.00756	0.15911	0.15695	0.00179	0.00263
ANDE		0.00844	0.00804	0.17875	0.16662	0.00185	0.00272

RT-ANDE		0.00405	0.00393	0.16799	0.15933	0.00090	0.00132
LE-ANDE		0.02066	0.01776	0.21799	0.19501	0.00429	0.00631
MXLE	300	0.00235	0.00134	0.12372	0.12083	0.00041	0.00060
CVOM		0.00253	0.00185	0.15414	0.14073	0.00049	0.00072
L-moment		0.00480	0.00421	0.12765	0.12603	0.00101	0.00149
ANDE		-0.00113	-0.00132	0.14436	0.13469	0.00028	0.00041
RT-ANDE		-0.00184	-0.00206	0.13449	0.12749	0.00044	0.00065
LE-ANDE		0.00230	0.00145	0.17603	0.15753	0.00042	0.00061

Table 2: Simulation results for parameters $\theta = 0.5$ and $\lambda = 0.8$

	n	$BIAS_{(\theta)}$	$BIAS_{(\lambda)}$	$RMSE_{(\theta)}$	$RMSE_{(\lambda)}$	D(abs)	D(max)
MXLE	50	0.03345	0.01650	0.16557	0.14737	0.01115	0.01629
CVOM		0.01924	0.00936	0.21546	0.16172	0.00642	0.00937
L-moment		0.03054	0.02507	0.16300	0.15418	0.01282	0.01870
ANDE		0.01599	0.01109	0.18807	0.15171	0.00621	0.00907
RT-ANDE		0.01217	0.00810	0.17077	0.14411	0.00464	0.00678
LE-ANDE		0.03669	0.02133	0.24494	0.17532	0.01304	0.01908
MXLE	100	0.01451	0.00702	0.11202	0.10085	0.00482	0.00706
CVOM		0.00959	0.00507	0.14610	0.11144	0.00330	0.00484
L-moment		0.01568	0.01299	0.11130	0.10585	0.00666	0.00972
ANDE		0.00737	0.00477	0.12856	0.10421	0.00278	0.00406
RT-ANDE		0.00918	0.00634	0.11828	0.09987	0.00357	0.00522
LE-ANDE		0.01576	0.00899	0.16492	0.11935	0.00560	0.00820
MXLE	200	0.01002	0.00552	0.07808	0.06956	0.00352	0.00515
CVOM		0.00707	0.00398	0.09938	0.07573	0.00251	0.00367
L-moment		0.00855	0.00719	0.07638	0.07337	0.00367	0.00536
ANDE		0.00321	0.00176	0.09171	0.07455	0.00113	0.00165
RT-ANDE		0.00110	-0.00009	0.08547	0.07271	0.00020	0.00030
LE-ANDE		0.00630	0.00311	0.11611	0.08448	0.00212	0.00310
MXLE	300	0.00583	0.00378	0.06414	0.05797	0.00220	0.00322
CVOM		0.00547	0.00344	0.08248	0.06329	0.00204	0.00298
L-moment		0.00494	0.00411	0.06020	0.05788	0.00211	0.00308
ANDE		0.00117	0.00051	0.07289	0.05950	0.00038	0.00055
RT-ANDE		0.00010	-0.00021	0.06784	0.05797	0.00004	0.00005
LE-ANDE		0.00352	0.00167	0.09273	0.06750	0.00116	0.00171

Table 3: Simulation results for parameters $\theta = 0.7$ and $\lambda = 0.7$

	n	$BIAS_{(\theta)}$	$BIAS_{(\lambda)}$	$RMSE_{(\theta)}$	$RMSE_{(\lambda)}$	$D_{(abs)}$	$D_{(max)}$
MXLE	50	0.04704	0.02261	0.19687	0.11747	0.01502	0.02200
CVOM		0.04299	0.02065	0.24119	0.12967	0.01374	0.02013
L-moment		0.02763	0.01523	0.19791	0.12600	0.00950	0.01389
ANDE		0.02184	0.01149	0.22493	0.12745	0.00734	0.01075
RT-ANDE		0.02444	0.01264	0.21413	0.12391	0.00815	0.01192
LE-ANDE		0.04831	0.02232	0.29671	0.15296	0.01513	0.02218
MXLE	100	0.01503	0.00511	0.13867	0.08566	0.00419	0.00614
CVOM		0.00862	0.00303	0.17976	0.09725	0.00244	0.00357
L-moment		0.01607	0.00915	0.13305	0.08538	0.00563	0.00825
ANDE		0.01024	0.00540	0.15179	0.08608	0.00346	0.00507
RT-ANDE		0.00839	0.00423	0.14241	0.08372	0.00278	0.00407
LE-ANDE		0.02123	0.00952	0.19050	0.09872	0.00662	0.00971
MXLE	200	0.00838	0.00380	0.09383	0.05793	0.00264	0.00387
CVOM		0.00691	0.00307	0.1086	0.06563	0.00216	0.00316
L-moment		0.00955	0.00555	0.09300	0.05971	0.00339	0.00497
ANDE		-0.00151	-0.00080	0.10628	0.06042	0.00051	0.00075
RT-ANDE		0.00127	0.00050	0.09830	0.05792	0.00038	0.00055
LE-ANDE		0.00524	0.00222	0.13188	0.06885	0.00160	0.00235
MXLE	300	0.00039	-0.00033	0.07453	0.04567	0.00004	0.00006
CVOM		0.00436	0.00221	0.09410	0.05126	0.00145	0.00212
L-moment		-0.00029	-0.00057	0.07471	0.04817	0.00023	0.00034
ANDE		0.00105	0.00052	0.08763	0.04996	0.00034	0.00050
RT-ANDE		-0.00009	-0.00032	0.08025	0.04749	0.00012	0.00017
LE-ANDE		0.00502	0.00215	0.10959	0.05739	0.00154	0.00226

7 The MXLE for the BHE-AFT model

In this section, we apply the maximum likelihood method to estimate the parameters of the AFT for the BHE distribution. We give a detailed description of the method as well as the score functions and the elements of the FIM.

7.1 The MXLE derivations

Let $x_1, \dots, x_n$ be a RS from the AFT for the BHE model with parameters θ, λ and β. Let $\underline{\mathbf{V}} = (\theta, \lambda, \beta_0, \beta_1)^\intercal$ be the 4×1 parameter vector. For determining the MXLE of $\underline{\mathbf{V}}$, we have the log-likelihood function

$$\ell(x; \underline{\mathbf{V}}) = \sum_{i=1}^{n} \log[\lambda exp(-\beta^T \mathrm{z}_i)] - \lambda\theta \sum_{i=1}^{n} x_i exp(-\beta^T \mathrm{z})$$
$$+ \sum_{i=1}^{n} \log\left[1 + \theta\left(1 + \lambda x_i exp(-\beta^T \mathrm{z})\right)\right] - 2 \sum_{i=1}^{n} \log[1 + \lambda x_i exp(-\beta^T \mathrm{z})].$$

The score vector

$$\mathbf{I}_{(\underline{\mathbf{V}})} = \frac{\partial}{\partial \underline{\mathbf{V}}} \ell(x; \underline{\mathbf{V}}) = \left(\frac{\partial}{\partial \theta} \ell(x; \underline{\mathbf{V}}), \frac{\partial}{\partial \lambda} \ell(x; \underline{\mathbf{V}}), \frac{\partial}{\partial \beta_0} \ell(x; \underline{\mathbf{V}}), \frac{\partial}{\partial \beta_1} \ell(x; \underline{\mathbf{V}})\right)^\intercal$$

is given by

$$\mathbf{I}_{(\theta)} = -\lambda \sum_{i=1}^{n} x_i exp(-\beta^T \mathrm{z}) + \sum_{i=1}^{n} \frac{1 + \lambda x_i exp(-\beta^T \mathrm{z})}{1 + \theta\left(1 + \lambda x_i exp(-\beta^T \mathrm{z})\right)},$$

$$\mathbf{I}_{(\lambda)} = \frac{n}{\lambda} - \theta \sum_{i=1}^{n} x_i exp(-\beta^T \mathrm{z})$$
$$+\theta \sum_{i=1}^{n} \frac{x_i}{1 + \theta\left(1 + \lambda x_i exp(-\beta^T \mathrm{z})\right)} exp(-\beta^T \mathrm{z}) - 2 \sum_{i=1}^{n} \frac{x_i}{1 + \lambda x_i exp(-\beta^T \mathrm{z})} exp(-\beta^T \mathrm{z}),$$

$$\mathbf{I}_{(\beta_0)} = \theta\lambda \sum_{i=1}^{n} x_i exp(-\beta^T \mathrm{z})$$
$$-\theta\lambda \sum_{i=1}^{n} \frac{x_i}{1 + \theta[1 + \lambda x_i exp(-\beta^T \mathrm{z})]} exp(-\beta^T \mathrm{z}) + 2\lambda \sum_{i=1}^{n} \frac{x_i}{1 + \lambda x_i exp(-\beta^T \mathrm{z})} exp(-\beta^T \mathrm{z}) - 1,$$

$$\mathbf{I}_{(\beta_1)} = -\sum_{i=1}^{n} \mathrm{z}_i + \theta\lambda \sum_{i=1}^{n} \mathrm{z}_i x_i exp(-\beta^T \mathrm{z})$$
$$+2\lambda \sum_{i=1}^{n} \frac{\mathrm{z}_i x_i}{1 + \lambda x_i exp(-\beta^T \mathrm{z})} exp(-\beta^T \mathrm{z}) - \theta\lambda \sum_{i=1}^{n} \frac{\mathrm{z}_i x_i}{1 + \theta[1 + \lambda x_i exp(-\beta^T \mathrm{z})]} exp(-\beta^T \mathrm{z}).$$

Setting the nonlinear system of equations $\mathbf{I}_{(\theta)} = 0, \mathbf{I}_{(\lambda)} = 0, \mathbf{I}_{(\beta_0)} = 0$ and $\mathbf{I}_{(\beta_1)} = 0$ and solving them simultaneously yields the MXLE $\underline{\widehat{\mathbf{V}}} = (\hat{\theta}, \hat{\lambda}, \widehat{\beta_0}, \widehat{\beta_1})^\intercal$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. Since, we can not find the explicit formulas for the MXLEs of the parameters, we use numerical methods such as the Newton Raphson method, the Monte Carlo method, the BB algorithm or others.

7.2 Assessing the BHE-AFT model via a simulation study

We carry out an important study by simulation using the R programming software. In the following, we present the results obtained by means of numerical method (the method of Newton Raphson). Suppose that the AFT for the BHE distribution is considered. The data is iterated $N = 5000$ times, with $\theta = 2.5, \lambda = 1.89, \beta_0 = 0.5, \beta_1 = 0.38$ as values of the parameters. Using BZB algorithm (see Ravi (2009)) in R software for calculating the averages of the simulated values of the MXLEs $\hat{\theta}, \hat{\lambda}, \hat{\beta}_0, \hat{\beta}_1$ parameters and their mean squared errors (MSE), sample sizes are $n = 15,\ n = 30,$

$n = 50,\ n = 150,\ n = 300$ and $n = 500$. Table 4 lists the square mean errors for the parameters' MXLEs (SME). These methods' results are definitive, as can be seen in the table. For confirming the fact that the MXLEs are $\sqrt{n}$ -consistent, we use the simulation results in Figure 1, we can be seen that all estimates converge faster than $n^{-0.5}$.

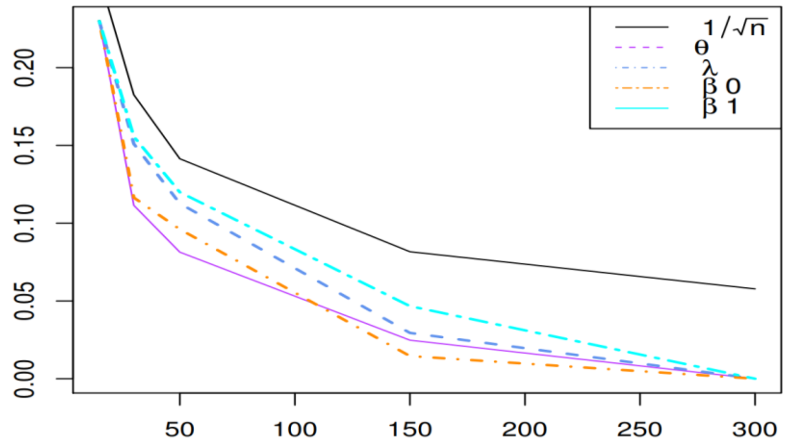

Figure 1: Simulated average absolute errors for MXLEs $\hat{\theta}, \hat{\lambda}, \hat{\beta}_0$ and $\hat{\beta}_1$ vs their true values as a function of sample size n. Number of runs for every n was $N = 5000$.

8 Validation of the BHE-AFT model

Any traditional test, such as Pearson's chi-square, Kolmogorov-Smirnov statistic, Anderson Darling statistic, and other statistics, can be used to validate the selection of model employed in analysis in the case of a well-defined distribution. However, when the parameters are unknown and must be estimated from the sample, the classical tests are no longer appropriate, and the test statistical distributions rely on the model put forth and the estimation technique utilised. In case of complete data, various techniques are used to verify the ANDEquacy of mathematical models to data from observation. The most ommon tests are those based on Pearson's Chi-square statistics. Nevertheless, these can not be applied in all situations, especially when the data is censored or when the the parameters of the model are unknown. Nikulin (1973) and Rao and Robson (1974) each independently presented a statistic for the whole data that is now known as the NIKRR statistic. At the limit, this statistic, which is based on the MXLEs on the initial data, likewise exhibits a Chi-square distribution. For more details on the construction of these statistics, we can see Voinov et al. (2013) and Goual et al. (2019).These methods were used to adapt observations to the distribution of Lomax inverse Weibull (Goual et al., 2020), the Burr XII inverse Rayleigh model (Goual et al., 2019), and the Lindley *exp*onentiated model (Goual et al., 2019). In this section, we build a modified chi-square type test based on the NIKRR test statistic for the BHE model.

Table 4: $MLEs(\hat{\theta}, \hat{\lambda}, \hat{\beta}_0, \hat{\beta}_1)$ of the parameters and their mean squared errors.

N = 5000	n = 15	n = 30	n = 50	n = 150	n = 300	n = 500
$\hat{\theta}$	2.5696	2.5629	2.5612	2.5580	2.5566	2.4992
SME	2.9145×10^{-3}	2.4007×10^{-3}	2.2343×10^{-3}	1.9871×10^{-3}	1.8087×10^{-3}	5.6424×10^{-4}
$\hat{\lambda}$	1.9318	1.9149	1.9067	1.8889	1.8826	1.8911
SME	5.5383×10^{-3}	3.5428×10^{-3}	2.7556×10^{-3}	1.2543×10^{-3}	7.8799×10^{-4}	2.0873×10^{-4}
$\hat{\beta}_0$	0.5085	0.5046	0.5039	0.5011	0.5006	0.4988
SME	2.0453×10^{-3}	1.6179×10^{-3}	1.4261×10^{-3}	1.1747×10^{-3}	1.0710×10^{-3}	6.3154×10^{-4}
$\hat{\beta}_1$	0.3807	0.3835	0.3838	0.3837	0.3837	0.3802
SME	1.1364×10^{-3}	4.5229×10^{-4}	2.7133×10^{-4}	9.2021×10^{-5}	6.0664×10^{-5}	4.2128×10^{-6}

8.1 The NIKRR statistic test for the BHE-AFT model

To test the hypothesis H_0 according to which $T_1, T_2, \cdots, T_n$, an n-sample comes from a parametric family $F_{\underline{\mathbf{V}}}(t)$

$$H_0: \Pr\{T_i \leq t\} = F_{\underline{\mathbf{V}}}(t), \quad t \in \mathbb{R},$$

where $\underline{\mathbf{V}} = (\underline{\mathbf{V}}_1, \underline{\mathbf{V}}_2, \cdots, \underline{\mathbf{V}}_s)^T$ represents the vector of unknown parameters, Nikulin (1973) and Rao and Robson (1974) proposed $\mathcal{K}^2$ the NIKRR statistic defined as below. Observations $T_1, T_2, \cdots, T_n$ are grouped in $\mathscr{r}$ subintervals $\mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_{\mathscr{r}}$ mutually disjoint $\mathbf{I}_j =]a_j\text{-}1; a_j]$; where $j = \overline{1; \mathscr{r}}$. The limits a_j of the intervals $\mathbf{I}_j$ are obtained such that

$$p_j(\underline{\mathbf{V}}) = p_j(\underline{\mathbf{V}}; a_{j-1}, a_j) = \int_{a_{j-1}}^{a_j} f_{\underline{\mathbf{V}}}(t)dt|_{(\ j=1,2,\cdots,\mathscr{r})},$$

so

$$a_j = F^{-1}\left(\frac{j}{\mathscr{r}}\right)|_{(j=1,\cdots,\mathscr{r}-1)}.$$

If $\nu_j = (\nu_1, \nu_2, \cdots, \nu_{\mathscr{r}})^T$ is the vector of frequencies obtained by the grouping of data in these $\mathbf{I}_j$ intervals

$$\nu_j = \textstyle\sum_{i=1}^{n} 1_{\{t_i \in \mathbf{I}_j\}}|_{(j=1,\ldots,\mathscr{r})}.$$

The NIKRR statistic is given by

$$\mathcal{K}^2(\underline{\widehat{\mathbf{V}}}_n) = X_n^2(\underline{\widehat{\mathbf{V}}}_n) + \frac{1}{n}\mathbf{L}^T(\underline{\widehat{\mathbf{V}}}_n)(\mathbf{I}(\underline{\widehat{\mathbf{V}}}_n) - \mathbf{J}(\underline{\widehat{\mathbf{V}}}_n))^{-1}\mathbf{L}(\underline{\widehat{\mathbf{V}}}_n),$$

where

$$X_n^2(\underline{\mathbf{V}}) = \left(P(\nu_1, p_1; \underline{\mathbf{V}}), P(\nu_2, p_2; \underline{\mathbf{V}}), \cdots, P(\nu_{\mathscr{r}}, p_{\mathscr{r}}; \underline{\mathbf{V}})\right)^T$$

where

$$P(\nu_1, p_1; \underline{\mathbf{V}}) = \frac{1}{\sqrt{np_1(\underline{\mathbf{V}})}}[\nu_1 - np_1(\underline{\mathbf{V}})],$$

$$P(\nu_2, p_2; \underline{\mathbf{V}}) = \frac{1}{\sqrt{np_2(\underline{\mathbf{V}})}}[\nu_2 - np_2(\underline{\mathbf{V}})],$$

$$P(\nu_{\mathscr{r}}, p_{\mathscr{r}}; \underline{\mathbf{V}}) = \frac{1}{\sqrt{np_{\mathscr{r}}(\underline{\mathbf{V}})}}[\nu_{\mathscr{r}} - np_{\mathscr{r}}(\underline{\mathbf{V}})]$$

and $\mathbf{J}(\underline{\mathbf{V}})$ is the information matrix for the grouped data defined by

$$\mathbf{J}(\underline{\mathbf{V}}) = B(\underline{\mathbf{V}})^T B(\underline{\mathbf{V}}),$$

with

$$B(\underline{\mathbf{V}}) = \left[\frac{1}{\sqrt{p_i}}\frac{\partial}{\partial\mu}p_i(\underline{\mathbf{V}})\right]_{\mathscr{r}\times s}|_{(i=1,2,\cdots,\mathscr{r}\ \text{and}\ k=1,\cdots,s)},$$

then

$$\mathbf{L}(\underline{\mathbf{V}}) = (\mathbf{L}_1(\underline{\mathbf{V}}), \ldots, \mathbf{L}_s(\underline{\mathbf{V}}))^T \ \text{with} \quad \mathbf{L}_k(\underline{\mathbf{V}}) = \sum_{i=1}^{\mathscr{r}} \frac{\nu_i}{p_i}\frac{\partial}{\partial \underline{\mathbf{V}}_k}p_i(\underline{\mathbf{V}}),$$

where $\mathbf{I}_n(\widehat{\underline{\mathbf{V}}_n})$ represents the estimated FIM and $\widehat{\underline{\mathbf{V}}_n}$ is the maximum likelihood estimator of the parameter vector. The $\mathcal{K}^2$ statistic follows a distribution of chi-square $\boldsymbol{\chi}^2_{\mathscr{r}-1}$ with $(\mathscr{r} - 1)$ degrees of freedom.

8.2 Simulation studies under the NIKRR statistic K^2

For simulation studies to make use of the NIKRR statistic, it is necessary to both generate data and carry out statistical analyses that are predicated on the NIKRR statistic. Both of these steps must be completed before the simulation studies can begin. A probability distribution and the data that has been seen are compared using these tests in order to determine the degree of similarity between the two. The NIKRR statistic is a test that determines whether or not a particular distribution accurately portrays the data. The test's name stands for the number of times the statistic was used. This is accomplished by analyzing the data in relation to the distribution that is under scrutiny. tests to identify whether or not a given distribution accurately represents the data that have been seen, in addition to tests to establish whether or not a distribution is a good fit for the data. tests to assess whether or not a particular distribution accurately captures the data that have been seen. They give researchers and statisticians with aid in generating well-informed decisions regarding the selection of models and the selection of probability distributions for use in a variety of applications by assisting with the selection of models and the selection of probability distributions. These choices can be made in regards to the models that are used and the probability distributions that are used.

Consider a sample $T_{1:n}$ where $T = T_{1:n} = (T_1, T_2, \cdots, T_n)^T$. If these data are distributed in accordance with the BHE model, then $P\{T_{1:n} \leq t\} = F_{\underline{\mathbf{V}}}(t)$; with unknown parameters $\underline{\mathbf{V}} = (\theta, \lambda, \beta_0, \beta_1)^T$, by fitting the NIKRR statistic created in the preceding section, a chi-square goodness-of-fit test is created. The MXLEs $\widehat{\underline{\mathbf{V}}_n}$ of the unknown parameters of the AFT-BHE model are computed on the initial data. Since, the statistic $\mathcal{K}^2$ not dependent on the parameters, we can therefore use the estimated Fisher information matrix (FIM) $I_n(\widehat{\underline{\mathbf{V}}_n})$.

All the components of the statistic $\mathcal{K}^2$, for the distribution BHE are provided, therefore $\mathcal{K}^2$ can be deduced easily. In order to support the results obtained in this work, a numerical simulation is performed. Therefore, in order to test the null hypothesis H_0 of the BHE model, we calculated 5000 sample data simulations ($n = 15$, $n = 30$, $n = 50$, $n = 150$, $n = 300$ and $n = 500$) from BHE distribution, after calculating the value of the criterion statistic $\mathcal{K}^2$, we count the number of rejected cases of the null hypothesis H_0. When $\mathcal{K}^2 > \mathcal{K}^2(r)$, the significance is different level α ($\alpha = 0.01, \alpha = 0.05, \alpha = 0.1$). The simulation results of the significance level of $\mathcal{K}^2$ and its theoretical value are shown in Table 5 below. It can be seen that the calculated empirical level value is very close to the corresponding theoretical level value. Therefore, we conclude that the proposed test is very suitable for the BHE distribution.

Table 5: Empirical levels and corresponding theoretical levels ($\epsilon = 0.01, 0.05, 0.1$)

N = 5000	n = 15	n = 30	n = 50	n = 150	n = 300	n = 500
α = 0.01	0.005	0.013	0.016	0.019	0.017	0.018
α = 0.05	0.037	0.049	0.053	0.059	0.051	0.054
α = 0.1	0.096	0.097	0.103	0.103	0.109	0.105

It is used to prove that the $\mathcal{K}^2$ statistics follow in the limit; a chi-squared distribution; the degree of freedom is $k = r - 1$. We calculate $N = 10000$ times, under the null hypothesis H_0, with different parameter values of parameters BHE $\underline{\mathbf{V}} = (\theta, \lambda, \beta_0, \beta_1)^T$, and different r interval values, versus the chi-squared distribution with k degrees of freedom. Their histograms are shown in Figure 2,, compared with the chi-square distribution with k degrees of freedom.

Figure 2 shows the statistical distribution of $\mathcal{K}^2$ for various parameter values and k grouping units. The restriction is based on the chi-square with k degrees of freedom within the simulated statistical error. The same findings are achieved for various parameter values and various intervals of equal probability grouping. As a result, the generalised chi-square $\mathcal{K}^2$ statistic's limit distribution is distribution-free.

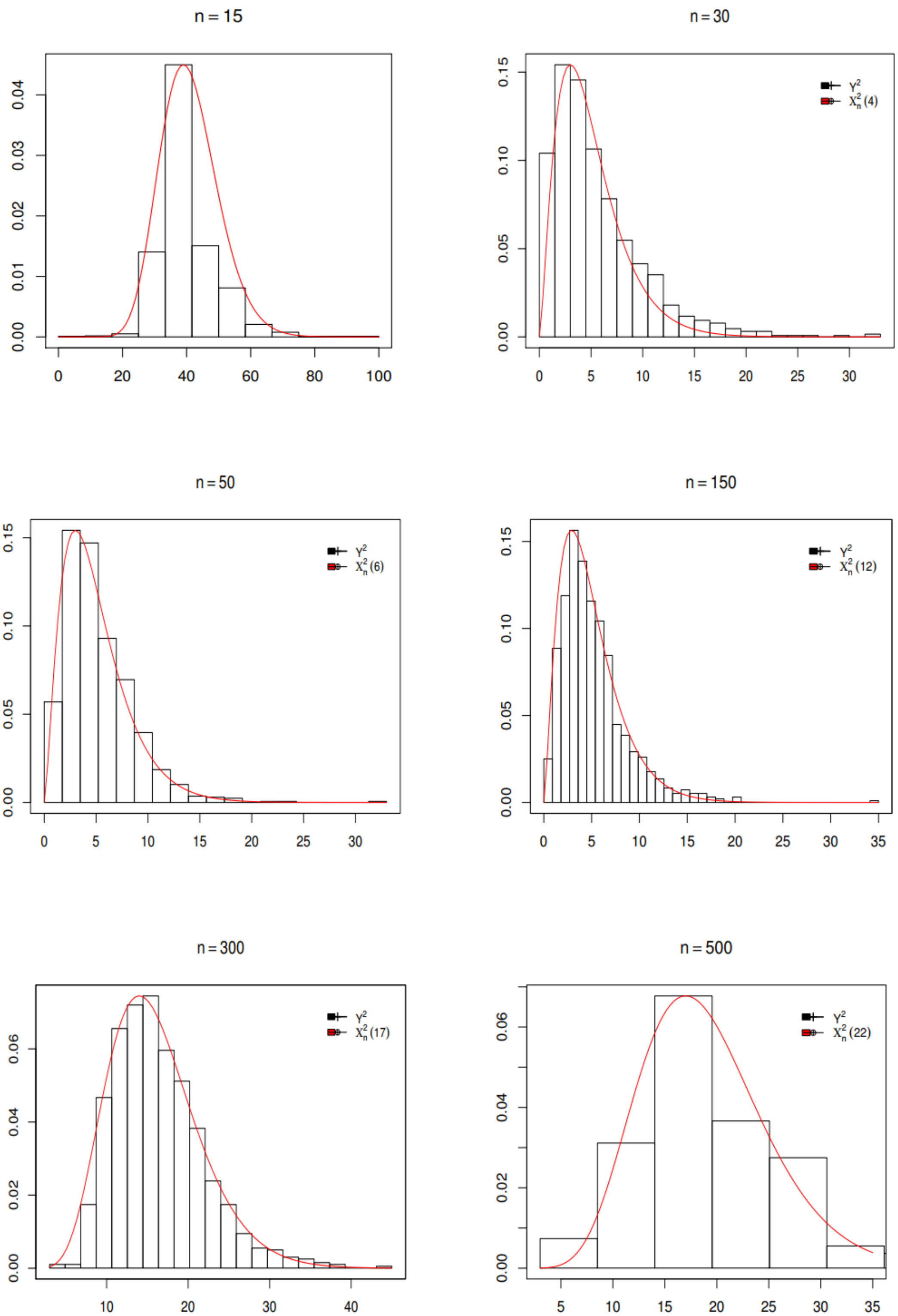

Figure 2: Simulated distribution of the Y^2 statistic under the null hypothesis H_0, with different parameters of $\hat{\mathbf{\Psi}}$ versus the chi-squared distribution with $\mathscr{k}$ degrees of freedom, with $N = 10000$.

8.3 Applications to real data

We take into account the following real data sets and confirm the presumption that their distribution is consistent with the BHE model in order to demonstrate the applicability of the proposed modified chi-square goodness-of-fit test.

8.3.1 Electric insulating fluid data

The failure times of 76 electrical insulating fluids tested at voltages ranging from 26 to 38 kilovolts are provided in Lawless (2003), from which this information was derived. Bagdonavicius and Nikulin (2011) used this data and examined its fit with the exponential and Weibull AFT power-rule models. In this part, we evaluate how well these data fit our suggested BHE model. The data are:

Voltage level (z_i)	n_i	Breakdown time x_i
26	3	5.79,1579.52,2323.7
28	5	68.85,426.07,110.29,108.29,1067.6
30	11	17.05,22.66,21.01,175.88,139.07,144.12,
		20.46,43.40,194.90,47.30,7.74
32	15	0.40,82.85,9.88,89.29,215.10,2.75,0.79,
		15.93,3.91,0.27,0.69,100.58,27.80,13.95,53.24
34	19	0.96,4.15,0.19,0.78,8.01,31.75,7.35,6.50,8.27,33.91,
		32.52,3.16,4.85,2.78,4.67,1.31,12.06,36.71,72.89
36	15	1.97,0.59,2.58,1.69,2.71,25.50,0.35,0.99,
		3.99,3.67,2.07,0.96,5.35,2.90,13.77
38	8	0.47,0.73,1.40,0.74,0.39,1.13,0.09,2.38

1- In case of $\varphi(z) = z$ log linear assumption:
Using R statistical software (the BB package) we find the values of the MXLEs of BHE distribution parameters:

$$\hat{\lambda} = 0.05072, \hat{\theta} = 0.00638, \hat{\beta}_0 = 0.05072, \hat{\beta}_1 = -0.03033,$$

we choose $\mathcal{r} = 8$ intervals and the estimated FIM can be expressed as :

$$I(\underline{\hat{\mathbf{V}}}) = \begin{pmatrix} 344.21671 & 131.60389 & -6.67623 & -179.14691 \\ 131.60389 & 106.72439 & -6.34703 & -207.22261 \\ -6.67623 & -6.34704 & 0.32198 & 10.51235 \\ -179.14691 & -207.22261 & 10.51235 & 347.00979 \end{pmatrix},$$

and then the NIKRR statistic : $\mathcal{K}^2 = 15.64587$. For the critical value : $\alpha = 0.01$, we find $\mathcal{K}^2 < \boldsymbol{\chi}^2_{0.01}(7) = 18.47531$.

2- In case of $\varphi(z) = \log(z)$ power-rule assumption:
We find the values of the MXLEs of the BHE distribution parameters:

$$\hat{\lambda} = 8.53485, \hat{\theta} = 0.00416, \hat{\beta}_0 = 0.51921, \hat{\beta}_1 = 1.05652,$$

we take $\mathcal{r} = 8$ intervals and the estimated FIM can be:

$$I(\underline{\hat{\mathbf{V}}}) = \begin{pmatrix} -700.37417 & 1.04630 & -8.93004 & -29.28962 \\ 1.04630 & 0.00394 & -0.03542 & -0.12322 \\ -8.93004 & -0.03542 & 0.30232 & 1.05171 \\ -29.28962 & -0.12322 & 1.05171 & 3.66217 \end{pmatrix}$$

the NIKRR statistic is $\mathcal{K}^2 = 12.62459$. For the critical values : $\alpha = 0.01$ and $\alpha = 0.05$, we find

$$\mathcal{K}^2 < \boldsymbol{\chi}^2_{0.01}(7) = 18.47531$$

and

$$\mathcal{K}^2 < \boldsymbol{\chi}^2_{0.05}(7) = 14.06714$$

respectively.

3- In case of $\varphi(z) = 1/z$ arrehnius model:
We fit these data by the BHE model. Using R statistical software (the BB package) we find the values of the MXLEs of the BHE distribution parameters :

$$\hat{\lambda} = 2.45748, \hat{\theta} = 0.01388, \hat{\beta}_0 = 3.30941, \hat{\beta}_1 = 1.98948,$$

we take $r = 8$ intervals and the estimated FIM expressed as :

$$I(\underline{\hat{\mathbf{V}}}) = \begin{pmatrix} -113.58757 & 2.27628 & -5.59395 & -0.20885 \\ 2.27628 & 0.06663 & -0.13973 & -0.00446 \\ -5.59395 & -0.13973 & 0.34338 & 0.01097 \\ -0.20885 & -0.00446 & 0.01097 & 0.00035 \end{pmatrix},$$

the NIKRR statistic is: $\mathcal{K}^2 = 18.05936$. For the critical value : $\alpha = 0.01$, we find $\mathcal{K}^2 < \boldsymbol{\chi}^2_{0.01}(7) = 18.47531$. We can assume that electric insulating fluid data of Lawless (2003) correspond appropriately to the BHE model.

8.3.2 Body fat data set

The data of Neter et al. (1996) provides information on ($n = 20$) body fat, triceps skinfold thickness, thigh circumference, and mid-arm circumference for twenty healthy females aged 20 to 34. The data are

z_{i1} (triceps skinfold measurement)	z_{i2} (thigh circumference)	x_i(body-fat)
19.5, 24.7, 30.7	43.1, 49.8, 51.9	11.9, 22.8, 18.7
29.8, 19.1, 25.6	54.3, 42.2, 53.9	20.1, 12.9, 21.7
31.4, 27.9, 22.1	58.5, 52.1, 49.9	27.1, 25.4, 21.3
25.5, 31.1, 30.4	53.5, 56.6, 56.7	19.3, 25.4, 27.2
18.7, 19.7, 14.6, 29.5	46.5, 44.2, 42.7, 54.4	11.7, 17.8, 12.8, 23.9
27.7, 30.2, 22.7, 25.2	55.3, 58.6, 48.2, 51.0	22.6, 25.4, 14.8, 21.1

For $\varphi(z) = z$ as a log linear assumption: We fit these data by the BHE model. Using R statistical software (the BB package) we find the values of the MXLEs of BHE distribution parameters :

$$\hat{\lambda} = 10.01547, \hat{\theta} = 0.0938, \hat{\beta}_0 = -1.50143, \hat{\beta}_1 = 0.0068, \hat{\beta}_2 = -0.16031,$$

we take $r = 4$ intervals and the estimated FIM expressed as

$$I(\underline{\hat{\mathbf{V}}}) = \begin{pmatrix} -1.27654 & 0.00515 & -0.05150 & -1.13224 & -2.44240 \\ 0.00515 & 0.00911 & -0.02890 & -0.68339 & -1.42468 \\ -0.05150 & -0.02890 & 0.28907 & 6.83392 & 14.24688 \\ -1.13224 & -0.68339 & 6.83392 & 168.76973 & 343.64429 \\ -2.44240 & -1.42468 & 14.24688 & 343.64429 & 709.74424 \end{pmatrix},$$

and then the NIKRR statistic : $\mathcal{K}^2 = 6.170162$. For different critical values : $\alpha = 0.01$, $\alpha = 0.05$ and $\alpha = 0.1$, we find

$$\mathcal{K}^2 < \boldsymbol{\chi}^2_{0.01}(3) = 11.34487, \mathcal{K}^2 < \boldsymbol{\chi}^2_{0.05}(3) = 7.81472$$

and

$$\mathcal{K}^2 < \boldsymbol{\chi}^2_{0.1}(3) = 6.25138$$

respectively.

8.3.3 Johnson's data set

Johnson (1996) used a dataset with a response variable (the estimated percentage of body fat) and 13 continuous covariates (age, weight, height, and 10 measurements of the body circumference) in 252 males to illustrate some problems with multiple regression analysis. The aim was to predict percentage body fat from the covariates. These dataset is available on the 'mfp' package in R software.

Variable	Name	Details	Variable	Name	Details
z_1	age	Age (years)	z_8	thigh	Circumference (cm)
z_2	weight	Weight (lb)	z_9	knee	Circumference (cm)

z_3	height	Height (in)	z_{10}	ankle	Circumference (cm)
z_4	neck	Circumference (cm)	z_{11}	biceps	Circumference (cm)
z_5	chest	Circumference (cm)	z_{12}	forearm	Circumference (cm)
z_6	ab	Circumference (cm)	z_{13}	wrist	Circumference (cm)
z_7	hip	Circumference (cm)	x	pcfat	Body fat (%)

In our case, we used two covariates density (Density determined from underwater weighing gm/cm^3) and age (years). We consider the log linear assumption $(\varphi(z) = z)$ and we fit this data by the BHE model. The values of the MXLE of BHE distribution parameters :

$$\hat{\lambda} = 1.61415, \hat{\theta} = 2.81693, \hat{\beta}_0 = -2.07192, \hat{\beta}_1 = 0.86928, \hat{\beta}_2 = 2.15662.$$

We take $r = 15$ intervals and the estimated FIM $I(\underline{\hat{\mathbf{V}}})$ expressed as :

$$I(\underline{\hat{\mathbf{V}}}) = \begin{pmatrix} -7.28659 & 0.16146 & -1.61463 & -1.70273 & -44.50611 \\ 0.16146 & 0.00483 & -0.04377 & -0.04617 & -1.64298 \\ -1.61463 & -0.04377 & 0.43777 & 0.46172 & 16.42980 \\ -1.70273 & -0.04617 & 0.46172 & 0.48713 & 17.31200 \\ -44.50611 & -1.64298 & 16.42980 & 17.312008 & 662.80368 \end{pmatrix},$$

The NIKRR statistic test: $\mathcal{K}^2 = 3.31332$. For different critical values : $\alpha = 0.01, \alpha = 0.05$ and $\alpha = 0.1$, we find

$$\mathcal{K}^2 < \boldsymbol{\chi}^2_{0.01}(14) = 29.14124, \mathcal{K}^2 < \boldsymbol{\chi}^2_{0.05}(14) = 23.68479$$

and

$$\mathcal{K}^2 < \boldsymbol{\chi}^2_{0.1}(14) = 21.06414,$$

respectively. This data can be fitted by our proposed BHE model with the log linear assumption $(\varphi(z) = z)$. One can affirm that our proposed BHE model can be an appropriate distribution of this data.

9 Conclusions

Validation of chosen models for any statistical analysis is necessary if we want to obtain reliable results. This is why methods and techniques of adjustment tests are in perpetual development. When the distribution is specified, we can use any conventional test, however, to accept a composite hypothesis when parameters are unknown and must be estimated from the sample in a censoring data case, these tests are more suitable and the distributions of the test statistics depend on the estimation method used and the proposed model. In this paper we presented a new exponential model called the Burr-Hatke exponential. Two truncated moments, the hazard function, and the conditional expectation of a function of the random variable are used to offer some conclusions for characterizing the BHE distribution. Different estimation methods are considered for assessing the finite sample behavior including the maximum likelihood, Cramer-von-Mises, Anderson Darling, right tail-Anderson Darling, left tail-Anderson Darling and method of L-moments. Simulation studies for comparing the estimation methods are performed. A new Burr-Hatke exponential accelerated failure time model is presented as a parametric accelerated life model when the baseline survival function belongs to BHE model. In both the complete and right censored data instances, we provide a novel modified chi-square test for the Burr-Hatke exponential accelerated failure time model. The validity of the Burr-Hatke exponentially accelerated failure time model is investigated using the Nikulin-Rao-Robson theoretical global. The maximum likelihood method is considered in this. For evaluating the Burr-Hatke exponential accelerated failure time model and evaluating the efficacy of the Nikulin-Rao-Robson test statistic, respectively, two simulation studies are presented. Additionally, three actual data sets are taken into account to demonstrate the Nikulin-Rao-Robson test statistic's effectiveness in validation.

For the electric insulating fluid data, we have the following results:
- Under the log linear assumption: $(\mathcal{K}^2 = 15.64587) < (\boldsymbol{\chi}^2_{0.01}(7) = 18.47531)$, decision: accept $H_0|\alpha = 0.01$.
- Under the power-rule assumption: $(\mathcal{K}^2 = 12.62459) < (\boldsymbol{\chi}^2_{0.01}(7) = 18.47531)$, decision: accept $H_0|\alpha = 0.01$. Also, $(\mathcal{K}^2 = 12.62459) < (\boldsymbol{\chi}^2_{0.05}(7) = 14.06714)$, decision: accept $H_0|\alpha = 0.05$;
- Under the arrehnius model: $(\mathcal{K}^2 = 18.05936) < (\boldsymbol{\chi}^2_{0.01}(7) = 18.47531)$, decision: accept $H_0|\alpha = 0.01$.

For the body fat data set, we have the following results:
- For $\alpha = 0.01$: $(\mathcal{K}^2 = 6.170162) < (\boldsymbol{\chi}^2_{0.01}(3) = 11.34487)$, decision: accept $H_0|\alpha = 0.01$;
- For $\alpha = 0.05$: $(\mathcal{K}^2 = 6.170162) < (\boldsymbol{\chi}^2_{0.04}(3) = 7.81472)$, decision: accept $H_0|\alpha = 0.05$
- For $\alpha = 0.1$: $(\mathcal{K}^2 = 6.170162) < (\boldsymbol{\chi}^2_{0.1}(3) = 6.25138)$, decision: accept $H_0|\alpha = 0.1$.

For the Johnson's data set:
• For $\alpha = 0.01$: $(\mathcal{K}^2 = 3.31332) < (\chi^2_{0.01}(14) = 29.14124)$, decision: accept $H_0|\alpha = 0.01$;
• For $\alpha = 0.05$: $(\mathcal{K}^2 = 3.31332) < (\chi^2_{0.04}(14) = 23.68479)$, decision: accept $H_0|\alpha = 0.05$;
• For $\alpha = 0.1$: $(\mathcal{K}^2 = 3.31332) < (\chi^2_{0.1}(14) = 21.06414)$, decision: accept $H_0|\alpha = 0.1$.
Below, we suggest some future points:

I. Accelerated failure time estimation under some discrete distributions. Estimating AFT models under discrete distributions typically involves using appropriate regression techniques. The specific estimation method may vary based on the distribution chosen and the software package used. Common estimation methods for AFT models include maximum likelihood estimation, moment-based estimation, and Bayesian estimation. In practice, when dealing with discrete survival data and AFT models, it's crucial to select the appropriate distribution based on the characteristics of your data and the research question you want to answer. Additionally, software tools like R, Python (with libraries like lifelines or survival), or specialized survival analysis software can help you implement and estimate AFT models for discrete survival data.

II. Accelerated failure time estimation under some heavy tailed distributions. Estimating AFT models under heavy-tailed distributions typically involves using appropriate regression techniques, often with maximum likelihood estimation or moment-based methods. The choice of the distribution should be based on the characteristics of the data and the research question at hand. Keep in mind that heavy-tailed distributions tend to capture extreme values more accurately than lighter-tailed distributions like the exponential or normal distribution. However, the choice of distribution should be guided by the theoretical considerations, domain knowledge, and model fit criteria. Analyzing survival times under heavy-tailed distributions can be especially relevant in fields where rare, extreme events are of interest, such as finance, insurance, and environmental science.

III. Accelerated failure time estimation under some extreme value distributions. Estimating AFT models under extreme value distributions typically involves using regression techniques, often with MXLE or moment-based methods. The choice of the distribution should be based on the characteristics of the data, the behavior of the extreme values, and the specific research question. AFT models under extreme value distributions are particularly useful when the focus is on modeling and predicting extreme events, such as rare natural disasters, extreme financial market events, or the durability of products that experience rare but critical failures. The selection of the appropriate extreme value distribution should be guided by domain knowledge and model fit criteria.

IV. Accelerated failure time estimation under some bivariate distributions. The bivariate normal distribution is an extension of the univariate normal distribution to two dimensions. It is often used in AFT models to describe the joint survival times of two events when their logarithms follow a bivariate normal distribution. AFT models using bivariate distributions allow you to capture the dependence between two survival times. The choice of the distribution should be guided by the nature of the dependence and the specific application. These models have applications in fields such as finance, actuarial science, reliability engineering, and medical research, where understanding the joint behavior of two events or variables is critical for making informed decisions and predictions.

V. Accelerated failure time estimation under some multivariate distributions. The choice of a multivariate distribution for AFT modeling should be guided by the nature of the interdependencies between the events and the goals of the analysis. It is essential to select an appropriate distribution that accurately captures the underlying data and relationships among the variables. These models are widely used in various fields, including epidemiology, finance, actuarial science, and engineering, where understanding the joint survival times of multiple events or individuals is crucial for making informed decisions and predictions.

VI. In survival analysis, estimating survival times under mixed distributions typically involves considering a mixture of two or more component distributions. Mixed distributions are used to account for situations where the data comes from different subpopulations or has a complex structure that cannot be adequately represented by a single distribution. The AFT models can be used to estimate survival times under mixed distributions by modeling the effects of covariates on the time-to-event while considering the mixture components. The choice of a specific mixed distribution for AFT modeling should be guided by the underlying characteristics of the data and the research question. Mixed distribution models are valuable when dealing with heterogeneous populations, complex data structures, or the presence of distinct subgroups within the dataset. Accurate estimation under mixed distributions can help provide a better understanding of survival times in real-world applications in fields such as medicine, engineering, and finance.

VII. Accelerated failure time estimation under some weighted distributions. The choice of the weighted distribution should be guided by the research question and the nature of the data. Weighted AFT models are useful in scenarios where the survival analysis needs to account for varying importance of different observations in a systematic and principled manner.

VIII. explore the applicability of the novel exponential model to different fields and domains. Investigate whether the model can be extended or adapted to various types of data beyond its original context.

IX. Assess the robustness of the accelerated failure time estimation methods under different conditions, such as the presence of outliers, censored data, or non-proportional hazards. Develop methods to handle such challenges effectively.

X. Conduct extensive simulation studies to evaluate the performance of the novel model and estimation methods under controlled conditions. Investigate how sample size, censoring, and other factors impact the accuracy and precision of parameter estimates.

XI. Compare the novel exponential model with existing models in terms of goodness-of-fit, predictive accuracy, and interpretability. explore methods to statistically validate the model's assumptions.

XII. Apply the novel model and estimation methods to real-world data sets from different fields, such as healthcare, finance, engineering, and social sciences. Investigate whether the model provides valuable insights and predictive power in these domains.

XIII. Investigate the use of Bayesian methods for parameter estimation in the novel exponential model. explore the advantages and limitations of Bayesian approaches compared to classical frequentist methods.

XIV. Develop techniques for variable selection and model simplification within the context of the novel exponential model. This could help in identifying the most influential covariates and improving model interpretability.

XV. explore the integration of machine learning techniques, such as deep learning or ensemble methods, with accelerated failure time estimation. Investigate whether these approaches can enhance prediction accuracy.

XVI. Conduct sensitivity analysis to assess the impact of variations in model assumptions and estimation methods on the results. Provide guidance on when and how to use the model in practice.

XVII. Develop user-friendly software packages or tools for implementing the novel exponential model and estimation methods. This can facilitate its adoption by researchers and practitioners in various fields.

XVIII. Consider extending the novel exponential model to incorporate time-varying covariates, frailty models, or other complex features that may be relevant in specific applications.

XIX. Investigate methods for combining results from multiple studies that apply the novel model. Develop meta-analytical approaches to synthesize findings and assess the overall impact of the model across different research contexts.

XX. explore the ethical and regulatory considerations associated with the use of the novel model, particularly in fields like healthcare and finance where the model may have significant implications for decision-making and policy.

These research points should help advance the understanding, applicability, and robustness of the accelerated failure time estimation for the novel exponential model, contributing to the field of survival analysis and statistics.

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