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Remarks on Seven New Publications Based on Concept of Sub-Independence

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Abstract

Chesneau and Palacios considered the infinite decomposability of the Geometric (Chesneau and Palacios(2021b), (paper 1)), and Gamma, Laplace and n-Laplace (Chesneau and Palacios(2021a), (paper 2)) of two (as well as n) independent random variables. They obtained, very nicely, certain important results on the decomposability concept. Yanev published a paper entitled "Exponential and Hyperexponential Distributions: Some Characterizations" (Yanev(2020a), (paper 3)) as well as a paper entitled "On Arnold-Villasenor Conjectures for Characterizing Exponential Distribution Based on Sample of Size Three" (Yanev(2020b), (paper 4)). In both papers, Yanev considered the distribution of the sum or a linear combination of the independent random variables. Yanev obtained certain nice results in these two papers under the assumption of independence of the summands. Roozegar and Bazyani published a paper entitled "Exact Distribution of Random Weighted Convolution of Some Beta Distributions Through an Integral Transform" (Roozegar and Bazyari(2017), (paper 5)), in which they considered the exact distribution of the weighted average of n independent beta random variables and provided a new integral transformation with some of its mathematical properties. Ahmad et al.(2021) considered "Compound Negative Binomial Distribution as the Sum of Independent Laplace Variates" (paper 6) and discussed infinite divisibility of the underlining distribution. Furthermore, Marques et al.(2015) considered the distribution of the linear combinations of independent Gumbel random variables and obtain, very nicely, certain important results (paper 7). In this short note, we like to show that the very strong assumption of "independence" can be replaced with a much weaker assumption of "sub-independence" in all aforementioned papers. This short note may be helpful to the other investigators dealing with the random variables which are not necessary independent, but could be sub-independent.

Key Words: Independence; Sub-Independence; Decomposability Concept; Exponential Distribution; Hyperexponential Distribution; Weighted Average.

1. Introduction

To make this very short note self contained, we will copy some parts of our previous work, Hamedani(2013), here.

We may in some occasions have asked ourselves if there is any concept between "uncorrelatedness" and "independence" of two random variables. It seems that the concept of "sub-independence" is the one: it is much stronger than uncorrelatedness and much weaker than independence. The notion of sub-independence seems important in the sense that under usual assumptions, Khintchine's Law of Large Numbers and Lindeberg-Levy's Central Limit Theorem as well as other important theorems in probability and statistics hold for a sequence of *s.i.* (sub-independent) random variables. While sub-independence can be substituted for independence in many cases, it is difficult, in general, to find conditions under which the former implies the latter. Even in the case of two discrete identically distributed rv' s (random variables) X and Y, the joint distribution can assume many forms consistent with sub-independence.
Limit theorems as well as other well-known results in probability and statistics are often based on the distribution of the sums of independent (and often identically distributed) random variables rather than the joint distribution of the summands. Therefore, the full force of independence of the summands will not be required. In other words, it is the convolution of the marginal distributions which is needed, rather than the joint distribution of the summands which, in the case of independence, is the product of the marginal distributions. The concept of sub-independence, which is weaker than that of independence, is shown to be sufficient to yield the conclusions of these theorems and results. This is precisely the reason for the statement: "why assume independence when you can get by with sub-independence".
The concept of sub-independence can help to provide solution for some modeling problems where the variable of interest is the sum of a few components. Examples include household income, the total profit of major firms in an industry, and a regression model $Y = g(X) + \varepsilon$ where $g(X)$ and ε are uncorrelated, however, they may not be independent. For example, in Bazargan et al.(2007), the return value of significant wave height (Y) is modeled by the sum of a cyclic function of random delay D, $\hat{g}(D)$, and a residual term $\hat{\varepsilon}$. They found that these two components are at least uncorrelated but not independent and used sub-independence to compute the distribution of the return value.
Let X and Y be two *rv 's* (random variables) with the joint and marginal *cdf 's* (cumulative distribution functions) $F_{X,Y}$, F_X and F_Y respectively. Then X and Y are said to be independent if and only if

$$F_{X,Y}(x,y) = F_X(x) F_Y(y) \ , \qquad for\ all \quad (x,y) \in \mathbb{R}^2, \tag{1.1}$$

or equivalently, if and only if

$$\varphi_{X,Y}(s,t) = \varphi_X(s)\varphi_Y(t) \ , \qquad for\ all \quad (s,t) \in \mathbb{R}^2, \tag{1.2}$$

where $\varphi_{X,Y}(s,t)$, $\varphi_X(s)$ and $\varphi_Y(t)$, respectively, are the corresponding joint and marginal *cf 's* (characteristic functions). Note that (1.1) and (1.2) are also equivalent to

$$\begin{gathered} P(X \in A\ and\ Y \in B) = P(X \in A) P(Y \in B) \ , \\ for\ \ all\ \ Borel\ \ sets\ \ A\ , B. \end{gathered} \tag{1.3}$$

The concept of sub-independence, as far as we have gathered, was formally introduced by Durairajan (1979) and developed by Hamedani in the past 40 years, stated as follows: The *rv 's* X and Y with *cdf 's* F_X and F_Y are *s.i.* (sub-independent) if the *cdf* of $X+Y$ is given by

$$F_{X+Y}(z) = (F_X * F_Y)(z) = \int_{\mathbb{R}} F_X(z-y)\, dF_Y(y), \qquad z \in \mathbb{R}, \tag{1.4}$$

or equivalently if and only if

$$\varphi_{X+Y}(t) = \varphi_{X,Y}(t,t) = \varphi_X(t)\varphi_Y(t), \qquad for\ all\ \ t \in \mathbb{R}, \tag{1.5}$$

or equivalently if and only if

$$M_{X+Y}(t) = M_X(t)\, M_Y(t), \qquad for\ all\ \ t \in \mathbb{R}, \tag{1.6}$$

where $M_X(t)$ is the moment generating function of the random variable X.

The drawback of the concept of sub-independence in comparison with that of independence has been that the former does not have an equivalent definition in the sense of (1.3) , which some believe, to be the natural definition of independence. We found such a definition which is stated below. We shall give the definitions for the discrete case (Definition 1.1) and continuous case (Definition 1.2).

Let $(X,Y) : \Omega \to \mathbb{R}^2$ be a discrete random vector with the range $\Re(X,Y) = \{(x_i, y_j) : i,\ j = 1, 2, ...\}$ (finitely or infinitely countable).

Consider the events

$$A_i = \{\omega \in \Omega :\ X(\omega) = x_i\}\ ,\ B_j = \{\omega \in \Omega :\ Y(\omega) = y_j\}$$

and

$$A^z = \{\omega \in \Omega :\ X(\omega) + Y(\omega) = z\}\ ,\ z \in \Re(X+Y)\,.$$

Definition 1.1. The discrete $rv\,'s$ X and Y are $s.i.$ if for every $z \in \Re(X+Y)$

$$P(A^z) = \sum_{i\ ,\ j\ ,\ x_i\ +\ y_j\ =\ z} P(A_i)\, P(B_j)\,. \tag{1.7}$$

To see that (1.7) is equivalent to (1.5), suppose X and Y are $s.i.$ via (1.5) , then

$$\sum_i \sum_j e^{it(x_i+y_j)}\ f(x_i, y_j) = \sum_i \sum_j e^{it(x_i+y_j)}\ f_X(x_i)\ \ f_Y(y_j)\,,$$

where f, f_X and f_Y are probability functions of (X,Y), X and Y respectively. Let $z \in \Re(X+Y)$, then

$$e^{itz} \sum_{i\ ,\ j\ ,\ x_i\ +\ y_j\ =\ z} f(x_i, y_j) = e^{itz} \sum_{i\ ,\ j\ ,\ x_i\ +\ y_j\ =\ z} f_X(x_i)\ \ f_Y(y_j)\,,$$

which implies (1.7) .

For the continuous case, we observe that the half-plane $H = \{(x,y) :\ x + y < 0\}$ can be expressed as a countable disjoint union of rectangles:

$$H = \cup_{i=1}^{\infty} E_i \times F_i,$$

where E_i and F_i are intervals. Now, let $(X,Y):\Omega\to\mathbb{R}^2$ be a continuous random vector and for $c\in\mathbb{R}$, let

$$A_c=\{\omega\in\Omega:X(\omega)+Y(\omega)<c\}$$

and

$$A_i^{(c)}=\left\{\omega\in\Omega:X(\omega)-\frac{c}{2}\in E_i\right\},\ B_i^{(c)}=\left\{\omega\in\Omega:Y(\omega)-\frac{c}{2}\in F_i\right\},$$

Definition 1.1. The continuous random variables X and Y are *s.i.* if for every $c\in\mathbb{R}$

$$P(A_c)=\sum_{i=1}^{\infty}P\left(A_i^{(c)}\right)P\left(B_i^{(c)}\right), \tag{1.8}$$

To see that (1.8) is equivalent to (1.4), observe that (LHS of (1.8))

$$P(A_c)=P(X+Y<c)=P((X,Y)\in H_c), \tag{1.9}$$

where $H_c=\{(x,y):x+y<c\}$. Now, if X and Y are *s.i.* then

$$P(A_c)=(P_X\times P_Y)(H_c)$$

where P_X *and* P_Y are probability measures on $\mathbb{R}$ defined by

$$P_X(B)=P(X\in B)\ \text{ and } P_Y(B)=P(Y\in B),$$

and $P_X\times P_Y$ is the product measure.

We also observe that (RHS of (1.8))

$$\begin{aligned}\sum_{i=1}^{\infty}P\left(A_i^{(c)}\right)P\left(B_i^{(c)}\right)&=\sum_{i=1}^{\infty}P\left(X-\frac{c}{2}\in E_i\right)P\left(Y-\frac{c}{2}\in F_i\right)\\&=\sum_{i=1}^{\infty}P\left(X\in E_i+\frac{c}{2}\right)P\left(Y\in F_i+\frac{c}{2}\right)\\&=\sum_{i=1}^{\infty}P_X\ \times P_Y\ \left(E_i+\frac{c}{2}\right)\times\left(F_i+\frac{c}{2}\right).\end{aligned} \tag{1.10}$$

Now, (1.9) and (1.10) will be equal if $H_c=\cup_{i=1}^{\infty}\left\{\left(E_i+\frac{c}{2}\right)\times\left(F_i+\frac{c}{2}\right)\right\}$, which is true since the points in H_c are obtained by shifting each point in H over to the right by $\frac{c}{2}$ units and then up by $\frac{c}{2}$ units.
If X and Y are *s.i.*, then unlike independence, X and αY are not necessarily *s.i.* for any real $\alpha\neq 1$. This demonstrates how weak the concept of sub-independence is in comparison with that of independence. Please observe the following simple example.

Example 1.1. Let X and Y have the joint cf given by

$$\begin{aligned}&\varphi_{X,Y}(t_1,t_2)=\exp\left\{-\left(t_1^2+t_2^2\right)/2\right\}[1+\beta\ t_1t_2(t_1-t_2)^2\times\\&\exp\left\{\left(t_1^2+t_2^2\right)/4\right\}],\qquad (t_1,t_2)\in\mathbb{R}^2.\end{aligned}$$

where β is an appropriate constant. We know that the characteristic function is the Fourier Transform of probability density function (*pdf*)). Therefore, the corresponding joint *pdf* is given by

$$f(x,y)=\frac{1}{2\pi}\exp\left\{-\left(x^2+y^2\right)/2\right\}[1-16\beta\, p(x,y)\times \exp\left\{-\left(x^2+y^2\right)/2\right\}], \qquad (x,y)\in\mathbb{R}^2.$$

where $p(x,y)=\left\{6xy-2x^2-2y^2+4x^2y^2-2x^3y-2xy^3+1\right\}$.

Then X and Y are $s.i.$ standard normal random variables, and hence $X+Y$ is normal with mean 0 and variance 2, but X and $-Y$ are not $s.i.$ and consequently $X-Y$ does not have a normal distribution.

The concept of sub-independence defined above can be extended to n (>2) random variables as follows.

Definition 1.3. The $rv\,'s$ $X_1,X_2,...,X_n$ are $s.i.$ if for each subset $\{X_{\alpha_1},X_{\alpha_2},...,X_{\alpha_r}\}$ of $\{X_1,X_2,...,X_n\}$

$$\varphi_{X_{\alpha_1},...,X_{\alpha_r}}(t,...,t)=\Pi_{i=1}^{r}\varphi_{X_{\alpha_i}}(t), \quad for\ all \quad t\in\mathbb{R}.$$

Definition 1.4. The $rv\,'s$ $X_1,X_2,...,X_n$ are max-sub-independent $(m.s.i.)$ if for each subset $\{X_{\alpha_1},X_{\alpha_2},...,X_{\alpha_r}\}$ of $\{X_1,X_2,...,X_n\}$

$$P\left(X_{\alpha_1}\le x,...,X_{\alpha r}\le x\right)=\Pi_{i=1}^{r}P\left(X_{\alpha_1}\le x\right), \qquad x\in\mathbb{R}.$$

2. Remarks

I) If the $rv's$ X and Y are $s.i.$ with distributions $MG_0(p)$ and $G(p/(1+p))$, respectively (using the notation of paper 1) the moment generating function of $X+Y$ is

$$M_{X+Y}(t)=M_X(t)\,M_Y(t), \qquad t\in\mathbb{R}.$$

which is used in the proof of the Proposition 1 in paper 1. Similarly, in this paper, the assumption of independence in Theorem 1, Proposition 2 and Proposition 3 can be replaced with that of sub-independence.

II) Considering paper 2, the assumption of independence in Proposition 3.2, Propositions 3.3, Proposition 4.1, Corollary 4.2 and Proposition 4.3 can be replaced with that of sub-independence.

III) For a detailed treatment of the concept of sub-independence, we refer the interested readers to Hamedani(2013). Interested readers can also refer to Hamedani and Rao(2015).

IV) On page 2 of paper 3, equation (2), $S_n=\sum_{i=1}^{n}\mu_iX_i$ is defined under the assumption of the independence of the random variables X_i $,i=1,2,...,n$. This assumption can be replaced with that of μ_iX_i, $i=1,2,...,n$ are $s.i.$.

V) On page 178 of paper 4, equation (1.1), $\sum_{j=1}^{3}\frac{1}{j}X_j$ is considered under the assumption of the independence of the random variables X_j, $j=1,2,3$. This assumption can be replaced with that of $\frac{1}{j}X_j$, $j=1,2,3$ are $m.s.i.$.

VI) On page 800 of paper 5, equation (1.1), $S_n=\sum_{i=1}^{n}R_iX_i$ is considered under the assumption of the independence of n random variables , $n\ge 2$. This assumption can be replaced with that of $S_n=\sum_{i=1}^{n}R_iX_i$, $n\ge 2$ are $s.i.$.

VII) Throughout paper 6, the assumption of independence can be replaced with that of $s.i.$ as the authors prove their results using the characteristic function of the sum of the random variables.

VIII)

a) Theorem 1 of paper 7 can be stated as follows:

Let $X_j \sim Gumbel\,(\mu_j, \sigma_j)$, with $\mu_j \in \mathbb{R}$ and $\sigma_j \in \mathbb{R}_+^*$ and let $\alpha_j X_j$, with $\alpha_j \in \mathbb{R}$, $j = 1, 2, ..., p$ be *s.i.*. The exact characteristic function of $W = \sum_{j=1}^{p} \alpha_j X_j$ can be written as $\Phi_W(t) = \Phi_{W_1}(t)\,\Phi_{W_2}(t)$, where for any $\gamma \in \mathbb{N}\;/\;\{1\}$
$\Phi_{W_1}(t) = \Pi_{j=1}^{p} \frac{\Gamma(\gamma - it\sigma_j\alpha_j)}{\Gamma(\gamma)}$, $\quad t \in \mathbb{R}$ and $\Phi_{W_2}(t) = \left\{ \Pi_{j=1}^{p} \Pi_{k=0}^{\gamma-2} \left(\frac{1+k}{\sigma_j\alpha_j} \right) \left(\frac{1+k}{\sigma_j\alpha_j} - it \right)^{-1} \right\} \exp\left\{ it \sum_{j=1}^{p} \mu_j\alpha_j \right\}$, $t \in \mathbb{R}$.

b) In corollary 1 of paper 7, the assumption of independence of the random variables $X_1, X_2, ..., X_p$ can be replaced with the assumption that $\alpha_1 X_1, \alpha_2 X_2, ..., \alpha_p X_p$ are *s.i.*.

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