Pakistan Journal of Statistics and Operation Research

The Beta Power Muth Distribution: Regression Modeling, Properties and Data Analysis

Lazhar Benkhelifa[1]

[1] Department of Mathematics and Informatics, Larbi Ben M'Hidi University, Oum El Bouaghi, 04000, Algeria, benkhelifa.lazhar@univ-oeb.dz

Abstract

In this paper, we propose a new flexible lifetime distribution. The proposed distribution will be referred to as beta power Muth distribution. It can be used to model increasing, decreasing, bathtub shaped or upside-down bathtub hazard rates. Some properties of the new model are obtained including moments, quantile function and moments of the order statistics. The unknown model parameters are estimated by the maximum likelihood method of estimation. A Monte Carlo simulation study is carried out to assess the performance of the maximum likelihood estimates. Two reliability data sets are applied to illustrate the usefulness and flexibility of the proposed model. In addition, we introduce a new location-scale regression model based on the logarithm of the proposed distribution and provide a real data application.

Key Words: Beta distribution, Muth distribution, data analysis, maximum likelihood estimation, regression modeling.

Mathematical Subject Classification: 60E05, 62E15, 62F10.

1. Introduction

In reliability and lifetime analysis, the hazard function, also called hazard rate, failure rate or instantaneous failure rate, has a crucial role to characterize real lifetime data. Several real-life data sets have a non-monotone hazard rates such as the bathtub shapes and upside-down bathtub (unimodal) hazard rates. The most popular traditional distributions do not provide a good fit for modeling this kind of the data sets. Hence, many parametric probability distributions have been introduced to analyze real data sets with non-monotone hazard rates.

One of the most recent models is that of Jodrá et al. (2017) who have proposed a new two-parameter lifetime distribution with bathtub-shaped and increasing failure rate called the power Muth (PM) distribution.distribution, where the Muth distribution was first proposed by Muth (1977) in the context of reliability theory. A random variable Y is said to follow the Muth distribution, with shape parameter $\alpha \in]0,1]$, if its probability density function (pdf) has the form:

$$g_Y(y) = (e^{\alpha y} - \alpha) e^{\left\{\alpha y - \frac{1}{\alpha}(e^{\alpha y} - \alpha)\right\}}, \qquad y > 0.$$

Since the Muth distribution was introduced, it has been largely overlooked in the literature until the paper of Leemis and McQueston (2008), where they reffered its relation with the exponential distribution. After seven years, some mathematical properties of the Muth distribution are derived, for the first time, by Jodrá et al. (2015). Then, Jodrá et al. (2017) extend the Muth distribution, by using the transformation $X = \beta Y^{1/\gamma}$, where $\beta > 0$, $\gamma > 0$ and Y have the Muth distribution with parameter α =1. The pdf of X is

 DOI: http://dx.doi.org/10.18187/pjsor.v18i1.3529

$$g(x) = \frac{\gamma}{\beta^{\gamma}} x^{\gamma-1}\left(e^{\left(\frac{x}{\beta}\right)^{\gamma}} - 1\right) e^{\left\{\left(\frac{x}{\beta}\right)^{\gamma} - \left(e^{\left(\frac{x}{\beta}\right)^{\gamma}} - 1\right)\right\}}, \tag{1}$$

and the cumulative distribution function (cdf) of *X* is

$$\mathrm{G(x)} = 1 - e^{\left\{(x/\beta)^{\gamma} - \left(e^{(x/\beta)^{\gamma}} - 1\right)\right\}}, \tag{2}$$

where $x > 0$, β is a scale parameter and γ is a shape parameter. Also, Jodrá et al. (2017) studied various statistical properties of this distribution and showed that it gives the best fit for two real data sets than many other distributions.

The PM distribution does not provide enough flexibility for analyzing different types of lifetime data. To increase the flexibility for modelling purposes, it will be useful to consider further alternatives to this distribution. Therefore, the goal of this paper is to introduce a new four-parameter generalization that can capture decreasing, increasing, unimodal (upside-down bathtub) and bathtub-shaped hazard rate functions. The new distribution will be called the beta PM (BPM) distribution. We study some properties of the new model, give maximum likelihood estimation of the parameters, derive the elements of the observed information matrix and apply this model to real-life data sets. Also, based on this distribution, we present a new regression model.

The rest of the paper is organized as follows. In Section 2, we define the BPM model. In Sections 3, 4 and 5 we explicit expressions for the quantile function, moments and the moments of the order statistics respectively. In Section 6, we discuss maximum likelihood estimation of the model parameters. In Section 7, we conduct a simulation study to check the performance of the maximum likelihood estimates. A new regression model and residual analysis are presented in Section 8. Three applications are given in Section 9. Conclusions are given in Section 10.

2. The model definition

Eugene et al. (200**2**) proposed the beta-generated family of distributions by using the beta random variable. For an arbitrary baseline cdf *G(x)*, the cdf of the beta generalized family is defined by

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)}\int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt, \tag{3}$$

where $a > 0$and $b > 0$ are two shape parameters, $I_y(a,b) = \frac{B_y(a,b)}{B(a,b)}$ is the incomplete beta function ratio, $B_y(a,b) = \int_0^y t^{a-1}(1-t)^{b-1}dt$ is the incomplete beta function, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the complete beta function and $\Gamma(.)$ is the gamma function. The pdf corresponding to (3) is

$$f(x) = \frac{g(x)}{B(a,b)}[G(x)]^{a-1}[1-G(x)]^{b-1}, \tag{4}$$

where $g(x) = dG(x)/dx$ is the baseline pdf. By using Equation (3), several authors have defined and studied many new distributions. For example, Eugene et al. (2002) introduced the beta-normal distribution. Nadarajah and Kotz (2005) defined and studied the beta exponential. Famoye et al. (2005) introduced the beta-Weibull distribution. Singla et al. (2012) defined the beta generalized Weibull distribution. Shakhatreh et al. (2016) introduced the beta generalized linear exponential distribution. Benkhelifa (2017) proposed the beta generalized Gompertz distribution. Awodutire et al. (2020) introduced the beta type I generalized half logistic distribution. Benkhelifa (2021) proposed the beta reduced modified Weibull distribution.

By substituting (2) in (3), the cdf of the BPM distribution with four parameters (*β > 0, γ > 0*, *a > 0* and *b > 0*) can be defined by

$$F(x) = I_{1-e^{\left\{(x/\beta)^{\gamma} - \left(e^{(x/\beta)^{\gamma}} - 1\right)\right\}}}(a,b) = \frac{1}{B(a,b)}\int_0^{1-e^{\left\{(x/\beta)^{\gamma} - \left(e^{(x/\beta)^{\gamma}} - 1\right)\right\}}} t^{a-1}(1-t)^{b-1}dt. \tag{5}$$

DOI: http://dx.doi.org/10.18187/pjsor.v18i1.3529

The pdf of the BPM distribution is given by

$$f(x)=\frac{\gamma x^{\gamma-1}\left(e^{(x/\beta)^{\gamma}}-1\right)}{\beta^{\gamma}B(a,b)}e^{b\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}\left(1-e^{\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}\right)^{a-1}. \tag{6}$$

It is clear that the PM distribution with parameters β and γ is a special sub-model for *a=b=1*. For *b=1*, we obtain the exponentiated PM (EPM) distribution, which is proposed by Irshad et al. (2021). Hereafter, a random variable X with pdf (6) will be denoted by $X \sim BPM(\beta,\gamma,a,b)$.

Figure 1 shows some possible shapes of the pdf (6) of the BPM distribution for some parameter values of β,γ,a and b. Then, we observe that the density function (6) can take various forms depending on the parameter values. It is evident that the BPM distribution is much more flexible than the PM distribution.

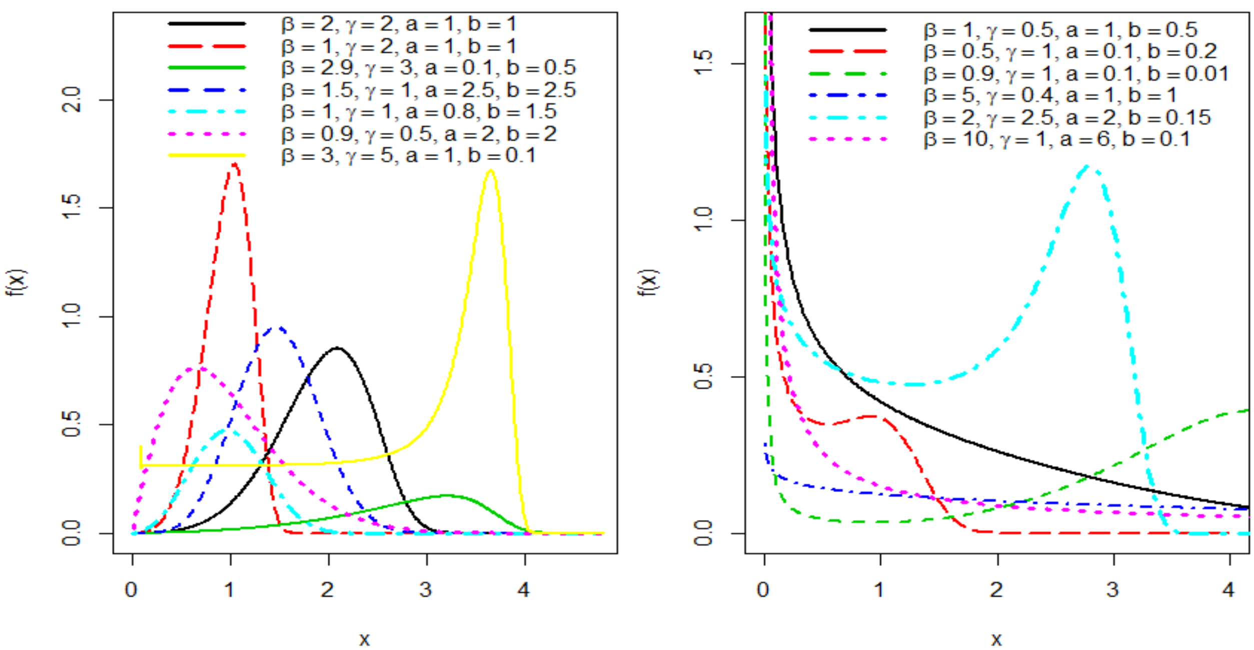

Figure 1: Plots of the BPM density for some parameter values.

The hazard function of the BPM distribution is given by

$$h(x)=\frac{\gamma x^{\gamma-1}\left(e^{(x/\beta)^{\gamma}}-1\right)e^{b\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}\left(1-e^{\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}\right)^{a-1}}{\beta^{\gamma}B(a,b)\left(1-I_{1-e^{\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}}(a,b)\right)}.$$

In Figure 2, we plot the hazard rate function of the BPM distribution for selected values of β,γ,a and b. We observe that the hazard rate function of the BPM distribution can be increasing, decreasing, and bathtub shaped or unimodal shaped depending on the values of the parameters. Therefore, the BPM distribution is quite flexible and can be used to fit various types of data sets in different domains.

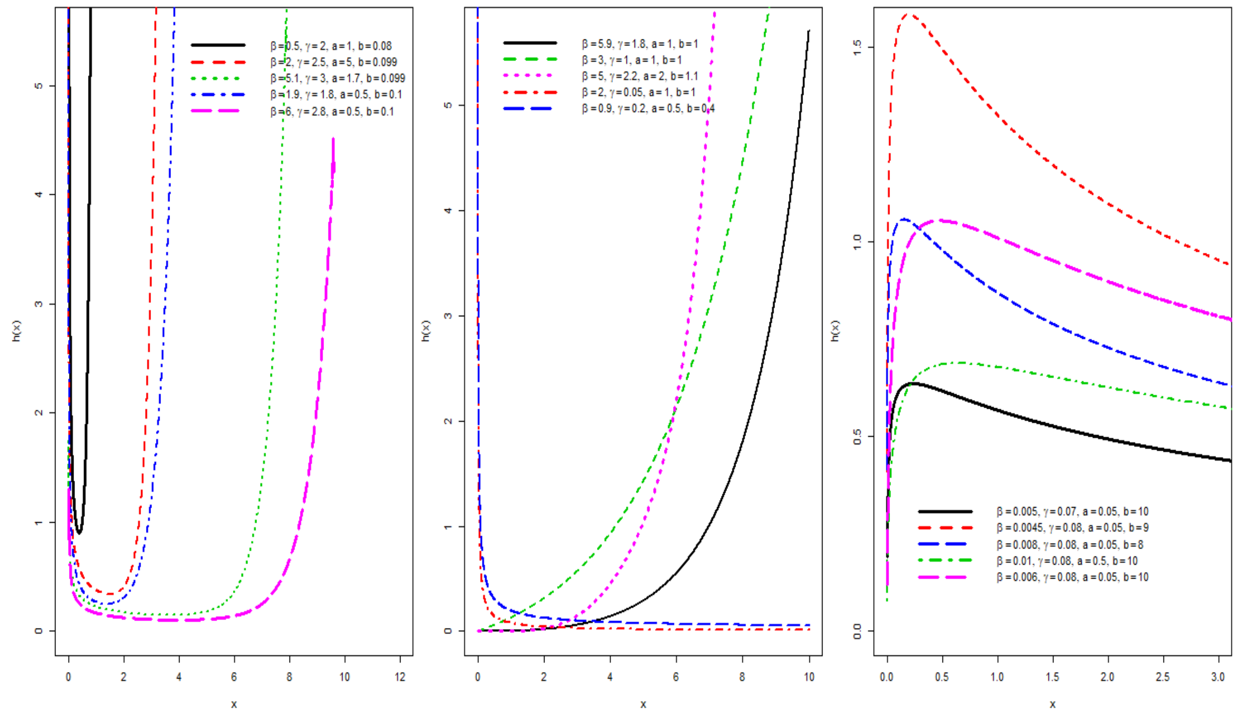

Figure 2: Plots of the hazard rate function of the BPM distribution for some parameter values.

3. Quantile function

From Proposition 3 of Jodrá et al. (2017), we have the quantile function of the PM distribution

$$Q_{PM}(u) = \beta\left[\ln\left\{W_{-1}\left(\frac{u-1}{e}\right)\right\}\right]^{1/\gamma}, \qquad 0 < u < 1,$$

where W_{-1} is the negative branch of the Lambert W function. Then, by inverting BPM cdf (5), we obtain the quantile function of the BPM distribution as follows

$$Q(u) = \beta\left[\ln\left\{W_{-1}\left(\frac{Q_{a,b}(u)-1}{e}\right)\right\}\right]^{1/\gamma},\ 0 < u < 1, \tag{7}$$

where $Q_{a,b}(u)$ is the uth quantile of a beta distribution with parameters a and b. Therefore, it is easy to simulate the BPM distribution. Let V be a beta random variable with parameters $a > 0$ and $b > 0$. Then, the random variable

$$X = \beta\left[\ln\left\{W_{-1}\left(\frac{V-1}{e}\right)\right\}\right]^{1/\gamma}, \tag{8}$$

follows the BPM distribution. From Equation (8), we can generate a random variable X having the BPM distribution when the parameters are known. The median can be derived from (7) by setting u=1/2.

4. Moments

Here, we give the moments of the BPM distribution. We can determine the skewness, kurtosis and the expected life time of a device in lifetime data. The following theorem gives the rth moment of the BPM distribution in terms of the generalized integro-exponential function, which is defined by (see Milgram, 1985):

$$E_s^k(z) = \frac{1}{\Gamma(k+1)},$$

where $z \in \mathbb{R}$, $s \in \mathbb{R}$ and $k > -1$.

Theorem 1. If $X \sim BPM(\beta, \gamma, a, b)$, then the *r*th moment of *X* is given by

$$E(X^r) = \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} \frac{\beta^r e^{(j+b)}}{(j+b)B(a,b)} E_{(j+b)+1}^{\frac{r}{\gamma}-1}(j+b).$$

Proof. The *r*th moment of *X* is

$$E(X^r) = \int_0^{\infty} x^r f(x) dx.$$

From (6) we have

$$E(X^r) = \frac{\gamma}{\beta^\gamma B(a,b)} \int_0^{\infty} x^{r+\gamma-1} \left(e^{(x/\beta)^\gamma} - 1\right) e^{b\left\{(x/\beta)^\gamma - \left(e^{(x/\beta)^\gamma} - 1\right)\right\}} \left(1 - e^{\left\{(x/\beta)^\gamma - \left(e^{(x/\beta)^\gamma} - 1\right)\right\}}\right)^{a-1} dx.$$

By making use of the binomial series expansion, if $\eta > 0$ is real non-integer and $|z| < 1$,

$$(1-z)^{\eta-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\eta-1}{j} z, \tag{9}$$

we obtain

$$E(X^r) = \frac{\gamma}{\beta^\gamma B(a,b)} \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} \int_0^{\infty} x^{r+\gamma-1} \left(e^{(x/\beta)^\gamma} - 1\right) e^{(j+b)\left\{(x/\beta)^\gamma - \left(e^{(x/\beta)^\gamma} - 1\right)\right\}} dx.$$

By setting $u = e^{(x/\beta)^\gamma}$ in the above integral, we get

$$E(X^r) = \frac{\beta^r}{B(a,b)} \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} e^{(j+b)} \int_1^{\infty} (\ln u)^{\frac{r}{\gamma}} (u-1) u^{(j+b)-1} e^{-(j+b)u} du.$$

Then

$$E(X^r) = \frac{\beta^r}{B(a,b)} \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} e^{(j+b)} \left[\int_1^{\infty} (\ln u)^{\frac{r}{\gamma}} u^{(j+b)} e^{-(j+b)u} du - \int_1^{\infty} (\ln u)^{\frac{r}{\gamma}} u^{(j+b)-1} e^{-(j+b)u} du\right].$$

From the generalized integro-exponential function we have

$$E(X^r) = \frac{\beta^r}{B(a,b)} \sum_{j=0}^{\infty} (-1)^j \binom{a-1}{j} e^{(j+b)} \Gamma\left(\frac{r}{\gamma} + 1\right) \left[E_{-(j+b)}^{r/\gamma}(j+b) - E_{-(j+b)+1}^{r/\gamma}(j+b)\right] \cdot$$

From Equation (2.4) of Milgram (1985), we have

$$E_s^k(z) = \frac{z E_{s-1}^k(z) - E_s^{k-1}(z)}{1-s}, z > 0, s \neq 1, k \geq 0.$$

Therefore, we obtain the desired result.

5. Moments of order statistics

In this section, the *p*th moment of the *i*th order statistic for the BPM distribution is given which have some applications in reliability and lifetime analysis. Let $X_{1,}, \dots, X_n$ be a simple random sample from BPM distribution and let $X_{1:n}, \dots, X_{n:n}$ denote the order statistics obtained from this sample. The following theorem gives the *p*th moment of the *i*th order statistic $X_{i:n}$.

Theorem 2. Then the *p*th moment of $X_{i:n}$ is given by

$$E\left(X_{i:n}^{p}\right)=\sum_{l=0}^{n-i}\sum_{j,k=0}^{\infty}\frac{(-1)^{k+l}\binom{n-i}{l}\binom{2a+j-1}{k}e^{(k+b)}\beta^{p}d_{j}}{(k+b)\Gamma\left(\frac{p}{\gamma}+1\right)B(i,n-i+1)\{B(a,b)\}^{i+l}}E_{-(k+b)+1}^{\frac{p}{\gamma}-1}(k+b),$$

where $d_0=a_0^s$ and $d_m=(ma_0)^{-1}\sum_{q=1}^{m}[q(s+1)-m]a_q\,d_{m-q}$, $\mathrm{m}\geq 1$.

Proof. By the definition, the *p*th moment of $X_{i:n}$ is

$$E\left(X_{i:n}^{p}\right)=\int_{0}^{\infty}x^{p}\,f_{i:n}(x)dx,$$

where $f_{i:n}(x)$ is the pdf of $X_{i:n}$ given by

$$f_{i:n}(x)=\frac{f(x)}{B(i,n-i+1)}\sum_{l=0}^{n-i}(-1)^{l}\binom{n-i}{l}\{F(x)\}^{i+l-1},$$

where $F(x)$ and $f(x)$ are given by (5) and (6), respectively. By using (9), the BPM cdf function (5) for $b>0$ real non-integer can be rewritten as

$$F(x)=\frac{1}{B(a,b)}\sum_{j=0}^{\infty}(-1)^{j}\binom{b-1}{j}\int_{0}^{1-e^{\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}}t^{a+j-1}dt$$

$$=\frac{1}{B(a,b)}\sum_{j=0}^{\infty}\frac{(-1)^{j}\binom{b-1}{j}}{(a+j)}\left(1-e^{\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}\right)^{(a+j)}.$$

From Gradshteyn and Ryzhik (2000), we have

$$\left(\sum_{r=0}^{\infty}a_{r}u^{r}\right)^{s}=\sum_{r=0}^{\infty}d_{r}u^{r},$$

where $d_0=a_0^s$ and $d_m=(ma_0)^{-1}\sum_{q=1}^{m}[q(s+1)-m]a_q\,d_{m-q}$, $\mathrm{m}\geq 1$. Therefore

$$\{F(x)\}^{i+l-1}=\frac{1}{\{B(a,b)\}^{i+l-1}}\sum_{j=0}^{\infty}d_{j}\left(1-e^{\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}\right)^{(a+j)}.$$

Then

$$f_{i:n}(x)=\frac{\gamma x^{\gamma-1}\left(e^{(x/\beta)^{\gamma}}-1\right)e^{\left\{b(x/\beta)^{\gamma}-b\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}}{\beta^{\gamma}B(i,n-i+1)}\sum_{l=0}^{n-i}\sum_{j=0}^{\infty}\frac{(-1)^{l}\binom{n-i}{l}}{\{B(a,b)\}^{i+l}}d_{j}\left(1-e^{\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}\right)^{(a+j)+a-1}.$$

By using (9), we get

$$f_{i:n}(x)=\frac{\gamma}{\beta^{\gamma}B(i,n-i+1)}\sum_{l=0}^{n-i}\sum_{j,k=0}^{\infty}\frac{(-1)^{k+l}\binom{n-i}{l}\binom{2a+j-1}{k}}{\{B(a,b)\}^{i+l}}d_{j}x^{\gamma-1}\left(e^{(x/\beta)^{\gamma}}-1\right)e^{(k+b)\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}.$$

Therefore

$$E\left(X_{i:n}^{p}\right)=\frac{\gamma}{\beta^{\gamma}B(i,n-i+1)}\sum_{l=0}^{n-i}\sum_{j,k=0}^{\infty}\frac{(-1)^{k+l}\binom{n-i}{l}\binom{2a+j-1}{k}d_{j}}{\{B(a,b)\}^{i+l}}\int_{0}^{\infty}x^{p+\gamma-1}\left(e^{(x/\beta)^{\gamma}}\right.$$

$$\left.-1\right)e^{(k+b)\left\{(x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right\}}dx.$$

Similarly, as a proof of Theorem 1, we obtain $E\left(X_{i:n}^{p}\right)$.

6. Estimation

Here, the maximum likelihood estimates (MLEs) of the parameters of the BPM distribution are presented. Let $x_1, \dots, x_n$ be n random observations from the BPM distribution with unknown parameter vector $\xi = (\beta, \gamma, a, b)^T$. Then, the log-likelihood function, denoted by $\ell(\xi)$, for ξ, is given by

$$\ell(\xi) = n\ln\gamma + n\gamma\ln\beta - n\ln[B(a,b)] + (\gamma-1)\sum_{i=1}^{n}\ln(x_i) + \sum_{i=1}^{n}\ln\left(e^{(x_i/\beta)^\gamma}-1\right) + \frac{b}{\beta^\gamma}\sum_{i=1}^{n}{x_i}^{\gamma}$$
$$- b\sum_{i=1}^{n}\left(e^{(x_i/\beta)^\gamma}-1\right) + (a-1)\sum_{i=1}^{n}\ln\left(1-e^{\left\{(x_i/\beta)^\gamma-\left(e^{(x_i/\beta)^\gamma}-1\right)\right\}}\right).$$

Therefore, the log-likelihood equations are obtained, by taking the partial derivatives of $\ell(\xi)$ with respect to β, γ, a and b, as follows

$$U_\beta(\xi) = \frac{n\gamma}{\beta} - \frac{\gamma}{\beta^{\gamma+1}}\sum_{i=1}^{n}\frac{x_i^\gamma e^{(x_i/\beta)^\gamma}}{e^{(x_i/\beta)^\gamma}-1} - \frac{b\gamma}{\beta^{\gamma+1}}\sum_{i=1}^{n}x_i^\gamma$$
$$+\frac{b\gamma}{\beta^{\gamma+1}}\sum_{i=1}^{n}x_i^\gamma e^{(x_i/\beta)^\gamma} - \frac{\gamma(a-1)}{\beta^{\gamma+1}}\sum_{i=1}^{n}\frac{x_i^\gamma\left(e^{(x_i/\beta)^\gamma}-1\right)e^{\left\{(x_i/\beta)^\gamma-\left(e^{(x_i/\beta)^\gamma}-1\right)\right\}}}{1-e^{\left\{(x_i/\beta)^\gamma-\left(e^{(x_i/\beta)^\gamma}-1\right)\right\}}},$$

$$U_\gamma(\xi) = \frac{n}{\gamma} - n\ln\beta + \sum_{i=1}^{n}\ln(x_i)$$
$$+\frac{1}{\beta^\gamma}\sum_{i=1}^{n}\frac{x_i^\gamma\ln(x_i/\beta)\,e^{(x_i/\beta)^\gamma}}{e^{(x_i/\beta)^\gamma}-1} + \frac{b}{\beta^\gamma}\sum_{i=1}^{n}x_i^\gamma\ln(x_i/\beta)\left(1-e^{(x_i/\beta)^\gamma}\right)$$
$$-\frac{(a-1)}{\beta^\gamma}\sum_{i=1}^{n}\frac{x_i^\gamma\ln(x_i/\beta)\left(1-e^{(x_i/\beta)^\gamma}\right)e^{\left\{(x_i/\beta)^\gamma-\left(e^{(x_i/\beta)^\gamma}-1\right)\right\}}}{1-e^{\left\{(x_i/\beta)^\gamma-\left(e^{(x_i/\beta)^\gamma}-1\right)\right\}}},$$

$$U_a(\xi) = -n[\psi(a)+\psi(a+b)] + \sum_{i=1}^{n}\ln\left(1-e^{\left\{(x_i/\beta)^\gamma-\left(e^{(x_i/\beta)^\gamma}-1\right)\right\}}\right),$$

and

$$U_b(\xi) = -n[\psi(b)+\psi(a+b)] + \frac{1}{\beta^\gamma}\sum_{i=1}^{n}x_i^\gamma - \sum_{i=1}^{n}\left(e^{(x_i/\beta)^\gamma}-1\right),$$

where $\psi(\cdot)$ is the digamma function.

Solving the non-linear likelihood equations $U_\beta(\xi)=0, U_\gamma(\xi)=0, U_a(\xi)=0$ and $U_b(\xi)=0$ simultaneously, we obtain the MLE $\hat{\xi} = \left(\hat{\beta}, \hat{\gamma}, \hat{a}, \hat{b}\right)^T$. To construct approximate confidence intervals, we use the multivariate normal $N_4\left(0, J^{-1}(\xi)\right)$ distribution, where $J^{-1}(\xi)$ is the inverse of the expected information matrix evaluated at ξ. The information matrix given by

$$J(\xi) = -\begin{pmatrix} U_{\beta\beta} & U_{\beta\gamma} & U_{\beta a} & U_{\beta b} \\ U_{\beta\gamma} & U_{\gamma\gamma} & U_{\gamma a} & U_{\gamma b} \\ U_{\beta a} & U_{\gamma a} & U_{aa} & U_{ab} \\ U_{\beta b} & U_{\gamma b} & U_{ab} & U_{bb} \end{pmatrix},$$

whose elements are given in Appendix B. So, the approximate confidence intervals for β, γ, a and b are given, respectively, by

$$\hat{\beta} \pm Z_{\frac{\omega}{2}}\sqrt{var\left(\hat{\beta}\right)}, \qquad \hat{\gamma} \pm Z_{\frac{\omega}{2}}\sqrt{var(\hat{\gamma})}, \qquad \hat{a} \pm Z_{\frac{\omega}{2}}\sqrt{var(\hat{a})} \quad and \quad \hat{b} \pm Z_{\frac{\omega}{2}}\sqrt{var\left(\hat{b}\right)},$$

where *var*(·) is the diagonal element of $J^{-1}(\xi)$ corresponding to each parameter and $Z_{\frac{\omega}{2}}$ is the quantile 100(1-ω/2)% of the standard normal distribution.

7. Monte Carlo simulation study

In this section, we conduct a simulation study to check the performance of the MLEs of the parameters of the BPM distribution. From Equation (8), we generate random samples of sizes n =20, 50,100,200 and 500 from the BPM distribution using the following sets of parameters:

- Set I: $\beta = 0.2,\ \gamma = 0.5, a = 2.5, b = 5,$
- Set II: $\beta = 2.5,\ \gamma = 1, a = 2, b = 1,$
- Set III: $\beta = 5,\ \gamma = 0.2, a = 2, b = 2.$

The simulation is performed via the statistical software **R** through the command *mle*. The number of Monte Carlo replications made was *N*=1000. The evaluation of the performance is based on the bias and the mean squared errors (MSE) defined as follows:

$$Bias = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon)\ and\ MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon)^2,$$

where $\epsilon = \beta, \gamma, a$ and b. The results of our simulation study are summarized in Table 1. We can see that the bias and MSE of the MLEs decrease when the sample size increases, as expected. This verifies the consistency properties of the MLEs, i.e., we can conclude that the maximum likelihood method performs well for estimating the parameters of the BPM distribution.

Table 1: Monte Carlo simulation results for the BPM bias and MSEs.

		Set I		Set II		Set III	
Sample size	Parameter	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
n=20	β	0.5622	0.3469	0.9820	0.0268	0.0658	0.0195
	γ	0.5418	0.3348	0.1808	1.3121	0.9343	1.2206
	a	-0.8340	0.7457	0.1658	0.1787	0.6632	2.9043
	b	0.4960	0.2925	0.3241	0.7857	0.8042	1.6549
n=50	β	0.4996	0.2496	0.8433	0.0159	0.0226	0.0065
	γ	0.3911	0.2842	0.1434	0.6799	0.1474	0.3497
	a	-0.6893	0.5903	0.0498	0.0566	0.3276	2.1792
	b	0.3811	0.2254	0.1752	0.1827	0.2269	0.3907
n=100	β	0.4261	0.2201	0.2230	0.0104	0.0158	0.0062
	γ	-0.1893	0.0573	0.1353	0.4343	0.0338	0.0654
	a	0.0243	0.0010	0.0019	0.0231	0.2480	1.2747
	b	-0.0764	0.0106	0.1373	0.1063	0.0611	0.1206
n=200	β	0.2330	0.0462	0.0073	0.0100	0.0133	0.0057
	γ	-0.1448	0.0373	0.0424	0.1470	0.0134	0.0255
	a	0.0161	0.0006	-0.0073	0.0118	0.2289	0.6935
	b	-0.0584	0.0073	0.1074	0.0630	0.0106	0.0553
n=500	β	0.1172	0.0228	0.0012	0.0007	-0.0209	0.0015
	γ	0.0011	0.00042	0.0279	0.0576	-0.0066	0.0094
	a	-0.0030	0.003	-0.0138	0.0058	0.0238	0.3746
	b	0.0011	0.0004	0.0756	0.0369	-0.0207	0.0246

8. The LBPM Regression Model

In some practical applications, the lifetimes are affected by many explanatory variables and for this reason the regression models are widely used to estimate univariate survival functions for censored data. Among them, the location-scale regression model is distinguished since it is frequently used in clinical trials. In this section, we introduce a new location-scale regression model based on the logarithm of the BPM distribution. If $X \sim BPM(\beta, \gamma, a, b)$ then the random variable Y $= \ln X$ has the log-BPM (LBPM) distribution. The density function of Y, replacing $\sigma = 1/\gamma$ and $\mu = \ln \beta$, is given by

$$f(y) = \frac{e^{\frac{y-\mu}{\sigma}}\left(e^{e^{\frac{y-\mu}{\sigma}}}-1\right)}{\sigma B(a,b)} e^{b\left\{e^{\frac{y-\mu}{\sigma}}-\left(e^{e^{\frac{y-\mu}{\sigma}}}-1\right)\right\}}\left(1-e^{\left\{e^{\frac{y-\mu}{\sigma}}-\left(e^{\frac{y-\mu}{\sigma}}-1\right)\right\}}\right)^{a-1}. \tag{10}$$

where $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the scale parameter and $a > 0$ and b > 0 are the shape parameters. The corresponding survival function is

$$S(y) = 1 - I_{1-exp\left\{e^{\frac{y-\mu}{\sigma}}-\left(e^{e^{\frac{y-\mu}{\sigma}}}-1\right)\right\}}(a,b). \tag{11}$$

Then, the pdf of the standardized random variable $Z = \frac{y-\mu}{\sigma}$ is given by

$$f(z) = \frac{e^{z}\left(e^{e^{z}}-1\right)}{\sigma B(a,b)} e^{b\left\{e^{z}-\left(e^{e^{z}}-1\right)\right\}}\left(1-e^{\{e^{z}-(e^{z}-1)\}}\right)^{a-1}. \tag{12}$$

The linear location-scale regression model linking the response variable y_i and the explanatory variable vector $v_i^T = (v_{i1}, \dots, v_{ip})$ is given by

$$y_i = v_i^T\theta + \sigma z_i, \qquad i = 1, \dots, n,$$

where the random error z_i has density function (12) and $\theta = (\theta_1, \dots, \theta_p)^T$ is the unknown vector of regression coefficients. The parameter $\mu_i = v_i^T\theta$ is the location of y_i. The location parameter vector $\mu = (\mu_1, \dots, \mu_p)^T$ is represented by a linear model μ=Vθ, where $V = (v_1, \dots, v_p)^T$ is a known model matrix. The LBPM regression model contains LEPM regression for (b = 1) and LPM for (a = b = 1) regression models as special sub-models.

Suppose $(y_1, v_1), \dots, (y_1, v_1)$ is sample of n independent observations, where the random response is defined by: $y_i = min\{\ln(x_i), \ln(c_i)\}$. Let F and C be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. The log-likelihood function for the vector of parameters $\tau = (a, b, \sigma, \theta^T)^T$ is given by:
$\ell(\xi) = \sum_{i\in F}\ln(f(y_i)) + \sum_{i\in C}\ln(S(y_i))$, where $f(y_i)$ is the density function (10) and $S(y_i)$ is the survival function (11) of Y_i. The log-likelihood function for τ is:

$$\begin{aligned}\ell(\tau) = -r[\ln\sigma + \ln B(a,b)] + \sum_{i\in F}\ln(t_i) + \sum_{i\in F}\ln(e^{t_i}-1) + b\sum_{i\in F}\left(e^{t_i} - e^{e^{t_i}} + 1\right)\\ + (a-1)\sum_{i\in F}\ln\left(1-e^{\left\{e^{t_i}-e^{e^{t_i}}+1\right\}}\right) + \sum_{i\in C}\ln\left(1 - I_{1-exp\left(e^{t_i}-e^{e^{t_i}}+1\right)}(a,b)\right),\end{aligned}$$

where r is the number of uncensored observations (failures) and $t_i = e^{z_i}$. We can obtain the MLE $\hat{\tau}$ of τ by maximizing the loglikelihood function $\ell(\tau)$. To compute this estimate, we use the procedure *optim* in **R** software.

8.1. Residual Analysis

Residual analysis has an important role in judging the adequacy of the fitted model. To study departures from error assumptions and the presence of outliers, we consider here residual analysis based on the martingale and modified deviance residuals.

8.1.1. Martingale residual

The martingale residual is defined in counting processes, for more details see Fleming and Harrington (1994). The martingale residuals for LBPM model is

$$r_{M_i} = \begin{cases} 1 + \ln\left(1 - I_{1-exp\left(e^{t_i^*} - e^{e^{t_i^*}} + 1\right)}(\hat{a}, \hat{b})\right) & if \ \ i \in F, \\ \ln\left(1 - I_{1-exp\left(e^{t_i^*} - e^{e^{t_i^*}} + 1\right)}(\hat{a}, \hat{b})\right) & if \ \ i \in C, \end{cases}$$

where $t_i^* = \frac{y_i - \hat{\mu}}{\hat{\sigma}}$ with $\hat{\mu} = v_i^T \hat{\theta}$.

8.1.2. Modified Deviance Residual

The main drawback of the martingale residual is that when the fitted model is correct, it is not symmetrically distributed about zero. To overcome this problem, modified deviance residual was proposed by Therneau et al. (1990). Th modified deviance residual for LBPM model is

$$r_{D_i} = \begin{cases} sign(r_{M_i})\left(-2\left[r_{M_i} + \ln(1 - r_{M_i})\right]\right)^{1/2} & if \ \ i \in F, \\ sign(r_{M_i})(-2r_{M_i})^{1/2} & if \ \ i \in C, \end{cases}$$

where r_{M_i} is the martingale residual.

9. Data Analysis

In order to show the flexibility of the BPM distribution we use two reliability real data sets with different shapes. For these data sets, we compare the fit of the BPM distribution with the PM, EPM (Irshad et al., 2021),, beta generalized Weibull (BGW) (Singla et al., 2012), beta Weibull (BW) (Famoye et al., 2005), Kumaraswamy Weibull (KW) (Corderio et al., 2010), McDonald Weibull (Mc-W) (Corderio et al. 2014), transmuted Weibull (TW) (Aryall and Tsokos, 2011), beta generalized Gompertz (BGG) (Benkhelifa, 2017), alpha power Weibull (APW) (Nassar et al., 2017), new generalized odd log-logistic flexible Weibull (GOLLFW) (Prataviera et al., 2018), exponentiated additive Weibull distribution (EAW) (Ahmad and Ghazal, 2020). The pdf's of these distributions are given in Appendix A.

In order to verify which distribution fits better to the data sets, we compute the values of the log-likelihood functions (-2$\hat{\ell}$), Akaike information criterion (AIC), consistent Akaike information criteria (CAIC), Bayesian information criterion (BIC) and the Kolmogorov-Smirnov (K-S) statistic with corresponding p-value. The better model corresponds to smaller values of these measures and high p-value.

9.1. First data set: *devices failure time data*

The first data set is given by Aarset (1987) and represents the time to first failure of 50 devices (in weeks). This data set is: 0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86. The TTT-plot of this data set, in Figure 3(a), shows a convex shape followed by a concave shape. This corresponds to a bathtub shaped hazard rate function. So, the BPM distribution is appropriate for modeling the first data.

Table 2 presents the values of the MLEs of the parameters for all fitted distributions. We see from Table 3 that the BPM distribution has the smallest values of -2$\hat{\ell}$, AIC, BIC, CAIC and K-S and largest p-value. Hence, the BPM distribution gives an excellent fit than the others models for the first data set. In addition, we plot the histogram of this data set and the fitted pdfs in Figure 4(a). This Figure shows that the BPM pdf provides a closer fit to the histogram than other distribution. The plots of the estimated cdfs and empirical cumulative function are displayed in Figure 4(b). These plots reveal that the BPM cdf is the closest curve to the empirical cumulative function. So, we conclude that the BPM model is the best.

The variance-covariance matrix for the estimated parameters of the BPM distribution is given by

$$J^{-1}(\hat{\xi}) = \begin{pmatrix} 2.31703 & 0.02339 & 0.00624 & 0.02355 \\ 0.02339 & 0.00567 & -0.00024 & -0.00034 \\ 0.00624 & -0.00024 & 0.00025 & 0.00017 \\ 0.02355 & -0.00034 & 0.00017 & 0.00053 \end{pmatrix}.$$

So, the approximate 95% confidence intervals for the parameters β, γ, *a* and b, are [51.5405, 57.5074], [2.4914, 2.7866], [0.0539, 0.1161], [0.0409, 0.1307] respectively.

Table 2: MLEs of the model parameters and the corresponding standard errors given in parentheses.

Model	β	γ	λ	*a*	b
BPM	54.5240(1.522)	2.6390(0.0753)	-	0.0850(0.01587)	0.0858(0.0229)
PM	33.5676(5.3343)	0.4254(0.0557)	-	-	-
EPM	82.809(2.6231)	2.4668(0.0776)	-	0.1377(0.0203)	-
BGW	1.9343(0.0197)	0.0337(0.1774)	0.0023(0.1868)	3.5284(0.0006)	0.1456(0.0728)
BW	5.1699(0.0064)	-	0.0217(0.0239)	0.0899(0.1123)	0.0726(0.1214)
KW	1.8942(0.5097)	-	0.0028(0.0071)	0.1395(0.0724)	0.1369(0.0698)
Mc-W	1.7122(0.3520)	1.8145(0.1028)	0.0169(0.0016)	0.2786(0.1772)	0.0525(0.1186)
TW	0.8958(0.1286)	-	0.0386(0.0230)	-0.3279(0.2592)	-
BGG	0.0037(0.0071)	2.5748(2.9461)	0.0688(0.0211)	0.1172(0.1231)	0.0983(0.1161)
APW	0.8355(0.1372)	-	0.0586(0.1372)	4.5265(4.0567)	-
GOLLFW	29.757(7.32×10^{-5})	0.1058(2.76×10^{-2})	-	0.0383(3.25×10^{-3})	0.1870(8.02×10^{-3})
EAW	1.5554(0.4801)	1.2450(0.1315)	0.1051(0.0312)	0.0019(0.0041)	7.8701(0.01369)

Table 3: The statistics: -2$\hat{\ell}$, AIC, BIC, CAIC, K-S and p-value.

Model	-2$\hat{\ell}$	AIC	BIC	CAIC	K-S	p-value
BPM	429.3273	437.3273	444.9754	438.2162	0.1200	0.4675
PM	476.6327	480.6327	484.4567	480.888	0.19217	0.0498
EPM	452.7223	458.7223	464.4584	459.2441	0.2053	0.0295
BGW	461.6844	471.6844	481.2445	473.048	0.1628	0.1414
BW	457.7916	465.7916	473.4397	466.6805	0.1397	0.2826
KW	461.7876	469.7876	477.4357	470.6765	0.1526	0.1941
Mc-W	460.388	470.388	479.9481	471.7516	0.1739	0.0971
TW	480.7334	486.7334	492.4695	487.2551	0.1841	0.0673
BGG	439.0648	449.0648	458.6249	450.4284	0.1276	0.3893
APW	479.2431	485.2431	490.9792	485.7648	0.1749	0.0936
GOLLFW	440.1259	448.1259	455.7740	449.0148	0.1493	0.2149
EAW	461.1976	471.1976	480.7577	472.5613	0.1456	0.2392

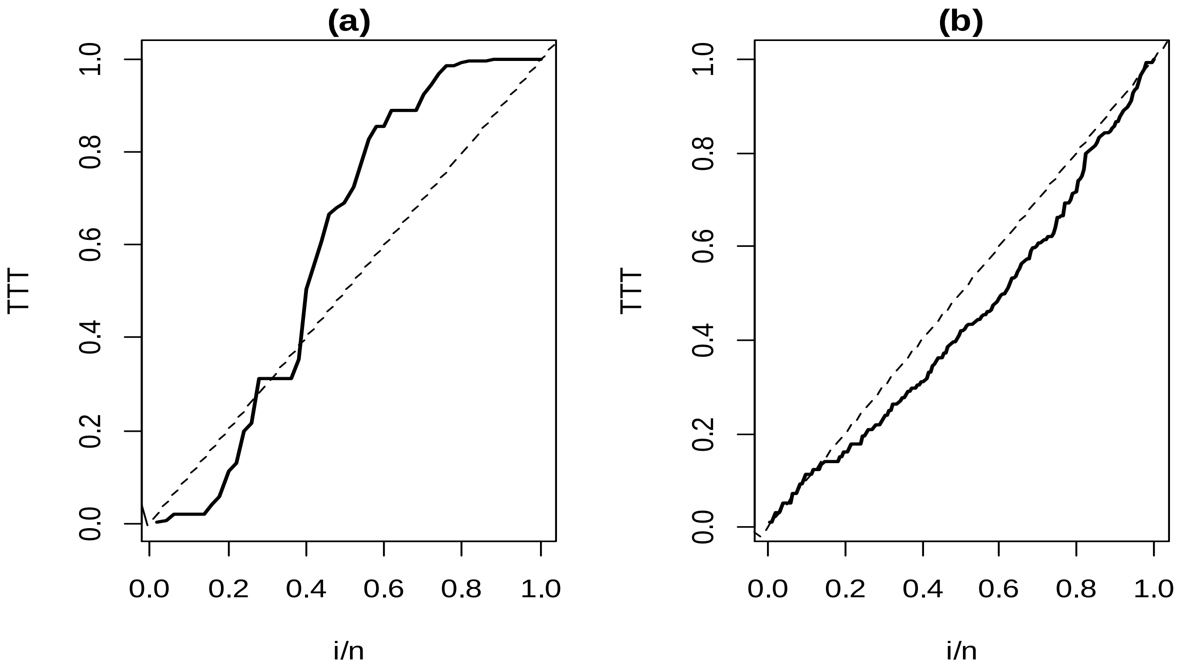

Figure 3: TTT-plot for (a) Aarset data (b) number of successive failures data.

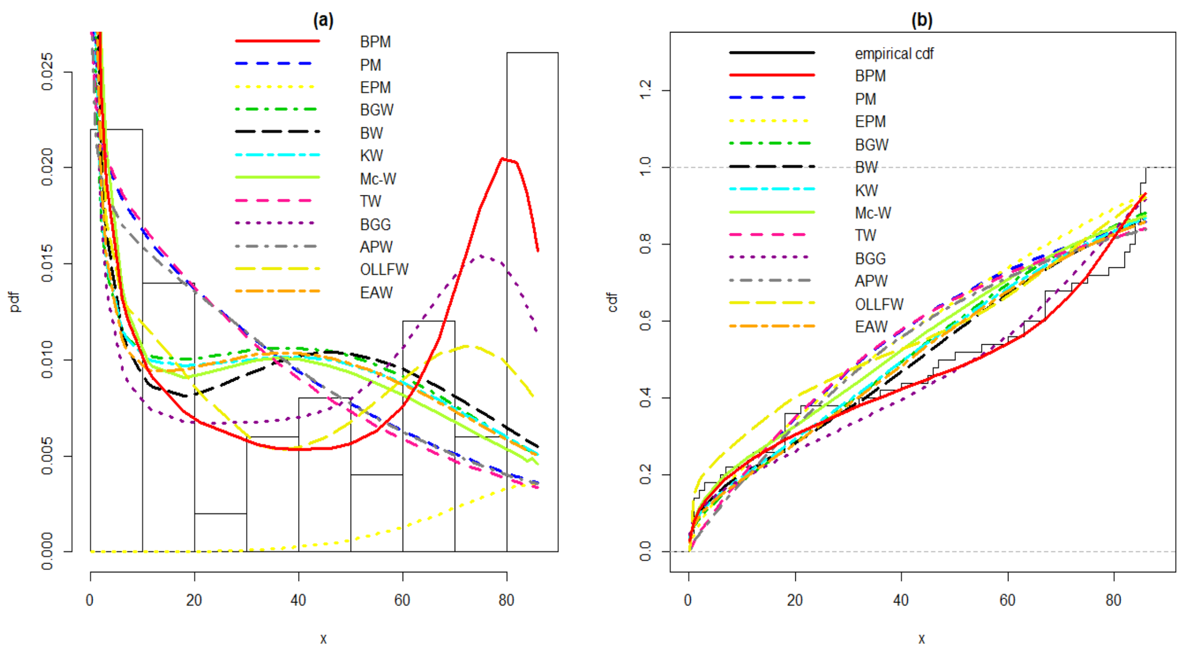

Figure 4: (a) Plots of the histogram and the fitted densities (b) Plots of the empirical cdf and estimated cdfs of Aarest data.

9.2. Second data set: *Number of successive failures data*

The data have been presented by Proschan (1963). The data set is: 194, 413, 90, 74, 55, 23, 97, 50, 359, 50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71. This data set has an upside-down bathtub shaped failure rate function as shown by the scaled TTT-plot, which has a concave shape followed by a convex shape; see Figure 3(b). The BPM distribution is appropriate for modeling these data.

Table 4 gives the MLEs of the parameters of all models used here for the second data set. The values of -2$\hat{\ell}$, AIC, BIC, CAIC, K-S and its p-value are listed in Table 5. From this Table, we can see that the BPM distribution as the best fit for the second data set. Figures 5(a), 5(b) illustrate the pdfs and empirical cdfs, respectively, of the comparative models to show the over fitting of the BPM distribution.

The estimated variance-covariance matrix of the BPM distribution for the this data set is

$$J^{-1}(\hat{\xi}) = \begin{pmatrix} 0.79321 & 0.00764 & -1.98644 & 0.08687 \\ 0.00764 & 0.00045 & -0.03481 & -0.00061 \\ -1.98644 & -0.03481 & 6.48624 & -0.15314 \\ 0.08687 & -0.00061 & -0.15314 & 0.01519 \end{pmatrix}.$$

Then, the approximate 95% confidence intervals for the parameters β, γ, *a* and b, are, respectively, [0, 0.6518], [0.1558, 0.2395], [0.22331, 0.2068], [0, 0.4112].

Table 4: MLEs of the model parameters and the corresponding standard errors given in parentheses.

Model	β	γ	λ	*a*	b
BPM	0.9062(0.8906)	0.1976(0.0213)	-	5.2150(2.5468)	0.1697(0.1232)
PM	63.6606(6.1905)	0.3648(0.0204)	-	-	-
EPM	3.8633(4.7341)	0.1616(0.0381)	-	5.1627(2.7911)	-
BGW	0.4288(0.2270)	0.6925(1.4808)	0.1005(0.3520)	6.3630(13.5748)	5.2195(16.5406)
BW	0.4010(0.1856)	-	0.1279(0.2949)	4.6125(3.8441)	5.2263(17.7615)
KW	0.7284(0.0911)	-	0.2923(0.8001)	2.9535(1.8606)	0.1656(0.2089)
Mc-W	0.6004(0.1272)	19.9958(3.0166)	0.1445(0.1258)	2.6628(1.3279)	0.7559(0.1167)
TW	0.9668(0.0565)	-	0.0099(0.0037)	0.4663(0.2808)	-
BGG	0.0336(1.82$\times 10^{-2}$)	10.266(7.35$\times 10^{-4}$)	2.9$\times 10^{-6}$(1$\times 10^{-3}$)	0.144(3.69$\times 10^{-2}$)	0.248(1.06$\times 10^{-1}$)
APW	0.4765(4.76$\times 10^{-2}$)	-	0.3054(4.02$\times 10^{-2}$)	197.44(2.12$\times 10^{-5}$)	-
GOLLFW	1.6681(0.0718)	2.9655(0.7605)	-	0.0019(0.0004)	2.4440(0.7314)
EAW	0.3394(0.1710)	0.0030(0.0253)	1.0060(0.2716)	0.5481(0.5842847)	6.4281(6.4167)

Table 5: The statistics: $-2\hat{\ell}$, AIC, BIC, CAIC, K-S and p-value.

Model	$-2\hat{\ell}$	AIC	BIC	CAIC	K-S	p-value
BPM	2064.106	2072.106	2085.052	2072.325	0.0361	0.9669
PM	2080.665	2084.665	2091.138	2084.73	0.0706	0.3065
EPM	2066.035	2072.035	2081.745	2072.166	0.0447	0.8475
BGW	2066.094	2076.094	2092.276	2076.423	0.0442	0.8556
BW	2066.151	2074.151	2087.097	2074.37	0.0443	0.8529
KW	2064.732	2072.732	2085.677	2072.950	0.0395	0.9307
Mc-W	2066.23	2076.23	2092.412	2076.560	0.0455	0.8298
TW	2070.853	2076.853	2086.563	2076.984	0.0465	0.8095
BGG	2064.774	2074.774	2089.956	2074.103	0.0381	0.9587
APW	2071.108	2077.108	2086.817	2077.238	0.0504	0.7259
GOLLFW	2106.803	2114.803	2127.749	2115.022	0.0969	0.05816
EAW	2065.315	2075.315	2091.497	2075.644	0.04231	0.8894

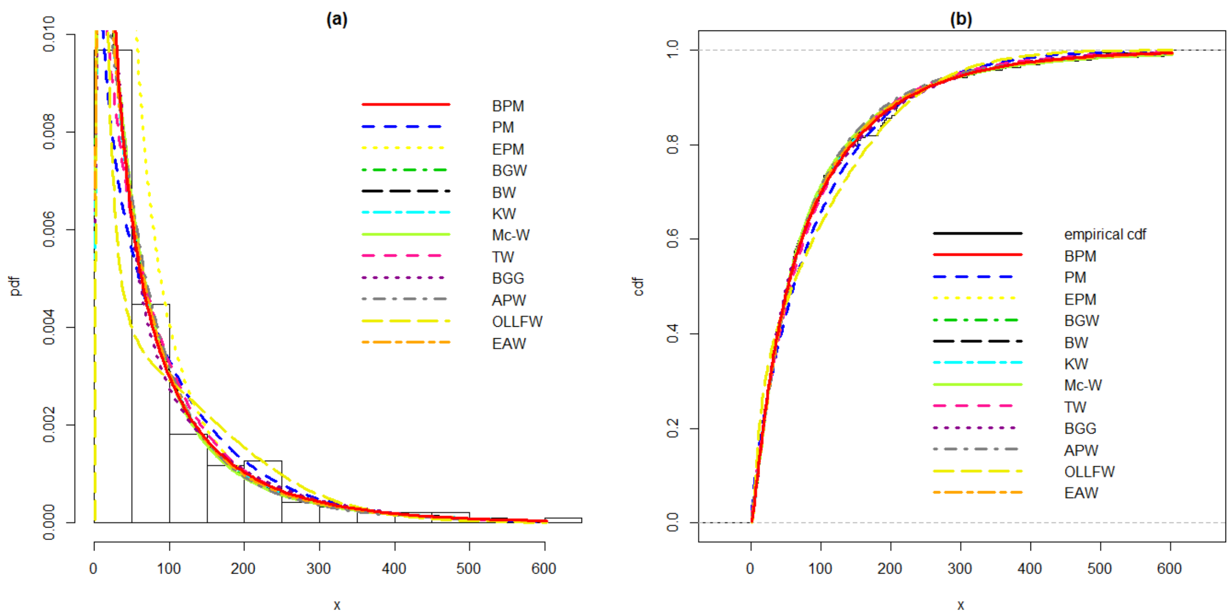

Figure 5: (a) Plots of the histogram and the fitted densities (b) Plots of the empirical cdf and estimated cdfs of successive failures data.

9.3. Third data set: *HIV survival data*

This data set is reported in Hosmer and Lemeshow (1999) and also it is available in *Bolstad2* package of **R** software. The sample size is n=100 on HIV+ subjects belonging to an health maintenance organization, where the goal is to evaluate the survival time of these subjects. Alizadeh et al. (2017) adopted the log-odd power Cauchy-Weibull (LOPCW) regression model to analyse this data set. We use the same data set to prove the flexibility of LBPM regression model, where the aim of this study is to relate the survival time (y) with the history of drug use (v). The variables are: y_i observed survival time (in months), $cens_i$: censoring indicator (0= alive at study end or lost to follow up, 1= death due to AIDS or AIDS related factors) and v_{i1} (1= yes, 0= no) represents the history of the drug use. The regression model fitted to the data set is given by

$$y_i = \theta_{0.} + \theta_1 v_i + \sigma z_i,$$

where y_i has the LBPM density (10) for i=1,…,100. We compare the LBPM regression model with the LPM, LEPM, LOPCW and log-Weibull (LW) regression models. Table 6 gives the MLEs, their approximate standard errors and p-values obtained from the fitted these models, $-2\hat{\ell}$, AIC, BIC and CAIC statistics. These results indicate that the LBPM regression model has the lowest values of the $-2\hat{\ell}$, AIC, BIC and CAIC. Therefore, it is clear that the LBPM model provides an adequate fit to HIV survival data. We can observe that the explanatory variable v_{i1} is significant at the level of 1%.

Figure 6(a) displays the modified deviance residuals (see Section 8) against the index of the observations. Figure 6(b) gives the normal probability plot with generated envelope. We conclude that none of observed values appear as possible outliers. Therefore, the fitted model is is very suitable for this data set.

Table 6: MLEs of the model parameters standard errors in (.) and p-values in [.] and the values of $-2\hat{\ell}$, AIC, BIC and CAIC.

Model	a	b	σ	θ_0	θ_1	$-2\hat{\ell}$	AIC	BIC	CAIC
LBPM	53.559	0.500	9.624	-4.7662	-0.826	241.853	251.853	264.878	252.491
	(77.715)	(0.079)	(3.613)	(3.674)	(0.270)				
				[0.098]	[0.0014]				
LEPM	46.161		12.737	-6.316	-0.851	281.759	289.759	300.180	290.180
	96.609		7.370	7.273	0.270				
				[0.1936]	[0.0011]				
LPM			2.703	2.666	-1.000	298.915	304.915	312.730	305.165
			(0.221)	(0.175)	(0.235)				
				[<0.001]	[<0.001]				
LOPCW		3.287	3.784	3.843	-1.001	286.660	294.661	305.080	295.081
		(2.921)	(3.348)	(1.206)	(0.252)				
				[<0.001]	[<0.001]				
LW			1.071	3.003	-1.052	292.875	298.875	306.690	299.125
			(0.088)	(0.166)	(0.239)				
				[<0.001]	[<0.001]				

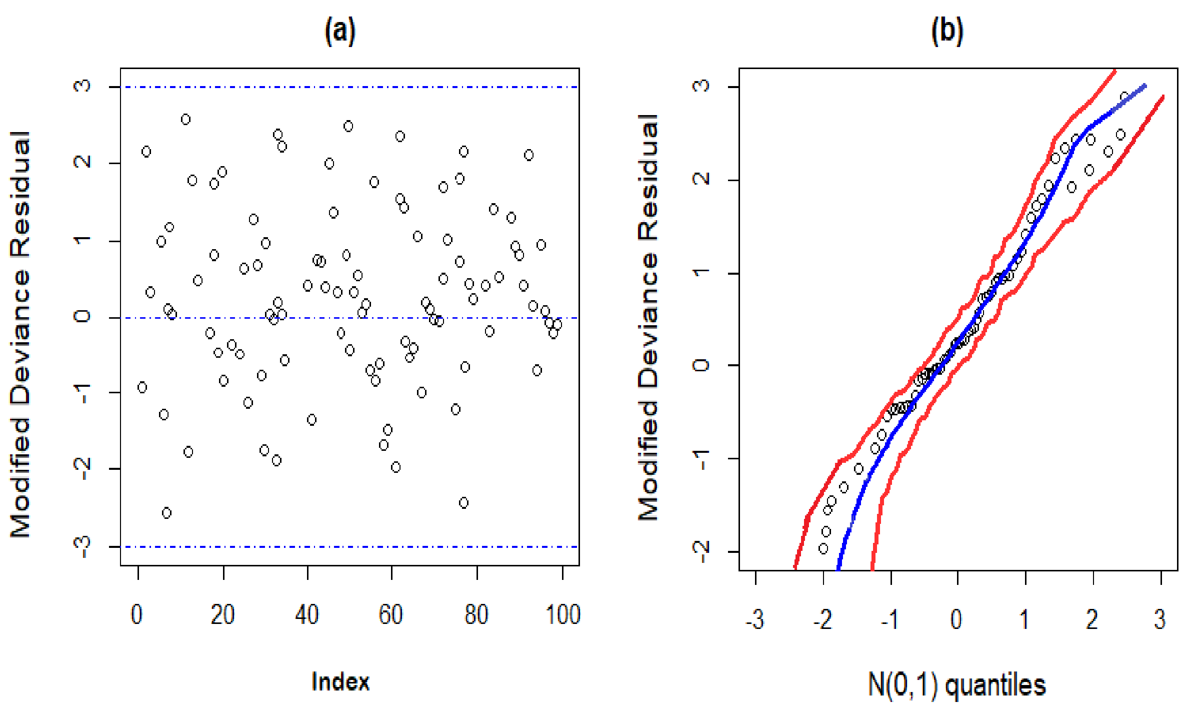

Figure 6: (a) Index plot of the modified deviance residuals and (b) Normal probability plot for the modified deviance residuals with envelope from the fitted LBPM regression model.

10. Conclusions

We have introduced a new four-parameter model called the beta power Muth distribution. This distribution has as sub-models the power Muth and exponentiated power Muth distributions. We have derived explicit expressions for the moments, quantile function and moments of the order statistics associated with the proposed model. We have used the maximum likelihood method to estimate the model parameters. We have presented two examples involving reliability data sets. One of the data sets has a bathtub shaped failure rate and the other has an upside-down bathtub shaped failure rate function. For these data sets, our model provides the best fit than other competitive models. Further, we have defined the LBPM regression model and shown that this model gives an adequate fit to HIV survival data.

Appendix A.

In this appendix, we give the pdf of each distribution used in the application section.

- EPM distribution

$$f(x)=\frac{a\gamma x^{\gamma-1}\left(e^{(x/\beta)^{\gamma}}-1\right)e^{b\left((x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right)}}{\beta^{\gamma}}\left(1-e^{\left((x/\beta)^{\gamma}-\left(e^{(x/\beta)^{\gamma}}-1\right)\right)}\right)^{a-1},$$

  where $x, \beta, \gamma, a > 0$.

- BGW distribution

$$f(x)=\frac{\beta\gamma\lambda x^{\beta-1}e^{-\lambda x^{\beta}}}{B(a,b)}\left(1-e^{-\lambda x^{\beta}}\right)^{\gamma a-1}\left(1-\left(1-e^{-\lambda x^{\beta}}\right)^{\gamma}\right)^{b-1},$$

  where x, β, γ, λ, a, b>0.

- BW distribution

$$f(x)=\frac{\beta\lambda x^{\beta-1}}{B(a,b)}\left(1-e^{-\lambda x^{\beta}}\right)^{a-1}e^{-b\lambda x^{\beta}},$$

  where $x, \beta, \lambda, a, b > 0$.

- Mc-W distribution

$$f(x)=\frac{\beta\gamma\lambda x^{\beta-1}e^{-\lambda x^{\beta}}}{B(a/\gamma,b)}\left(1-e^{-\lambda x^{\beta}}\right)^{a-1}\left(1-\left(1-e^{-\lambda x^{\beta}}\right)^{\gamma}\right)^{b-1},$$

  where $x, \beta, \gamma, \lambda, a, b > 0$.

- TW distribution

$$f(x)=\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}\left(1+a-2ae^{-\lambda x^{\beta}}\right),$$

  where $x, \beta, \lambda, a > 0$.

- BGG distribution

$$f(x)=\frac{\beta\gamma e^{\lambda x}e^{-\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}}{B(a,b)}\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}\right)^{\gamma a-1}\left(1-\left(1-e^{-\frac{\beta}{\lambda}\left(e^{\lambda x}-1\right)}\right)^{\gamma}\right)^{b-1},$$

  where $x, \beta, \gamma, \lambda, a, b > 0$.

- APW distribution

$$f(x)=\frac{\ln a}{a-1}\lambda\beta x^{\beta-1}e^{-\lambda x^{\beta}}a^{1-e^{-\lambda x^{\beta}}},$$

  where $x, a > 0, a \neq 1, \lambda, \beta > 0$.

- GOLLFW distribution

$$f(x) = \beta\gamma\left(a + \frac{b}{x^2}\right) exp\left[\left(ax - \frac{b}{x}\right) - \kappa_{ab}(x)\right](1 - exp[-\kappa_{ab}(x)])^{\gamma\beta-1}\big[1 - \{1 - exp[-\kappa_{ab}(x)]\}^{\beta}\big]^{\gamma-1}\Big\{\big[1 - exp[-\kappa_{ab}(x)]\big]^{\gamma\beta} + \big[1 - \{1 - exp[-\kappa_{ab}(x)]\}^{\beta}\big]^{\gamma}\Big\}^{-2},$$

where $x, \beta, \gamma, a, b > 0$ and $\kappa_{ab}(x) = exp\left(ax - \frac{b}{x}\right)$.

- EAW distribution

$$f(x) = b\left(a\beta x^{\beta-1} + \lambda\gamma x^{\lambda-1}\right)\left[1 - exp\left(-ax^{\beta} - \gamma x^{\lambda}\right)\right]^{b-1} exp\left(-ax^{\beta} - \gamma x^{\lambda}\right),$$

where $x, \beta, \gamma, \lambda, a, b > 0$.

Appendix B.

Let $t_i = \frac{x_i}{\beta}$. The elements of the observed information matrix are:

$$U_{\beta\beta} = \frac{n\gamma}{\beta^2} + \frac{\gamma(\gamma+1)}{\beta^{\gamma+2}}\sum_{i=1}^{n}\frac{x_i^{\gamma}e^{t_i^{\gamma}}}{e^{t_i^{\gamma}} - 1} - \frac{\gamma^2}{\beta^{2\gamma+2}}\sum_{i=1}^{n}\frac{x_i^{2\gamma}e^{t_i^{\gamma}}}{\left(e^{t_i^{\gamma}} - 1\right)^2} + \frac{b\gamma(\gamma+1)}{\beta^{\gamma+2}}\sum_{i=1}^{n}x_i^{\gamma} - \frac{b\gamma(\gamma+1)}{\beta^{\gamma+2}}\sum_{i=1}^{n}x_i^{\gamma}\, e^{t_i^{\gamma}} - \frac{b\gamma^2}{\beta^{2\gamma+2}}\sum_{i=1}^{n}x_i^{2\gamma}\, e^{t_i^{\gamma}} + \frac{(a-1)\gamma(\gamma+1)}{\beta^{\gamma+2}}\sum_{i=1}^{n}\frac{x_i^{\gamma}\left(e^{t_i^{\gamma}} - 1\right)e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}}{1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}} + \frac{(a-1)\gamma^2}{\beta^{2\gamma+2}}\sum_{i=1}^{n}\frac{x_i^{2\gamma}e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}\left(e^{2t_i^{\gamma}} - e^{t_i^{\gamma}} + 1\right)}{1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}} + \frac{(a-1)\gamma^2}{\beta^{2\gamma+2}}\sum_{i=1}^{n}\frac{x_i^{2\gamma}\left(1 - e^{t_i^{\gamma}}\right)^2 e^{2\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}}{\left(1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}\right)^2},$$

$$U_{\beta a} = \frac{\gamma}{\beta^{\gamma+1}}\sum_{i=1}^{n}\frac{x_i^{\gamma}\left(e^{t_i^{\gamma}} - 1\right)e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}}{1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}},$$

$$U_{\beta b} = -\frac{\gamma}{\beta^{\gamma+1}}\sum_{i=1}^{n}x_i^{\gamma} + \frac{\gamma}{\beta^{\gamma+1}}\sum_{i=1}^{n}x_i^{\gamma}\, e^{t_i^{\gamma}},$$

$$U_{\gamma a} = \frac{1}{\beta^{\gamma}}\sum_{i=1}^{n}\frac{x_i^{\gamma}\ln(t_i)\left(1 - e^{t_i^{\gamma}}\right)e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}}{1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}},$$

 DOI: http://dx.doi.org/10.18187/pjsor.v18i1.3529

$$U_{\gamma b} = -\frac{1}{\beta^{\gamma}} \sum_{i=1}^{n} x_i^{\gamma} \ln(t_i) \left(1 - e^{t_i^{\gamma}}\right),$$

$$\begin{aligned} U_{\beta\gamma} = & -\frac{n}{\beta} - \frac{\gamma}{\beta^{\gamma+1}} \sum_{i=1}^{n} \frac{x_i^{\gamma} \ln(t_i)\, e^{t_i^{\gamma}}}{e^{t_i^{\gamma}} - 1} + \frac{b\gamma}{\beta^{\gamma+1}} \sum_{i=1}^{n} x_i^{\gamma} \ln(t_i) \left(1 + e^{t_i^{\gamma}}\right) - \frac{b}{\beta^{\gamma+1}} \sum_{i=1}^{n} x_i^{\gamma} \\ & + \frac{1}{\beta^{2\gamma+1}} \sum_{i=1}^{n} \frac{\beta^{\gamma} x_i^{\gamma} e^{t_i^{\gamma}} + \gamma x_i^{2\gamma} e^{t_i^{\gamma}} \ln(t_i)}{\left(e^{t_i^{\gamma}} - 1\right)^2} - \frac{\gamma}{\beta^{2\gamma+1}} \sum_{i=1}^{n} \frac{x_i^{2\gamma} e^{2t_i^{\gamma}} \ln(t_i)}{\left(e^{t_i^{\gamma}} - 1\right)^2} \\ & + \frac{b}{\beta^{2\gamma+1}} \sum_{i=1}^{n} \left[\beta^{\gamma} x_i^{\gamma} e^{t_i^{\gamma}} + \gamma x_i^{2\gamma} e^{t_i^{\gamma}} \ln(t_i)\right] + \frac{\gamma(a-1)}{\beta^{\gamma+1}} \sum_{i=1}^{n} \frac{x_i^{\gamma} \ln(t_i) \left(1 - e^{t_i^{\gamma}}\right) e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}}{1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}} \\ & - \frac{(a-1)}{\beta^{\gamma+1}} \sum_{i=1}^{n} \frac{x_i^{\gamma} \left(1 + e^{\left(2t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}\right)}{1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}} \\ & + \frac{\gamma(a-1)}{\beta^{2\gamma+1}} \sum_{i=1}^{n} \frac{x_i^{2\gamma} \ln(t_i) \left(e^{t_i^{\gamma}} + \left(1 - e^{t_i^{\gamma}}\right)^2\right) e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}}{1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}} \\ & + \frac{\gamma(a-1)}{\beta^{2\gamma+1}} \sum_{i=1}^{n} \frac{x_i^{2\gamma} \ln(t_i)\, (1 - e^{t_i})^{2\gamma} e^{2\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}}{\left(1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}\right)^2}, \end{aligned}$$

$$\begin{aligned} U_{\gamma\gamma} = & -\frac{n}{\gamma^2} - \frac{\ln\beta}{\beta^{\gamma}} \sum_{i=1}^{n} \frac{x_i^{\gamma} \ln(t_i)\, e^{t_i^{\gamma}}}{e^{t_i^{\gamma}} - 1} + \frac{1}{\beta^{\gamma}} \sum_{i=1}^{n} x_i^{\gamma} \ln(t_i) e^{t_i^{\gamma}} \frac{\left[\ln(x_i)\left(e^{t_i^{\gamma}} - 1\right) - t_i^{\gamma} \ln(t_i)\right]}{\left(e^{t_i^{\gamma}} - 1\right)^2} - \frac{b\ln\beta}{\beta^{\gamma}} \sum_{i=1}^{n} x_i^{\gamma} \ln(t_i) \\ & + \frac{b}{\beta^{\gamma}} \sum_{i=1}^{n} x_i^{\gamma} \ln(x_i) \ln(t_i) + \frac{b\ln\beta}{\beta^{\gamma}} \sum_{i=1}^{n} x_i^{\gamma} e^{t_i^{\gamma}} \ln(t_i) \\ & - \frac{b}{\beta^{\gamma}} \sum_{i=1}^{n} x_i^{\gamma} e^{t_i^{\gamma}} \ln(t_i) \left[\ln(x_i) + t_i^{\gamma} \ln(t_i)\right] - \frac{(a-1)\ln\beta}{\beta^{\gamma}} \sum_{i=1}^{n} \frac{x_i^{\gamma} \ln(t_i) \left(1 - e^{t_i^{\gamma}}\right) e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}}{1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}} \\ & + \frac{a-1}{\beta^{\gamma}} \sum_{i=1}^{n} \frac{x_i^{2\gamma} [\ln(t_i)]^2 t_i^{\gamma} \left(1 - e^{t_i^{\gamma}}\right)^2 e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}}{\left(1 - e^{\left(t_i^{\gamma} - e^{t_i^{\gamma}} + 1\right)}\right)^2}, \end{aligned}$$

$$U_{aa} = -n[\psi'(a) - \psi'(a+b)], U_{ab} = n[\psi'(a+b)] \; and \;\; U_{bb} = -n[\psi'(b) - \psi'(a+b)],$$

where $\psi'(\cdot)$ is the trigamma function.

References

1. Aarset, M. V. (1987). How to identify a bathtub hazard rate. IEEE Transactions on Reliability, 36(1), 106-108.
2. Ahmad, A. A. Ghazal M. G. M. (2020). Exponentiated additive Weibull distribution. Reliability Engineering and System Safety 193, 106663.
3. Awodutire P. O., Nduka E. C., Ijomah M. A. (2020). The beta type I generalized half logistic distribution: properties and application. Asian Journal of Probability and Statistics, 6(2), 27-41.
4. Alizadeh M., Altun E., Cordeiro G. M., Rasekhi M. (2018). The odd power cauchy family of distributions: properties, regression models and applications. Journal of Statistical Computation and Simulation, 88(4), 785-807.
5. Aryall G.R., Tsokos C.P. (2011). Transmuted Weibull distribution: A generalization of the Weibull probability distribution. European Journal of Pure and Applied Mathematics, 2, 89-102.
6. Benkhelifa L. (2017). The beta generalized Gompertz distribution. Applied Mathematical Modelling, 52, 341-357.
7. Benkhelifa L. (2021). The beta reduced modified Weibull distribution with applications to reliability data. Journal of Reliability and Statistical Studies, 14(1), 323-352.
8. Cordeiro G. M., Ortega E.M., Nadarajah S. (2010). The Kumaraswamy Weibull distribution with application to failure data. Journal of The Franklin Institute, 347, 1399-1429.
9. Corderio G.M., Hashimoto E.H., Ortega, E.M.M. (2014). The McDonald Weibull model. Statistics, 48, 256-278.
10. Eugene N., Lee C., Famoye F. (2002). Beta-normal distribution and its applications. Commun. Stat. Theor. M., 31, 497-512.
11. Famoye F., Lee C., Olumolade O. (2005). The beta-Weibull distribution. J. Stat. Theor. Applic., 4(2), 121-136.
12. Fleming T.R., Harrington D.P., Counting Process and Survival Analysis, John Wiley, New York, NY, USA, 1994.
13. Gradshteyn I.S., Ryzhik I.M. Table of Integrals, Series and Products. Academic Press, New York, 2000.
14. Hosmer D.W., Lemeshowik S. Applied survival analysis: regression modeling of time to event data. New York: JohnWiley Son, 1999.
15. Irshad M. R., Maya R., Krishna A. (2021). Exponentiated power Muth distribution and associated inference. Journal of the Indian Society for Probability and Statistics, 22, 265–302.
16. Jodrá P., Gómez H.W., Jiménez-Gamero M.D., Alba-Fernández M.V. (2017). The power Muth distribution. Math. Model. Anal., 22(2), 186-201.
17. Jodrá P. J, Jiménez-Gamero M.D., Alba-Fernández M.V. (2015). On the Muth distribution. Math. Model. Anal., 20(3), 291-310.
18. Leemis L.M., McQueston J.T. (2008). Univariate distribution relationships. Am. Stat., 62(1), 45-53.
19. Milgram M.S. (1985). The generalized integro-exponential function. Math. Comp., 44(170), 443-458.
20. Muth E.J. (1977). Reliability models with positive memory derived from the mean residual life function. In C.P. Tsokos and I. Shimi (Eds.), The Theory and Applications of Reliability, volume 2, pp. 401-435, New York,. Academic Press, Inc.
21. Nadarajah S., Kotz S.(2005). The beta exponential distribution. Reliab. Eng. Syst. Saf., 91, 689-697.
22. Nassar M., Alzaatreh A., Mead M., Abo-Kasem O. (2017). Alpha power Weibull distribution: properties and applications, Commun. Stat.-Theory Methods, 46, 10236-10252.
23. Prataviera F., Ortega E.M.M., Cordeiro G.M., Pescim R.R., Verssania B.A.W. (2018). A new generalized odd log-logistic flexible Weibull regression model with applications in repairable systems. Reliab Eng Syst Saf, 176,13-26.
24. Proschan F. (1963). Theoretical explanation of observed decreasing failure rate. Technometrics; 5, 375-383.
25. Shakhatreh M.K., Yusuf A., Mugdadi A. (2016). The beta generalized linear exponential distribution. Statistics, 50(6), 1346-1362.
26. Singla N., Jain K., Sharma S. K.(2012). The beta generalized Weibull distribution: properties and applications. Reliab. Eng. Syst. Saf., 102, 5-15.
27. Therneau T.M., Grambsch, P.M., Fleming, T.R. (1990). Martingale-based residuals for survival models. Biometrika, 77, 147-160.