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A Two Parameters Rani Distribution: Estimation and Tests for Right Censoring Data with an Application

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Abstract

In this paper, we developed a new distribution, namely the two parameters Rani distribution (TPRD). Some statistical properties of the proposed distribution are derived including the moments, moment-generating function, reliability function, hazard function, reversed hazard function, odds function, the density function of order statistics, stochastically ordering, and the entropies. The maximum likelihood method is used for model parameters estimation. Following the same approach suggested by Bagdonavicius and Nikulin (2011), modified chi squared goodness-of-fit tests are constructed for right censored data and some tests for right data is considered. An application study is presented to illustrate the ability of the suggested model in fitting aluminum reduction cells sets and the strength data of glass of the aircraft window.

Key Words: Two parameters Rani distribution; Modified chi-square test; Reliability analysis; Rényi entropy; Censored data; Stochastic ordering

Mathematical Subject Classification: 62Exx

1. Introduction

Shanker (2017) proposed a distribution of one parameter called as a Rani distribution (RD) with probability density function (pdf) given by

$$f_{RD}(x;\theta)=\frac{\theta^5}{\theta^5+24}\left(\theta+x^4\right)\mathrm{e}^{-\theta x}, x>0,\theta>0, \tag{1}$$

and the cumulative distribution function (cdf) of the RD is defined as

$$F_{RD}(x;\theta)=1-\left(1+\frac{\theta x(\theta^3x^3+4\theta^2x^2+12x\theta+24}{24+\theta^5}\right)\mathrm{e}^{-\theta x}, x>0,\theta>0. \tag{2}$$

Shanker showed the flexibility of the RD in modeling real life time data. In the literature, many authors suggested different new distributions for modeling and fitting various lifetime data in several fields such as medicine, industry, biology, nursing, agriculture, insurance and other fields. These distributions are generated using various methods. As an example, recently, Alhyasat et al. (2020) proposed power size biased two-parameter Akash distribution. Alsmairan and Al-Omari (2020) introduced weighted Suja distribution. Al-Omari and Garaibah (2018) proposed Topp-Leone Mukherjee-Islam distribution. Al-Omari, and Alsmairan (2019) proposed length-biased Suja distribution. Usman et al. (2019) introduced the Marshall-Olkin Length biased exponential distribution. Al-Omari et al. (2019) suggested a size biased Ishita distribution. Garaibah and **Al**-Omari(2019) proposed the transmuted Ishita distribution .

In this paper, we developed the RD by adding a new parameter to the RD in order to improve its flexibility in fitting real data. Also, we investigated the problem of right censored data with some well-known tests for the

TPRD. Aidi and Seddik-Ameur (2016) considered Chi-square tests for generalized exponential AFT distributions with censored data.

The maximum likelihood estimates are investigated in the case of censored data and also, the estimated information matrix is presented. The structure of a modified chi-square goodness-of-fit test for the new TPRD when the data are right censored is proposed. For these purposes, we used the criteria tests considered by Bagdonavicius and Nikulin (2011), Bagdonavicius et al. (2013). This statistic test is based on the maximum likelihood estimators on initial data, and follows by chi-square distribution. In order to confirm the usefulness of the proposed goodness-of-fit test, and the new model, an important simulation study was carried out. Theoretical results obtained from this study are applied to a real data set from reliability.

The rest of this paper is organized as follows: In Section 2, the two parameters Rani distribution is presented with its pdf and cdf and some plots are involved. Some statistical properties of the TPRD including the moments, variance, skewness kurtosis, the *r*th moment and the moment generating function are given in Section 3, also some simulations are considered. Section 4, is devoted to the distribution reliability analysis. The order statistics and stochastic ordering are presented in Section 5. In Section 6, the entropies of the distribution are derived. The TPRD parameters estimation is considered in Section 7 and the statistic for right censored data is given in Section 8. Criteria test for TPRD distribution are presented in Section 9. A simulation study and an application are given in Section 10 and 11, respectively. Finally, the paper is concluded in Section 12.

2. The suggested model

In this section, we introduce the pdf of the two parameters Rani distribution (TPRD) as

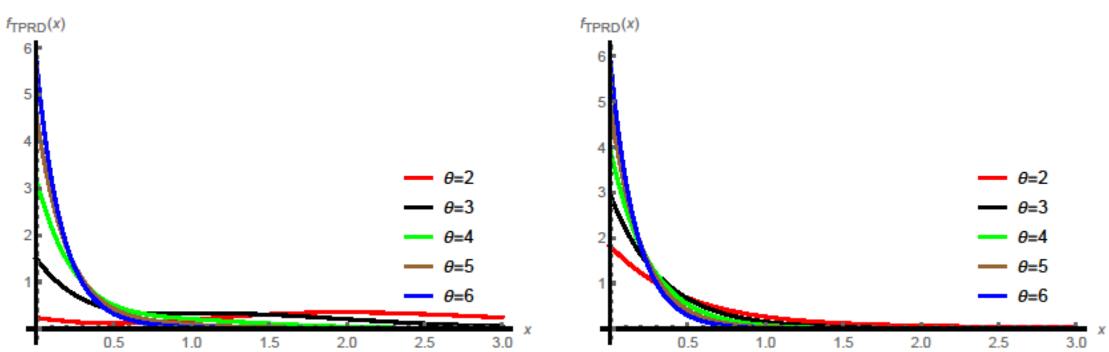
$$f_{TPRD}(x;\theta,\alpha)=\frac{\theta^5}{\alpha\theta^5+24}\left(\alpha\theta+x^4\right)e^{-\theta x}, x>0,\theta>0,\alpha>0. \quad (3)$$

It is of interest to note here that $\int_0^\infty f_{TPRD}(x;\theta,\alpha)dx=1$ and the TPRD reduces to the RD when $\alpha=1$.

Also, it can be noted that $f_{TPRD}(x;\theta,\alpha)=p\, f_1(x;\theta)+(1-p)\, f_2(x;\theta)$, which is a mixture of two well-known distributions $f_1(x;\theta)=\theta e^{-\theta x}$ and $f_2(x;\theta)=\frac{\theta^5}{24}x^4 e^{-\theta x}$ with mixing factor $p=\frac{\alpha\theta^5}{\alpha\theta^5+24}$ and $1-p=\frac{24}{\alpha\theta^5+24}$. The corresponding cdf of the TPRD is

$$F_{TPRD}(x;\theta,\alpha)=1-\left(1+\frac{24\theta x+\theta^2x^2\left[12+x\theta(4+x\theta)\right]}{24+\alpha\theta^5}\right)e^{-\theta x}, x>0,\theta>0,\alpha>0. \quad (4)$$

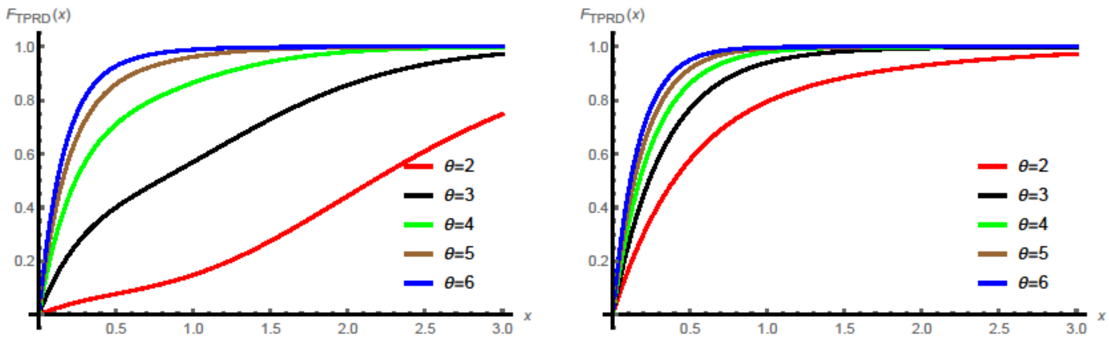
Plots of the pdf and cdf of the TPRD distribution are presented in Figures (1) and (2), respectively, for various distribution parameters on the interval $x\in[0,3]$.



$\alpha=0.1$ $\alpha=8$

Figure 1: Graphs of cdf of the TPRD for $\theta=2,3,4,5,6$ and $\alpha=0.1,8$.

Figure (1) shows that the TPRD distribution is non symmetric and skewed to the right. Also, the skewness of the distribution depends on the values of the parameters.



$\alpha = 0.1$ $\alpha = 8$

Figure 2: Graphs of cdf of the TPRD for $\theta = 2,3,4,5,6$ and $\alpha = 0.1, 8$.

Figure (2) illustrate the behavior of the TPRD with the parameters values. For example, for fixed value of $\theta = 2$, with large value of $\alpha = 8$ the curve goes to 1 faster than $\alpha = 0.1$.

3. Moments of the TPRD distribution

In this section, we presented some statistical properties of the TPRD as the moments, the *r*th moment, coefficient of variation, coefficient of skewness and coefficient of kurtosis.

Theorem 2: Let $X \sim f_{TPRD}(x;\theta,\alpha)$. Then the *r*th moment of *X* is

$$E\left(X_{TPRD}^{r}\right) = \frac{\alpha\theta^{5}\Gamma(r+1)+\Gamma(r+5)}{\theta^{r}\left(\alpha\theta^{5}+24\right)}, \quad r = 1,2,3,... \tag{5}$$

Theorem 3: If $X \sim f_{TPRD}(x;\theta,\alpha)$, then the moment generating function, $M_X(t)$, of *X* is

$$M_X(t) = \frac{\theta^{5}\left(\alpha\theta\left(\theta - t\log(e)\right)^{4}+24\right)}{\left(\alpha\theta^{5}+24\right)\left(\theta - t\log(e)\right)^{5}}. \tag{6}$$

From Equation (5), the first and second moments of the TPRD distribution, respectively, are given as

$$E\left(X_{TPRD}\right) = \frac{\alpha\theta^{5}+120}{\alpha\theta^{6}+24\theta} \text{ and } E\left(X_{TPRD}^{2}\right) = \frac{2\alpha\theta^{5}+720}{\alpha\theta^{7}+24\theta^{2}}.$$

Hence, the variance of the distribution is

$$\mathrm{Var}(X_{TPRD}) = \sigma_{TPRD}^{2} = \frac{\alpha\theta^{5}\left(\alpha\theta^{5}+528\right)+2880}{\theta^{2}\left(\alpha\theta^{5}+24\right)^{2}}, \theta > 0. \tag{7}$$

The coefficient of skewness of the suggested TPRD distribution is given by

$$Sk_{TPRD} = \frac{2\left\{\alpha\theta^{5}\left[\alpha\theta^{5}\left(\alpha\theta^{5}+1512\right)+1728\right]+69120\right\}}{\left[\alpha\theta^{5}\left(\alpha\theta^{5}+528\right)+2880\right]^{3/2}}. \tag{8}$$

The coefficients of kurtosis and variation for the TPRD, respectively, are given by

$$Ku_{TPRD} = \frac{9\left(\alpha\theta^{5}\left\{\alpha\theta^{5}\left[\alpha\theta^{5}\left(\alpha\theta^{5}+2656\right)+58752\right]+1234944\right\}+3870720\right)}{\left[\alpha\theta^{5}\left(\alpha\theta^{5}+528\right)+2880\right]^{2}}, \tag{9}$$

and

$$Cv_{TPRD} = \frac{\sqrt{2880+\alpha\theta^{5}\left(528+\alpha\theta^{5}\right)}}{120+\alpha\theta^{5}}. \tag{10}$$

In Table (1), some simulated values of the measures $\mu_{TPRD}, \sigma^2_{TPRD}$, Cv_{TPRD}, Sk_{TPRD}, and Ku_{TPRD} are presented for various values of the distribution parameters θ and α.

Table 1: The mean, variance, coefficients of variation, skewness and kurtosis for the TPRD with different values of the parameters

θ	α	μ_{TPRD}	σ^2_{TPRD}	Cv_{TPRD}	Sk_{TPRD}	Ku_{TPRD}
2	0.1	2.2647	1.2440	0.5493	0.5818	3.6521
↑	0.2	2.0790	1.3055	0.6280	0.5694	3.3737
↑	0.3	1.9286	1.3344	0.6919	0.6267	3.2710
↑	0.4	1.8044	1.3452	0.7455	0.7029	3.2691
↑	0.5	1.7000	1.3454	0.7914	0.7827	3.3277
↑	0.6	1.6111	1.3391	0.8312	0.8610	3.4241
↑	0.7	1.5345	1.3289	0.8660	0.9359	3.5452
↑	0.8	1.4677	1.3164	0.8969	1.0069	3.6826
↑	0.9	1.4091	1.3024	0.9243	1.0739	3.8309
5	1	0.2061	0.2146	1.0413	2.3475	12.2179
↑	2	0.2031	0.2075	1.0218	2.1992	10.9145
↑	3	0.2020	0.2050	1.0148	2.1393	10.3558
↑	4	0.2015	0.2038	1.0112	2.1070	10.0485
↑	5	0.2012	0.2030	1.0090	2.0869	9.8545
↑	6	0.2010	0.2025	1.0075	2.0731	9.7210
↑	7	0.2009	0.2022	1.0065	2.0631	9.6236
↑	8	0.2008	0.2019	1.0057	2.0555	9.5493
↑	9	0.2007	0.2017	1.0051	2.0495	9.4908
α	θ	μ_{TPRD}	σ^2_{TPRD}	Cv_{TPRD}	Sk_{TPRD}	Ku_{TPRD}
3	0.1	50.0000	22.3607	0.4472	0.8944	4.2000
↑	0.2	24.9992	11.1809	0.4472	0.8942	4.1998
↑	0.3	16.6626	7.4563	0.4475	0.8928	4.1985
↑	0.4	12.4872	5.5987	0.4484	0.8876	4.1939
↑	0.5	9.9689	4.4929	0.4507	0.8741	4.1812
↑	0.6	8.2692	3.7690	0.4558	0.8458	4.1531
↑	0.7	7.0253	3.2702	0.4655	0.7975	4.0987
↑	0.8	6.0533	2.9175	0.4820	0.7292	4.0048
↑	0.9	5.2501	2.6642	0.5075	0.6521	3.8625
5	1	4.3103	2.5678	0.5957	0.5603	3.4677
↑	2	0.7609	0.9133	1.2003	2.2731	9.3385
↑	3	0.3592	0.3918	1.0908	2.6086	14.0718
↑	4	0.2547	0.2614	1.0263	2.2358	11.2470
↑	5	0.2012	0.2030	1.0090	2.0869	9.8545
↑	6	0.1671	0.1677	1.0037	2.0362	9.3592
↑	7	0.1430	0.1433	1.0017	2.0170	9.1689
↑	8	0.1251	0.1252	1.0009	2.0087	9.0873
↑	9	0.1111	0.1112	1.0005	2.0049	9.0486

From Table (1) it can be noted that the mean and variance of the TPRD distribution are decreasing as α increasing for various fixed values of θ.

4. Reliability

The reliability (survival) and hazard rate functions of the TPRD distribution are, respectively given by

$$\begin{aligned} R_{TPRD}(x;\theta,\alpha) &= 1-F_{TPRD}(x;\theta,\alpha) \\ &= \left(1+\frac{24\theta x+\theta^2x^2\left[12+x\theta(4+x\theta)\right]}{24+\alpha\theta^5}\right)\mathrm{e}^{-\theta x}, \end{aligned} \tag{11}$$

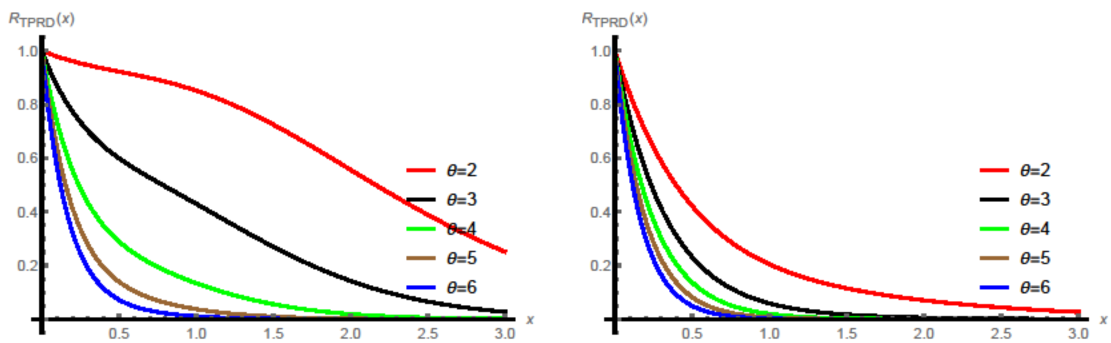
and

$$\begin{aligned} H_{TPRD}(x;\theta,\alpha) &= \frac{f_{TPRD}(x;\theta,\alpha)}{1-F_{TPRD}(x;\theta,\alpha)} \\ &= \frac{\theta^5\left(\alpha\theta+x^4\right)}{\alpha\theta^5+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x+24}. \end{aligned} \tag{12}$$

The cumulative hazard rate function of the TPRD is

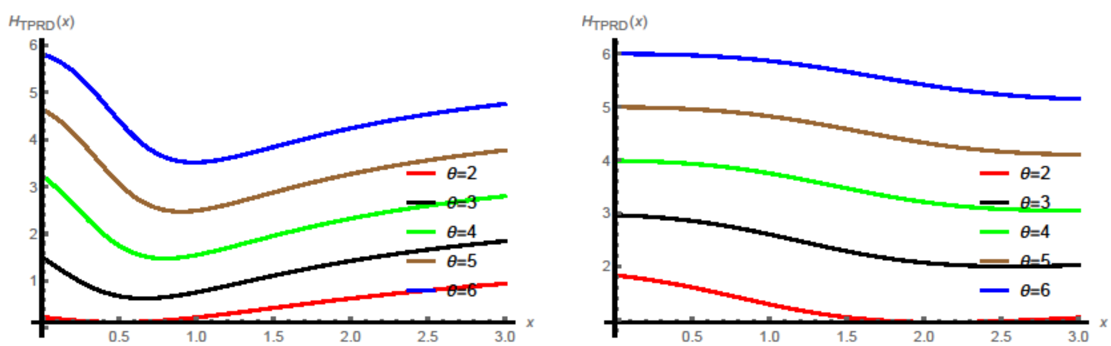
$$\begin{aligned} CH_{TPRD}(t;\theta,\alpha) &= \int_0^t \frac{f_{TPRD}(x;\theta,\alpha)}{1-F_{TPRD}(x;\theta,\alpha)}dx = -\ln R_{TPRD}(t;\theta,\alpha) \\ &= \theta t-\ln\left(1+\frac{24\theta t+\theta^2t^2\left[12+t\theta(4+t\theta)\right]}{24+\alpha\theta^5}\right). \end{aligned} \tag{13}$$

Figures (3) and (4) showed the plots of the $R_{TPRD}(x;\theta,\alpha)$ and $H_{TPRD}(x;\theta,\alpha)$ of the TPRD for $\theta=2,3,4,5,6$ and $\alpha=0.1,8$.



$\alpha=0.1$ $\alpha=8$

Figure 3: The reliability function of the TPRD for $\theta=2,3,4,5,6$ and $\alpha=0.1,8$.



$\alpha=0.1$ $\alpha=8$

Figure 4: The hazard function of the TPRD for $\theta=2,3,4,5,6$ and $\alpha=0.1,8$.

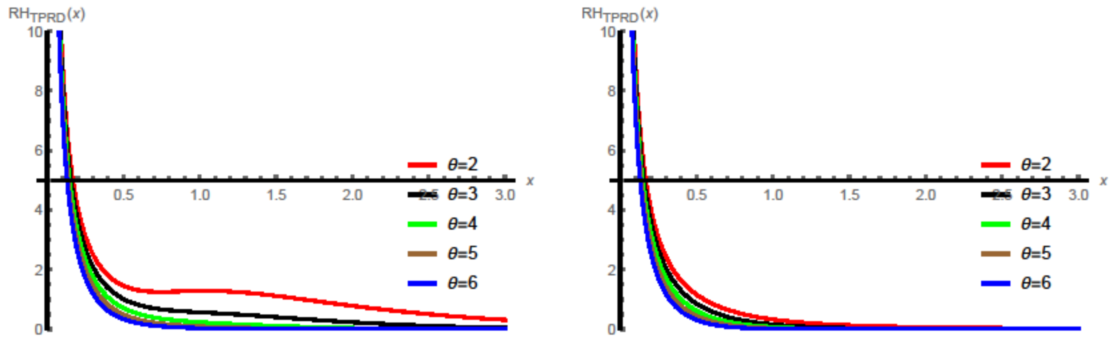
The reversed hazard rate function of the TPRD denoted by $RH_{LBR}(x;\theta,\alpha)$ and odds function denoted by $OO_{LBR}(x;\theta,\alpha)$, respectively, are defined as

$$RH_{TPRD}(x;\theta,\alpha)=\frac{f_{TPRD}(x;\theta,\alpha)}{F_{TPRD}(x;\theta,\alpha)}=\frac{\theta^5\left(\alpha\theta+x^4\right)}{\left(\alpha\theta^5+24\right)e^{\theta x}-\theta\left(\alpha\theta^4+\theta^3x^4+4\theta^2x^3+12\theta x^2+24x\right)-24}, \tag{14}$$

and

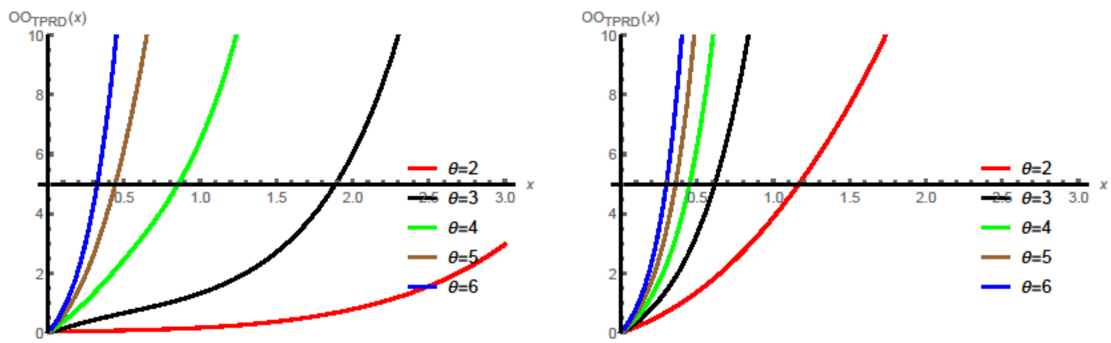
$$OO_{TPRD}(x;\theta,\alpha)=\frac{F_{TPRD}(x;\theta,\alpha)}{1-F_{TPRD}(x;\theta,\alpha)}=\frac{\left(\alpha\theta^5+24\right)e^{\theta x}}{\alpha\theta^5+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x+24}-1. \tag{15}$$

Figures (5) and (6) showed the $RH_{TPRD}(x;\theta,\alpha)$ and $OO_{TPRD}(x;\theta,\alpha)$ of the TPRD for $\theta=2,3,4,5,6$ and $\alpha=0.1,8$



$\alpha=0.1$ $\alpha=8$

Figure 5: The reversed hazard function of the TPRD for $\theta=2,3,4,5,6$ and $\alpha=0.1,8$.



$\alpha=0.1$ $\alpha=8$

Figure 6: The odds function of the TPRD for $\theta=2,3,4,5,6$ and $\alpha=0.1,8$.

5. Order Statistics and Stochastic Ordering

Let $X_{(1:n)},X_{(2:n)},...,X_{(n:n)}$ be the order statistics of the random sample $X_1,X_2,...,X_n$ chosen from a pdf $f(x)$ and a cdf $F(x)$. The pdf of the *j*th order statistics say $X_{(i:n)}$, is given by

$$f_{(j:n)}(x)=\frac{n!}{(j-1)!(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j}f(x),\ \text{for } j=1,2,...,n. \tag{16}$$

Substituting the pdf and cdf of the TPRD in Equation (16) to obtain the pdf of $X_{(j:n)}$ as

$$f_{(j:n)}(x;\theta,\alpha)=\frac{\theta^5n!e^{-\theta x}\left(\alpha\theta+x^4\right)}{\left(\alpha\theta^5+24\right)\Gamma(j)\Gamma(-j+n+1)}\times A\times B, \tag{17}$$

$$A=\left(1-\frac{e^{-\theta x}\left(\alpha\theta^5+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x+24\right)}{\alpha\theta^5+24}\right)^{j-1},$$

$$B=\left(\frac{e^{-\theta x}\left(\alpha\theta^5+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x+24\right)}{\alpha\theta^5+24}\right)^{m-j}.$$

Also, the pdf of smallest and largest order statistics $X_{(1:n)}$ and $X_{(n:n)}$, respectively, are given by

$$f_{(1:n)}(x;\theta,\alpha)=\frac{n\theta^5 e^{-\theta n x}\left(\alpha\theta^5+\theta^4x^4+4\theta^3x^3+12\theta^2x^2+24\theta x+24\right)^{n-1}}{\left(\alpha\theta^5+24\right)^n\left(\alpha\theta+x^4\right)^{-1}}, \tag{18}$$

$$f_{(n:n)}(x;\theta,\alpha)=\frac{\theta^5 n e^{-\theta x}\left[1-\begin{pmatrix}\alpha\theta^5+\theta^4x^4+4\theta^3x^3\\+12\theta^2x^2+24\theta x+24\end{pmatrix}\left(\alpha\theta^5+24\right)^{-1}e^{-\theta x}\right]^{n-1}}{\left(\alpha\theta^5+24\right)\left(\alpha\theta+x^4\right)^{-1}}. \tag{19}$$

The stochastic ordering can be used to compare two positive continuous random variables. A random variable X is smaller than a random variable Y in

1) Mean residual life order denoted by $X \underset{MRLO}{\le} Y$, if $m_X(x)\le m_Y(x)$ for a x.

2) Likelihood ratio order denoted by $X \underset{LRO}{\le} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

3) Hazard rate order denoted by $X \underset{HRO}{\le} Y$, if $h_X(x)\ge h_Y(x)$ for a x.

4) Stochastic order denoted by $X \underset{SO}{\le} Y$, if $F_X(x)\ge F_Y(x)$ for a x.

Shaked and Shanthikumar (1994) deduced that $X \underset{LRO}{\le} Y \Rightarrow X \underset{HRO}{\le} Y \Rightarrow X \underset{MRLO}{\le} Y$.

$$\Downarrow$$
$$X \underset{SO}{\le} Y$$

Theorem 4: Let $X \sim f_X(x;\theta,\alpha)$ and $Y \sim f_Y(x;\mathrm{F},\mathrm{W})$, then if ($\mathrm{W}<\alpha$ and $\mathrm{F}=\theta$) or ($\mathrm{F}<\theta$ and $\mathrm{W}=\alpha$), we have $X \underset{LRO}{\le} Y$, and hence $X \underset{HRO}{\le} Y$, $X \underset{MRLO}{\le} Y$ and $X \underset{SO}{\le} Y$.

Proof:

Let $X \sim f_X(x;\theta,\alpha)$ and $Y \sim f_Y(x;\mathrm{F},\mathrm{W})$. Hence,

$$f(x;\theta,\alpha)=\frac{\theta^5}{\alpha\theta^5+24}\left(\alpha\theta+x^4\right)e^{-\theta x} \text{ and } f_Y(x;\mathrm{F},\mathrm{W})=\frac{\mathrm{F}^5}{\mathrm{W}\mathrm{F}^5+24}\left(x^4+\mathrm{W}\mathrm{F}\right)\mathrm{e}^{-\mathrm{F}x}.$$

Therefore,

$$\frac{f_X(x;\theta,\alpha)}{f_Y(x;\mathrm{F},\mathrm{W})}=\frac{\frac{\theta^5}{\alpha\theta^5+24}\left(\alpha\theta+x^4\right)e^{-\theta x}}{\frac{\mathrm{F}^5}{\mathrm{W}\mathrm{F}^5+24}\left(x^4+\mathrm{W}\mathrm{F}\right)\mathrm{e}^{-\mathrm{F}x}}=\frac{\theta_1^6\left(\theta_2^5+120\right)\left(\theta_1+x^4\right)}{\theta_2^6\left(\theta_1^5+120\right)\left(\theta_2+x^4\right)}e^{-(\theta_1-\psi)x},$$

and

$$\log\frac{f_X(x;\theta,\alpha)}{f_Y(x;\mathrm{F},\mathrm{W})}=\log\left(\frac{\theta^5\left(\mathrm{W}\mathrm{F}^5+24\right)\left(\alpha\theta+x^4\right)}{\mathrm{F}^5\left(\alpha\theta^5+24\right)\left(x^4+\mathrm{W}\mathrm{F}\right)}e^{-(\theta-\mathrm{F})x}\right)$$
$$=\log\left(\frac{\alpha\theta+x^4}{x^4+\mathrm{W}\mathrm{F}}\right)-x(\theta-\mathrm{F})+\log\left(\frac{\theta^5\left(\mathrm{W}\mathrm{F}^5+24\right)}{\mathrm{F}^5\left(\alpha\theta^5+24\right)}\right)$$

and its derivative with respect to x is

$$\frac{d}{dx}log\left(\frac{f_X(x;\theta,\alpha)}{f_Y(x;\mathrm{F},\mathrm{W})}\right)=F-\theta+\frac{x^4+F\mathrm{W}}{x^4+\alpha\theta}\left(\frac{4x^3}{x^4+F\mathrm{W}}-\frac{4x^3\left(x^4+\alpha\theta\right)}{\left(x^4+F\mathrm{W}\right)^2}\right).$$

Now, if $\mathrm{W}<\alpha$ and $\mathrm{F}=\theta$ or $\mathrm{F}<\theta$ and $\mathrm{W}=\alpha$, $\frac{d}{dx}log\left(\frac{f_X(x;\theta,\alpha)}{f_Y(x;\mathrm{F},\mathrm{W})}\right)<0$, which implies that $X \underset{LRO}{\le} Y$, and hence $X \underset{HRO}{\le} Y$, $X \underset{MRLO}{\le} Y$ and $X \underset{SO}{\le} Y$.

6. Entropies of the TPRD

The Rényi entropy is defined as $\Re(\lambda)=\frac{1}{1-\lambda}\log\left(\int_0^\infty f(x)^\lambda dx\right)$, where $\lambda>0$ and $\lambda\neq 1$.

Theorem 5: Let $X \sim f_{TPRD}(x;\theta,\alpha)$, then the and Rényi entropy of X is defined as

$$\Re(\lambda)=\frac{1}{1-\lambda}\log\left(\sum_{j=0}^{\lambda}\binom{\lambda}{j}\frac{\theta^{6\lambda-j}\alpha^{\lambda-j}}{(\alpha\theta^5+24)^\lambda}\frac{(4j)!}{(\theta\lambda)^{4j+1}}\right). \tag{20}$$

Proof: To prove (20) let

$$\begin{aligned}\Re(\lambda)&=\frac{1}{1-\lambda}\log\left(\int_0^\infty\left(\frac{\theta^5}{\alpha\theta^5+24}\right)^\lambda(\alpha\theta+x^4)^\lambda e^{-\theta\lambda x}dx\right)\\&=\frac{1}{1-\lambda}\log\left(\int_0^\infty\left(\frac{\theta^5}{\alpha\theta^5+24}\right)^\lambda\sum_{j=0}^{\lambda}\binom{\lambda}{j}(\alpha\theta)^{\lambda-j}(x^4)^j e^{-\theta\lambda x}dx\right)\\&=\frac{1}{1-\lambda}\log\left(\sum_{j=0}^{\lambda}\binom{\lambda}{j}\frac{\theta^{6\lambda-j}\alpha^{\lambda-j}}{(\alpha\theta^5+24)^\lambda}\int_0^\infty x^{4j}e^{-\theta\lambda x}dx\right)\\&=\frac{1}{1-\lambda}\log\left(\sum_{j=0}^{\lambda}\binom{\lambda}{j}\frac{\theta^{6\lambda-j}\alpha^{\lambda-j}}{(\alpha\theta^5+24)^\lambda}\frac{(4j)!}{(\theta\lambda)^{4j+1}}\right).\end{aligned}$$

The q-entropy, say $Q_q(x)$ is given by $Q_q(x)=\frac{1}{q-1}\log\left(1-\int_{-\infty}^\infty f(x)^q dx\right), q>0, q\neq 1$. For the TPRD an explicit expression of the $Q_q(x)$ is defined as

$$Q_q(x)=\frac{1}{q-1}\log\left(1-\sum_{j=0}^{q}\binom{q}{j}\frac{\theta^{6q-j}\alpha^{q-j}}{(\alpha\theta^5+24)^q}\frac{(4j)!}{(\theta q)^{4j+1}}\right). \tag{21}$$

7. Parameter estimation

7.1 Maximum likelihood estimation

Here, the parameters of the TPRD distribution are estimated using the method of maximum likelihood. Let $x_1, x_2, \ldots x_n$ be a random sample distributed according to the TPRD distribution, the likelihood function is obtained by

$$l(x;\gamma)=\prod_{i=1}^n f(x_i;\gamma)=\prod_{i=1}^n\frac{\theta^5}{\alpha\theta^5+24}\left(\alpha\theta+x_i^4\right)e^{-\theta x_i}, \text{ where } \gamma=\alpha,\theta. \tag{22}$$

The logarithm of Equation (22) is

$$l_n(\gamma)=\sum_{i=1}^n \ln f(x,\gamma)=5n\ln(\theta)-\theta\sum_{i=1}^n x_i+\sum_{i=1}^n\ln\left(\alpha\theta+x_i^4\right)-\sum_{i=1}^n\ln\left(\alpha\theta^5+24\right).$$

The maximum likelihood estimators $\hat{\alpha}$ and $\hat{\theta}$ of the unknown parameters α and θ are derived from the nonlinear following score equations:

$$\frac{\partial L}{\partial\alpha}=\sum_{i=1}^n\frac{\theta}{\alpha\theta+x_i^4}-\sum_{i=1}^n\frac{\theta^5}{\alpha\theta^5+24}, \text{ and } \frac{\partial L}{\partial\theta}=\frac{5n}{\theta}-\sum_{i=1}^n\frac{5\alpha\theta^4}{\left(24+\alpha\theta^5\right)}+\sum_{i=1}^n\frac{\alpha}{\alpha\theta+x_i^4}-\sum_{i=1}^n x_i. \tag{23}$$

7.2 Estimation under right-censored data

The hypothesizing test will be discussed under complete and censored data, however, the TPRD is only defined for complete data. Since the MLE is usually considered for right-censored data, let us consider $X_1, X_2, \ldots, X_n$ be a random right censored sample obtained from the TPRD distribution with the parameter vector $=(\alpha,\theta)^T$. The censoring time τ is fixed. So, the observation X_i is equal to $X_i=(x_i,\delta_i)$ where

$$\delta_i = \begin{cases} 0 & if\ \ x_i\ is\ a\ censoring\ time \\ 1 & if\ \ x_i\ is\ a\ failure\ time \end{cases}$$

In this case, the log-likelihood is obtained as follow

$$L_n(\gamma) = \sum_{i=1}^{n} \delta_i \ln f(x_i,\gamma) + \sum_{i=1}^{n} (1-\delta_i) \ln R(x_i,\gamma)$$
$$= \sum_{i=1}^{n} \delta_i \left[5\ln(\theta) - \theta x_i + \ln\left(\alpha\theta + x_i^4\right) - \ln\left(\alpha\theta^5 + 24\right) \right] - \sum_{i=1}^{n} (1-\delta_i)[\theta x_i - \ln(1+\varphi_i)], \qquad (24)$$

where $\varpi_i = 24\theta x_i + \theta^2 x_i^2 \left[12 + x_i\theta(4+\theta x_i) \right]$, $\varphi_i = \dfrac{24\theta x_i + \theta^2 x_i^2 \varpi_i}{24+\alpha\theta^5}$ with $v_i = \theta^3 x_i^3 + 4\theta^2 x_i^2 + 12\theta x_i + 24,$ and $s_i = \theta^3 x_i^3 + 3\theta^2 x_i^2 + 6\theta x_i + 6$.

The maximum likelihood estimators $\hat{\alpha}$ and $\hat{\theta}$ of the unknown parameters α and θ are derived from the nonlinear following score equations:

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \delta_i \left[\frac{\theta}{\alpha\theta + x_i^4} - \frac{\theta^5}{\alpha\theta^5 + 24} \right] - \sum_{i=1}^{n} (1-\delta_i) \frac{\theta^5 \varpi_i}{\left(24+\alpha\theta^5\right)^2 (1+\varphi_i)}, \qquad (25)$$

$$\frac{\partial L}{\partial \theta} = \sum_{i=1}^{n} \delta_i \left[\frac{5}{\theta} - \frac{5\alpha\theta^4}{\left(24+\alpha\theta^5\right)} + \frac{\alpha}{\alpha\theta + x_i^4} - x_i \right] - \sum_{i=1}^{n} (1-\delta_i) \left[x_i - \frac{4x_i\left(24+\alpha\theta^5\right)s_i - 5\alpha\theta^5 x_i v_i}{\left(24+\alpha\theta^5\right)^2 (1+\varphi_i)} \right]. \qquad (26)$$

Monte Carlo technique or other iterative methods can be used to determine the values of$\hat{\alpha}$ and $\hat{\theta}$.

8. Test statistic for right censored data

Let $X_1,\dots,X_n$ be *n* i.i.d. random variables grouped into r classes I_i. To assess the adequacy of a parametric model F_0 we have

$$H_0 : P(X_i \le x) = F_0(x;\gamma), x \ge 0,\ \ \gamma = (\gamma_1,\dots,\gamma_s)^T \in \Theta \subset R^s$$

when the data are right censored and the parameter vector γ is unknown, Bagdonavicius and Nikulin (2011) proposed a statistic test Y^2 based on the vector

$$Z_j = \frac{1}{\sqrt{n}}\left(U_j - e_j\right),\ j = 1,2,..,r,\ with\, r > s.$$

This one represents the differences between observed and expected numbers of failures (U_j and e_j) to fall into these grouping intervals $I_j = (p_{j-1}, p_j]$ with $p_0 = 0,\ p_r = \tau$, where τ is a finite time. The authors considered p_j as random data functions such as the r intervals chosen have equal expected numbers of failures e_j.

The statistic test Y^2 is defined by

$$Y^2 = Z^T \hat{\Sigma}^- Z = \sum_{i=1}^{r} \frac{\left(U_j - e_j\right)^2}{U_j} + Q, \qquad (27)$$

where $Z = (Z_1,\dots,Z_r)^T$ and $\hat{\Sigma}^-$ is a generalized inverse of the covariance matrix $\hat{\Sigma}$ and

$$Q = W^T \hat{G}^- W, A_j = \frac{U_j}{n}, U_j = \sum_{i:X_i \in I_j} \delta_i,\ W = (W_1,..,W_s)^T, \hat{G} = \left[\ \hat{g}_{ll'}\ \right]_{s\times s}, \hat{g}_{ll'} = \hat{i}_{ll'} - \sum_{j=1}^{r} \hat{C}_{lJ} \hat{G}_{l'J} \hat{A}_J^{-1},$$

$$\hat{C}_{lj}=\frac{1}{n}\sum_{i:X_i\in I_j}\delta_i\frac{\partial lnH(x_i,\hat{\gamma})}{\partial\gamma},\hat{i}_{ll'}=\frac{1}{n}\sum_{i=1}^{n}\delta_i\frac{\partial lnH(x_i,\hat{\gamma})}{\partial\gamma_l}\frac{\partial lnH(x_i,\hat{\gamma})}{\partial\gamma_{l'}},\hat{W}_l=\sum_{j=1}^{r}\hat{C}_{lJ}\hat{A}_J^{-1}Z_j,l,l'=1,..,s,$$

where $\hat{\gamma}$ is the maximum likelihood estimator of γ on initial non-grouped data.

Under the null hypothesis H_0, the limit distribution of the statistic Y^2 is a chi-square with $r=rank(\Sigma)$ degrees of freedom. The description and applications of modified chi-square tests are discussed in Voinov et al. (2013). The interval limits p_j for grouping data into j classes I_j are considered as data functions and defined by

$$\hat{p}_j=H^{-1}\left(\frac{E_j-\sum_{l=1}^{i-1}\Lambda(x_l,\hat{\gamma})}{n-i+1},\hat{\gamma}\right),\quad \hat{p}_j=\max\left(X_{(n)},\tau\right),\quad (28)$$

where $\Lambda(x_i,\gamma)$ is the cumulative hazard function, such as the expected failure times e_j to fall into these intervals are $e_j=\frac{E_r}{r}$for any j, with $E_r=\sum_{i=1}^{n}\Lambda(x_i,\gamma)$. The distribution of this statistic test Y_n^2 is chi-square (see Voinov et al., 2013). For goodness of fit tests one can see Al-Omari and Zamanzade (2017, 2016) for goodness of fit-tests for Laplace and Rayleigh distributions using ranked set sampling.

9. Criteria test for TPRD distribution

For testing the null hypothesis H_0 that data belong to the TPRD model, we construct a modified chi-squared type goodness-of-fit test based on the statistic Y^2. Suppose that τ is a finite time, and observed data are grouped into $r>s$ sub-intervals $I_j=(p_{j-1},p_j]$ of $[0,\tau]$. The limit intervals p_j are considered as random variables such that the expected numbers of failures in each interval I_j are the same, so the expected numbers of failures e_j are obtained as

$$E_j=-\frac{j}{r-1}\sum_{i=1}^{n}\ln[(1+\varphi_i)e^{-\theta x_i}],\ j=1,..,r-1.$$

9.1 Estimated matrix $\widehat{W}$ and $\widehat{C}$

The components of the estimated matrix $\widehat{\boldsymbol{W}}$ are derived from the estimated matrix $\widehat{\boldsymbol{C}}$ which is given by:

$$\hat{C}_{1j}=\frac{1}{n}\sum_{i:x_i\in I_j}^{n}\delta_i\left[\frac{\theta}{\alpha\theta+x_i^4}-\frac{\theta^5}{\alpha\theta^5+24}+\frac{\theta^5\varpi_i}{(24+\alpha\theta^5)^2(1+\varphi_i)}\right],\qquad (29)$$

$$\hat{C}_{2j}=\frac{1}{n}\sum_{i:x_i\in I_j}^{n}\delta_i\left[\frac{5}{\theta}-\frac{5\alpha\theta^4}{(24+\alpha\theta^5)}+\frac{\alpha}{\alpha\theta+x_i^4}-\frac{4x_i(24+\alpha\theta^5)s_i-5\alpha\theta^5x_iv_i}{(24+\alpha\theta^5)^2(1+\varphi_i)}\right],\qquad (30)$$

and $\hat{W}_l=\sum_{j=1}^{r}\hat{C}_{lJ}\hat{A}_J^{-1}Z_j,l,l'=1,2,\ j=1,2,...,r.$

9.2 Estimated matrix $\widehat{G}$

The estimated matrix $\hat{G}=[\hat{g}_{ll'}]_{2\times2}$ is defined by

$$\hat{g}_{ll'}=\hat{i}_{ll'}-\sum_{j=1}^{r}\hat{C}_{lJ}\hat{G}_{l'J}\hat{A}_J^{-1},\qquad (31)$$

where $\hat{i}_{ll'}=\frac{1}{n}\sum_{i=1}^{n}\delta_i\frac{\partial lnh(x_i,\hat{\gamma})}{\partial\gamma_l}\frac{\partial lnh(x_i,\hat{\gamma})}{\partial\gamma_{l'}},ll'=1,2.$ Therefore, the quadratic form of the test statistic can be obtained easily:

$$Y_n^2(\hat{\gamma}) = \sum_{j=1}^{r} \frac{\left(U_j - e_j\right)^2}{U_j} + \hat{W}^T \left[\hat{i}_{ll'} - \sum_{j=1}^{r} \hat{C}_{lJ} \hat{G}_{l'J} \hat{A}_J^{-1} \right]^{-1} \hat{W} . \quad (32)$$

10. Simulations

In this section, a simulation study is conducted to investigate the efficiency of several estimators considered in this study.

10.1 Maximum likelihood estimation

We generated $N = 10{,}000$ right censored samples with different sizes ($n = 25, 50, 130, 350, 500$) from the TPRD model with parameters $\alpha = 0.7$ *and* $\theta = 1.5$. Using R statistical software and the Barzilai-Borwein (BB) algorithm (Ravi, 2009), we calculate the maximum likelihood estimators of the unknown parameters and their mean squared errors (MSE). The results are given in Table 2.

Table 2. Mean simulated values of MLEs $\hat{\gamma}$ their corresponding square mean errors

$N = 10{,}000$	$n = 25$	$n = 50$	$n = 130$	$n = 350$	$n = 500$
$\hat{\alpha}$	0.7182	0.7156	0.7123	0.7063	0.7025
	(0.0098)	(0.0063)	(0.0031)	(0.0022)	(0.0015)
$\hat{\theta}$	1.5634	1.5421	1.5334	1.5137	1.5053
	(0.0078)	(0.0067)	(0.0043)	(0.0028)	(0.0019)

The maximum likelihood estimated parameter values, presented in Table 2, agree closely with the true parameter values. Also, as the sample sizes are increasing the estimation of the parameters be more efficient.

10.2 Criteria test Y_n^2

For testing the null hypothesis H_0 that right censored data become from TPRD model, we compute the criteria statistic $Y_n^2(\gamma)$ as defined above for 10,000 simulated samples from the hypothesized distribution with different sizes (30, 50,150, 350, 500).Then, we calculate empirical levels of significance, when $Y^2 > \chi_\varepsilon^2(r)$, corresponding to theoretical levels of significance ($\varepsilon = 0.10, \varepsilon = 0.05, \varepsilon = 0.01$) We choose $r = 4$. The results are reported in Table 3.

Table 3. Simulated levels of significance for $Y_n^2(\gamma)$ test for TPRD model against their theoretical values ($\varepsilon = 0.01, 0.05, 0.10$)

$N = 10{,}000$	$n_1 = 30$	$n_2 = 50$	$n_3 = 150$	$n_4 = 350$	$n_5 = 500$
$\varepsilon = 1\%$	0.0062	0.0077	0.0088	0.0096	0.0105
$\varepsilon = 5\%$	0.0432	0.0443	0.0652	0.0478	0.0496
$\varepsilon = 10\%$	0.0933	0.0962	0.0976	0.0991	0.1002

The null hypothesis H_0 for which simulated samples are fitted by TPRD distribution, is widely validated for the different levels of significance. Therefore, the test proposed in this work, can be used to fit data from this new distribution.

11. Applications

To show the flexibility of the proposed distribution and the usefulness of the criteria test Y_n^2, we analyze censored and uncensored real data sets. Using model selection criteria (NLL, AIC, CAIC and BIC) and Y_n^2, we show that the TPRD distribution fits data better than some alternatives such as the lognormal, gamma and Weibull distributions.

Example 1: This example concerned a right censored dataset from the aluminum reduction cells study of Whitmore (1983). He considered the times of failures of 20 aluminum reduction cells. Failures times, in units of 1000 days are given below:

0.468, 0.725, 0.838, 0.853, 0.965, 1.139, 1.142, 1.304,1.317,1.427, 1.554, 1.658, 1.764 , 1.776, 1.990 , 2.010 , 2.224 , 2.279* , 2.244* , 2.286* .

We use the statistic test provided above to verify if these data are modeled by TPRD distribution, and at that end, we first calculate the maximum likelihood estimators of the unknown parameters $\gamma = (\alpha, \theta)^T = (2.165, 4.956)^T$. Data are grouped into $r = 4$ intervals I_j. We give the necessary calculus in the following Table 4.

Table 4. Values of $p_j, e_j, U_j, \hat{C}_{1j}, \hat{C}_{2j}$.

p_j	0.953	1.456	2.00	2.286
U_j	4	6	5	5
e_j	0.7432	0.7432	0.7432	0.7432
$\hat{C}_{1j}$	1.5261	2.9417	3.0458	1.0568
$\hat{C}_{2j}$	2.0762	3.1653	1.4263	1.0856

Then, we obtain the value of the statistic test Y_n^2 as

$$Y_n^2 = X^2 + Q = 4.667 + 2.596 = 7.263.$$

For significance level ε = 0.05, the critical value $\chi_4^2 = 9.4877$ is superior than the value of

$Y_n^2 = 7.263$, so we can say that the proposed model TPRD fits these data.

Example 2:

We considered a complete sample representing the strength data of glass of the aircraft window reported by Fuller et al. (1994): 18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

After calculating the maximum likelihood estimators of the unknown parameters, we use classical criteria (NLL, AIC, CAIC, BIC) to select the best model which describes these data. From the results given in Table 5, we can see that values of the different criteria for the TPRD distribution are the smallest. So we can say that the TPRD distribution fits these data better than the lognormal, gamma and Weibull distributions.

Table 5. model selection criteria scores

Distribution	α	θ	-NLL	AIC	CAIC	BIC
TPRD	0.842	2.013	62.513	129.026	129.4545	131.8939
Lognormal	1.061	2.931	65.626	135.252	135.6805	138.1199
Gamma	1.442	0.052	64.197	132.394	132.8225	135.2619
Weibull	1.306	0.034	64.026	132.052	132.4805	134.9199

12. Conclusions

A new continuous two-parameter lifetime distribution called as the two parameters Rani (TPRD) distribution. Some statistical properties includes the mean, variance, moment generating function, the rth moment, the coefficients of variation, skewness and kurtosis are obtained. Also, the distributions of order statistics and the stochastic ordering are studied, reliability analysis, maximum likelihood estimation and goodness of fit tests of the distribution are proved based on right censoring data. The Rényi entropy and q-entropy are derived. Real data sets are considered for illustration.

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