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A Two Parameter Ratio-Product-Ratio Estimator in Post Stratification

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Abstract

In this paper we consider a two parameter ratio-product-ratio estimator for estimating population mean in case of post stratification following the estimator due to Chami et al (2012). The bias and mean squared error of proposed estimator are obtained to the first degree of approximation. We derive conditions under which the proposed estimator has smaller mean squared error than the sample mean $\bar{y}_{ps}$, ratio estimator $\bar{y}_{R(ps)}$ and product estimators $\bar{y}_{P(ps)}$. Empirical studies gives insight on the magnitude of the efficiency of the estimator developed.

Key Words: Bias, Mean squared error, Post stratification, Auxiliary information.

Mathematical Subject Classification: 62D05

1. Introduction:

The principal aim of statistical surveys is to obtain information about of interest. To increase the precision of the estimates we use information on the auxiliary variable. Stratification is one of the most widely used techniques, requires the size of the strata as well as sampling frame for each stratum. But in many situations sampling frame is not available. For example in households surveys it is possible to know the number of families added in a locality but which families belong to which locality up to date sampling frame of different strata may not be possible. In this type of situation, post stratification technique is used. The post stratification was first introduced by Holt and Smith (1979) and Ige and Tripathi (1989). It is known that when the auxiliary information is used at the estimation stage, the ratio estimator is best among a wide class of estimators when the relation between *y* and *x*, the variate under study and auxiliary variate, respectively is a straight line through the origin and the variance of *y* about this line is proportional to *x*. In such a situation the ratio estimator is as good as regression estimator. In many practical situations, the regression line does not pass well as that of regression estimator. Keeping this fact in view and also due to the stronger intuitive appeal, statisticians are more inclined towards the use of the ratio and the product estimators and hence a large amount of work has been carried out towards the modification of ratio and product estimators, for instance, see Singh (1986), Singh and Espejo (2003) etc.. Motivated by this Ige and Tripathi have suggested an improved version of combined/separate ratio and product estimators in post stratified sampling and studied their properties under large sample approximation. Ige and Tripathi (1989) have conducted an empirical study in support of their studies. Tuteja et al (1995) have extended the study of Ige and Tripathi (1989) based on post stratification and auxiliary information. Recently Chouhan (2012) proposed class of ratio type estimators using various known parameters of auxiliary variates in case of post stratification. Keeping in this view we have suggested a two parameter ratio-product-ratio estimator for estimating population mean in case of post stratification adapting the estimator due to Chami et al (2012).

For a finite population of size N, we are interested in estimating the population mean $\bar{Y}$ of main variable y. Use of auxiliary information has been in practice for improving the efficiency of estimator(s). Usually, auxiliary information is easily available with study variate with little extra cost and efforts. Let us consider a finite population $U=(U_1,U_2,...,U_N)$. A sample of size n is drawn from population U using simple random sampling without replacement (*SRSWOR*). After selecting the sample, it is observed that which units belong to h^{th} stratum. Let n_h be the size of the sample falling in h^{th} stratum such that $\sum_{h=1}^{L} n_h = n$. Here it is assumed that n is so large that the probability of n_h being zero is very small.

Let y_{hi} be the observation on i^{th} unit that fall in h^{th} stratum for study variate y and x_{hi} be the observation on i^{th} unit that fall in n_h stratum for auxiliary variate x, then,

$\bar{X}_h = \frac{1}{N_h}\sum_{h=1}^{L} x_{hi}$: h^{th} stratum mean for the auxiliary variate x,

$\bar{Y}_h = \frac{1}{N_h}\sum_{h=1}^{L} y_{hi}$: h^{th} stratum mean for the study variate y,

$\bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h$: Population mean of the auxiliary variate x,

$\bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h$: Population mean of the study variate y,

In case of post stratification, usual unbiased estimator of population mean $\bar{Y}$ is defined as

$\bar{y}_{ps} = \sum_{h=1}^{L} W_h \bar{y}_h$.

where $W_h = \frac{N_h}{N}$ is the weight of the h^{th} stratum and

$\bar{y}_h = \frac{1}{n_h}\sum_{i=1}^{n_h} y_{hi}$ is sample mean of n_h sample units that falls in the h^{th} stratum.

We denote the population variances/mean squares of Y and X as

$S_{yh}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i-\bar{Y})^2$, $S_{xh}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i-\bar{X})^2$ respectively.

Furthermore, we define the coefficient of variation of Y and X as

$C_{yh} = \frac{S_{yh}}{\bar{Y}}$, $C_{xh} = \frac{S_{xh}}{\bar{X}}$ respectively.

Using the results from Stephen (1945), the variance/*MSE* of $\bar{y}_{ps}$ to the first degree of approximation is obtained as

$$Var(\bar{y}_{ps}) = \gamma \sum_{h=1}^{L} W_h S_{yh}^2 , \tag{1.1}$$
$$= MSE(\bar{y}_{ps})$$

where $S_{yh}^2 = \frac{1}{N_h - 1}\sum_{i=1}^{N_h}\left(y_{hi} - \bar{Y}_h\right)^2$, $\gamma = \left(\frac{1-f}{n}\right)$ and $f = \frac{n}{N}$ is the sampling fraction.

Ige and Tripathi (1989) defined classical ratio and product type estimators for estimating the population mean $\bar{Y}$ in case of post stratification as

$$\hat{\bar{Y}}_{ps}^{R} = \bar{y}_{ps}\left(\frac{\bar{X}}{\bar{x}_{ps}}\right), \tag{1.2}$$

and

$$\hat{\bar{Y}}_{ps}^{P} = \bar{y}_{ps}\left(\frac{\bar{x}_{ps}}{\bar{X}}\right). \tag{1.3}$$

where $\bar{x}_{ps} = \sum_{h=1}^{L} W_h \bar{x}_h$ is the unbiased estimate of population means in case of post stratification, $\bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h$ is the known population mean of the auxiliary variate x and $\bar{x}_h$ is the mean of the sample of size n_h that fall in the h^{th} stratum.

MSE of the Ige and Tripathi (1989) estimators $\hat{\bar{Y}}_{ps}^{R}$ and $\hat{\bar{Y}}_{ps}^{P}$ are

$$MSE\left(\hat{\bar{Y}}_{ps}^{R}\right) = \gamma \sum_{h=1}^{L} W_h \left(S_{yh}^2 + R^2 S_{xh}^2 - 2RS_{yxh}\right), \tag{1.4}$$

and

$$MSE\left(\hat{\bar{Y}}_{ps}^{P}\right) = \gamma \sum_{h=1}^{L} W_h \left(S_{yh}^2 + R^2 S_{xh}^2 + 2RS_{yxh}\right). \tag{1.5}$$

where $R = \frac{\bar{Y}}{\bar{X}}$.

Following Khoshnevisan et al (2007), Onyeka (2012) proposed a class of estimators for population mean $\bar{Y}$ in case of post stratified sampling as

$$\bar{y}_{pss} = \bar{y}_{ps}\left(\frac{a\bar{X} + b}{\alpha\left(a\bar{x}_{ps} + b\right) + (1-\alpha)\left(a\bar{X} + b\right)}\right)^{g}, \tag{1.6}$$

where (α, δ) are real constants and $a(\neq 0)$, b are either constants or functions of known population parameters of the auxiliary variates such as standard deviation $\sigma_x(\text{or}\, S_x)$, coefficient of variation C_x, Skewness $\beta_1(x)$, kurtosis $\beta_2(x)$ and correlation coefficient ρ_{yx} between *y* and *x*.

To the first degree of approximation, the *MSE* of $\bar{y}_{pss}$ is given by

$$MSE\left(\bar{y}_{pss}\right) = \gamma \sum_{h=1}^{L} W_h \left[S_{yh}^2 + \alpha^2 \lambda^2 g^2 R^2 S_{xh}^2 - 2\alpha\lambda g R S_{yxh}\right], \tag{1.7}$$

where $\lambda = \frac{a\bar{X}}{a\bar{X} + b}$.

It is to be mentioned that Onyeka (2012) generated five new ratio-type estimators for $\bar{Y}$:

$t_{R_1}=\bar{y}_{ps}\frac{(\bar{X}+C_x)}{(\bar{x}_{ps}+C_x)}$, $t_{R_2}=\bar{y}_{ps}\frac{(\bar{X}+\beta_2(x))}{(\bar{x}_{ps}+\beta_2(x))}$, $t_{R_3}=\bar{y}_{ps}\frac{(\bar{X}\beta_2(x)+C_x)}{(\bar{x}_{ps}\beta_2(x)+C_x)}$, $t_{R_4}=\bar{y}_{ps}\frac{(\bar{X}C_x+\beta_2(x))}{(\bar{x}_{ps}C_x+\beta_2(x))}$ and $\hat{\bar{Y}}_{R(ps)ST1}=\bar{y}_{ps}\frac{(\bar{X}+\rho_{yx})}{(\bar{x}_{ps}+\rho_{yx})}$ for $(\alpha,g,a,b)=(1,1,1,C_x),(1,1,1,\beta_2(x)),(1,1,\beta_2(x),C_x),(1,1,C_x,\beta_2(x)),(1,1,1,\rho_{yx})$;

respectively and eight product-type estimators for $\bar{Y}$:

$t_{p_1}=\bar{y}_{ps}\frac{(\bar{x}_{ps}+C_x)}{(\bar{X}+C_x)}$, $t_{p_2}=\bar{y}_{ps}\frac{(\bar{x}_{ps}+\beta_2(x))}{(\bar{X}+\beta_2(x))}$, $t_{p_3}=\bar{y}_{ps}\frac{(\bar{x}_{ps}+\rho_{yx})}{(\bar{X}+\rho_{yx})}$, $t_{p_4}=\bar{y}_{ps}\frac{(\bar{x}_{ps}+\sigma_x)}{(\bar{X}+\sigma_x)}$, $t_{p_5}=\bar{y}_{ps}\frac{(\bar{x}_{ps}\beta_2(x)+C_x)}{(\bar{X}\beta_2(x)+C_x)}$, $t_{p_6}=\bar{y}_{ps}\frac{(\bar{x}_{ps}C_x+\beta_2(x))}{(\bar{X}C_x+\beta_2(x))}$, $t_{p_7}=\bar{y}_{ps}\frac{(\bar{x}_{ps}\beta_1(x)+\sigma_x)}{(\bar{X}\beta_1(x)+\sigma_x)}$ and $t_{p_8}=\bar{y}_{ps}\frac{(\bar{x}_{ps}\beta_2(x)+\sigma_x)}{(\bar{X}\beta_2(x)+\sigma_x)}$ for $(\alpha,g,a,b)=(1,-1,1,C_x),(1,-1,1,\beta_2(x)),(1,-1,1,\rho_{yx}),(1,-1,1,\sigma_x),\ (1,-1,\beta_2(x),C_x),(1,-1,C_x,\beta_2(x))$ and $(1,-1,\beta_2(x),\sigma_x)$ respectively.

Motivated by Bhal and Tuteja (1991), Tailor et al (2017) suggested the following combined ratio-type and product-type exponential estimators for population mean $\bar{Y}$ of y in post stratified sampling as

$$\hat{\bar{Y}}_{\text{Re}}^{ps}=\bar{y}_{ps}\exp\left(\frac{\bar{X}-\bar{x}_{ps}}{\bar{X}+\bar{x}_{ps}}\right), \tag{1.8}$$

and

$$\hat{\bar{Y}}_{Pe}^{ps}=\bar{y}_{ps}\exp\left(\frac{\bar{x}_{ps}-\bar{X}}{\bar{X}+\bar{x}_{ps}}\right). \tag{1.9}$$

To the first degree of approximation, the *MSE*s of $\hat{\bar{Y}}_{\text{Re}}^{ps}$ and $\hat{\bar{Y}}_{Pe}^{ps}$ are respectively given by

$$MSE\left(\hat{\bar{Y}}_{\text{Re}}^{ps}\right)=\gamma\sum_{h=1}^{L}W_h\left(S_{yh}^2+\frac{1}{4}R^2S_{xh}^2-RS_{yxh}\right), \tag{1.10}$$

and

$$MSE\left(\hat{\bar{Y}}_{Pe}^{ps}\right)=\gamma\sum_{h=1}^{L}W_h\left(S_{yh}^2+\frac{1}{4}R^2S_{xh}^2+RS_{yxh}\right). \tag{1.11}$$

Bacanli and Aksu (2012) proposed the following separate ratio estimators for the population mean $\bar{Y}$ of y in post stratified sampling as

$$\bar{y}_{Rs_1}^{ps}=\sum_{h=1}^{L}\bar{y}_hW_h\frac{(\bar{X}_h+C_{xh}^*)}{(\bar{x}_h+C_{xh}^*)} \tag{1.12}$$

$$\bar{y}_{Rs_2}^{ps}=\sum_{h=1}^{L}\bar{y}_hW_h\frac{(\bar{X}_h+\beta_{2h}(x))}{(\bar{x}_h+\beta_{2h}(x))} \tag{1.13}$$

$$\bar{y}_{Rs_3}^{ps}=\sum_{h=1}^{L}\bar{y}_hW_h\frac{(\bar{X}_h\beta_{2h}(x)+C_{xh}^*)}{(\bar{x}_h\beta_{2h}(x)+C_{xh}^*)}, \tag{1.14}$$

$$\bar{y}_{Rs_4}^{ps}=\sum_{h=1}^{L}\bar{y}_hW_h\frac{(\bar{X}_hC_{xh}^*+\beta_{2h}(x))}{(\bar{x}_hC_{xh}^*+\beta_{2h}(x))}, \tag{1.15}$$

where $C_{xh}^*=\frac{S_{xh}}{\bar{X}_h}$ is the coefficient of variation of *x* in stratum *h*.

Motivated by Khoshnevisan et al (2007), we define a class of separate ratio-type estimators for population mean $\bar{Y}$ in post stratified sampling as

$$\hat{\bar{Y}}_{SR}^{ps}=\sum_{h=1}^{L}w_h\bar{y}_h\left(\frac{a_h\bar{X}_h+b_h}{\alpha_h\left(a_h\bar{x}_h+b_h\right)+(1-\alpha_h)\left(a_h\bar{X}_h+b_h\right)}\right)^{g_h}, \quad (1.16)$$

where (α_h,g_h) are real constants and $(a_h(\neq 0),b_h)$ are either constants or functions of known parameters of the auxiliary variable x in the h^{th} stratum of the population such as standard deviation $\sigma_{xh}(\text{or}\, S_{xh})$, coefficient of variation C_{xh}, skewness $\beta_{1h}(x)$, kurtosis $\beta_{2h}(x)$ and correlation coefficient ρ_{yxh} of the h^{th} stratum.

To the first degree of approximation the *MSE* of $\hat{\bar{Y}}_{SR}^{ps}$ is given by

$$MSE\left(\hat{\bar{Y}}_{SR}^{ps}\right)=\gamma\sum_{h=1}^{L}W_h\left[S_{yh}^2+\alpha_h^2\lambda_h^2g_h^2R_h^2S_{xh}^2-2\alpha_h\lambda_hg_hR_hS_{yxh}\right], \quad (1.17)$$

where $\lambda_h=\dfrac{a_h\bar{X}_h}{\left(a_h\bar{X}_h+b_h\right)}$.

The *MSE*s of the estimators $y_{Rs_1}^{ps},y_{Rs_2}^{ps},y_{Rs_3}^{ps}$ and $y_{Rs_4}^{ps}$ can be easily obtained from (1.17) just by putting $(\alpha_h,g_h,a_hb_h)=(1,1,1,C_{xh}^*),(1,1,1,\beta_{2h}(x)),(1,1,\beta_{2h}(x),C_{xh}^*)$ and $(1,1,C_{xh}^*,\beta_{2h}(x))$ respectively.

In this paper we have developed a two parameter ratio-product-ratio estimator in post stratification on the line of Chami et al (2012). Combined as well as separate estimators are proposed and their properties are studied under large sample approximation. Numerical examples are given in support of the present study.

2. Proposed two parameter combined ratio-product-ratio type estimator

Taking motivation from Holt and Smith (1979), Ige and Tripathi (1989) and Chami et al. (2012) for estimating the population mean $\bar{Y}$ in case of post stratification we suggest the following two parameter ratio-product-ratio estimator

$$\bar{y}_{ps(\eta,\delta)}=\bar{y}_{ps}\left[\eta\left\{\frac{(1-\delta)\bar{x}_{ps}+\delta\bar{X}}{\delta\bar{x}_{ps}+(1-\delta)\bar{X}}\right\}+(1-\eta)\left\{\frac{\delta\bar{x}_{ps}+(1-\delta)\bar{X}}{(1-\delta)\bar{x}_{ps}+\delta\bar{X}}\right\}\right], \quad (2.1)$$

where (η,δ) are real constants. The aim of this paper is to derive values for these constants (η,δ) such that the bias and mean squared error (*MSE*) of $\bar{y}_{ps(\eta,\delta)}$ is minimal.

It is to be mentioned that $\bar{y}_{ps(\eta,\delta)}=\bar{y}_{ps(1-\eta,1-\delta)}$ that is the estimator $\bar{y}_{ps(\eta,\delta)}$ is invariant under a point reflection through the point $(\eta,\delta)=\left(\frac{1}{2},\frac{1}{2}\right)$. The proposed estimator $\bar{y}_{ps(\eta,\delta)}$ boils down to

the known estimators of the population mean $\bar{Y}$:

(i) $\bar{y}_{ps(\eta,\delta)}=\bar{y}_{ps}$ for $(\eta,\delta)=\left(\frac{1}{2},\frac{1}{2}\right)$

that is, $\bar{y}_{ps\left(\frac{1}{2},\frac{1}{2}\right)}=\bar{y}_{ps}$.

(ii) $\bar{y}_{ps(\eta,\delta)}=\bar{y}_{ps}\dfrac{\bar{x}_{ps}}{\bar{X}}$ (product estimator) for $(\eta,\delta)=(1,0)$ and $(0,1)$.

(iii) $\bar{y}_{ps(\eta,\delta)}=\bar{y}_{ps}\dfrac{\bar{X}}{\bar{x}_{ps}}$ (ratio estimator) for $(\eta,\delta)=(0,0)$ and $(1,1)$.

To obtain the bias and *MSE* of the proposed estimator $\bar{y}_{ps(\eta,\delta)}$, we write

$\bar{y}_{ps} = \bar{Y}(1+e_0)$, $\bar{x} = \bar{X}(1+e_1)$ such that $E(e_i) = 0$ for i = 1,2.

and to the first degree of approximation (i.e. up to order n^{-1});

$$E\left(e_0^2\right) = \frac{\gamma}{\bar{Y}^2}\sum_{h=1}^{L} W_h S_{yh}^2 \ , \ E\left(e_1^2\right) = \frac{\gamma}{\bar{X}^2}\sum_{h=1}^{L} W_h S_{xh}^2 \ , E\left(e_0 e_1\right) = \frac{\gamma}{\bar{Y}\bar{X}}\sum_{h=1}^{L} W_h S_{yxh}.$$

Now expressing (2.1) in terms of e_i's, we have

$$\bar{y}_{ps(\eta,\delta)} = \bar{Y}\left(1+e_0\right)\left[\eta\left\{\frac{(1-\delta)\bar{X}(1+e_1)+\delta\bar{X}}{\delta\bar{X}(1+e_1)+(1-\delta)\bar{X}}\right\} + (1-\eta)\left\{\frac{\delta\bar{X}(1+e_1)+(1-\delta)\bar{X}}{(1-\delta)\bar{X}(1+e_1)+\delta\bar{X}}\right\}\right]$$

$$= \bar{Y}\left(1+e_0\right)\left[\eta\left\{\frac{1+e_1-\delta e_1}{1+\delta e_1}\right\} + (1-\eta)\left\{\left[\frac{1+\delta e_1}{1+e_1-\delta e_1}\right]\right\}\right]. \tag{2.2}$$

We assume that $|e_1| < \min\left\{\frac{1}{|\delta|}, \frac{1}{|1-\delta|}\right\}$, so that $(1+\delta e_1)^{-1}$ and $(1+(1-\delta)e_1)^{-1}$ are expandable.

Now expanding the right hand side of $\bar{y}_{ps(\eta,\delta)}$ at (2.2) and neglecting terms of e_i's having power greater than two we have

$$\bar{y}_{ps(\eta,\delta)} \cong (1+e_0)\bar{Y}\left[1-(1-2\eta)(1-2\delta)e_1 + (1-2\delta)(1-\eta-\delta)e_1^2\right]$$

or

$$\left(\bar{y}_{ps(\eta,\delta)} - \bar{Y}\right) \cong \bar{Y}\left[e_0 - (1+e_0)\left\{(1-2\eta)(1-2\delta)e_1 - (1-\eta-\delta)(1-2\delta)e_1^2\right\}\right] \tag{2.3}$$

Taking expectations of both sides of (2.3), we get the bias of $\bar{y}_{ps(\eta,\delta)}$ to the first degree of approximation as

$$B\left(\bar{y}_{ps(\eta,\delta)}\right) = \frac{\gamma(1-2\delta)R}{\bar{X}}\sum_{h=1}^{L} W_h S_{xh}^2\left[(1-\eta-\delta) - (1-2\eta)C\right] \tag{2.4}$$

where $C = \frac{\sum_{h=1}^{L} W_h \beta_h S_{xh}^2}{R\sum_{h=1}^{L} W_h S_{xh}^2}$ and $\beta_h = \frac{S_{yxh}}{S_{xh}^2}$.

Equating (2.4) to zero, we obtain

$$(1-2\delta) = 0 \text{ or } \left\{1-\eta-\delta-(1-2\eta)C\right\} = 0$$

$$\text{i.e. } \delta = \frac{1}{2} \text{ or } \delta = 1-\eta-C+2\eta C. \tag{2.5}$$

The suggested ratio-product-ratio estimator $\bar{y}_{ps(\eta,\delta)}$, substituted with the values of δ from (2.5), becomes an (approximately) unbiased estimator for the population mean $\bar{Y}$. In the three dimensional parameters space $(\eta,\delta,C) \in R^3$, these unbiased estimators lie on a plain $\left(\text{in the case } \delta = \frac{1}{2}\right)$. When the sample size n is sufficiently large (i.e. n approaches the population size N) the bias of $\bar{y}_{ps(\eta,\delta)}$ at (2.4) is negligible.

- **Mean Squared Error of** $\bar{y}_{ps(\eta,\delta)}$

Squaring both sides of (2.3) and neglecting terms of e_i's having power greater than two, we have

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$$\left(\bar{y}_{ps(\eta,\delta)} - \bar{Y}\right)^2 = \bar{Y}^2\left[e_0^2 + (1-2\eta)^2(1-2\delta)^2 e_1^2 - 2(1-2\eta)(1-2\delta)e_0 e_1\right] \tag{2.6}$$

Taking expectations of both sides of (2.6) we get the mean squared error of $\bar{y}_{ps(\eta,\delta)}$ to the first degree of approximation as

$$MSE\left(\bar{y}_{ps(\eta,\delta)}\right) = \gamma\bar{Y}^2\left[\sum_{h=1}^{L} W_h C_{yh}^2 + (1-2\eta)(1-2\delta)\sum_{h=1}^{L} W_h C_{xh}^2\{(1-2\eta)(1-2\delta) - 2C\}\right], \tag{2.7}$$

where $C_{yh}^2 = \frac{S_{yh}^2}{\bar{Y}^2}$ and $C_{xh}^2 = \frac{S_{xh}^2}{\bar{X}^2}$.

Taking the gradient $\nabla = \left(\frac{\partial}{\partial\eta}, \frac{\partial}{\partial\delta}\right)$ of equation (2.7), we get

$$\nabla MSE_1\left(\bar{y}_{ps(\eta,\delta)}\right) = 4\gamma\bar{Y}^2\sum_{h=1}^{L} W_h C_{xh}^2 (1-2\eta)(1-2\delta)\left[(1-2\eta)(1-2\delta) - C\right] \tag{2.8}$$

Inserting (2.8) to zero to get the critical points, we obtain the following solutions

$$\eta = \frac{1}{2}, \delta = \frac{1}{2} \tag{2.9}$$

$$C = (1-2\eta)(1-2\delta) \tag{2.10}$$

One can see that the critical point in (2.9) is a saddle point unless C=0, in which we get a local minimum. However, the critical points obtained by (2.10) are always minima, for a given C, (2.10) is the equation of hyperbola symmetric through $(\eta,\delta) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Putting $(\eta,\delta) = \left(\frac{1}{2}, \frac{1}{2}\right)$ into the estimator $\bar{y}_{ps(\eta,\delta)}$ gives the unbiased estimator $\bar{y}_{ps}$ of $\bar{Y}$. Thus we get the *MSE* of $\bar{y}_{ps}$:

$$MSE\left(\bar{y}_{ps\left(\frac{1}{2},\frac{1}{2}\right)}\right) = MSE\left(\bar{y}_{ps}\right) = \gamma\sum_{h=1}^{L} W_h S_{yh}^2. \tag{2.11}$$

Substituting (2.10) into the estimator, an asymptotically optimum estimator (*AOE*) $\bar{y}_{ps(\eta,\delta)}^{(o)}$ is obtained. Thus the *MSE* of $\bar{y}_{ps(\eta,\delta)}^{(o)}$ (or the minimum *MSE* of $\bar{y}_{ps(\eta,\delta)}$) to the first degree of approximation is given by

$$MSE\left(\bar{y}_{ps(\eta,\delta)}^{(o)}\right) = MSE_{\min}\left(\bar{y}_{ps(\eta,\delta)}\right) = \gamma\sum_{h=1}^{L} W_h S_{yh}^2\left(1-\rho^2\right), \tag{2.12}$$

where $\rho = \frac{\left(\sum_{h=1}^{L} W_h S_{yxh}\right)}{\sqrt{\left\{\left(\sum_{h=1}^{L} W_h S_{yh}^2\right)\left(\sum_{h=1}^{L} W_h S_{xh}^2\right)\right\}}}$.

3. Comparison of *MSE*s and Selection of Parameters

In this section we present the comparison of the proposed estimator $\bar{y}_{ps(\eta,\delta)}$ with usual unbiased estimator $\bar{y}_{ps}$, ratio estimator $\hat{\bar{y}}_{ps}^{R}$ and product estimator $\hat{\bar{y}}_{ps}^{P}$.

3.1 Comparing the *MSE* of the Usual Unbiased Estimator $\bar{y}_{ps}$ to the Suggested Estimator $\bar{y}_{ps(\eta,\delta)}$

From (1.1) and (2.7) we have

$$MSE\left(\bar{y}_{(ps)}\right)-MSE\left(\bar{y}_{ps(\eta,\delta)}\right)=\gamma \bar{Y}^2(1-2\eta)(1-2\delta)\{2C-(1-2\eta)(1-2\delta)\}\sum_{h=1}^{L}W_h C_{yh}^2$$

which is non-negative if

$$(1-2\eta)(1-2\delta)\sum_{h=1}^{L}W_h C_{xh}^2\{2C-(1-2\eta)(1-2\delta)\}>0$$

That is if either

(i) $\eta>\frac{1}{2},\delta>\frac{1}{2}$ and $C>\frac{(1-2\eta)(1-2\delta)}{2}$,

(ii) $\eta<\frac{1}{2},\delta>\frac{1}{2}$ and $C<\frac{(1-2\eta)(1-2\delta)}{2}$,

(iii) $\eta>\frac{1}{2},\delta<\frac{1}{2}$ and $C<\frac{(1-2\eta)(1-2\delta)}{2}$, or

(iv) $\eta<\frac{1}{2},\delta<\frac{1}{2}$ and $C>\frac{(1-2\eta)(1-2\delta)}{2}$.

It is well known that when $-\frac{1}{2}\le C\le\frac{1}{2}$, the post stratified sample mean $\bar{y}_{ps}$ is preferred.

Therefore combining the conditions (i)-(iv) with the condition $-\frac{1}{2}\le C\le\frac{1}{2}$, we derive the following explicit ranges.

(i) if $0<C\le\frac{1}{2}$ and $\delta>\frac{1}{2}$, then $\frac{1}{2}<\eta<\left(\frac{2\delta+2C-1}{2(2\delta-1)}\right)$ (from (i))

(ii) if $0<C\le\frac{1}{2}$ and $\delta<\frac{1}{2}$, then $\left(\frac{2\delta+2C-1}{2(2\delta-1)}\right)<\eta<\frac{1}{2}$ (from (vi))

(iii) if $-\frac{1}{2}\le C<0$ and $\delta>\frac{1}{2}$, then $\left(\frac{2\delta+2C-1}{2(2\delta-1)}\right)<\eta<\frac{1}{2}$ (from (ii))

(iv) if $-\frac{1}{2}\le C<0$ and $\delta<\frac{1}{2}$, then $\frac{1}{2}<\eta<\left(\frac{2\delta+2C-1}{2(2\delta-1)}\right)$ (from (iii))

Thus the proposed ratio-product-ratio estimator $\bar{y}_{ps(\eta,\delta)}$ is more efficient than usual unbiased estimator $\bar{y}_{ps}$ as long as the conditions (i)-(iv) are satisfied.

3.2 Comparing the *MSE* of the Ratio Estimator $\bar{y}_{ps}^R$ to the *MSE* of Proposed Estimator $\bar{y}_{ps(\eta,\delta)}$

It is well known that the ratio estimator $\bar{y}_{ps}^R$ is more efficient than the usual unbiased estimator $\bar{y}_{ps}$ if

$$C>\frac{1}{2} \tag{3.1}$$

From (1.4) and (2.7) we have

$$MSE\left(\bar{y}_{(ps)}^R\right)-MSE\left(\bar{y}_{ps(\eta,\delta)}\right)=4\gamma R^2\{2\eta\delta-\eta-\delta\}\{C-1-(2\eta\delta-\eta-\delta)\}\sum_{h=1}^{L}W_h S_{xh}^2$$

which is non-negative if

$$\{2\eta\delta-\eta-\delta\}\{C-1-(2\eta\delta-\eta-\delta)\}>0 \qquad (3.2)$$

That is if either

(i) $C-1>2\eta\delta-\eta-\delta>0$, or

(ii) $C-1<2\eta\delta-\eta-\delta<0$.

Hence from solution (i), where C>1, we have the following

(i) If $\delta<\frac{1}{2}$, then $\frac{(\delta+C-1)}{(2\delta-1)}<\eta<\frac{\delta}{2\delta-1}$,

(ii) If $\delta>\frac{1}{2}$, then $\frac{\delta}{2\delta-1}<\eta<\frac{(\delta+C-1)}{(2\delta-1)}$.

Further from solution (ii), where $\frac{1}{2}<C<1$, we obtain the following

(i) If $\delta<\frac{1}{2}$, then $\frac{\delta}{2\delta-1}<\eta<\frac{(\delta+C-1)}{(2\delta-1)}$,

(ii) If $\delta>\frac{1}{2}$, then $\frac{(\delta+C-1)}{(2\delta-1)}<\eta<\frac{\delta}{2\delta-1}$.

3.3 Comparing the *MSE* of the Product Estimator $\bar{y}_{ps}^{P}$ to the *MSE* of Proposed Estimator $\bar{y}_{ps(\eta,\delta)}$

It is well known that the product estimator $\bar{y}_{ps}^{P}$ is more efficient than the usual unbiased estimator $\bar{y}_{ps}$ if

$$C<-\frac{1}{2} \qquad (3.3)$$

From (1.5) and (2.7) we have

$$MSE\left(\bar{y}_{(ps)}^{P}\right)-MSE\left(\bar{y}_{ps(\eta,\delta)}\right)=4\gamma R^2\{1+2\eta\delta-\eta-\delta\}\{C-(2\eta\delta-\eta-\delta)\}\sum_{h=1}^{L}W_h S_{xh}^2$$

which is non-negative if

$$[\{1+2\eta\delta-\eta-\delta\}\{C-(2\eta\delta-\eta-\delta)\}]>0 \qquad (3.4)$$

That is if either

(i) $C>2\eta\delta-\eta-\delta>-1$ (if both the factors in (3.4) are positive) or

(ii) $C<2\eta\delta-\eta-\delta<-1$ (if both the factors in (3.4) are negative).

Hence from solution (i), where C >1, we have the following

(i) If $\delta<\frac{1}{2}$, then $\frac{(\delta+C-1)}{(2\delta-1)}<\eta<\frac{\delta}{2\delta-1}$

(ii) If $\delta>\frac{1}{2}$, then $\frac{\delta}{2\delta-1}<\eta<\frac{(\delta+C-1)}{(2\delta-1)}$.

Further from solution (ii), where $\frac{1}{2}<C<1$, we obtain the following

(i) If $\delta<\frac{1}{2}$, then $\frac{\delta}{2\delta-1}<\eta<\frac{(\delta+C-1)}{(2\delta-1)}$

(ii) If $\delta>\frac{1}{2}$, then $\frac{(\delta+C-1)}{(2\delta-1)}<\eta<\frac{\delta}{2\delta-1}$.

We are only interested in the case $C < -\frac{1}{2}$, we get from (i)

$$-\frac{1}{2} > C > 2\eta\delta - \eta - \delta > -1 . \tag{3.5}$$

We mention that this implies $-1 < C < -\frac{1}{2}$, and the range for η and δ , where these inequalities hold are explicitly given by the following two situations:

(i) If $\delta < \frac{1}{2}$, than $\frac{(\delta + C)}{(2\delta - 1)} < \eta < \frac{(\delta - 1)}{(2\delta - 1)}$

(ii) If $\delta > \frac{1}{2}$, then $\frac{(\delta - 1)}{(2\delta - 1)} < \eta < \frac{(\delta + C)}{(2\delta - 1)}$.

In situation (ii), where $C < -1$, the following range of η and δ can be obtained.

(i) If $\delta < \frac{1}{2}$, than $\frac{(\delta - 1)}{(2\delta - 1)} < \eta < \frac{(\delta + C)}{(2\delta - 1)}$

(ii) If $\delta > \frac{1}{2}$, then $\frac{(\delta + C)}{(2\delta - 1)} < \eta < \frac{(\delta - 1)}{(2\delta - 1)}$.

3.4 Comparison of the Proposed Class of Estimators $\bar{y}_{ps(\eta,\delta)}$ with Onyeka (2012) Class of Estimators $\bar{y}_{pss}$

From (1.6) and (2.7) we have

$MSE(\bar{y}_{pss}) - MSE(\bar{y}_{ps(\eta,\delta)}) = \gamma \sum_{h=1}^{L} W_h \left[R^2 \left\{ \alpha^2 \lambda^2 g^2 - (1-2\eta)^2 (1-2\delta)^2 \right\} S_{xh}^2 - 2R\{\alpha\lambda g - (1-2\eta)(1-2\delta)\} S_{yxh} \right]$ which is positive if

$$\sum_{h=1}^{L} W_h \left[R\{\alpha\lambda g - (1-2\eta)(1-2\delta)\}\{\alpha\lambda g - (1-2\eta)(1-2\delta)\} - 2\{\alpha\lambda g - (1-2\eta)(1-2\delta)\}\beta_h \right] S_{xh}^2 > 0$$

i.e. if $\{\alpha\lambda g - (1-2\eta)(1-2\delta)\}\{\alpha\lambda g - (1-2\eta)(1-2\delta)\} - 2\{\alpha\lambda g - (1-2\eta)(1-2\delta)\} C > 0$

i.e. if

$$\left.\begin{array}{ll} \text{either} & \alpha\lambda g < (1-2\eta)(1-2\delta) < \{2C - \alpha\lambda g\} \\ \text{or} & \{2C - \alpha\lambda g\} < (1-2\eta)(1-2\delta) < \alpha\lambda g \end{array}\right\} \tag{3.6}$$

or equivalently,

$$\min.[\alpha\lambda g, (2C - \alpha\lambda g)] < (1-2\eta)(1-2\delta) < \max.[\alpha\lambda g, (2C - \alpha\lambda g)] \tag{3.7}$$

Let $\eta = \eta_0, \alpha = \alpha_0, g = g_0, a = a_0 (\neq 0), b = b_0$ and $\lambda = \lambda_0$ be pre-assigned. Then the proposed class of estimators $\bar{y}_{ps(\eta,\delta)}$ is better than Onyeka (2012) class of estimators if

$$\left.\begin{array}{l} \text{either} \frac{1}{2}\left[1 + \frac{(2C - \lambda_0\alpha_0 g_0)}{(2\eta_0 - 1)}\right] < \delta < \frac{1}{2}\left[1 + \frac{\lambda_0\alpha_0 g_0}{(2\eta_0 - 1)}\right] \\ \text{or } \frac{1}{2}\left[1 + \frac{\lambda_0\alpha_0 g_0}{(2\eta_0 - 1)}\right] < \delta < \frac{1}{2}\left[1 + \frac{(2C - \lambda_0\alpha_0 g_0)}{(2\eta_0 - 1)}\right] \end{array}\right\} \tag{3.8}$$

or equivalently,

$$\min.\left[\frac{1}{2}\left\{1 + \frac{(2C - \lambda_0\alpha_0 g_0)}{(2\eta_0 - 1)}\right\}, < \frac{1}{2}\left\{1 + \frac{\lambda_0\alpha_0 g_0}{(2\eta_0 - 1)}\right\}\right] < \delta < \max.\left[\frac{1}{2}\left\{1 + \frac{(2C - \lambda_0\alpha_0 g_0)}{(2\eta_0 - 1)}\right\}, < \frac{1}{2}\left\{1 + \frac{\lambda_0\alpha_0 g_0}{(2\eta_0 - 1)}\right\}\right] \tag{3.9}$$

Thus for given values of (η,α,g,a,b) the range of δ can be easily calculated from (3.9). Similarly for given values of (α,g,a,b) the range of η can be computed from the following inequality:

$$\min.\left[\frac{1}{2}\left\{1+\frac{(2C-\lambda_0\alpha_0 g_0)}{(2\delta_0-1)}\right\},<\frac{1}{2}\left\{1+\frac{\lambda_0\alpha_0 g_0}{(2\delta_0-1)}\right\}\right]<\eta<\max.\left[\frac{1}{2}\left\{1+\frac{(2C-\lambda_0\alpha_0 g_0)}{(2\delta_0-1)}\right\},<\frac{1}{2}\left\{1+\frac{\lambda_0\alpha_0 g_0}{(2\delta_0-1)}\right\}\right] \quad (3.10)$$

It is to be mentioned that the conditions under which the suggested class of estimators $\bar{y}_{ps(\eta,\delta)}$ is more efficient than the members of the Onyeka (2012) class of estimators $\bar{y}_{pss}$ just by putting the appropriate values of the constants (α,g,a,b) in (3.9) for given $\eta=\eta_0$ and in (3.10)for given $\delta=\delta_0$.

3.5 Comparison of the proposed class of estimators $\bar{y}_{ps(\eta,\delta)}$ with Tailor et al (2017) Ratio-type Exponential Estimator $\hat{\bar{Y}}_{\mathrm{Re}}^{ps}$

From (1.10) and (2.7) we have

$$MSE\left(\hat{\bar{Y}}_{\mathrm{Re}}^{ps}\right)-MSE\left(\bar{y}_{ps(\eta,\delta)}\right)=\gamma R^2\left\{\frac{1}{2}-(1-2\eta)(1-2\delta)\right\}\left[\left\{\frac{1}{2}+(1-2\eta)(1-2\delta)\right\}-2C\right]\sum_{h=1}^{L}W_h S_{xh}^2$$

which is positive if

$$\left[\frac{1}{2}-(1-2\eta)(1-2\delta)\right]\left[(1-2\eta)(1-2\delta)-2C+\frac{1}{2}\right]>0$$

i.e. if

$$\left.\begin{array}{l}\text{either}\left(2C-\frac{1}{2}\right)<(1-2\eta)(1-2\delta)<\frac{1}{2}\\ \text{or }\frac{1}{2}<(1-2\eta)(1-2\delta)<\left(2C-\frac{1}{2}\right)\end{array}\right\} \quad (3.11)$$

or equivalently,

$$\min.\left\{\frac{1}{2},\left(2C-\frac{1}{2}\right)\right\}<(1-2\eta)(1-2\delta)<\max.\left\{\frac{1}{2},\left(2C-\frac{1}{2}\right)\right\} \quad (3.12)$$

For given $\eta=\eta_0$ the range of δ in which the proposed class of estimators $\bar{y}_{ps(\eta,\delta)}$ is more efficient than Tailor et al (2017) ratio-type exponential estimator $\hat{\bar{Y}}_{\mathrm{Re}}^{ps}$ is:

$$\left.\begin{array}{l}\text{either }\frac{1}{2}\left[1+\frac{1}{2(2\eta_0-1)}\right]<\delta<\frac{1}{2}\left[1+\frac{1}{(2\eta_0-1)}\left(2C-\frac{1}{2}\right)\right]\\ \text{or }\frac{1}{2}\left[1+\frac{1}{(2\eta_0-1)}\left(2C-\frac{1}{2}\right)\right]<\delta<\frac{1}{2}\left[1+\frac{1}{2(2\eta_0-1)}\right]\end{array}\right\} \quad (3.13)$$

or equivalently,

$$\min.\left[\frac{1}{2}\left(1+\frac{1}{2(2\eta_0-1)}\right),\frac{1}{2}\left\{1+\frac{\left(2C-\frac{1}{2}\right)}{(2\eta_0-1)}\right\}\right]<\delta<\max.\left[\frac{1}{2}\left(1+\frac{1}{2(2\eta_0-1)}\right),\frac{1}{2}\left\{1+\frac{\left(2C-\frac{1}{2}\right)}{(2\eta_0-1)}\right\}\right] \quad (3.14)$$

In similar way for given $\delta=\delta_0$, the range of η under which the suggested class of estimator $\bar{y}_{ps(\eta,\delta_0)}$ is more efficient than the Tailor et al (2017) ratio-type estimator $\hat{\bar{Y}}_{\mathrm{Re}}^{ps}$ is:

$$\left.\begin{array}{l}\text{either } \frac{1}{2}\left[1+\frac{1}{2(2\delta_0-1)}\right]<\eta<\frac{1}{2}\left[1+\frac{1}{(2\delta_0-1)}\left(2C-\frac{1}{2}\right)\right] \\ \text{or } \frac{1}{2}\left[1+\frac{1}{(2\delta_0-1)}\left(2C-\frac{1}{2}\right)\right]<\eta<\frac{1}{2}\left[1+\frac{1}{2(2\delta_0-1)}\right]\end{array}\right\} \quad (3.15)$$

or equivalently,

$$\min.\left[\frac{1}{2}\left(1+\frac{1}{2(2\delta_0-1)}\right),\frac{1}{2}\left\{1+\frac{\left(2C-\frac{1}{2}\right)}{(2\delta_0-1)}\right\}\right]<\eta<\max.\left[\frac{1}{2}\left(1+\frac{1}{2(2\delta_0-1)}\right),\frac{1}{2}\left\{1+\frac{\left(2C-\frac{1}{2}\right)}{(2\delta_0-1)}\right\}\right] \quad (3.16)$$

Thus from (3.14) and (3.16) we can calculate the range of δ for given $\eta=\eta_0$ and the range of η for given $\delta=\delta_0$.

3.6 Comparison of the proposed class of estimators $\bar{y}_{ps(\eta,\delta)}$ with Tailor et al (2017) Product-type Exponential Estimator $\hat{\bar{Y}}_{Pe}^{ps}$

From (1.11) and (2.7) we have

$$MSE\left(\hat{\bar{Y}}_{\mathrm{Re}}^{ps}\right)-MSE\left(\bar{y}_{ps(\eta,\delta)}\right)=\gamma R^2\left\{\frac{1}{2}+(1-2\eta)(1-2\delta)\right\}\left[\left\{\frac{1}{2}+2C-(1-2\eta)(1-2\delta)\right\}\right]\sum_{h=1}^{L}W_hS_{xh}^2$$

which is positive if

$$\left[\frac{1}{2}+(1-2\eta)(1-2\delta)\right]\left[\frac{1}{2}+2C-(1-2\eta)(1-2\delta)\right]>0$$

i.e. if

$$\left.\begin{array}{l}\text{either}-\frac{1}{2}<(1-2\eta)(1-2\delta)<\left(\frac{1}{2}+2C\right) \\ \text{or } \left(\frac{1}{2}+2C\right)<(1-2\eta)(1-2\delta)<-\frac{1}{2}\end{array}\right\} \quad (3.17)$$

or equivalently

$$\min.\left[-\frac{1}{2},\left(\frac{1}{2}+2C\right)\right]<(1-2\eta)(1-2\delta)<\max.\left[-\frac{1}{2},\left(\frac{1}{2}+2C\right)\right]. \quad (3.18)$$

Let $\eta=\eta_0$ be pre-assigned constant. Then the range of δ in which the suggested class of estimators $\bar{y}_{ps(\eta,\delta)}$ is more efficient than Tailor et al (2017) product-type exponential estimator $\hat{\bar{Y}}_{Pe}^{ps}$ is given by:

$$\left.\begin{array}{l}\text{either } \frac{1}{2}\left[1+\frac{1}{(2\eta_0-1)}\left(\frac{1}{2}+2C\right)\right]<\delta<\frac{1}{2}\left[1-\frac{1}{2(2\eta_0-1)}\right] \\ \text{or } \frac{1}{2}\left[1-\frac{1}{2(2\eta_0-1)}\right]<\delta<\frac{1}{2}\left[1+\frac{1}{(2\eta_0-1)}\left(\frac{1}{2}+2C\right)\right]\end{array}\right\} \quad (3.19)$$

or equivalently,

$$\min.\left[\frac{1}{2}\left\{1+\frac{\left(\frac{1}{2}+2C\right)}{(2\eta_0-1)}\right\},\frac{1}{2}\left(1+\frac{1}{2(2\eta_0-1)}\right)\right]<\delta<\max.\left[\frac{1}{2}\left\{1+\frac{\left(\frac{1}{2}+2C\right)}{(2\eta_0-1)}\right\},\frac{1}{2}\left(1+\frac{1}{2(2\eta_0-1)}\right)\right] \quad (3.20)$$

Further if the values of $\delta=\delta_0$ is given, then the range of η in which the suggested class of estimators $\bar{y}_{ps(\eta,\delta_0)}$ is better than the Tailor et al (2017) product-type exponential estimator $\hat{\bar{Y}}_{Pe}^{ps}$ is given by

$$\left.\begin{array}{l}\text{either } \frac{1}{2}\left[1+\frac{1}{(2\delta_0-1)}\left(\frac{1}{2}+2C\right)\right]<\eta<\frac{1}{2}\left[1-\frac{1}{2(2\delta_0-1)}\right]\\ \text{or } \frac{1}{2}\left[1-\frac{1}{2(2\delta_0-1)}\right]<\eta<\frac{1}{2}\left[1+\frac{1}{(2\delta_0-1)}\left(\frac{1}{2}+2C\right)\right]\end{array}\right\} \quad (3.21)$$

or equivalently,

$$\min.\left[\frac{1}{2}\left(1-\frac{1}{2(2\delta_0-1)}\right),\frac{1}{2}\left\{1+\frac{\left(\frac{1}{2}+2C\right)}{(2\delta_0-1)}\right\}\right]<\eta<\max.\left[\frac{1}{2}\left(1-\frac{1}{2(2\delta_0-1)}\right),\frac{1}{2}\left\{1+\frac{\left(\frac{1}{2}+2C\right)}{(2\delta_0-1)}\right\}\right] \quad (3.22)$$

Thus from (3.21) and (3.22) we can easily get the range of δ for given $\eta=\eta_0$ and the range of η for given $\delta=\delta_0$.

4. Comparison with Other Estimators

In *SRSWOR* scheme, Chami et al (2012) proposed a ratio-product-ratio estimator for population mean $\bar{Y}$ as

$$\bar{y}_{(\alpha,\beta)}=\alpha\left\{\frac{(1-\beta)\bar{x}+\beta\bar{X}}{\beta\bar{x}+(1-\beta)\bar{X}}\right\}\bar{y}+(1-\alpha)\left\{\frac{\beta\bar{x}+(1-\beta)\bar{X}}{(1-\beta)\bar{x}+\beta\bar{X}}\right\}\bar{y} \quad (4.1)$$

where α,β are real constants, $\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i$ and $\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i$.

To the first degree of approximation, the bias and mean squared error of $\bar{y}_{(\alpha,\beta)}$ are respectively given by

$$B\left(\bar{y}_{(\alpha,\beta)}\right)=\gamma\bar{Y}(1-2\beta)\left[1-\alpha-\beta-(1-2\alpha)C^*\right]C_x^2 \quad (4.2)$$

$$MSE\left(\bar{y}_{(\alpha,\beta)}\right)=\gamma\bar{Y}^2\left[C_y^2+C_x^2(1-2\alpha)(1-2\beta)\{(1-2\alpha)(1-2\beta)-2C^*\}\right], \quad (4.3)$$

where $C^*=\frac{\rho_{yx}C_y}{C_x},C_y=\frac{S_y}{\bar{Y}},C_x=\frac{S_x}{\bar{X}},\rho_{yx}=\frac{S_{yx}}{S_yS_x}$,

$S_y^2=\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})^2, S_x^2=\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{X})^2$ and $S_{yx}=\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})(x_i-\bar{X})$.

It is well known identity that

$$S_y^2=\frac{1}{N-1}\left[\sum_{h=1}^{L}(N_h-1)S_{yh}^2+\sum_{h=1}^{L}N_h\left(\bar{Y}_h-\bar{Y}\right)^2\right]$$

$$\cong\left[\sum_{h=1}^{L}W_hS_{yh}^2+\sum_{h=1}^{L}W_h\left(\bar{Y}_h-\bar{Y}\right)^2\right] \text{ (neglecting terms } \frac{1}{N_h} \text{ and } \frac{1}{N}\text{)} \quad (4.4)$$

$$S_x^2 \cong \left[\sum_{h=1}^{L} W_h S_{xh}^2 + \sum_{h=1}^{L} W_h \left(\bar{X}_h - \bar{X}\right)^2\right] \tag{4.5}$$

$$S_{yx} \cong \left[\sum_{h=1}^{L} W_h S_{yxh} + \sum_{h=1}^{L} W_h \left(\bar{Y}_h - \bar{Y}\right)\left(\bar{X}_h - \bar{X}\right)\right] \tag{4.6}$$

Using (4.4), (4.5), (4.6) and from (2.7) and (4.3), we have

$$MSE\left(\bar{y}_{(\alpha=\eta,\beta=\delta)}\right) - MSE\left(\bar{y}_{ps(\eta,\delta)}\right) = \gamma \sum_{h=1}^{L} W_h \left\{\left(\bar{Y}_h - \bar{Y}\right) - (1-2\eta)(1-2\delta)\left(\bar{X}_h - \bar{X}\right)\right\}^2$$
$$\geq 0 \tag{4.7}$$

Expression (4.7) clearly indicates that the post-stratified estimator $\bar{y}_{ps(\eta,\delta)}$ has lesser mean squared error than the corresponding estimator $\bar{y}_{(\alpha,\beta)}$ in simple random sampling if $\alpha = \eta$ and $\beta = \delta$.

The *MSE* of $\bar{y}_{(\alpha,\beta)}$ at (4.3) is minimized for

$$MSE_{\min}\left(\bar{y}_{(\alpha,\beta)}\right) = \gamma S_y^2\left(1-\rho_{yx}^2\right) \tag{4.8}$$

which equals to the approximate *MSE* of the regression estimator

$$\bar{y}_{lr} = \bar{y} + \hat{\beta}_{yx}\left(\bar{X} - \bar{x}\right) \tag{4.9}$$

where $\hat{\beta}_{yx} = \frac{s_{yx}}{s_x^2}$ is the sample regression coefficient of *y* on *x*, $s_{yx} = \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i - \bar{y}\right)\left(x_i - \bar{x}\right)$ and $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$.

From (2.12) and (4.8) we have

$$MSE_{\min}\left(\bar{y}_{(\alpha,\beta)}\right) - MSE_{\min}\left(\bar{y}_{ps(\eta,\delta)}\right) = \gamma\left[S_y^2\left(1-\rho_{yx}^2\right) - \sum_{h=1}^{L} W_h S_{yh}^2\left(1-\rho^2\right)\right] \tag{4.10}$$

Using (4.4), (4.5) and (4.6) we have

$$MSE_{\min}\left(\bar{y}_{(\alpha,\beta)}\right) - MSE_{\min}\left(\bar{y}_{ps(\eta,\delta)}\right) = \gamma\left[\left(\rho^2 - \rho_{yx}^2\right)\sum_{h=1}^{L} W_h S_{yh}^2 + \left(1-\rho_{yx}^2\right)\sum_{h=1}^{L} W_h\left(\bar{Y}_h - \bar{Y}\right)^2\right] \tag{4.11}$$

which is non-negative if

$$\left(\rho^2 - \rho_{yx}^2\right) > 0$$
$$\text{i.e. if } |\rho| \geq \left|\rho_{yx}\right| \tag{4.12}$$

Thus the proposed post-stratified estimator $\bar{y}_{ps(\eta,\delta)}$ is more efficient than the corresponding estimator $\bar{y}_{(\alpha,\beta)}$ in *SRSWOR* scheme at their optimum conditions as long as the condition $|\rho| \geq \left|\rho_{yx}\right|$ is satisfied.

5. The Proposed Separate Ratio-Product-Ratio Estimator Using Auxiliary Information in Post Stratified Sampling

For estimating the population mean $\bar{Y}$ of the study variable *y* in post stratified sampling, we define the following separate two-parameter ratio-product-ratio estimator

$$\bar{y}_{ts} = \sum_{h=1}^{L} W_h \bar{y}_{(\eta_h,\delta_h)}\,, \tag{5.1}$$

where

$$\bar{y}_{(\eta_h,\delta_h)} = \bar{y}_h\left[\eta_h\left\{\frac{(1-\delta_h)\bar{x}_h+\delta_h\bar{X}_h}{\delta_h\bar{x}_h+(1-\delta_h)\bar{X}_h}\right\}+(1-\eta_h)\left\{\frac{\delta_h\bar{x}_h+(1-\delta_h)\bar{X}_h}{(1-\delta_h)\bar{x}_h+\delta_h\bar{X}_h}\right\}\right] \tag{5.2}$$

and (η_h,δ_h) are real constants.

Using the results from Stephan (1945) for $E\left(n_h^{-1}\right)$ to terms of order n^{-1}, the bias and *MSE* of $\bar{y}_{ts}$ are respectively given by

$$B(\bar{y}_{ts}) = \gamma\sum_{h=1}^{L} W_h(1-2\delta_h)\left(\frac{1}{\bar{X}_h}\right)\left[(1-\eta_h-\delta_h)R_h S_{xh}^2-(1-2\eta_h)S_{yxh}\right], \tag{5.3}$$

$$MSE(\bar{y}_{ts}) = \gamma\sum_{h=1}^{L} W_h\left[S_{yh}^2+(1-2\eta_h)(1-2\delta_h)\{(1-2\eta_h)(1-2\delta_h)R_h^2S_{xh}^2-2R_hS_{yxh}\}\right], \tag{5.4}$$

where $R_h = \dfrac{\bar{Y}_h}{\bar{X}_h}$.

The $MSE(\bar{y}_{ts})$ is minimized for

$$C_h = (1-2\eta_h)(1-2\delta_h)\,, \tag{5.5}$$

where $C_h = \rho_{yxh}\dfrac{C_{yh}}{C_{xh}}$ and $\rho_{yxh} = \dfrac{S_{yxh}}{S_{yh}S_{xh}}$.

Thus the resulting minimum *MSE* of $\bar{y}_{ts}$ is given by

$$MSE_{\min}(\bar{y}_{ts}) = \gamma\sum_{h=1}^{L} W_h S_{yh}^2\left(1-\rho_{yxh}^2\right), \tag{5.6}$$

Now we established the following theorem.

Theorem 5.1- To the first degree of approximation,

$$MSE_{\min}(\bar{y}_{ts}) \ge \gamma\sum_{h=1}^{L} W_h S_{yh}^2\left(1-\rho_{yxh}^2\right)$$

with equality holding if

$$(1-2\eta_h)(1-2\delta_h) = C_h\,.$$

Inserting $(\eta_h,\delta_h)=(1,1)$ and (0,1) in (5.1) we get the separate ratio and separate product estimators in post stratified sampling respectively as

$$\hat{\bar{Y}}_{ps}^{R(s)} = \sum_{h=1}^{L} W_h\bar{y}_h\left(\frac{\bar{X}_h}{\bar{x}_h}\right) \tag{5.7}$$

and

$$\hat{\bar{Y}}_{ps}^{P(s)} = \sum_{h=1}^{L} W_h\bar{y}_h\left(\frac{\bar{x}_h}{\bar{X}_h}\right) \tag{5.8}$$

Substitution of $(\eta_h,\delta_h)=(1,1)$ and (0,1) yield the *MSE*s of $\hat{\bar{Y}}_{ps}^{R(s)}$ and $\hat{\bar{Y}}_{ps}^{P(s)}$ to the first degree of approximation respectively as

$$MSE\left(\hat{\bar{Y}}_{ps}^{R(s)}\right)=\gamma\sum_{h=1}^{L}W_h\left[S_{yh}^2+R_h^2S_{xh}^2-2R_hS_{yxh}\right] \tag{5.9}$$

$$MSE\left(\hat{\bar{Y}}_{ps}^{P(s)}\right)=\gamma\sum_{h=1}^{L}W_h\left[S_{yh}^2+R_h^2S_{xh}^2+2R_hS_{yxh}\right] \tag{5.10}$$

It is observed from (1.1), (5.4), (5.9) and (5.10) that

(i) the separate ratio estimator $\hat{\bar{Y}}_{ps}^{R(s)}$ is better than $\bar{y}_{ps}$ if

$$\frac{\sum_{h=1}^{L}W_hR_h\beta_{yxh}S_{xh}^2}{\sum_{h=1}^{L}W_hR_h^2S_{xh}^2}>\frac{1}{2} \tag{5.11}$$

(ii) the separate product estimator $\hat{\bar{Y}}_{ps}^{P(s)}$ is more efficient than $\bar{y}_{ps}$ if

$$\frac{\sum_{h=1}^{L}W_hR_h\beta_{yxh}S_{xh}^2}{\sum_{h=1}^{L}W_hR_h^2S_{xh}^2}<-\frac{1}{2} \tag{5.12}$$

(iii) the suggested class of separate estimators $\bar{y}_{ts}$ is better than $\bar{y}_{ps}$ if

$$\min.\{0,2C_h\}<\theta_h<\max.\{0,2C_h\},\forall h=1,2,...L. \tag{5.13}$$

where $C_h=\rho_{yxh}\dfrac{C_{yh}}{C_{xh}}$ and $\theta_h=(1-2\eta_h)(1-2\delta_h)$.

(iv) the suggested class of separate estimator $\bar{y}_{ts}$ is more efficient than separate ratio estimator $\hat{\bar{Y}}_{ps}^{R(s)}$ if

$$\min.\{1,(2C_h-1)\}<\theta_h<\max.\{1,(2C_h-1)\},\forall h=1,2,...L. \tag{5.14}$$

(v) the proposed class of separate estimator $\bar{y}_{ts}$ is more efficient than separate product estimator $\hat{\bar{Y}}_{ps}^{P(s)}$ if

$$\min.\{-1,(2C_h+1)\}<\theta_h<\max.\{-1,(2C_h+1)\},\forall h=1,2,...L. \tag{5.15}$$

Motivated by Bahl and Tuteja (1991), we consider the separate ratio-type exponential and the separate product-type exponential estimators for population mean $\bar{Y}$ in post stratified sampling respectively as

$$\hat{\bar{Y}}_{\mathrm{Re}(s)}^{ps}=\sum_{h=1}^{L}W_h\exp\left(\frac{\bar{X}_h-\bar{x}_h}{\bar{X}_h+\bar{x}_h}\right) \tag{5.16}$$

$$\hat{\bar{Y}}_{Pe(s)}^{ps}=\sum_{h=1}^{L}W_h\exp\left(\frac{\bar{x}_h-\bar{X}_h}{\bar{X}_h+\bar{x}_h}\right) \tag{5.17}$$

To the first degree of approximation, the *MSE*s of $\hat{\bar{Y}}_{\mathrm{Re}(s)}^{ps}$ and $\hat{\bar{Y}}_{Pe(s)}^{ps}$ are respectively given by

$$MSE\left(\hat{\bar{Y}}_{\mathrm{Re}(s)}^{ps}\right)=\gamma\sum_{h=1}^{L}W_h\left(S_{yh}^2+\frac{1}{4}R_h^2S_{xh}^2-R_hS_{yxh}\right) \tag{5.18}$$

$$MSE\left(\hat{\bar{Y}}_{Pe(s)}^{ps}\right)=\gamma\sum_{h=1}^{L}W_h\left(S_{yh}^2+\frac{1}{4}R_h^2S_{xh}^2+R_hS_{yxh}\right) \tag{5.19}$$

It is observed form (1.1), (5.9), (5.10), (5.16) and (5.17) that

(i) the separate ratio-type exponential estimator $\hat{\bar{Y}}_{\mathrm{Re}(s)}^{ps}$ is more efficient than the usual unbiased estimator $\bar{y}_{ps}$ if

$$\frac{\sum_{h=1}^{L} W_h R_h \beta_{yxh} S_{xh}^2}{\sum_{h=1}^{L} W_h R_h^2 S_{xh}^2} > \frac{1}{4} \tag{5.20}$$

(ii) the separate product-type exponential estimator $\hat{\bar{Y}}_{Pe(s)}^{ps}$ is more efficient than the usual unbiased estimator $\bar{y}_{ps}$ if

$$\frac{\sum_{h=1}^{L} W_h R_h \beta_{yxh} S_{xh}^2}{\sum_{h=1}^{L} W_h R_h^2 S_{xh}^2} < -\frac{1}{4} \tag{5.21}$$

(iii) the separate ratio-type exponential estimator $\hat{\bar{Y}}_{\mathrm{Re}(s)}^{ps}$ is more efficient than the separate ratio estimator $\hat{\bar{Y}}_{ps}^{R(s)}$ if

$$\frac{\sum_{h=1}^{L} W_h R_h \beta_{yxh} S_{xh}^2}{\sum_{h=1}^{L} W_h R_h^2 S_{xh}^2} < \frac{3}{4} \tag{5.22}$$

(iv) the separate product-type exponential estimator $\hat{\bar{Y}}_{Pe(s)}^{ps}$ is more efficient than the separate product estimator $\hat{\bar{Y}}_{ps}^{P(s)}$ if

$$\frac{\sum_{h=1}^{L} W_h R_h \beta_{yxh} S_{xh}^2}{\sum_{h=1}^{L} W_h R_h^2 S_{xh}^2} > -\frac{3}{4} \tag{5.23}$$

Combining (5.20) and (5.22) we get the separate ratio-type exponential estimator $\hat{\bar{Y}}_{\mathrm{Re}(s)}^{ps}$ is more efficient than $\bar{y}_{ps}$ and $\hat{\bar{Y}}_{ps}^{R(s)}$ if

$$\frac{1}{4} < \frac{\sum_{h=1}^{L} W_h R_h \beta_{yxh} S_{xh}^2}{\sum_{h=1}^{L} W_h R_h^2 S_{xh}^2} < \frac{3}{4} \tag{5.24}$$

Also combining (5.21) and (5.23) it is observed that the separate product-type exponential estimator $\hat{\bar{Y}}_{Pe(s)}^{ps}$ is better than $\bar{y}_{ps}$ and $\hat{\bar{Y}}_{ps}^{P(s)}$ if

$$-\frac{3}{4}<\frac{\sum_{h=1}^{L}W_hR_h\beta_{yxh}S_{xh}^2}{\sum_{h=1}^{L}W_hR_h^2S_{xh}^2}<-\frac{1}{4} \tag{5.25}$$

It is observed from (5.4), (5.18) and (5.19) that the proposed class of separate estimators $\bar{y}_{ts}$ performs better than the separate ratio-type exponential estimator $\hat{\bar{Y}}_{\mathrm{Re}(s)}^{ps}$ if

$$\min.\left\{\frac{1}{2},\left(2C_h-\frac{1}{2}\right)\right\}<\theta_h<\max.\left\{\frac{1}{2},\left(2C_h-\frac{1}{2}\right)\right\}\forall h=1,2,\ldots,L. \tag{5.26}$$

and $\bar{y}_{ts}$ is better than separate product-type exponential estimator $\hat{\bar{Y}}_{Pe(s)}^{ps}$ if

$$\min.\left\{-\frac{1}{2},\left(2C_h+\frac{1}{2}\right)\right\}<\theta_h<\max.\left\{-\frac{1}{2},\left(2C_h+\frac{1}{2}\right)\right\}\forall h=1,2,\ldots,L. \tag{5.27}$$

Further from (1.17) and (5.4) we have

$$MSE\left(\hat{\bar{Y}}_{SR}^{ps}\right)-MSE(\bar{y}_{ts})=\gamma\sum_{h=1}^{L}W_hR_h^2S_{xh}^2\{\alpha_h\lambda_h g_h-(1-2\eta_h)(1-2\delta_h)\}[\{\alpha_h\lambda_h g_h-(1-2\eta_h)(1-2\delta_h)\}-2C_h]$$

which is non-negative if

$$\sum_{h=1}^{L}W_hR_h^2S_{xh}^2\{\alpha_h\lambda_h g_h-(1-2\eta_h)(1-2\delta_h)\}[\{\alpha_h\lambda_h g_h-(1-2\eta_h)(1-2\delta_h)\}-2C_h]>0$$

i.e. if

$$\{\alpha_h\lambda_h g_h-(1-2\eta_h)(1-2\delta_h)\}[\{\alpha_h\lambda_h g_h-(1-2\eta_h)(1-2\delta_h)\}-2C_h]>0\,\forall h=1,2,\ldots,L;$$

i.e. if

$$\left.\begin{array}{l}\text{either}\,(2C_h-\alpha_h\lambda_h g_h)<(1-2\eta_h)(1-2\delta_h)<\alpha_h\lambda_h g_h\\ \text{or}\;\alpha_h\lambda_h g_h<(1-2\eta_h)(1-2\delta_h)<(2C_h-\alpha_h\lambda_h g_h)\end{array}\right\}\forall h=1,2,\ldots,L; \tag{5.28}$$

or equivalently,

$$\min.[\alpha_h\lambda_h g_h,(2C_h-\alpha_h\lambda_h g_h)]<(1-2\eta_h)(1-2\delta_h)<\max.[\alpha_h\lambda_h g_h,(2C_h-\alpha_h\lambda_h g_h)]\forall h=1,2,\ldots,L; \tag{5.29}$$

Let $\alpha_h=\alpha_{h0},g_h=g_{h0},a_h=a_{h0},b_h=b_{h0},\lambda_h=\lambda_{h0}\text{and}\,\eta_h=\eta_{h0}$ be given. Then the range of δ_h under which the proposed class of estimators $\bar{y}_{ts}$ is more efficient than the class of estimators $\hat{\bar{Y}}_{SR}^{ps}$:

$$\left.\begin{array}{l}\text{either}\,\frac{1}{2}\left[1+\frac{\alpha_{h0}\lambda_{h0}g_{h0}}{(2\eta_{h0}-1)}\right]<\delta_h<\frac{1}{2}\left[1+\frac{(2C_h-\alpha_{h0}\lambda_{h0}g_{h0})}{(2\eta_{h0}-1)}\right]\\ \text{or}\,\frac{1}{2}\left[1+\frac{(2C_h-\alpha_{h0}\lambda_{h0}g_{h0})}{(2\eta_{h0}-1)}\right]<\delta_h<\frac{1}{2}\left[1+\frac{\alpha_{h0}\lambda_{h0}g_{h0}}{(2\eta_{h0}-1)}\right]\end{array}\right\}\forall h=1,2,\ldots,L; \tag{5.30}$$

or equivalently

$$\min.\left[\frac{1}{2}\left(1+\frac{\alpha_{h0}\lambda_{h0}g_{h0}}{(2\eta_{h0}-1)}\right),\frac{1}{2}\left\{1+\frac{(2C_h-\alpha_{h0}\lambda_{h0}g_{h0})}{(2\eta_{h0}-1)}\right\}\right]<\delta_h<\max.\left[\frac{1}{2}\left(1+\frac{\alpha_{h0}\lambda_{h0}g_{h0}}{(2\eta_{h0}-1)}\right),\frac{1}{2}\left\{1+\frac{(2C_h-\alpha_{h0}\lambda_{h0}g_{h0})}{(2\eta_{h0}-1)}\right\}\right]\forall h=1,2,\ldots,L; \tag{5.31}$$

Similarly for given $\alpha_h=\alpha_{h0},g_h=g_{h0},a_h=a_{h0},b_h=b_{h0},\lambda_h=\lambda_{h0}\text{and}\,\delta_h=\delta_{h0}$, the range of η_h in which the suggested class of estimators $\bar{y}_{ts}$ is more efficient than the class of estimators $\hat{\bar{Y}}_{SR}^{ps}$ if

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$$\left.\begin{array}{l}\text{either}\ \frac{1}{2}\left[1+\frac{\alpha_{h0}\lambda_{h0}g_{h0}}{(2\delta_{h0}-1)}\right]<\eta_h<\frac{1}{2}\left[1+\frac{(2C_h-\alpha_{h0}\lambda_{h0}g_{h0})}{(2\delta_{h0}-1)}\right]\\ \text{or}\ \frac{1}{2}\left[1+\frac{(2C_h-\alpha_{h0}\lambda_{h0}g_{h0})}{(2\delta_{h0}-1)}\right]<\eta_h<\frac{1}{2}\left[1+\frac{\alpha_{h0}\lambda_{h0}g_{h0}}{(2\delta_{h0}-1)}\right]\end{array}\right\}\forall h=1,2,\ldots,L; \quad (5.32)$$

or equivalently

$$\min.\left[\frac{1}{2}\left(1+\frac{\alpha_{h0}\lambda_{h0}g_{h0}}{(2\delta_{h0}-1)}\right),\frac{1}{2}\left\{1+\frac{(2C_h-\alpha_{h0}\lambda_{h0}g_{h0})}{(2\delta_{h0}-1)}\right\}\right]<\eta_h<\max.\left[\frac{1}{2}\left(1+\frac{\alpha_{h0}\lambda_{h0}g_{h0}}{(2\delta_{h0}-1)}\right),\frac{1}{2}\left\{1+\frac{(2C_h-\alpha_{h0}\lambda_{h0}g_{h0})}{(2\delta_{h0}-1)}\right\}\right]$$

$$\forall h=1,2,\ldots,L; \quad (5.33)$$

From (5.31) and (5.33) the ranges of δ_h (for given $\eta_h=\eta_{h0}$) and η_h (for given $\delta_h=\delta_{h0}$) under which the suggested class of estimators $\bar{y}_{ts}$ is more efficient than the class of estimators $\hat{\bar{Y}}_{SR}^{ps}$ and hence the members (proposed by Bacanli and Aksu (2012)) of the $\hat{\bar{Y}}_{Rs}^{ps}$ -family of estimators.

Now from (4.3) and (5.4) we have

$$MSE\left(\bar{y}_{ps(\eta,\delta)}\right)-MSE(\bar{y}_{ts})=\sum_{h=1}^{L}\left[W_hS_{xh}^2(R\theta-R_h\theta_h)^2+2(R\theta-R_h\theta_h)\left\{R_h\theta_hS_{xh}^2+S_{yxh}\right\}\right] \quad (5.34)$$

where $\theta=(1-2\eta)(1-2\delta)$.

Expression (5.34) shows that the difference between $MSE\left(\bar{y}_{ps(\eta,\delta)}\right)$ and $MSE(\bar{y}_{ts})$ depends on the magnitude of the $(R\theta-R_h\theta_h)$ and $\left(R_h\theta_hS_{xh}^2+S_{yxh}\right)$. The term $\left(R_h\theta_hS_{xh}^2+S_{yxh}\right)$ will be zero if $\theta_h=-C_h$. It follows that the separate class of estimators $\bar{y}_{ts}$ is more efficient than the combined class of estimators $\bar{y}_{ps(\eta,\delta)}$ when $\theta_h=-C_h$. Also both the estimators $\bar{y}_{ps(\eta,\delta)}$ and $\bar{y}_{ts}$ are equally efficient if $R\theta=R_h\theta_h$, $\forall h=1,2,\ldots,L.$

From (2.12) and (5.6) we have

$$MSE_{\min}\left(\bar{y}_{ps(\eta,\delta)}\right)-MSE_{\min}(\bar{y}_{ts})=\gamma\sum_{h=1}^{L}W_hS_{xh}^2\left(\beta_h-\beta^*\right)^2 \quad (5.35)$$

where $\beta^*=\frac{\sum_{h=1}^{L}W_hS_{yxh}}{\sum_{h=1}^{L}W_hS_{xh}^2}$.

which is always positive provided $\beta_h=\beta^*$,

Thus we conclude that unless the regression coefficient is same from stratum to stratum in the separate class of estimators $\bar{y}_{ts}$ is better than the combined class of estimators at optimum condition.

6. Empirical Study

To exhibit the performance of the proposed estimators, two data sets have been considered. Description of data set is given below:

Data set 1-[**Source: Chouhan (2012)**]

y: Productivity (MT/Hectare), x: Production in '000 Tons

$N_1 = 10, N_2 = 10, n_1 = 4, n_2 = 4, \bar{Y}_1 = 1.70, \bar{Y}_2 = 3.67, \bar{X}_1 = 10.41, \bar{X}_2 = 289.14$

$S_{y1} = 0.50, S_{y2} = 0.50, S_{x1} = 3.53, S_{x2} = 111.61, S_{yx1} = 1.60, S_{yx2} = 144.87.$

Data set 2-[**Source: Murthy (1967), p-228**]

y: Output, x: Fixed capital

$N_1 = 5, N_2 = 5, n_1 = 2, n_2 = 2, \bar{Y}_1 = 1925.8, \bar{Y}_2 = 315.6, \bar{X}_1 = 214.4, \bar{X}_2 = 333.8$

$S_{y1} = 615.92, S_{y2} = 340.38, S_{x1} = 74.87, S_{x2} = 66.35, S_{yx1} = 39360.68, S_{yx2} = 22356.50.$

Table 6.1 gives the *PRE* of different estimators of $\bar{Y}$ with respect to usual unbiased estimator $\bar{y}_{ps}$.

Table 6.2 gives the optimum values of δ for given η for both the data sets.

Table 6.3 gives the *PRE* of the proposed combined ratio-product-ratio estimator $\bar{y}_{ps(\eta,\delta)}$ with respect to $\bar{y}_{ps}$, $\bar{y}_{ps}^{R}$ and $\bar{y}_{\text{Re}}^{ps}$ for different values of (η, δ).

Table 6.4 gives the optimum values of δ_h for given η_h for both the data sets

Table 6.5 gives the *PRE* of the proposed separate ratio-product-ratio estimator $\bar{y}_{ts}$ with respect to usual estimator $\bar{y}_{ps}$, ratio-type estimator $\hat{\bar{Y}}_{ps}^{R(s)}$ and ratio-type exponential estimator $\hat{\bar{Y}}_{\text{Re}(s)}^{ps}$ for different values of (η_h, δ_h).

Table 6.1: *PRE* of different estimators with respect to usual unbiased estimator $\bar{y}_{ps}$.

Estimator	Data Set 1	Data Set 2
$\bar{y}_{ps}$	100	100
$\hat{\bar{Y}}_{ps}^{R}$	225.19	313.75
$\hat{\bar{Y}}_{ps}^{R(s)}$	593.5	245.11
$\hat{\bar{Y}}_{\text{Re}}^{ps}$	364.41	173.94
$\hat{\bar{Y}}_{\text{Re}(s)}^{ps}$	309.64	211.11
$\bar{y}_{ps(\eta,\delta)}^{(o)}$	432.41	432.07
$\bar{y}_{ts}^{(o)}$	643.41	470.29

* $\bar{y}_{ps(\eta,\delta)}^{(o)}$ and $\bar{y}_{ts}^{(o)}$ stand for optimum estimators in the classes of estimators $\bar{y}_{ps(\eta,\delta)}$ and $\bar{y}_{ts}$ respectively.

Table 6.2: Optimum values of δ for given η.

η	2	1.75	1.50	1.25	1	0.75	0.25	0
δ (data 1)	0.60	0.63	0.66	0.71	0.82	1.15	-0.15	0.1
δ (data 2)	0.75	0.80	0.87	1	1.25	2	-1	-0.25

Table 6.3: *PRE* of the proposed combined ratio-product-ratio estimator with respect to $\bar{y}_{ps}$, $\hat{\bar{Y}}_{ps}^{R}$ and $\hat{\bar{Y}}_{\text{Re}}^{ps}$.

Data Set 1

(η,δ)	$\boldsymbol{PRE}\left(\bar{y}_{ps(\eta,\delta)},\bar{y}_{ps}\right)$	$\boldsymbol{PRE}\left(\bar{y}_{ps(\eta,\delta)},\hat{\bar{Y}}_{ps}^{R}\right)$	$\boldsymbol{PRE}\left(\bar{y}_{ps(\eta,\delta)},\hat{\bar{Y}}_{\mathrm{Re}}^{ps}\right)$
(1,1)	225.19	100	61.795
(2,0.61)	432.33	191.99	118.64
(1.75,0.65)	404.3	179.54	110.95
(2.10,0.58)	373.13	165.7	102.39
(1.75,0.63)	432.31	191.98	118.63
(1.40,0.70)	418.81	185.98	114.93
(1.75,0.60)	364.41	161.83	100
(1.50,0.66)	431.63	191.68	118.45
(1.50,0.70)	372.05	165.22	102.1
(2,0.60)	422.43	187.59	115.92
(1.25,0.71)	430.28	191.08	118.08
(1.25,0.75)	404.3	179.54	110.95
(1.50,0.65)	422.43	187.59	115.92
(1,0.82)	431.63	191.68	118.45
(1,0.85)	425.8	189.09	116.85
(1,0.75)	364.41	161.83	100
(0.75,1.15)	432.31	191.98	118.63
(0.75,1.20)	425.8	189.09	116.85
(0.75,1.10)	422.43	187.59	115.92
(0.25,-0.15)	432.31	191.98	118.63
(0.25,-0.20)	425.8	189.09	116.85
(0.25,-0.10)	422.43	187.59	115.92
(0,0.17)	432.33	191.99	118.64
(0,0.2)	422.43	187.59	115.92
(0,0.1)	372.05	165.22	102.1
(0.1,0.1)	431.63	191.68	118.45
(0.3,-0.2)	404.03	179.42	110.87
(0.75,1.30)	372.05	165.22	102.1

Data Set 2			
(η,δ)	$\boldsymbol{PRE}\left(\bar{y}_{ps(\eta,\delta)},\bar{y}_{ps}\right)$	$\boldsymbol{PRE}\left(\bar{y}_{ps(\eta,\delta)},\hat{\bar{Y}}_{ps}^{R}\right)$	$\boldsymbol{PRE}\left(\bar{y}_{ps(\eta,\delta)},\hat{\bar{Y}}_{\mathrm{Re}}^{ps}\right)$
(1,1)	313.75	100	180.38
(2,0.75)	432.03	137.7	248.38
(2,0.80)	384.34	122.5	220.96
(2,0.70)	379.43	120.94	218.14
(1.75,0.80)	432.03	137.7	248.38
(1.75,0.95)	398.12	126.89	228.88
(1.75,0.80)	393.73	125.49	226.36
(1.50,0.87)	431.57	137.55	248.11
(1.50,0.95)	384.34	122.5	220.96
(1.50,0.80)	379.43	120.94	218.14
(1.25,1)	432.03	137.7	248.38
(1.25,1.10)	384.34	122.5	220.96
(1.25,0.80)	280.55	89.417	161.29
(1,1.25)	432.03	137.7	248.38
(1,1.50)	319.36	101.79	183.6
(1,1.10)	379.43	120.94	218.14
(0.75,2)	432.03	137.7	248.38
(0.75,2.10)	426.84	136.04	245.39
(0.75,1.75)	393.73	125.49	226.36

(0.25**,-1)**	432.03	137.7	248.38
(0.25,-0.50)	313.75	100	180.38
(0.25,-1.25)	398.12	126.89	228.88
(0**,-0.25)**	432.03	137.7	248.38
(0,-0.50)	319.36	101.79	183.6
(0,-0.10)	379.43	120.94	218.14

Table 6.4: Optimum values of δ_h for given η_h.

$\eta_1 = \eta_2$		2	1.75	1.50	1.25	1	0.75	0.25	0
Data 1	δ_1	0.63	0.66	0.70	0.76	0.89	1.29	-0.30	0.11
	δ_2	0.65	0.68	0.73	0.81	0.96	1.42	-0.40	0.04
Data 2	δ_1	0.63	0.66	0.70	0.76	0.89	1.28	-0.30	0.11
	δ_2	1.40	1.57	1.84	2.19	3.19	5.87	-4.90	-2.20

Table 6.5: *PRE* of the proposed separate ratio-product-ratio estimator with respect to $\bar{y}_{ps}, \hat{\bar{Y}}_{ps}^{R(s)}$ and $\hat{\bar{Y}}_{\mathrm{Re}(s)}^{ps}$.

Data Set 1			
$(\eta_1, \eta_2, \delta_1, \delta_2)$	$PRE(\bar{y}_{ts}, \bar{y}_{ps})$	$PRE\left(\bar{y}_{ts}, \hat{\bar{Y}}_{ps}^{R(s)}\right)$	$PRE\left(\bar{y}_{ts}, \hat{\bar{Y}}_{\mathrm{Re}(s)}^{ps}\right)$
(1,1,1,1)	593.5	100	191.67
(2,2**,0.63,0.65)**	642.41	108.24	207.47
(2,2,0.60,0.65)	621.85	104.78	200.83
(2,2,0.65,0.65)	634.6	106.93	204.95
(1.75,1.75**,0.66,0.68)**	642.32	108.23	207.44
(1.75,1.75,0.60,0.68)	595.85	100.4	192.43
(1.75,1.75,0.65,0.70)	617.64	104.07	199.47
(1.50,1.50**,0.70,0.73)**	643.24	108.38	207.74
(1.50,1.50,0.65,0.73)	622.72	104.92	201.11
(1.50,1.50,0.75,0.75)	593.5	100	191.67
(1.25,1.25**,0.76,0.81)**	642.69	108.29	207.56
(1.25,1.25,0.70,0.81)	622.11	104.82	200.91
(1.25,1.25,0.76,0.84)	605.77	102.07	195.64
(1,1**,0.89,0.96)**	643.34	108.4	207.77
(1,1,0.89,0.96)	637.91	107.48	206.01
(1,1,0.89,0.96)	611.13	102.97	197.37
(0.75,0.75**,1.29,1.42)**	643.35	108.4	207.77
(0.75,0.75,1.25,1.40)	641.63	108.11	207.22
(0.75,0.75,1.35,1.50)	616.09	103.81	198.97
(0.25,0.25**,-0.30,-0.40)**	642.32	108.23	207.44
(0.25,0.25,-0.20,-0.35)	623.2	105	201.27
(0.25,0.25,-0.40,-0.50)	611.13	102.97	197.37
(0,0**,0.11,0.04)**	643.34	108.4	207.77
(0,0,0.05,0.01)	621.17	104.66	200.61
(0,0,0.15,0.09)	606.67	102.22	195.93

Data Set 2			
$(\eta_1, \eta_2, \delta_1, \delta_2)$	$PRE(\bar{y}_{ts}, \bar{y}_{ps})$	$PRE\left(\bar{y}_{ts}, \hat{\bar{Y}}_{ps}^{R(s)}\right)$	$PRE\left(\bar{y}_{ts}, \hat{\bar{Y}}_{\mathrm{Re}(s)}^{ps}\right)$
(1,1,1,1)	245.11	100	116.11
(2,2**,0.63,1.4)**	470.27	191.86	222.76

(2,2,0.60,1.25)	401.88	163.96	190.37
(2,2,0.65,1.50)	437.54	178.51	207.26
(1.75,1.75**,0.66,1.57)**	469.61	191.59	222.45
(1.75,1.75,0.60,1.50)	349.38	142.54	165.5
(1.75,1.75,0.70,1.65)	388.68	158.57	184.11
(1.50,1.50**,0.70,1.84)**	469.61	191.59	222.45
(1.50,1.50,0.65,1.80)	411.47	167.87	194.91
(1.50,1.50,0.75,1.90)	389.78	159.02	184.63
(1.25,1.25**,0.76,2.19)**	470.28	191.87	222.77
(1.25,1.25,0.70,2.20)	410.88	167.63	194.63
(1.25,1.25,0.80,2.35)	443.14	180.79	209.91
(1,1**,0.89,3.19)**	470.28	191.87	222.77
(1,1,0.85,3.10)	456.69	186.32	216.33
(1,1,0.95,3.25)	443.38	180.89	210.03
(0.75,0.75**,1.28,5.87)**	470.28	191.87	222.77
(0.75,0.75,1.25,5.85)	468.26	191.04	221.81
(0.75,0.75,1.35,5.90)	461.05	188.1	218.39
(0.25,0.25**,-0.30,-4.9)**	469.6	191.59	222.45
(0.25,0.25,-0.25,-4.5)	465.88	190.07	220.68
(0.25,0.25,-0.35,-5.00)	460.78	187.99	218.27
(0,0**,0.11,-2.2)**	470.27	191.86	222.76
(0,0,0.05,-2.00)	441.5	180.12	209.13
(0,0,0.15,-2.25)	456.9	186.41	216.43

It is observed from Tables 6.3 and 6.5 that the proposed combined and separate estimators are more efficient than the usual unbiased estimator $\bar{y}_{ps}$, ratio estimator $\hat{\bar{Y}}_{ps}^{R}\left(\hat{\bar{Y}}_{ps}^{R(s)}\right)$ and ratio-type exponential estimator $\hat{\bar{Y}}_{\text{Re}}^{ps}\left(\hat{\bar{Y}}_{\text{Re}(s)}^{ps}\right)$ with considerable gain in efficiency for both the data sets. It is further observed that the proposed estimators are more efficient than $\bar{y}_{ps}$, $\hat{\bar{Y}}_{ps}^{R}\left(\hat{\bar{Y}}_{ps}^{R(s)}\right)$ and $\hat{\bar{Y}}_{\text{Re}}^{ps}\left(\hat{\bar{Y}}_{\text{Re}(s)}^{ps}\right)$ even if the optimum value of $\delta(\delta_h)$ $(\text{or}\,\eta(\eta_h))$ for given values of $\eta(\eta_h)$ $(\text{or}\,\delta(\delta_h))$ depart from its exact optimum values. Thus there is enough scope of selecting the values of scalars (η,δ) $(\text{or}(\eta_h,\delta_h))$ for obtaining estimators better than $\bar{y}_{ps}$, $\hat{\bar{Y}}_{ps}^{R}\left(\hat{\bar{Y}}_{ps}^{R(s)}\right)$ and $\hat{\bar{Y}}_{\text{Re}}^{ps}\left(\hat{\bar{Y}}_{\text{Re}(s)}^{ps}\right)$.

We also note that from Table 6.1 that the proposed optimum separate estimator $\bar{y}_{ts}^{(o)}$, say is more efficient than $\bar{y}_{ps}$, $\hat{\bar{Y}}_{ps}^{R}$, $\hat{\bar{Y}}_{ps}^{R(s)}$, $\hat{\bar{Y}}_{\text{Re}}^{ps}$, $\hat{\bar{Y}}_{\text{Re}(s)}^{ps}$ and the optimum combined estimator $\bar{y}_{ps(\eta,\delta)}^{(o)}$.

Thus we recommend our proposed estimators for their use in practice.

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