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Characterizations of Some Probability Distributions with Completely Monotonic Density Functions

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Abstract

For a non-negative continuous random variable X, Chaudhry and Zubair (2002) introduced a probability distribution with a completely monotonic probability density function based on the generalized gamma function, and called it the Macdonald probability function. In this paper, we establish various basic distributional properties of Chaudhry and Zubair's Macdonald probability distribution. Since the percentage points of a given distribution are important for any statistical applications, we have also computed the percentage points for different values of the parameter involved. Based on these properties, we establish some new characterization results of Chaudhry and Zubair's Macdonald probability distribution by the left and right truncated moments, order statistics and record values. Characterizations of certain other continuous probability distributions with completely monotonic probability density functions such as Mckay, Pareto and exponential distributions are also discussed using the proposed characterization techniques.

Key Words: Characterizations; Completely Monotonic Functions; Order Statistics; Record Values; Truncated Moments.

Mathematical Subject Classification: 60E05, 62E10, 62E15, 62G30

1. Introduction

The notion of completely monotonic functions was introduced by Hausdorff (1921). Since then, many authors and researchers have discussed the properties and examples of the completely monotonic functions. For example, see Feller (1971), Widder (1941, 1971), Miller and Samko (2001), Chaudhry and Zubair (2002), Alzer and Berg (2002, 2006), Schilling et al. (2012), Guo (2016), and Aguech and Jedidi (2019), among others. As pointed out by Alzer and Berg (2002), "Completely monotonic functions have remarkable applications in different branches. For instance, they play a role in potential theory, probability theory, physics, numerical and asymptotic analysis, and combinatorics". For example, see Feller (1971) and Berg and Forst (1975) for applications in probability theory, and Ismail et al. (1986) for applications in numerical analysis, among others. Following Miller and Samko (2001), a function $f : (0, \infty) \rightarrow \Re$ is said to be completely monotonic, if it possesses derivatives $f^{(n)}(x)$ for all $n = 0, 1, 2, 3, \cdots$ and if

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1.1}$$

for all $x > 0$. Moreover, the limit $f^{(n)}(0) = \lim_{x\to 0^+} f^{(n)}(x)$, finite or infinite, exists. Also, it is known that a necessary and sufficient condition that $f(x)$ be completely monotonic is that $f(x) = \int_0^\infty e^{-xt}\, d\big(g(t)\big)$ where $g(t)$ is non-decreasing and the integral converges for $0 < x < \infty$; see Widder (1941, 1971). Hence we conclude that a non-identically zero completely monotonic function $f(x)$cannot vanish for any positive x. For further properties of completely monotonic functions and a list some examples of elementary functions that are completely monotonic, the interested readers are referred to the findings of Miller and Samko (2001), and references therein.

The objective of this paper is to discuss various basic distributional properties of a probability distribution with a completely monotonic probability density function introduced by Chaudhry and Zubair (2002), and based on these properties, to establish its some new characterization results by truncated moments, order statistics and record values. Chaudhry and Zubair (1994) introduced the following generalized gamma function:

$$\Gamma_x(\alpha) = \int_0^\infty t^{\alpha-1} \exp\left[-t - \frac{x}{t}\right] dt,\ 0 \le x < \infty,\ \alpha > 0. \tag{1.2}$$

In view of Gradshteyn and Rhyzhik (1980), (1.2) is expressed in terms of the Macdonald function (or the modified Bessel function of the second kind) by

$$\Gamma_x(\alpha) = 2(x)^{\frac{\alpha}{2}} K_\alpha\left(2\sqrt{x}\right). \tag{1.3}$$

Further, Chaudhry and Zubair (2002) showed that $\int_0^\infty \Gamma_x(\alpha)\, dx = \Gamma(\alpha + 1)$, where $\Gamma(.)$ denotes the gamma function. Using the definitions (1.2) and (1.3), for a non-negative continuous random variable X, Chaudhry and Zubair (2002) introduced a probability distribution with the following probability density function (pdf)

$$f_X(x) = \begin{cases} \frac{\Gamma_x(\alpha)}{\Gamma(\alpha+1)}, & (0 \le x < \infty,\ \alpha > 0), \\ 0, & \text{otherwise}, \end{cases} \tag{1.4}$$

$$= \begin{cases} \frac{2(x)^{\frac{\alpha}{2}} K_\alpha(2\sqrt{x})}{\Gamma(\alpha+1)}, & (0 \le x < \infty,\ \alpha > 0), \\ 0, & \text{otherwise}, \end{cases}$$

and called it the Macdonald probability density function. Since it is known from Chaudhry and Zubair (2002) that $\frac{\partial^n\{\Gamma_x(\alpha)\}}{\partial x^n} = (-1)^n \Gamma_x(\alpha - n),\ (Re(\alpha) > 0,\ n = 0, 1, 2, 3, \ldots)$, it follows that the pdf (1.4) is completely monotonic for all $x > 0$. For some selected values of the parameters, we have sketched the graph of the pdf (1.4) which is given in Figure 1. The effects of the parameters can easily be observed from Figure 1 that the distribution of the random variable X has a pdf which is decreasing and is positively right skewed with longer and heavier right tails.

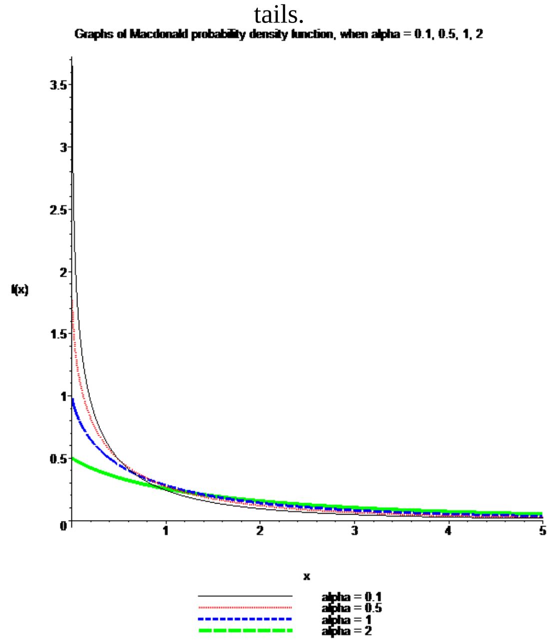

Figure 1: Plots of the pdf (1.4) for $\alpha = 0.1,\ 0.5,\ 1,\ 2$.

The organization of this paper is as follows. In Section 2, various basic distributional properties of Chaudhry and Zubair's Macdonald pdf (1.4) are given. In Section 3, we give our proposed new characterization results of Chaudhry and Zubair's Macdonald pdf (1.4) by truncated moments, order statistics and record values. Some concluding remarks are given in Section 4.

2. Distributional properties of Chaudhry and Zubair's Macdonald probability distribution

In this section, for the sake of completeness, we independently derive various basic distributional properties, viz., expressions for the cumulative distribution function (cdf), the survival and hazard functions, and the expressions for the kth moment and the kth incomplete moment. We also sketch the corresponding graphs of the cdf and the hazard function.

2.1. Cumulative distribution function:

The cumulative distribution function (cdf) corresponding to the pdf (1.4) is given by

$$F(x) = \int_0^x \left[\frac{\Gamma_u(\alpha)}{\Gamma(\alpha+1)}\right] du = \int_0^\infty \left[\frac{\Gamma_u(\alpha)}{\Gamma(\alpha+1)}\right] du - \int_x^\infty \left[\frac{\Gamma_u(\alpha)}{\Gamma(\alpha+1)}\right] du$$

$$= 1 - \frac{\Gamma_x(\alpha+1)}{\Gamma(\alpha+1)}, \tag{2.1}$$

which easily follows by evaluating the above integral in terms of the generalized gamma function, $\Gamma_x(\alpha)$, that is,

$$\int_x^\infty \left[\frac{\Gamma_u(\alpha)}{\Gamma(\alpha+1)}\right] du = \int_x^\infty \left[\frac{2(u)^{\frac{\alpha}{2}} K_\alpha(2\sqrt{u})}{\Gamma(\alpha+1)}\right] du = \frac{2(x)^{\frac{(\alpha+1)}{2}} K_{\alpha+1}(2\sqrt{x})}{\Gamma(\alpha+1)} = \frac{\Gamma_x(\alpha+1)}{\Gamma(\alpha+1)},$$ where

$\Gamma_x(\alpha+1) = \int_0^\infty t^\alpha \exp\left[-t - \frac{x}{t}\right] dt = 2(x)^{\frac{(\alpha+1)}{2}} K_{\alpha+1}(2\sqrt{x})$; see Chaudhry and Zubair (2002), and Gradshteyn and Rhyzhik (1980). Please note that since

$\frac{\partial^n\{\Gamma_x(\alpha)\}}{\partial x^n} = (-1)^n \Gamma_x(\alpha - n),\ (Re(\alpha) > 0,\ n = 0, 1, 2, 3, \ldots)$, it is easily verified that

$\frac{dF(x)}{dx} = \frac{d}{dx}\left(1 - \frac{\Gamma_x(\alpha+1)}{\Gamma(\alpha+1)}\right) = \frac{\Gamma_x(\alpha)}{\Gamma(\alpha+1)}$, which is the pdf (1.4) under question.

2.2. Survival and Hazard functions:

Using (1.4) and (2.1), we compute the corresponding survival function (sf) and hazard function (hf) which are respectively given by

$$S(x) = 1 - F_X(x) = \frac{\Gamma_x(\alpha+1)}{\Gamma(\alpha+1)}, \tag{2.2}$$

and

$$h(x) = \frac{f_X(x)}{1 - F_X(x)} = \frac{\Gamma_x(\alpha)}{\Gamma_x(\alpha+1)}. \tag{2.3}$$

For some selected values of the parameters, we have sketched the graphs of the cdf (2.1) and the hf (2.3), which are respectively given in Figures 2 and 3. The effects of the parameters can easily be observed from Figure 3 that the hazard function of distribution of the random variable X is a decreasing function and has a bathtub shape with longer and heavier right tails.

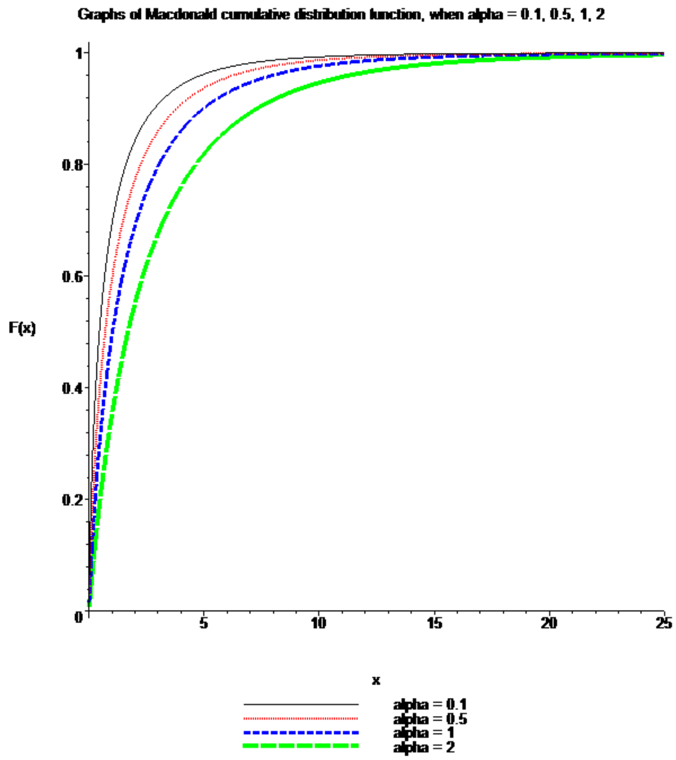

Figure 2: Plots the cdf (2.1) for $\alpha = 0.1, 0.5, 1, 2$.

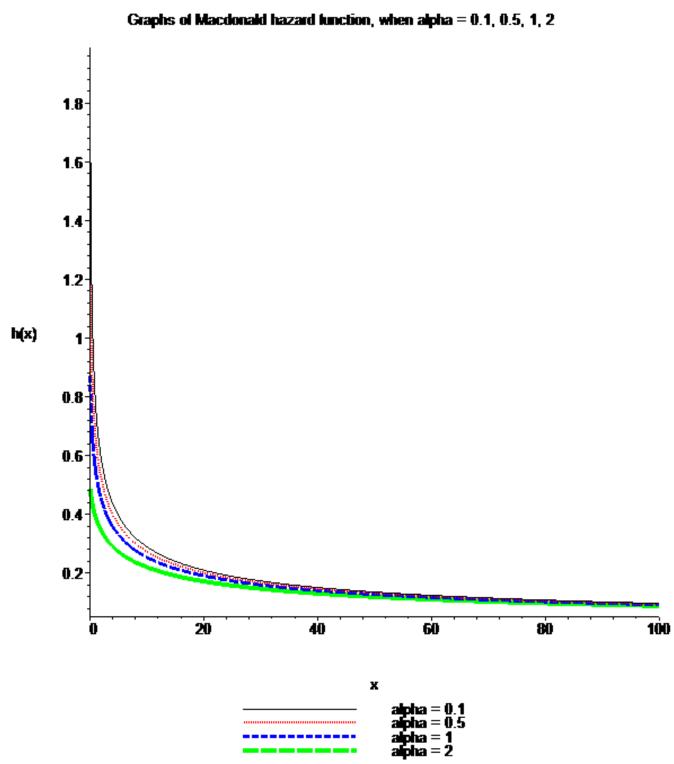

Figure 3: Plots of the hf (2.3) for $\alpha = 0.1, 0.5, 1, 2$.

2.3. Moments:

In what follows, we give the expressions for the kth moment and the kth incomplete moment.

2.3.1. The kth Moment:

Now, for some integer $k > 0$, using Prudnikov, et al. (1986), the kth moment is given by

$$E(X^k) = \int_0^\infty x^k f(x)dx = \int_0^\infty x^k \frac{2(x)^{\frac{\alpha}{2}} K_\alpha(2\sqrt{x})}{\Gamma(\alpha + 1)} dx. \tag{2.4}$$

Letting $\sqrt{x} = t$ in (2.4), we have

$$E(X^k) = \frac{4}{\Gamma(\alpha + 1)} \int_0^\infty u^{2k + \alpha + 1} K_\alpha(2t)dt,$$

from which, using Prudnikov, et al. (1986), the kth moment is given by

$$E(X^k) = \frac{\Gamma(k + 1)\Gamma(\alpha + k + 1)}{\Gamma(\alpha + 1)}, \alpha > 0. \tag{2.5}$$

2.3.2. First Moment:

When $k = 1$ in Eq. (2.5), the 1stmoment is given by

$$E(X) = \alpha + 1, \alpha > 0. \tag{2.6}$$

2.3.3. The kth Incomplete Moment:

For some integer $k > 0$, the kth incomplete moment is given by

$$I_x = \int_0^x u^k f(u)du = \int_0^x u^k \frac{2(u)^{\frac{\alpha}{2}} K_\alpha(2\sqrt{u})}{\Gamma(\alpha + 1)} du$$

$$= \frac{2}{\Gamma(\alpha + 1)} \int_0^x u^{k + \frac{\alpha}{2}} K_\alpha(2\sqrt{u})du. \tag{2.7}$$

Letting $2\sqrt{u} = t$ in (2.7) , we have

$$I_x = \frac{2}{\Gamma(\alpha + 1)} \int_0^{2\sqrt{x}} \left(\frac{t}{2}\right)^{2k + \alpha + 1} K_\alpha(t)dt,$$

from which, on using Miller (1988), we have

$$I_x = \frac{1}{\Gamma(\alpha + 1)} \left[\frac{2x^{k + 1 + \frac{\alpha}{2}}}{k + 1} K_\alpha(2\sqrt{x})\ {}_1F_2(1;\ k + 2,\ \alpha + k + 1;\ x) \right.$$

$$\left. + \frac{2x^{k + \frac{\alpha}{2} + \frac{3}{2}}}{(k + 1)\ (\alpha + k + 1)} K_{\alpha - 1}(2\sqrt{x})\ {}_1F_2(1;\ k + 2,\ \alpha + k + 2;\ x)\right]$$

$$= P_k(x), \text{ say,} \tag{2.8}$$

where $\alpha > 0$, and ${}_1F_2(a;\ b,\ c;\ z)$ denotes the generalized hypergeometric function.

2.3.4. The First Incomplete Moment:

When $k = 1$ in Eq. (2.8), the 1stincomplete moment is given by

$$P_1(x) = \frac{1}{\Gamma(\alpha+1)}\left[x^{2+\frac{\alpha}{2}}K_\alpha\left(2\sqrt{x}\right)\,{}_1F_2\,(1;\ 3,\ \alpha\ +\ 2;\ x)\right.$$

$$\left.+\ \frac{x^{\frac{\alpha+5}{2}}}{(\alpha+2)}K_{\alpha-1}\left(2\sqrt{x}\right)\,{}_1F_2(1;\ 3,\ \alpha\ +\ 3;\ x)\right],\ \alpha\ >\ 0. \qquad (2.9)$$

2.4. Percentiles

The percentage points of a given distribution are also important for any statistical applications, for example, we may be interested in knowing the median (50%), or 75% quartiles, or 95%, or 99% confidence levels, to assess the statistical significance of an observation whose distribution is known. For any $0 < p < 1$, the $100pth$ percentile or the quantile of order p of a distribution with the pdf $f_X(x)$ is defined as a number x_p such that the area under $f_X(x)$ to the left of x_p is p, that is, x_p is any solution of the equation $F(x_p) = \int_0^{x_p} f_X(u)du = p$, where $F(x_p)$ denotes the cdf corresponding to the given pdf $f_X(x)$. Thus, for Chaudhry and Zubair's distribution with the pdf (1.4) and cdf (2.1), solving the equation $F(x_p) = \int_0^{x_p} f_X(u)du = p$ numerically, we have computed the percentage points x_p for different sets of values of the parameter $\boldsymbol{\alpha}$**,** which are given in the Table 4.1.

Table 4.1: Percentage Points

	Percentiles					
Parameter$\boldsymbol{\alpha}$	0.75 (75 %)	0.80 (80 %)	0.85 (85 %)	0.90 (90 %)	0.95 (95 %)	0.99 (99 %)
0.1	1.2756	1.6059	2.0770	2.8257	4.3290	8.8707
0.2	1.4095	1.7649	2.2694	3.0668	4.6574	9.4180
0.3	1.5440	1.9238	2.4611	3.3065	4.9827	9.9571
0.4	1.6780	2.0827	2.6524	3.5450	5.3055	10.4898
0.5	1.8126	2.2415	2.8433	3.7825	5.6261	11.0169
0.6	1.9473	2.4003	3.0340	4.0192	5.9447	11.5388
0.7	2.0822	2.5590	3.2242	4.2551	6.2618	12.0560
0.8	2.2173	2.7178	3.4143	4.4904	6.5774	12.5700
0.9	2.3525	2.8766	3.6041	4.7252	6.8916	13.0781
1.0	2.4878	3.0355	3.7938	4.9595	7.2049	13.5883

3. Characterization Results

The problems of characterizations of probability distributions have been investigated by many authors and researchers. As pointed out by Nagaraja (2006), "A characterization is a certain distributional or statistical property of a statistic or statistics that uniquely determines the associated stochastic model". Similarly, according to Koudou and Ley (2014), "In probability and statistics, a characterization theorem occurs when a given distribution is the only one which satisfies a certain property". Furthermore, Koudou and Ley (2014) points out that "characterization theorems also deepen our understanding of the distributions under investigation and sometimes open unexpected paths to innovations which might have been uncovered otherwise".

In order to apply a particular probability distribution to some real world data, many authors and researchers recommend characterizing it first subject to certain conditions. See, for example, Galambos and Kotz (1978), Kotz and Shanbhag (1980), Nagaraja (2006), Koudou and Ley (2014), Ahsanullah and Shakil (2012, 2015), Ahsanullah et al. (2014, 2015, 2016), Ahsanullah (2017), and references therein. It appears from literature that no attention has been paid to the characterizations of the probability distribution with completely monotonic density function (1.4) of the probability distribution introduced by Chaudhry and Zubair (2002). Motivated by the importance of the characterizations of probability distributions, in this paper, we establish some new characterizations by truncated moments, order statistics and record values of the Chaudhry and Zubair's Macdonald probability distribution.

3.1. Characterization by Truncated Moment:

In this subsection, we provide two new characterization results of the Chaudhry and Zubair's Macdonald probability distribution by truncated moment. The first characterization result (Theorem 3.1) is based on a relation between left truncated moment and failure rate function. The second characterization result (Theorem 3.2) is based on a relation between right truncated moment and reversed failure rate function. For this, we will need the following assumption and lemmas.

Assumption 3.1.

Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\omega = inf\{x|F(x) > 0\}$, and $\delta = sup\{x \mid F(x) < 1\}$. We also assume that $f(x)$ is a differentiable for all x, and$E(X)$ exists.

Lemma 3.1.

If the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, and if $E(X|X \le x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_0^x \frac{u - g^/(u)}{g(u)} du$ is finite for $x > 0$, then $f(x) = ce^{\int_0^x \frac{u - g^/(u)}{g(u)} du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof.

For proof, see Shakil, et al. (2018**).**

Lemma 3.2.

If the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, and if $E(X|X \ge x) = \tilde{g}(x)r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$ and $\tilde{g}(x)$ is a continuous differentiable function of x with the condition that

$\int_x^\infty \frac{u + \left[\tilde{g}(u)\right]^/}{\tilde{g}(u)} du$ is finite for $x > 0$, then $f(x) = ce^{-\int_0^x \frac{u + \left[\tilde{g}(u)\right]^/}{\tilde{g}(u)} du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx = 1$.

Proof.

For proof, see Shakil, et al. (2018**).**

Theorem 3.1.

If the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, then

$$E(X|X \le x) = g(x) \frac{f(x)}{F(x)}, \text{ where}$$
$$g(x) = \frac{P_1(x)}{2(x)^{\frac{\alpha}{2}} K_\alpha(2\sqrt{x})} = \frac{P_1(x)}{\Gamma_x(\alpha)}, \tag{3.1}$$

and $P_1(x)$ is given by (2.9), if and only if X has the pdf

$$f_X(x) = \begin{cases} \frac{\Gamma_x(\alpha)}{\Gamma(\alpha + 1)}, & (0 \le x < \infty,\ \alpha > 0), \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

Suppose that $E(X|X \le x) = g(x) \frac{f(x)}{F(x)}$. Then, since $E(X|X \le x) = \frac{\int_0^x uf(u)du}{F(x)}$, we have

$$g(x) = \frac{\int_0^x uf(u)du}{f(x)}.$$

Now, if the random variable X satisfies the Assumption 3.1 and has the distribution with the pdf (1.4), then we have

$$g(x)=\frac{\int_0^x u\, f(u)du}{f(x)}=\frac{\int_0^x u\left[2(u)^{\alpha/2}K_\alpha\left(2\sqrt{u}\right)\right]du}{2(x)^{\alpha/2}K_\alpha\left(2\sqrt{x}\right)} = \frac{P_1(x)}{\Gamma_x(\alpha)},$$

where $P_1(x)$ is given by (2.9). Consequently, the proof of "if" part of the Theorem 3.1 follows from Lemma 3.1.

Conversely, suppose that

$$g(x) = \frac{P_1(x)}{2(x)^{\frac{\alpha}{2}}K_\alpha(2\sqrt{x})} = \frac{P_1(x)}{\Gamma_x(\alpha)},$$

where $P_1(x)$ is given by (2.9). Now, from Lemma 3.1, we have

$$g(x)=\frac{\int_0^x u\, f(u)du}{f(x)},$$

or

$$\int_0^x u\, f(u)du = f(x)g(x).$$

Differentiating the above equation with respect to respect to x, we obtain

$$xf(x) = f^{/}(x)g(x) + f(x)g^{/}(x),$$

from which, using the definition of the pdf (1.4) and noting that

$$\frac{\partial^n\{\Gamma_x(\alpha)\}}{\partial x^n} = (-1)^n\Gamma_x(\alpha - n),$$

we easily obtain

$$g^{/}(x) = x + g(x)\frac{\Gamma_x(\alpha-1)}{\Gamma_x(\alpha)},$$

or, $$\frac{x-g^{/}(x)}{g(x)} = -\frac{\Gamma_x(\alpha-1)}{\Gamma_x(\alpha)}.$$

Since, by Lemma 3.1, we have

$$\frac{x-g^{/}(x)}{g(x)} = \frac{f^{/}(x)}{f(x)}$$, see Shakil, et al. (2018)**,**

it follows that

$$\frac{f^{/}(x)}{f(x)} = -\frac{\Gamma_x(\alpha-1)}{\Gamma_x(\alpha)}. \tag{3.2}$$

Since $\frac{\partial\{\Gamma_x(\alpha)\}}{\partial x} = (-1)\Gamma_x(\alpha - 1)$, therefore, on integrating (3.2) with respect to x and simplifying, we easily have

$$ln\, f(x) = ln\big(c\Gamma_x(\alpha)\big),$$

or

$$f(x) = c\Gamma_x(\alpha),$$

where c is the normalizing constant to be determined. Thus, on integrating the above equation with respect to x from $x = 0$ to $x = \infty$, using the condition $\int_0^\infty f(x)dx = 1$ and noting that $\int_0^\infty \Gamma_x(\alpha)\,dx = \Gamma(\alpha + 1)$, see Chaudhry and Zubair (2002), we obtain $c = \frac{1}{\Gamma(\alpha + 1)}$, and thus $f_X(x) = \frac{\Gamma_X(\alpha)}{\Gamma(\alpha + 1)}$, $(0 \le x < \infty,\ \alpha > 0)$, which is the required pdf of the random variable X. This completes the proof of Theorem 3.1.

Theorem 3.2.

If the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, then

$$E(X|X \ge x) = \tilde{g}(x)\frac{f(x)}{1 - F(x)},$$

where

$$\tilde{g}(x) = \frac{\left(E(X) - g(x)f(x)\right)\left(\Gamma(\alpha + 1)\right)}{\Gamma_X(\alpha)},$$

where $g(x)$ is given by Eq. (3.1) and $E(X)$ is given by Eq. (2.6), if and only if X has the pdf

$$f_X(x) = \begin{cases} \frac{\Gamma_X(\alpha)}{\Gamma(\alpha + 1)}, & (0 \le x < \infty,\ \alpha > 0), \\ 0, & \text{otherwise.} \end{cases}$$

Proof.

Suppose that $E(X|X \ge x) = \tilde{g}(x)\frac{f(x)}{1 - F(x)}$. Then, since $E(X|X \ge x) = \frac{\int_x^\infty uf(u)du}{1 - F(x)}$, we have

$$\tilde{g}(x) = \frac{\int_x^\infty uf(u)du}{f(x)}.$$

Now, if the random variable X satisfies the Assumption 3.1 and has the distribution with the pdf (1), then we have

$$\tilde{g}(x) = \frac{\int_x^\infty u\, f(u)du}{f(x)} = \frac{\int_0^\infty u\, f(u)du - \int_0^x u\, f(u)du}{f(x)}$$

$$= \frac{\left(E(X) - g(x)f(x)\right)\left(\Gamma(\alpha + 1)\right)}{\Gamma_X(\alpha)}.$$

Consequently, the proof of “if” part of the Theorem 3.2 follows from Lemma 3.2.

Conversely, suppose that

$$\tilde{g}(x) = \frac{\left(E(X) - g(x)f(x)\right)\left(\Gamma(\alpha + 1)\right)}{\Gamma_X(\alpha)}.$$

Now, from Lemma 3.2, we have

$$\tilde{g}(x) = \frac{\int_x^\infty uf(u)du}{f(x)},$$

or

$$\int_x^\infty u\, f(u)du = f(x).\tilde{g}(x).$$

Differentiating the above equation with respect to respect to x, we obtain

$$-xf(x) = f^{/}(x).\tilde{g}(x) + f(x).\left(\tilde{g}(x)\right)^{/},$$

from which, using the definition of the pdf (1.4) and noting that

$$\frac{\partial^n\{\Gamma_x(\alpha)\}}{\partial x^n} = (-1)^n\Gamma_x(\alpha - n),$$

we easily obtain

$$\left(\tilde{g}(x)\right)^{/} = -x + \tilde{g}(x)\left(\frac{\Gamma_x(\alpha - 1)}{\Gamma_x(\alpha)}\right),$$

from which we obtain

$$\frac{x+\left(\tilde{g}(x)\right)^{/}}{\tilde{g}(x)} = \frac{\Gamma_x(\alpha - 1)}{\Gamma_x(\alpha)}.$$

Since, by Lemma 3.2, we have

$$\frac{f^{/}(x)}{f(x)} = -\frac{x + \left[\tilde{g}(x)\right]^{/}}{\tilde{g}(x)},$$

see Shakil, et al. (2018), it follows that

$$\frac{f^{/}(x)}{f(x)} = -\frac{\Gamma_x(\alpha - 1)}{\Gamma_x(\alpha)}. \tag{3.3}$$

Since $\frac{\partial\{\Gamma_x(\alpha)\}}{\partial x} = (-1)\Gamma_x(\alpha - 1)$, therefore, on integrating (3.3) with respect to x and simplifying, we easily have

$$ln\, f(x) = ln\left(c\Gamma_x(\alpha)\right),$$

or

$$f(x) = c\Gamma_x(\alpha),$$

where c is the normalizing constant to be determined. Thus, on integrating the above equation with respect to x from $x = 0$ to $x = \infty$, using the condition $\int_0^\infty f(x)dx = 1$ and noting that $\int_0^\infty \Gamma_x(\alpha)\, dx = \Gamma(\alpha + 1)$, see Chaudhry and Zubair (2002), we obtain $c = \frac{1}{\Gamma(\alpha + 1)}$, and thus $f_X(x) = \frac{\Gamma_x(\alpha)}{\Gamma(\alpha + 1)}$, $(0 \le x < \infty,\ \alpha > 0)$, which is the required pdf of the random variable X. This completes the proof of Theorem 3.2.

3.2. Characterizations by Order Statistics:

If $X_1, X_2, \ldots, X_n$ be the n independent copies of the random variable X with absolutely continuous distribution function $F(x)$ and pdf $f(x)$, and if $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ be the corresponding order statistics, then it is known from Ahsanullah et al. (2013), chapter 5, or Arnold et al. (2005), chapter 2, that $X_{j,n}|X_{k,n} = x$, for $1 \le k < j \le n$, is distributed as the $(j-k)th$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf $f_V(v|x)$ where $f_V(v|x) = \frac{f(v)}{1 - F(x)}$, $0 \le v < x$, and $X_{i.,n}|X_{k,n} = x$, $1 \le i < k \le n$, is distributed as ith order statistics from k independent observations from the random variable W having the pdf $f_W(w|x)$ where $f_W(w|x) = \frac{f(w)}{F(x)}, w < x$. Let $S_{k-1} = \frac{1}{k - 1}\left(X_{1,n} + X_{2,n} + \ldots + X_{k-1,n}\right)$, and $T_{k,n} = \frac{1}{n - k}\left(X_{k+1,n} + X_{k+2,n} + \ldots + X_{n.n}\right)$.

Theorem 3.3:

Suppose the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, then

$$E(S_{k-1}|X_{k,n} = x) = g(x)\tau(x),$$

where

$$\tau(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = \frac{P_1(x)}{2(x)^{\frac{\alpha}{2}}K_\alpha(2\sqrt{x})} = \frac{P_1(x)}{\Gamma_x(\alpha)},$$

where $P_1(x)$ is given by (2.9), if and only if X has the pdf

$$f_X(x) = \begin{cases} \frac{\Gamma_x(\alpha)}{\Gamma(\alpha+1)}, & (0 \le x < \infty,\ \alpha > 0), \\ 0, & \text{otherwise.} \end{cases}$$

Proof:

It is known that $E(S_{k-1}|X_{k,n} = x) = E(X|X \le x)$; see Ahsanullah et al. (2013), and David and Nagaraja (2003). Hence, by Theorem 3.1, the result follows.

Theorem 3.4:

Suppose the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, then

$$E(T_{k,n}|X_{k,n} = x) = \tilde{g}(x)\frac{f(x)}{1-F(x)},$$

where

$$\tilde{g}(x) = \frac{(E(X) - g(x)f(x))(\Gamma(\alpha+1))}{\Gamma_x(\alpha)},$$

where $g(x)$ is given by Eq. (3.1) and $E(X)$ is given by Eq. (2.6), if and only if X has the pdf

$$f_X(x) = \begin{cases} \frac{\Gamma_x(\alpha)}{\Gamma(\alpha+1)}, & (0 \le x < \infty,\ \alpha > 0), \\ 0, & \text{otherwise.} \end{cases}$$

Proof:

Since $E(T_{k,n}|X_{k,n} = x) = E(X|X \ge x)$, see Ahsanullah et al. (2013), and David and Nagaraja (2003), the result follows from Theorem 3.2.

3.3. Characterization by Upper Record Values:

For details on record values, see Ahsanullah (1995). Let$X_1, X_2, \ldots$ be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. If $Y_n = max(X_1, X_2, \ldots, X_n)$ for $n \ge 1$ and $Y_j > Y_{j-1},\ j > 1$, then X_j is called an upper record value of $\{X_n,\ n \ge 1\}$. The indices at which the upper records occur are given by the record times $\{U(n) > min(j|j > U(n+1),\ X_j > X_{U(n-1)},\ n > 1)\}$ and $U(1) = 1$. Let the nth upper record value be denoted by $X(n) = X_{U(n)}$.

Theorem 3.5:

Suppose the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = \infty$, then

$$E(X(n+1)|X(n) = x) = \tilde{g}(x)\frac{f(x)}{1 - F(x)},$$

where

$$\tilde{g}(x) = \frac{(E(X) - g(x)f(x))(\Gamma(\alpha + 1))}{\Gamma_x(\alpha)},$$

where $g(x)$ is given by Eq. (3.1) and $E(X)$ is given by Eq. (2.6), if and only if X has the pdf

$$f_X(x) = \begin{cases} \frac{\Gamma_x(\alpha)}{\Gamma(\alpha + 1)}, & (0 \leq x < \infty, \alpha > 0), \\ 0, & \text{otherwise.} \end{cases}$$

Proof:

It is known from Ahsanullah et al. (2013), and Nevzorov (2001) that $E(X(n+1)|X(n) = x) = E(X|X \geq x)$. Then, the result follows from Theorem 3.2.

Remark 1: **McKay Distribution:**

It should be noted if $Y = \frac{X^2}{4}$, where X has the Chaudhry and Zubair's Macdonald distribution with the pdf (1.4), then Yhas the McKay distribution with the pdf given by

$$f_Y(x) = \begin{cases} \frac{2^{\left(\frac{\alpha+2}{2}\right)}(x)^{\left(\frac{\alpha-2}{4}\right)}K_\alpha\left(2^{\frac{3}{2}}\sqrt[4]{x}\right)}{\Gamma(\alpha + 1)}, & (0 \leq x < \infty, \alpha > 0), \\ 0, & \text{otherwise,} \end{cases}$$

see McKay (1932) and Johnson, et al. (1994). Since, from (1.3), $\Gamma_x(\alpha) = 2(x)^{\frac{\alpha}{2}}K_\alpha(2\sqrt{x})$ and, as pointed out above, since $\frac{\partial^n\{\Gamma_x(\alpha)\}}{\partial x^n} = (-1)^n\Gamma_x(\alpha - n)$, obviously, for the above McKay distribution's pdf, we have $(-1)^n f^{(n)}(x) \geq 0$, and hence it is completely monotonic for all $x > 0$. In view of the fact that, by the above-said transformation, the McKay distribution can be obtained from the Chaudhry and Zubair's Macdonald distribution with the pdf (1.4), we can easily extend all the distributional properties and characterizations of the Chaudhry and Zubair's Macdonald distribution to the McKay distribution.

Remark 2: **Pareto Distribution:**

A continuous random variable X is said to have the Pareto distribution, if its pdf $f_X(x)$ is given by $f(x) = \frac{\alpha}{x^{\alpha+1}}$, where $x \geq 1$, $\alpha > 1$. For details on Pareto distribution, see Johnson et al. (1994). It is easily seen that, for the Pareto distribution's pdf, we have $(-1)^n f^{(n)}(x) \geq 0$, and hence it is completely monotonic for all $x \geq 1$. For the characterizations of the Pareto distribution by truncated moment, the interested readers are referred to Ahsanullah, et al. (2016), and by upper records, please refer to Ahsanullah and Shakil (2012).

Remark 3: **Exponential Distribution:**

A random variable X is said to have the exponential distribution if its pdf is given by $f(x) = \lambda exp(-\lambda x)$, where $x > 0, \lambda > 0$ and $\alpha > 0$, It is easy to see that , for the exponential distribution's pdf, we have $(-1)^n f^{(n)}(x) \geq 0$, and hence it is completely monotonic for all $x > 0$. For the characterizations of the exponential distribution by truncated moment, please refer to Ahsanullah and Shakil (2015), where the characterizations of the Boltzmann distribution by truncated moment have been discussed since the Boltzmann distribution and the exponential distribution coincide by simple transformation of the parameters.

4. Concluding Remarks

In this paper, we have considered, for a non-negative continuous random variable X, a probability distribution with a completely monotonic probability density function introduced by Chaudhry and Zubair (2002), called the Macdonald probability density function.. We have established some new characterization results of Chaudhry and Zubair's Macdonald probability distribution by truncated moments, order statistics and record values. Since the percentage points of a given distribution are important for any statistical applications, we have also computed the percentage points for different sets of values of the parameters. Characterizations of certain other continuous probability distributions with completely monotonic probability density functions such as Mckay, Pareto and exponential distributions are also discussed. We hope the findings of the paper will be quite useful for the practitioners in various fields of sciences.

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