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Marshall-Olkin Alpha Power Rayleigh Distribution: Properties, Characterizations, Estimation and Engineering applications

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Abstract

In this paper, we introduce a new there-parameter Rayleigh distribution, called the Marshall-Olkin alpha power Rayleigh (MOAPR) distribution. Some statistical properties of the MOAPR distribution are obtained. The proposed model is characterized based on truncated moments and reverse hazard function. The maximum likelihood and bootstrap estimation methods are considered to estimate the MOPAR parameters. A Monte Carlo simulation study is performed to compare the maximum likelihood and bootstrap estimation methods. Superiority of the MOAPR distribution over some well-known distributions is illustrated by means of two real data sets.

Key Words: Marshall-Olkin Alpha Power; Rayleigh distribution; Maximum likelihood estimation; Bootstrap estimation.

Mathematical Subject Classification: 60E05, 62E15.

1. Background

In many applied sciences such as medicine, insurance and engineering, among others, modeling and analyzing life testing of data is crucial. The statistical distributions have been used to describe real life phenomena and considerable effort has been expended in the development of the large classes of standard probability distributions and generating new flexible distributions. Many classes of distributions have been developed to describe different phenomena.

The cumulative distribution function (cdf) and probability density function (pdf) of Rayleigh distribution are given by

$$G(y;\lambda) = 1 - e^{-\frac{y^2}{2\lambda^2}}, \tag{1}$$

and

$$g(y;\lambda) = \frac{y}{\lambda^2} e^{-\frac{y^2}{2\lambda^2}}, \tag{2}$$

respectively.

Marshall and Olkin (1997) have suggested a new wider class of distributions called the extended Marshall-Olkin generated (MO-G) family. For any baseline cdf, F(y), the cdf and pdf of the MO-G have the forms

$$F(y;\theta)=\frac{G(y)}{\theta+(1-\theta)G(y)},\ ,\ \theta>0,\ y\in\Re, \tag{3}$$

and

$$f(y;\theta)=\frac{\theta g(y)}{[\theta+(1-\theta)G(y)]^2}\ ,\ \theta>0,\ y\in\Re, \tag{4}$$

respectively. The MO-G family offer a wide range of behavior than basic distribution from which they are derived. For more details, see Ghitany (2005), Ghitany et al. (2007), Okasha and Kayid (2016), Afify et al. (2018) and Ahmad and Almetwally (2020).

Definition 3: Mahdavi and Kundu (2017) proposed a transformation of the baseline cdf by adding a new parameter to obtain a family of distributions. The proposed method is called alpha power (AP) transformation. The cdf and pdf of the AP family are

$$F(y;\alpha)=\frac{\alpha^{G(y)}-1}{\alpha-1};\alpha\neq 1,\alpha>0\ y\in\Re, \tag{5}$$

and

$$f(y;\alpha)=\frac{\ln(\alpha)}{\alpha-1}\ \alpha^{G(y)}\ g(y);\alpha\neq 1,\alpha>0,\ y\in\Re, \tag{6}$$

respectively. A good deal of work have been proposed based on the AP transformation, for example, see Nassar et al. (2017), Elbatal et al. (2018), Dey et al. (2018), Dey et al. (2019), Hassan et al. (2019), Mead et al. (2019) and Basheer (2019).

Nassar et al. (2019) proposed a new flexibility family of distributions using the MO-G and AP classes called Marshall-Olkin alpha power-G (MOAP-G) family. The cdf and pdf of the MOAP-G family are as follows

$$F(y;\alpha,\theta)=\frac{\alpha^{G(y)}-1}{(\alpha-1)\left\{\theta+\frac{1-\theta}{\alpha-1}\left[\alpha^{G(y)}-1\right]\right\}},\ \alpha\neq 1,\alpha>0,\ y\in\Re, \tag{7}$$

and

$$f(y;\alpha,\theta)=\frac{\theta\ \ln(\alpha)}{\alpha-1}\frac{\alpha^{G(y)}\ g(y)}{\left\{\theta+\frac{1-\theta}{\alpha-1}\left[\alpha^{G(y)}-1\right]\right\}^2},\ y\in\Re, \tag{8}$$

respectively. Almetwally et al. (2021) introduced a new Weibull distribution by using MOAP family. Almetwally (2021) introduced a new extended Weibull distribution by using MOAP family based on Type I and Type II censored samples. Almongy et al. (2021) introduced a new extended Lomax distribution by using MOAP family.

In this paper, we study a new three-parameter distribution, called Marshall-Olkin alpha power Rayleigh (MOAPR) distribution which extends the Rayleigh distribution and provides more flexibility in modeling engineering data.

The rest of this paper is organized as follows: The MOAPR distribution is defined in Section 2. Some of its statistical properties are given in Section 3. We provide some characterizations of the MOAPR model in Section 4. The model parameters are estimated via the maximum likelihood and bootstrap methods in Section 5. In Section 6, the potentiality of the estimation approaches is assessed via simulation results. In Section 7, two applications to real data are discussed. Finally, some remarks are offered in Section 8.

2. The MOAPR Distribution

In this section, we will introduce a new extension of Rayleigh distribution by using the MOAP-G family and some of its special sub-models.

The MOAP-G family and Rayleigh distribution have been used to generate the MOAPR distribution. It is represented by the random variable $Y\sim MOAPR(\alpha,\theta,\lambda)$. From Equations (7, 8, 1 and 2), the cdf of the MOAPR is given by

$$F(y;\alpha,\theta,\lambda)=\frac{\alpha^{1-e^{-\frac{y^2}{2\lambda^2}}}-1}{(\alpha-1)}\left[\theta+\frac{1-\theta}{\alpha-1}\left(\alpha^{1-e^{-\frac{y^2}{2\lambda^2}}}-1\right)\right]^{-1};\alpha\neq 1,\alpha,\theta,\lambda>0. \tag{9}$$

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Its pdf takes the form

$$f(y;\alpha,\theta,\lambda)=\frac{\theta\ \ln(\alpha)}{\alpha-1}\alpha^{1-e^{-\frac{y^2}{2\lambda^2}}}\frac{y}{\lambda^2}e^{-\frac{y^2}{2\lambda^2}}\left[\theta+\frac{1-\theta}{\alpha-1}\left(\alpha^{1-e^{-\frac{y^2}{2\lambda^2}}}-1\right)\right]^{-2} \tag{10}$$

Fig. 1 depicts some possible shapes of the pdf of the MOAPR distribution for some selected parameter values.

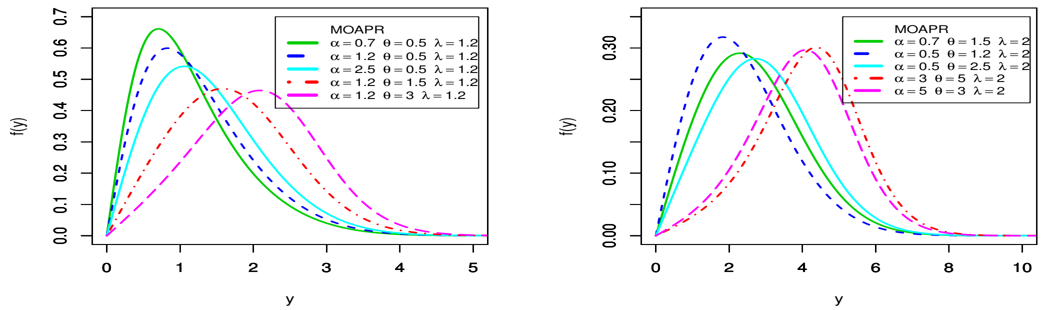

Figure 1: The MOAPR density plots for different parametric values.

The MOAPR distribution contains some special cases which are listed in Table 1.

Table 1: Special cases of the MOAPR distribution.

Distributions	α	θ	λ
Marshall Olkin alpha power chi square (new)	α	θ	1
alpha power chi square (new)	α	1	1
Marshall Olkin chi square (new)	1	θ	1
Marshall Olkin Rayleigh MirMostafaee et al. (2017)	1	θ	λ
alpha power Rayleigh Malik and Ahmad (2017)	α	1	λ
Rayleigh	1	1	λ
chi square	1	1	1

The survival function of the MOAPR distribution reduces to

$$S(y;\alpha,\theta,\lambda)=\frac{(\alpha-1)\theta-\theta\left(\alpha^{1-e^{-\frac{y^2}{2\lambda^2}}}-1\right)}{\theta(\alpha-1)+(1-\theta)\left(\alpha^{1-e^{-\frac{y^2}{2\lambda^2}}}-1\right)}. \tag{11}$$

Fig. 2 displays some shapes of the survival function of the MOAPR distribution for some selected parameter values.

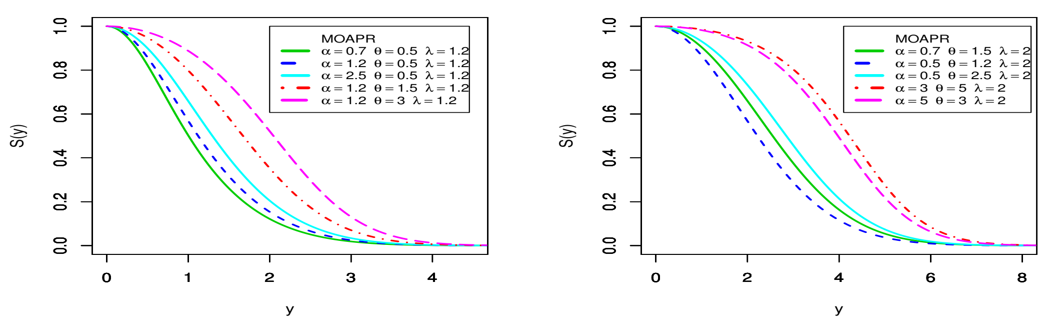

Figure 2: The MOAPR survival function plots for different parametric values.

The failure rate function of the MOAPR distribution has the form

$$fr(y;\alpha,\theta,\lambda)=\frac{(\alpha-1)\ln(\alpha)\alpha^{1-e^{-\frac{y^2}{2\lambda^2}}}\frac{y}{\lambda^2}e^{-\frac{y^2}{2\lambda^2}}}{\left(\alpha-\alpha^{1-e^{-\frac{y^2}{2\lambda^2}}}\right)\left[\theta(\alpha-1)+(1-\theta)\left(\alpha^{1-e^{-\frac{y^2}{2\lambda^2}}}-1\right)\right]}. \tag{12}$$

Fig. 3 displays some plots of the failure rate of the MOAPR distribution for some selected parameter values.

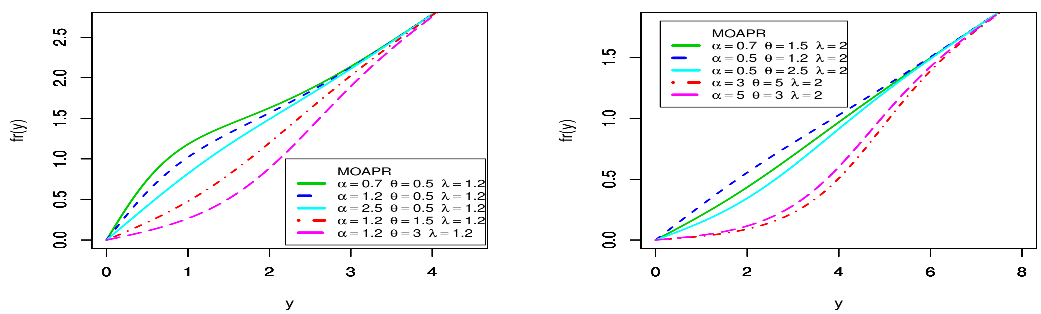

Figure 3: The MOAPR failure rate function plots for different parametric values.

3. Statistical Properties

In this section, we discuss some statistical properties of the MOAPR distribution such as quantile function, moments, moment generating function and related measures, and stress-strength model.

3.1. Quantiles Function

By inverting Equation (9), we have the quantile function of the MOAPR model

$$y_q=\left(-2\lambda^2\ln\left\{1-\frac{1}{\ln(\alpha)}\ln\left[1+\frac{q\theta(\alpha-1)}{1-q(1-\theta)}\right]\right\}\right)^{0.5};0<q<1. \tag{13}$$

From Equation (13), we can obtain the median (M) function or second quartile of the MOAPR distribution when q=0.5 as follow

$$M=\left(-2\lambda^2\ln\left\{1-\frac{1}{\ln(\alpha)}\ln\left[1+\frac{\theta(\alpha-1)}{2-(1-\theta)}\right]\right\}\right)^{0.5}. \tag{14}$$

From Equations (14 and 13), we can obtain the Galton skewness (SK).

3.2. Moments

If the random variable Y is distributed as MOAPR distribution, then its r^{th} moments around zero can be expressed as follows

$$\mu_r=\frac{\delta(\alpha,\theta)}{p+1}\left(\frac{\lambda}{\sqrt{p+1}}\right)^r 2^{\frac{r}{2}}\Gamma\left(1+\frac{r}{2}\right), \tag{15}$$

where

$$\delta(\alpha,\theta)=\frac{\theta\ln(\alpha)}{\alpha-1}\sum_{k=0}^{\infty}\binom{k+1}{k}\theta^{-2-k}\left(\frac{1-\theta}{\alpha-1}\right)^k\sum_{j=0}^{\infty}\binom{k}{j}\sum_{h=0}^{\infty}\frac{(\ln(\alpha)(j+1))^h}{h!}\sum_{p=0}^{\infty}\binom{h}{p}(-1)^{j+k+p}. \tag{16}$$

From Equation (15) with $r=1$, we obtain
The mean of a MOAPR random variable is: $\mu_1=\delta(\alpha,\theta)\frac{\lambda}{\sqrt{p+1}}\sqrt{\frac{\pi}{2}}$.

The variance of the MOAPR random variable is: $V(Y) = \delta(\alpha,\theta)\frac{\lambda^2}{p+1}\left(2-\sqrt{\frac{\pi}{2}}\right)$.

3.3. Moment Generating Function

The moment generating function of the MOAPR distribution is given by

$$M(t) = \frac{\delta(\alpha,\theta)}{p+1}\left\{1+\frac{\lambda}{\sqrt{p+1}}te^{-(\frac{\lambda}{\sqrt{p+1}})^2t^2}\sqrt{\frac{\pi}{2}}\left[erf(\left(\frac{\lambda t}{\sqrt{2(p+1)}}\right)+1\right]\right\}, \quad (17)$$

where $erf(y)$ is the Gauss error function: $erf(y) = \frac{2}{\pi}\int_0^y e^{-t^2}dt$.
The quantile function and moments are used to calculate the mean, median, variance, skewness and kurtosis of Y for different values of the parameters of the MOAPR distribution. The numerical values of these measures are computed by using the R program and are displayed in Table 2.

Table 2: Median, mean, variance, skewness and kurtosis of the MOAPR distribution for several values of α,θ and λ.

λ	θ	α	Median	Mean	Variance	Skewness	Kurtosis
0.5	0.5	0.25	0.3410	0.4008	0.0707	1.2959	5.1882
		0.75	0.4244	0.4818	0.0874	1.0285	4.1891
		2	0.5178	0.5646	0.0991	0.7877	3.5473
0.5	0.75	0.25	0.4050	0.4609	0.0809	1.0867	4.4402
		0.75	0.5004	0.5498	0.0975	0.8310	3.6618
		2	0.6011	0.6372	0.1077	0.6104	3.2108
0.5	3	0.25	0.6735	0.6999	0.1100	0.4805	3.0942
		0.75	0.7998	0.8095	0.1205	0.2571	2.8680
		2	0.9074	0.9049	0.1227	0.0937	2.8444
0.75	0.5	0.25	0.5115	0.6012	0.1591	1.2959	5.1882
		0.75	0.6366	0.7227	0.1967	1.0285	4.1891
		2	0.7767	0.8468	0.2230	0.7877	3.5473
0.75	3	0.25	1.0102	1.0498	0.2474	0.4805	3.0942
		0.75	1.1997	1.2142	0.2710	0.2571	2.8680
		2	1.3612	1.3573	0.2760	0.0937	2.8444
3	0.5	0.25	2.0461	2.4048	2.5456	1.2959	5.1882
		0.75	2.5463	2.8907	3.1467	1.0285	4.1891
		2	3.1067	3.3873	3.5681	0.7877	3.5473
3	3	0.25	4.0409	4.1993	3.9589	0.4805	3.0942
		0.75	4.7987	4.8569	4.3365	0.2571	2.8680
		2	5.4447	5.4293	4.4162	0.0937	2.8444

Table 2 shows that for fixed λ and θ, the mean, median and variance are increasing functions in α, while the skewness and kurtosis are decreasing functions in α. Also, for fixed α and θ, the mean, median and variance are increasing functions in λ, while the skewness and kurtosis are decreasing functions in λ. Further, for fixed λ and α, the mean, median and variance are increasing functions in θ, while the skewness and kurtosis are decreasing functions in θ.

3.4. Stress-Strength Model

Let X and Y be the independent strength and stress random variables observed from the MOAPR distribution, then the stress-strength reliability R is

$$R = P(X<Y) = \int_0^\infty f(y;\alpha_1,\theta_1,\lambda_1)\int_0^y f(x;\alpha_2,\theta_2,\lambda_2)dxdy = \frac{\Phi(\alpha,\theta)}{\sqrt{(p+1)}},$$

where

$$\Phi(\alpha,\theta)=\frac{\theta\ln(\alpha)}{(\alpha-1)^2}\sum_{k=0}^{\infty}\binom{3+k-1}{k}\theta^{-3-k}(\frac{1-\theta}{\alpha-1})^k\sum_{j=0}^{\infty}\binom{k+1}{j}\sum_{h=0}^{\infty}\frac{(\ln(\alpha)(k+2+j))^h}{h!}\sum_{p=0}^{\infty}\binom{h}{p}(-1)^{j+k+p}.$$

4. Characterization Results

This section is devoted to the characterizations of the MOAPR distribution in two directions : (i) based on a relationship between two truncated moments and (ii) in terms of the reverse hazard function. We present our characterizations (i) and (ii) in two subsections.

4.1. Characterizations Based on Truncated Moments

In this subsection we present characterizations of MOAPR distribution in terms of a simple relationship between two truncated moments. The first characterization result employs a theorem due to Glänzel [1] , see Theorem 1 of Appendix A. Note that the result holds also when the interval H is not closed. Moreover, it could be also applied when the cdf F does not have a closed form. As shown in [2], this characterization is stable in the sense of weak convergence. Due to the nature of the cdf of MOAPR, our characterizations may be the only possible ones.

Proposition 4.1.1. Let $Y:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $h\left(y\right)=\frac{\left[\theta+\frac{1-\theta}{\alpha-1}\left(\alpha^{1-e^{-y^2/2\lambda^2}}\right)\right]^2}{\alpha^{1-e^{-y^2/2\lambda^2}}}$ and $g\left(y\right)=h\left(y\right)e^{-y^2/2\lambda^2}$ for $y>0$. The random variable Y has pdf (10) if and only if the function ξ defined in Theorem 1 has the form

$$\xi\left(y\right)=\frac{1}{2}e^{-y^2/2\lambda^2},\quad y>0.$$

Proof. Let Y be a random variable with pdf (10), then

$$\left(1-F\left(y\right)\right)E\left[h\left(Y\right)\mid Y\geq y\right]=\frac{\theta\ln\left(\alpha\right)}{\alpha-1}e^{-y^2/2\lambda^2},\quad y>0,$$

and

$$\left(1-F\left(y\right)\right)E\left[g\left(Y\right)\mid Y\geq y\right]=\frac{\theta\ln\left(\alpha\right)}{2(\alpha-1)}e^{-y^2/\lambda^2},\quad y>0,$$

and finally

$$\xi\left(y\right)h\left(y\right)-g\left(y\right)=-\frac{1}{2}h\left(y\right)e^{-y^2/2\lambda^2}<0\quad for\ \ y>0.$$

Conversely, if ξ is given as above, then

$$s'\left(y\right)=\frac{\xi'\left(y\right)h\left(y\right)}{\xi\left(y\right)h\left(y\right)-g\left(y\right)}=\frac{y}{\lambda^2}\quad y>0,$$

and hence

$$s\left(y\right)=\frac{y^2}{2\lambda^2},\quad y>0.$$

Now, in view of Theorem 1, Y has density (10) .

Corollary 4.1.1. Let $Y:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $h\left(y\right)$ be as in Proposition 4.1.1. The pdf of Y is (10) if and only if there exist functions g and ξ defined in Theorem 1 satisfying the differential equation

$$\frac{\xi'(y)\,h(y)}{\xi(y)\,h(y)-g(y)}=\frac{y}{\lambda^2}\quad y>0.$$

Corollary 4.1.2. The general solution of the differential equation in Corollary 4.1.1 is

$$\xi(y)=e^{y^2/2\lambda^2}\left[-\int\frac{y}{\lambda^2}e^{-y^2/2\lambda^2}\left(h(x)\right)^{-1}g(x)+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 4.1.1 with $D=0$. It should, however, be noted that there are other triplets (h,g,ξ) satisfying the conditions of Theorem 1.

4.2. Characterization in Terms of Reverse Hazard Function

The reverse hazard function, f_r, of a twice differentiable distribution function, F , is defined as

$$f_r(y)=\frac{f(y)}{F(y)},\;\; x\in support\;of\;F.$$

In this subsection we present a characterization of the MOAPR distribution, for $\theta=1$, in terms of the reverse hazard function.

Proposition 4.2.1. Let $Y:\Omega\to(0,\infty)$ be a continuous random variable. The random variable Y has pdf (10), for $\theta=1$, if and only if its reverse hazard function $f_r(y)$ satisfies the following differential equation
.

$$f_r'(y)+\frac{y}{\lambda^2}f_r(y)=\frac{\ln(\alpha)}{\lambda^2}e^{-y^2/2\lambda^2}\frac{d}{dy}\left\{\frac{y\alpha^{1-e^{-y^2/2\lambda^2}}}{\alpha^{1-e^{-y^2/2\lambda^2}}-1}\right\},\; y>0,$$

with the boundary condition $\lim_{y\to\infty}f_r(y)=0$.

Proof. If Y has pdf (10), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dy}\left\{e^{y^2/2\lambda^2}r_F(x)\right\}=\frac{\ln(\alpha)}{\lambda^2}\frac{d}{dy}\left\{\frac{y\alpha^{1-e^{-y^2/2\lambda^2}}}{\alpha^{1-e^{-y^2/2\lambda^2}}-1}\right\},\;\; y>0,$$

from which we arrive at

$$f_r(y)=\frac{\ln(\alpha)}{\lambda^2}\left\{\frac{ye^{y^2/2\lambda^2}\alpha^{1-e^{-y^2/2\lambda^2}}}{\alpha^{1-e^{-y^2/2\lambda^2}}-1}\right\},\quad y>0,$$

which is the reverse hazard function corresponding to the pdf (10) for $\theta=1$.
Remark 4.2.1. We have used the fact that the reverse hazard function of a random variable Y uniquely determine the distribution of Y.

5. Parameter Estimation

In this section, we investigate the maximum likelihood estimation (MLE) of the unknown parameters of MOAPR model for a complete sample. Furthermore, we discuss bootstrap method to obtain the estimates of the MOAPR parameter by using MLE method.

Let $y_1, ..., y_n$ be a random sample of size n from the MOAPR distribution. Then, the likelihood function of the MOAPR distribution follows as

$$L(\Omega) = \frac{\theta^n\left[\ln(\alpha)\right]^n}{(\alpha-1)^n}\frac{\alpha^{n-e^{\frac{-1}{2\lambda^2}\sum_{i=0}^{n}{y_i}^2}}\frac{1}{\lambda^2}\sum_{i=0}^{n}y_i e^{\frac{-1}{2\lambda^2}\sum_{i=0}^{n}{y_i}^2}}{\prod_{i=0}^{n}\left[\theta+\frac{1-\theta}{\alpha-1}\left(\alpha^{1-e^{-\frac{{y_i}^2}{2\lambda^2}}}-1\right)\right]^2}.$$

The log-likelihood function reduces to

$$\begin{aligned}\ell(\Omega) &= n(\ln(\theta)+\ln\left[ln(\alpha)\right)-\ln(\alpha-1)]+\ln(\alpha)\left(n-e^{\frac{-1}{2\lambda^2}\sum_{i=0}^{n}{y_i}^2}\right)-\ln\lambda^2+\sum_{i=0}^{n}\ln(y_i)\\ &+\frac{-1}{2\lambda^2}\sum_{i=0}^{n}{y_i}^2-2\sum_{i=0}^{n}\ln\left[\theta+\frac{1-\theta}{\alpha-1}\left(\alpha^{1-e^{-\frac{{y_i}^2}{2\lambda^2}}}-1\right)\right].\end{aligned} \tag{18}$$

where Ω is a vector of the MOAPR parameters. The estimators of α, θ and λ are obtained by differentiating the log-likelihood equation (18) with respect to each parameter separately, as follows

$$\frac{\partial\ell(\Omega)}{\partial\alpha} = \frac{-n}{\alpha(\ln(\alpha))^2}-\frac{n}{\alpha-1}+\frac{1}{\alpha}\left(n-e^{\frac{-1}{2\lambda^2}\sum_{i=0}^{n}{y_i}^2}\right)-2\sum_{i=0}^{n}\frac{\frac{1-\theta}{\alpha-1}\alpha^{-e^{-\frac{{y_i}^2}{2\lambda^2}}}}{\theta+\frac{1-\theta}{\alpha-1}(\alpha^{1-e^{-\frac{{y_i}^2}{2\lambda^2}}}-1)},$$

$$\frac{\partial\ell(\Omega)}{\partial\theta} = \frac{n}{\theta}-2\sum_{i=0}^{n}\ln\left(1+\frac{-1}{\alpha-1}(\alpha^{1-e^{-\frac{{y_i}^2}{2\lambda^2}}}-1)\right)$$

and

$$\frac{\partial\ell(\Omega)}{\partial\lambda} = \ln(\alpha)\left(4\lambda^{-3}\sum_{i=0}^{n}{y_i}^2 e^{\frac{-1}{2\lambda^2}\sum_{i=0}^{n}{y_i}^2}\right)-\frac{2}{\lambda}+\frac{1}{\lambda^3}\sum_{i=0}^{n}{y_i}^2-2\frac{1-\theta}{\alpha-1}\sum_{i=0}^{n}\frac{e^{-\frac{{y_i}^2}{2\lambda^2}}\frac{-{y_i}^2}{\lambda^3}\alpha^{-e^{-\frac{{y_i}^2}{2\lambda^2}}}}{\theta+\frac{1-\theta}{\alpha-1}(\alpha^{1-e^{-\frac{{y_i}^2}{2\lambda^2}}}-1)}.$$

The bootstrap is a resampling method for statistical inference. We consider parametric bootstrap estimation based on the MLE method.

6. Simulation Study

In this section, we obtain the Monte Carlo simulation results that is performed to see the effectiveness of maximum likelihood estimators (MLEs) of the MOAPR parameters by R language. We mainly compare between MLEs and bootstrapped MLEs in terms of bias vales, the mean squared errors (MSE). We compute MLEs and bootstap MLEs using the Newton-Raphson method.

We start by building our model with generate all simulation controls. In this stage, we must do the following steps by order:

- Step 1: Suppose the following values for the parameter vector of the MOAPR distribution: $\alpha = 0.5, 1.5, 3, \theta = 0.5, 1.5, 3, \lambda = 0.5, 3$.
- Step 2: Choose sample sizes n=30, 70 and 150.
- Step 3: Solve differential equations for both estimation methods, to obtain the estimators of the MOAPR parameters.
- Step 4: Repeat this experiment (L) times. In each experiment use the same values of the parameters. It is certain that, the values of generating random are varying from experiment to experiment even though sample size (n) does not change.

Finally, we have L-values of bias and MSE, we compute the average biases and average MSEs over 10,000 runs. This number of runs will give the accuracy in the order Â±0.01 (see Karian and Dudewicz (1998)). The simulated results are presented in Tables 3, 4 and 5, where the MSE and bias are given in each cell.

The convergence of MLEs and Bootstrap estimates of the MOAPR parameters can be showed in Figures 4 and 5 for $\alpha = 1.5, \theta = 0.5, \lambda = 0.5$.

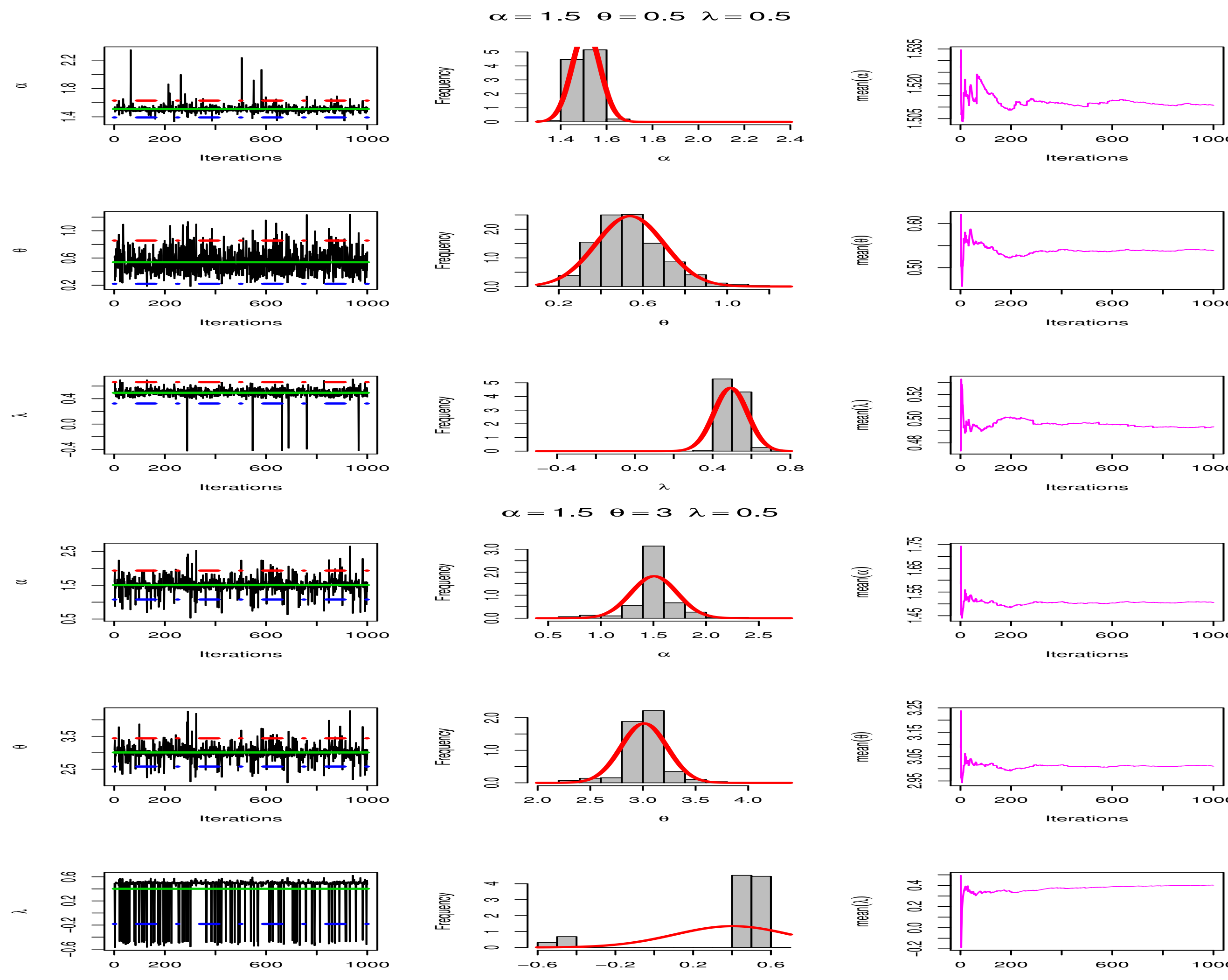

Figure 4: The convergence of MLE of the MOAPR parameters.

7. Applications to Engineering Data

In this section, we consider two different examples of real-life data sets. The MLE of the MOAPR parameters are reported in Tables 9 and 10. The MOAPR distribution is compared with other special models namely, Rayleigh, generalized Rayleigh (GR), Marshall-Olkin Rayleigh (MOR), alpha power Rayleigh (APR) and MOAPEx. This comparison was conducted using Kolmogorove-Smernove (KS) (with its p-value), Akaike information (AIC) and Bayesian information (BIC) criteria.
The first data set represents fatigue times of 6061-T6 aluminum coupons, and it consists of 101 observations with maximum stress per cycle 31,000 psi (Birnbaum and Saunders).
The second data represent the strengths of 1.5 cm glass fibres for 63 observations originally obtained by workers at the UK National Physical Laboratory. This data set is analyzed by Mansour et al. (2018) and Aldahlan and Afify (2018).
Tables 6 and 7 illustrate the ML estimates of the fitted models and the values of KS, p-value, AIC and BIC. The values

Table 3: Maximum likelihood and bootstrap estimates of the MOAPR parameters for $n = 30$ and different parametric values.

			$\lambda = 0.5$				$\lambda = 3$			
n=30			**MLE**		**Bootstrab**		**MLE**		**Bootstrab**	
α	θ		**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
0.5	0.5	α	0.1044	0.0723	0.1045	0.0110	0.0951	0.1505	0.0952	0.0092
		θ	0.1800	0.1782	0.1800	0.0325	0.3388	0.4748	0.3388	0.1152
		λ	-0.0461	0.0556	-0.0461	0.0022	-0.0791	0.5134	-0.0791	0.0068
	1.5	α	0.2017	0.2845	0.2019	0.0410	0.3763	0.7431	0.3766	0.1424
		θ	0.1466	0.2856	0.1467	0.0218	0.2520	0.7558	0.2521	0.0643
		λ	-0.0789	0.0900	-0.0789	0.0063	0.0072	0.3621	0.0071	0.0004
	3	α	0.2259	0.3739	0.2259	0.0514	0.4913	1.0156	0.4915	0.2424
		θ	0.1116	0.2103	0.1117	0.0127	0.1176	1.0640	0.1177	0.0149
		λ	-0.0986	0.1142	-0.0986	0.0098	-0.0035	0.2487	-0.0037	0.0003
1.5	0.5	α	0.0447	0.0326	0.0447	0.0020	0.1049	0.3649	0.1049	0.0114
		θ	0.1876	0.2329	0.1876	0.0354	0.2231	0.3081	0.2231	0.0500
		λ	-0.0591	0.0575	-0.0591	0.0035	-0.1002	0.4365	-0.1001	0.0105
	1.5	α	0.1488	0.2385	0.1488	0.0224	0.5741	2.1818	0.5734	0.3307
		θ	0.1913	0.5221	0.1916	0.0372	0.3228	1.1615	0.3229	0.1053
		λ	-0.0794	0.0787	-0.0796	0.0064	-0.0520	0.2036	-0.0521	0.0029
	3	α	0.0770	0.3908	0.0772	0.0063	0.6269	2.6025	0.6269	0.3952
		θ	0.1041	0.3928	0.1042	0.0113	0.4337	1.8672	0.4339	0.1899
		λ	-0.1489	0.1539	-0.1490	0.0223	-0.0263	0.1941	-0.0263	0.0009
3	0.5	α	0.0274	0.0601	0.0273	0.0008	-0.0355	0.9443	-0.0359	0.0022
		θ	0.1928	0.2413	0.1929	0.0374	0.2799	0.4850	0.2800	0.0788
		λ	-0.0449	0.0429	-0.0450	0.0021	-0.0924	0.3479	-0.0925	0.0089
	1.5	α	0.0568	0.1980	0.0569	0.0034	0.4321	3.0747	0.4315	0.1890
		θ	0.1818	0.6286	0.1818	0.0337	0.5355	1.8553	0.5358	0.2886
		λ	-0.0946	0.0970	-0.0947	0.0091	-0.0516	0.1747	-0.0517	0.0028
	3	α	0.1072	0.2422	0.1073	0.0118	0.5412	2.9632	0.5406	0.2949
		θ	0.1953	0.7190	0.1957	0.0390	0.6489	3.4480	0.6493	0.4246
		λ	-0.1176	0.1142	-0.1177	0.0140	-0.1943	1.1031	-0.1945	0.0389

Table 4: Maximum likelihood and bootstrap estimates of the MOAPR parameters for $n = 70$ and different parametric values.

			λ=0.5				$\lambda = 3$			
n=70			**MLE**		**Bootstrab**		**MLE**		**Bootstrab**	
α	θ		**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
0.5	0.5	α	0.0480	0.0689	0.0480	0.0024	0.0397	0.1057	0.0397	0.0017
		θ	0.0858	0.0639	0.0858	0.0074	0.1686	0.1791	0.1687	0.0286
		λ	-0.0195	0.0386	-0.0194	0.0004	0.0234	0.2493	0.0232	0.0008
	1.5	α	0.0562	0.0802	0.0563	0.0032	0.1479	0.1783	0.1480	0.0221
		θ	0.0441	0.0880	0.0441	0.0020	0.1183	0.3263	0.1182	0.0143
		λ	-0.0296	0.0474	-0.0297	0.0009	0.0432	0.2271	0.0430	0.0021
	3	α	0.0322	0.0855	0.0323	0.0011	0.1766	0.2567	0.1766	0.0314
		θ	0.0209	0.0452	0.0208	0.0005	0.0276	0.3315	0.0275	0.0011
		λ	-0.0622	0.0753	-0.0622	0.0039	0.0102	0.1979	0.0101	0.0003
1.5	0.5	α	0.0138	0.0345	0.0138	0.0002	0.0489	0.2993	0.0490	0.0027
		θ	0.0655	0.0567	0.0655	0.0043	0.0730	0.0669	0.0731	0.0054
		λ	-0.0201	0.0212	-0.0201	0.0004	-0.0255	0.1622	-0.0256	0.0008
	1.5	α	0.0005	0.0328	0.0006	0.0000	0.3034	0.6870	0.3035	0.0927
		θ	-0.0054	0.1063	-0.0054	0.0001	0.0866	0.3469	0.0868	0.0079
		λ	-0.0571	0.0622	-0.0571	0.0033	-0.0157	0.1087	-0.0157	0.0004
	3	α	0.0176	0.0570	0.0177	0.0004	0.2082	0.6279	0.2086	0.0441
		θ	0.0223	0.0641	0.0223	0.0006	0.1189	0.5292	0.1190	0.0147
		λ	-0.0898	0.0891	-0.0899	0.0082	-0.0102	0.0861	-0.0104	0.0002
3	0.5	α	0.0101	0.0947	0.0101	0.0002	0.0011	0.4842	0.0012	0.0005
		θ	0.0633	0.0562	0.0634	0.0041	0.0818	0.0800	0.0818	0.0068
		λ	-0.0323	0.0331	-0.0323	0.0011	-0.0297	0.1562	-0.0298	0.0010
	1.5	α	0.0053	0.0154	0.0053	0.0000	0.0263	0.8684	0.0268	0.0016
		θ	-0.0028	0.1265	-0.0028	0.0001	0.2260	0.4919	0.2261	0.0516
		λ	-0.0729	0.0769	-0.0728	0.0054	-0.0358	0.1839	-0.0360	0.0015
	3	α	0.0373	0.0527	0.0373	0.0014	0.1778	0.7784	0.1779	0.0324
		θ	0.0876	0.1584	0.0876	0.0078	0.1645	0.9509	0.1648	0.0281
		λ	-0.0806	0.0781	-0.0806	0.0066	-0.0182	0.1410	-0.0183	0.0005

Table 5: Maximum likelihood and bootstrap estimates of the MOAPR parameters for $n = 150$ and different parametric values.

			λ=0.5				$\lambda = 3$			
n=150			**MLE**		**Bootstrab**		**MLE**		**Bootstrab**	
α	θ		**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**	**Bias**	**MSE**
0.5	0.5	α	0.0016	0.0265	0.0015	0.0000	0.0071	0.0918	0.0071	0.0001
		θ	0.0518	0.0297	0.0518	0.0027	0.1290	0.1118	0.1290	0.0167
		λ	-0.0177	0.0305	-0.0177	0.0003	0.0452	0.1432	0.0449	0.0022
	1.5	α	0.0399	0.0460	0.0399	0.0016	0.0731	0.1298	0.0732	0.0055
		θ	0.0459	0.0377	0.0459	0.0021	0.1712	0.3796	0.1712	0.0297
		λ	-0.0274	0.0305	-0.0275	0.0008	0.0487	0.0990	0.0484	0.0024
	3	α	0.0597	0.0733	0.0598	0.0036	0.1356	0.1371	0.1356	0.0185
		θ	0.0279	0.0958	0.0279	0.0009	0.0013	0.3872	0.0013	0.0004
		λ	-0.0306	0.0357	-0.0307	0.0010	0.0163	0.0590	0.0160	0.0003
1.5	0.5	α	0.0108	0.0038	0.0108	0.0001	0.0204	0.1243	0.0205	0.0005
		θ	0.0386	0.0278	0.0385	0.0015	0.0499	0.0400	0.0499	0.0025
		λ	-0.0068	0.0074	-0.0068	0.0001	-0.0123	0.0894	-0.0127	0.0002
	1.5	α	0.0482	0.0632	0.0482	0.0024	0.1987	0.7600	0.1985	0.0401
		θ	0.0551	0.1040	0.0550	0.0031	0.0665	0.2182	0.0665	0.0046
		λ	-0.0277	0.0271	-0.0278	0.0008	-0.0119	0.0757	-0.0123	0.0002
	3	α	0.0059	0.0477	0.0059	0.0001	0.3585	1.3406	0.3586	0.1298
		θ	0.0108	0.0477	0.0108	0.0002	0.0342	0.4717	0.0340	0.0016
		λ	-0.0973	0.0984	-0.0976	0.0096	-0.0121	0.0948	-0.0123	0.0002
3	0.5	α	0.0093	0.0270	0.0093	0.0001	-0.0552	0.3292	-0.0551	0.0034
		θ	0.0404	0.0290	0.0403	0.0017	0.0636	0.0522	0.0635	0.0041
		λ	-0.0044	0.0046	-0.0044	0.0000	-0.0193	0.1045	-0.0196	0.0005
	1.5	α	0.0177	0.2196	0.0175	0.0005	0.2526	2.2283	0.2526	0.0660
		θ	0.0244	0.1100	0.0243	0.0007	0.1659	0.4252	0.1658	0.0279
		λ	-0.0264	0.0290	-0.0264	0.0007	-0.0106	0.0708	-0.0108	0.0002
	3	α	0.0689	0.0586	0.0689	0.0048	0.2777	1.7028	0.2773	0.0785
		θ	0.1328	0.1739	0.1327	0.0178	0.1726	0.7660	0.1724	0.0304
		λ	-0.0822	0.0791	-0.0822	0.0068	-0.0253	0.1584	-0.0256	0.0008

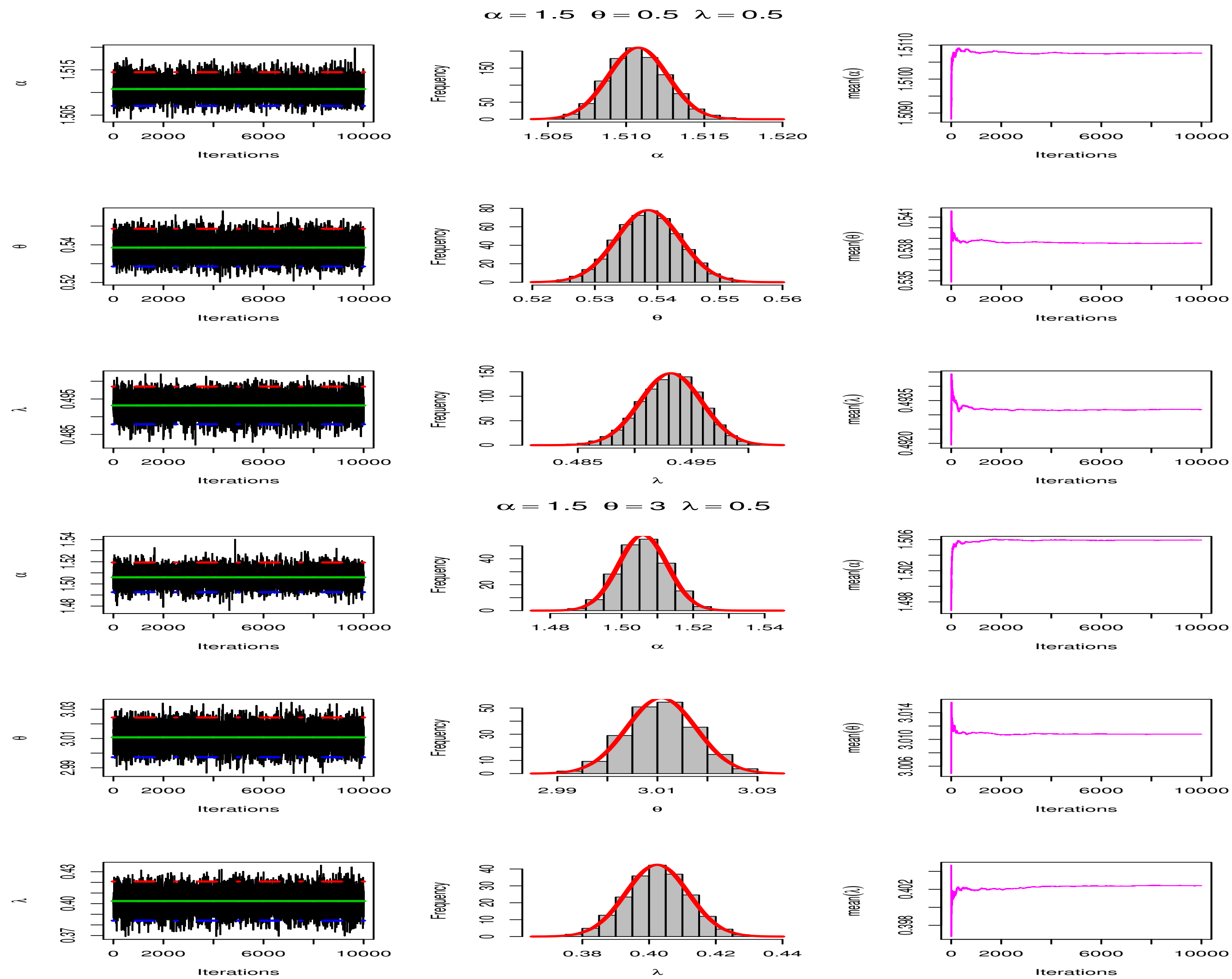

Figure 5: The convergence of bootstrap estimates of the MOAPR parameters.

in Tables 6 and 7 show that the MOAPR provides close fit to both data sets that the MOAPE, MOR, APR and R distributions. The fitted pdf, cdf, sf, PP plots of the MOAPR model are displayed in Figure 6, for the two data sets.

Table 6: ML estimates, KS, p-value, AIC and BIC for the MOAPR and other competing distributions for fatigue life data.

models	alpha	theta	lambda	KS	p-value	AIC	BIC
R			95.9756	0.37838	0.00000	1061.444	1064.059
			4.7782				
MOR		391.1671	39.0634	0.06664	0.76093	920.9695	926.1997
		367.2318	2.8201				
APR	313.8841		62.8342	0.16123	0.01048	952.4154	957.6457
	129.1449		1.8004				
MOAPE	225.6350	503.0672	0.0603	0.10482	0.21710	928.8386	936.684
	202.8139	0.0030	191.5924				
MOAPR	74.7386	41.0215	41.5889	0.05970	0.86429	920.4115	926.1826
	179.9659	33.5457	1.9685				

Table 7: ML estimates, KS, p-value, AIC and BIC for the MOAPR and other competing distributions for glass fibres data.

models	alpha	theta	lambda	KS	p-value	AIC	BIC
R	-	-	1.0883	0.33465	0.00000	101.3993	103.5424
			0.0686				
MOR	-	50.00095	0.5437	0.15664	0.09084	31.44394	31.64394
		20.73115	0.0262				
APR	148.193	-	0.7348	0.22924	0.00266	50.87395	55.16022
	82.12687		0.0291				
MOAPE	251.6759	5.2738	559.8229	0.15522	0.09607	38.31442	44.74382
	426.8927	0.4143	356.4862				
MOAPR	5.2977	55.2123	0.5029	0.11161	0.41252	31.22288	31.60523
	15.9633	67.4928	0.0286				

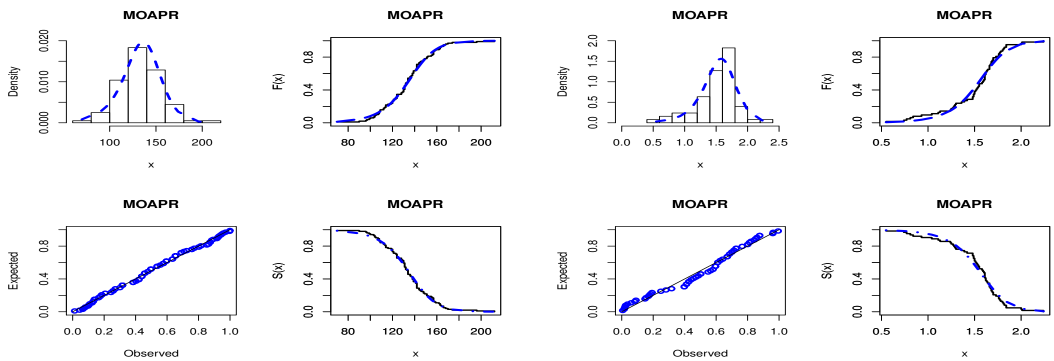

Figure 6: Fitted pdfs, cdfs, sfs, and PP plots of the MOAPR model (left) for fatigue life data and (right) for glass fibres data.

8. Conclusion

In this paper, we propose and study a new three-parameter model, called the Marshall-Olkin alpha power-Rayleigh (MOAPR) distribution to extend the Rayleigh distribution and provide more flexibility to analyze positive real data using the generated model. Some mathematical properties of the MOAPR are provided. The MOAPR generalizes the Marshall-Olkin Rayleigh, alpha power Rayleigh, Rayleigh and some other new models. The maximum likelihood is used to estimate the MOAPR along with parameters. Further, the bootstrap estimates are obtained. the MOAPR parameter by using MLE method. Simulation results are provided to assess the performance of the proposed maximum likelihood and bootstrap methods. Two real data applications of the MOAPR distribution are conducted to illustrate the flexibility of the proposed model.

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Appendix A

Theorem 1. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $d < b$ $(a = -\infty,\ b = \infty$ might as well be allowed$)$. Let $Y : \Omega \to H$ be a continuous random variable with the distribution function F and let h and g be two real functions defined on H such that

$$\mathbf{E}\left[g\left(Y\right)\ \mid Y \geq y\right] = \mathbf{E}\left[h\left(Y\right)\ \mid Y \geq y\right]\xi\left(x\right), \quad y \in H,$$

is defined with some real function ξ. Assume that $h, g \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi h = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions h, g and ξ , particularly

$$F(y) = \int_a^y C \left| \frac{\xi'(u)}{\xi(u)\, h(u) - g(u)} \right| \exp(-s(u))\, du \,,$$

where the function s is a solution of the differential equation $s' = \frac{\xi' h}{\xi h - g}$ and C is the normalization constant, such that $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel [2]), in particular, let us assume that there is a sequence $\{Y_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions h_n , g_n and ξ_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 1 and let $h_n \rightarrow h$, $g_n \rightarrow g$ for some continuously differentiable real functions h and g. Let, finally, X be a random variable with distribution F . Under the condition that $h_n(Y)$ and $g_n(Y)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence Y_n converges to Y in distribution if and only if ξ_n converges to ξ , where

$$\xi(y) = \frac{E[g(Y) \mid Y \geq y]}{E[h(Y) \mid Y \geq y]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions h, g and ξ , respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \rightarrow \infty$.

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions h, g and, specially, ξ should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose ξ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.