Pak.j.stat.oper.res. Vol.17 No. 4 2021 pp 1015-1035 DOI: http://dx.doi.org/10.18187/pjsor.v17i4.3471

Pakistan Journal of Statistics and Operation Research

A New Extreme Value Model with Different Copula, Statistical Properties and Applications

Hanaa Elgohari [1*] and Haitham M. Yousof [2]

* Corresponding Author

[1]Department of Applied Statistics, Faculty of Commerce, Mansoura University, Egypt,
hanaa_elgohary@mans.edu.eg
[2]Department of Statistics, Mathematics and Insurance, Benha University, Benha, Egypt,
haitham.yousof@fcom.bu.edu.eg

Abstract

In this article, we defined and studied a new distribution for modeling extreme value. Some of its mathematical properties are derived and analyzed. Simple types copula is employed for proposing many bivariate and multivariate type extensions. Method of the maximum likelihood estimation is employed to estimate the model parameters. Graphically, we perform the simulation experiments to assess of the finite sample behavior of the maximum likelihood estimations. Three applications are presented for measuring the flexibility of the new model is illustrated using three real data applications.

Key Words: Fréchet; Extreme Value Theory; Estimation; Clayton Copula; Renyi's entropy; Simulation; Farlie Gumbel Morgenstern family.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction and motivation

The Fréchet (Fr) model is one of the most important distributions in modeling extreme values. The Fr model was originally proposed by Fréchet (1927). It has many applications in ranging, accelerated life testing, earthquakes, the floods, the wind speeds, the horse racing, the rainfall, queues in supermarkets and sea waves. One can find more details about the Fr model in the literature for example: Nadarajah and Kotz (2003) investigated the exponentiated Fr distribution. Nadarajah and Kotz (2008) discussed the sociological models based on Fr random variables (RVs), Zaharim et al. (2009) applied the Fr distribution for analyzing the wind speed data, Barreto-Souza et al. (2011) (beta Fr distribution), Krishna et al. (2013) (Marshall--Olkin Fr distribution), Mahmoud and Mandouh (2013) (transmuted Fr distribution), Mead and Abd-Eltawab (2014) (Kumaraswamy Fr distribution), Mead et al. (2016) (beta exponential Fr distribution), among others.

In this article, we expanded the extreme value theory (EVT) theory by proposing and studying a new version of the Fr model called the generalized Odd-Burr generalized Fréchet (OB-Fr) model (for more details about EVT see Fréchet (1927) and Fisher and Tippett (1928)). The new model is derived based on compiling the standard F model with the Odd-Burr generalized (OB-G) family (see Alizadeh et al. (2016)). Straightforward types of copula are employed for proposing many bivariate OB-Fr (BvOB-Fr) and multivariate OB-Fr (MOB-Fr) type extensions. A RV T is said to have the Fr distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given by

$$\pi_{\alpha,\beta}(t) = \beta\alpha^{\beta}t^{-(\beta+1)}\,exp\left(-\alpha^{\beta}t^{-\beta}\right)\,|_{t\geq 0} \quad (1)$$

and

$$\Pi_{\alpha,\beta}(t) = exp\left(-\alpha^{\beta}t^{-\beta}\right)\,|_{t\geq 0}, \quad (2)$$

where $\beta > 0$ refers to the shape parameter.

For $\beta=2$ we get the Inverse Rayleigh (IR) model. Due to Alizadeh et al. (2016), the CDF of the OB-G family is given by

$$F_{a,b,\underline{\Upsilon}}(t)=1-\frac{\overline{\Pi}_{\underline{\Upsilon}}(t)^{ab}}{\left[\Pi_{\underline{\Upsilon}}(t)^{a}+\overline{\Pi}_{\underline{\Upsilon}}(t)^{a}\right]^{b}}, \tag{3}$$

where $\overline{\Pi}_{\underline{\Upsilon}}(t)=1-\Pi_{\underline{\Upsilon}}(t)$. The PDF corresponding to (3) is given by

$$f_{a,b,\underline{\Upsilon}}(t)=\frac{ab\pi_{\underline{\Upsilon}}(t)\Pi_{\underline{\Upsilon}}(t)^{a-1}\overline{\Pi}_{\underline{\Upsilon}}(t)^{ab-1}}{\left[\Pi_{\underline{\Upsilon}}(t)^{a}+\overline{\Pi}_{\underline{\Upsilon}}(t)^{a}\right]^{1+b}}. \tag{4}$$

For $b=1$, the OB-G family reduces to the Odd G (O-G) family (see Gleaton and Lynch (2006)). For $a=1$, the OB-G family reduces to the Proportional reversed hazard rate family (PRHR) (see Gupta and Gupta (2007)). In this work, we define and study a new Fréchet model based on OB-G family called generalized odd log-logistic F (OB-Fr) model. The OB-Fr survival function (SF) is given by

$$S_{\underline{P}}(t)=\frac{\left[1-exp\left(-\alpha^{\beta}t^{-\beta}\right)\right]^{ab}}{\left\{exp\left(-a\alpha^{\beta}t^{-\beta}\right)+\left[1-exp\left(-\alpha^{\beta}t^{-\beta}\right)\right]^{a}\right\}^{b}}, \tag{5}$$

where $S_{\underline{P}}(t)=1-F_{\underline{P}}(t)|_{(\underline{P}=a,b,\alpha,\beta)}$. For $b=1$, the OB-Fr reduces to the O-Fr. For $a=1$, the OB-Fr reduces to the PRHR-Fr. The PDF corresponding to (5) is given by

$$f_{\underline{P}}(t)=\frac{ab\beta t^{-(\beta+1)}\,exp\left(-a\alpha^{\beta}t^{-\beta}\right)\left[1-exp\left(-\alpha^{\beta}t^{-\beta}\right)\right]^{ab-1}}{\left\{exp\left(-a\alpha^{\beta}t^{-\beta}\right)+\left[1-exp\left(-\alpha^{\beta}t^{-\beta}\right)\right]^{a}\right\}^{1+b}}. \tag{6}$$

The hazard rate function (HRF) for the new model can be derived from $f_{\underline{P}}(t)/S_{\underline{P}}(t)$. The new model in (6) can be used in modeling extreme data such as the extreme floods, maximum sizes of ecological populations, the size of freak waves, the amounts of large insurance losses, equity risks, day to day market risk, side effects of drugs (e.g., Ximelagatran), survival time data, strengths and breaking stress experiments, large wildfires and repair times data. In mathematical analysis, the asymptotic analysis is used for describing the limiting behavior of some functions. Asymptotics derivations for the CDF, PDF and HRF can be obtained for the new model. The asymptotics of the CDF, PDF and HRF as $t\to V$ are given by

$$F_{\underline{P}}(t)\sim b\,exp\left(-a\alpha^{\beta}t^{-\beta}\right)\,|_{t\to 0}, f_{\underline{P}}(t)\sim ab\beta t^{-(\beta+1)}\,exp\left(-a\alpha^{\beta}t^{-\beta}\right)\,|_{t\to 0},$$

and

$$h_{\underline{P}}(t)\sim ab\beta t^{-(\beta+1)}\,exp\left(-a\alpha^{\beta}t^{-\beta}\right)\,|_{t\to 0}.$$

The asymptotics of CDF, PDF and HRF as $t\to\infty$ are given by

$$S_{\underline{P}}(t)\sim a^{b}\left[1-exp\left(-\alpha^{\beta}t^{-\beta}\right)\right]^{b}|_{t\to\infty}, f_{\underline{P}}(t)\sim ba^{b}\beta\frac{t^{-(\beta+1)}\,exp\left(-\alpha^{\beta}t^{-\beta}\right)}{\left[1-exp\left(-\alpha^{\beta}t^{-\beta}\right)\right]^{1-b}}|_{t\to\infty}$$

and

$$h_{\underline{P}}(t)\sim\frac{b\beta t^{-(\beta+1)}\,exp\left(-\alpha^{\beta}t^{-\beta}\right)}{1-exp\left(-\alpha^{\beta}t^{-\beta}\right)}|_{t\to\infty}.$$

For simulation of this new model, we obtain the quantile function (QF) of T (by inverting (5)), say $t_u=F^{-1}(u)$, as

$$t_u=\left\{-\frac{1}{\alpha^{\beta}}ln\left[\frac{\left(1-\overline{u}^{\frac{1}{b}}\right)^{\frac{1}{a}}}{\overline{u}^{\frac{1}{ab}}+\left(1-\overline{u}^{\frac{1}{b}}\right)^{\frac{1}{a}}}\right]\right\}^{-\frac{1}{\beta}}, \tag{7}$$

where $\overline{u}=1-u$. Equation (7) is used for simulating the new model (see Section 4). Many useful Fréchet extension can be cited and used in future comparisons see, for example, Korkmaz et al. (2017), Yousof et al. (2017), Haq et al. (2017), Korkmaz et al. (2017), Yousof et al. (2018b), Jahanshahi et al. (2019), Yousof et al. (2019a, b), Salah et al. (2020), Al-Babtain et al. (2020), Elsayed and Yousof (2020), Ibrahim et al. (2020) and
Yousof et al. (2020), Ibrahim et al. (2021), among others

2. Copula

In this section, we derive some new bivariate OB-Fr (BvOB-Fr) via

i. Farlie Gumbel Morgenstern (FGM) Copula.

ii. Modified Farlie Gumbel Morgenstern (FGM) Copula.
iii. Clayton Copula.
iv. Renyi's entropy.

The Multivariate OB-Fr (MOB-Fr) type is also presented. However, future works may be allocated to study these new models. For more details see Ali et al. (2021a, b), Elgohari and Yousof (2020b and 2021) and Shehata and Yousof (2021a, b).

2.1 FGM Copula

Consider the joint CDF of the FGM family (see Morgenstern (1956), Gumbel (1958) and Gumbel (1960)), then $C_\sigma(u,v)=uv(1+\sigma\overline{u}\overline{v})$ where the marginal function $u=F_1(t_1)$, $v=F_2(t_2)\in(0,1)$, $\sigma\in(-1,1)$ is a dependence parameter and for every $u,v\in(0,1)$, $C_\sigma(u,0)=C_\sigma(0,v)=0$ which is "grounded minimum" and $C_\sigma(u,1)=u$ and $C_\sigma(1,v)=v$ which is "grounded maximum". Then, setting

$$\overline{u}=\overline{u}_{\underline{P_1}}=\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right)^{a_1b_1}}{\left[\boldsymbol{Q}_{\alpha,\beta_1,t_1}^{a_1}+\left(1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right)^{a_1}\right]^{b_1}}|_{\underline{P_1}=a_1,b_1,\beta_1>0},$$

where $\boldsymbol{Q}_{\alpha,\beta_1,t_1}=exp\left(-\alpha^{\beta_1}t_1^{-\beta_1}\right)$ and

$$\overline{v}=\overline{v}_{\underline{P_2}}=1-\frac{\left[1-\boldsymbol{Q}_{\alpha,\beta_2,t_2}\right]^{a_2b_2}}{\left\{\boldsymbol{Q}_{\alpha,\beta_2,t_2}^{a_2}+\left[1-\boldsymbol{Q}_{\alpha,\beta_2,t_2}\right]^{a_2}\right\}^{b_2}}|_{\underline{P_2}=a_2,b_2,\beta_2>0},$$

where $\boldsymbol{Q}_{\alpha,\beta_2,t_2}=exp\left(-\alpha^{\beta_2}t_2^{-\beta_2}\right)$ then we have

$$F(t_1,t_2)=C(F_{\underline{P_1}}(t_1),F_{\underline{P_2}}(t_2))|_{(\alpha=\alpha_1=\alpha_2)}=\left\{1-\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right)^{a_1b_1}}{\left[\boldsymbol{Q}_{\alpha,\beta_1,t_1}^{a_1}+\left[1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right]^{a_1}\right]^{b_1}}\right\}$$
$$\times\left\{1-\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_2,t_2}\right)^{a_2b_2}}{\left[\boldsymbol{Q}_{\alpha,\beta_2,t_2}^{a_2}+\left(1-\boldsymbol{Q}_{\alpha,\beta_2,t_2}\right)^{a_2}\right]^{b_2}}\right\}\left(1+\sigma\left\{\begin{array}{c}\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right)^{a_1b_1}}{\left[\boldsymbol{Q}_{\alpha,\beta_1,t_1}^{a_1}+\left[1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right]^{a_1}\right]^{b_1}}\\\times\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_2,t_2}\right)^{a_2b_2}}{\left[\boldsymbol{Q}_{\alpha,\beta_2,t_2}^{a_2}+\left(1-\boldsymbol{Q}_{\alpha,\beta_2,t_2}\right)^{a_2}\right]^{b_2}}\end{array}\right\}\right).$$

The joint PDF can then be derived from

$$c_\sigma(u,v)=1+\sigma u^*v^*|_{(u^*=1-2u\text{ and }v^*=1-2v)}$$

or from

$$f(t_1,t_2)=f_{\underline{P_1}}(t_1)f_{\underline{P_2}}(t_2)c(F_{\underline{P_1}}(t_1),F_{\underline{P_2}}(t_2))|_{(\alpha=\alpha_1=\alpha_2)}.$$

2.2 Modified FGM Copula

Consider the following modified version of the bivariate FGM copula defined as (see Rodriguez-Lallena and Ubeda-Flores (2004))

$$C_\sigma(u,v)|_{\sigma\in[-1,1]}=uv[1+\sigma\Phi(u)\Psi(v)]=uv+\sigma\Phi^{\cdot}(u)\Psi^{\cdot}(v),$$

where $\Phi^{\cdot}(u)=u\Phi(u)$, and $\Psi^{\cdot}(v)=v\Psi(v)$. Where $\Phi(u)$ and $\Psi(v)$ are two absolutely continuous functions on $(0,1)$ with the following conditions:

1-The boundary condition:

$$\Phi(0)=\Phi(1)=\Psi(0)=\Psi(1)=0.$$

2-Let

$$\alpha=inf\left\{\frac{\partial}{\partial u}\Phi^{\cdot}(u)\ :\ A_1\right\}<0,\beta=sup\left\{\frac{\partial}{\partial u}\Phi^{\cdot}(u)\ :\ A_1\right\}<0,$$

$$\xi=inf\left\{\frac{\partial}{\partial v}\Psi^{\cdot}(v)\ :\ A_2\right\}>0,\eta=sup\left\{\frac{\partial}{\partial v}\Psi^{\cdot}(v)\ :\ A_2\right\}>0,$$

Then, $min(\alpha\beta, \xi\eta) \geq 1$, where

$$\frac{\partial}{\partial u}\Phi^{\cdot}(u) = \Phi(u) + u\frac{\partial}{\partial u}\Phi(u),$$

$$A_1 = \left\{u \;:\; u \in (0,1) | \frac{\partial}{\partial u}\Phi^{\cdot}(u) \;\; \text{exists}\right\},$$

and

$$A_2 = \left\{v \;:\; v \in (0,1) | \frac{\partial}{\partial v}\Psi^{\cdot}(v) \;\; \text{exists}\right\}.$$

BvOB-Fr-FGM (Type I) model

Here, we consider the following functional form for both $\Phi(u)$ and $\Psi(v)$ as

$$C_\sigma(u,v) = \sigma[\Phi^{\cdot}(u)\Psi^{\cdot}(v)] + \left\{1 - \frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_1,u}\right)^{a_1 b_1}}{\left[\boldsymbol{Q}_{\alpha,\beta_1,u}^{a_1} + \left(1-\boldsymbol{Q}_{\alpha,\beta_1,u}\right)^{a_1}\right]^{b_1}}\right\}\left\{1 - \frac{\left[1-\boldsymbol{Q}_{\alpha,\beta_2,v}\right]^{a_2 b_2}}{\left\{\boldsymbol{Q}_{\alpha,\beta_2,v}^{a_2} + \left[1-\boldsymbol{Q}_{\alpha,\beta_2,v}\right]^{a_2}\right\}^{b_2}}\right\},$$

where

$$\Phi^{\cdot}(u) = \frac{u\left(1-\boldsymbol{Q}_{\alpha,\beta_1,u}\right)^{a_1 b_1}}{\left[\boldsymbol{Q}_{\alpha,\beta_1,u}^{a_1} + \left(1-\boldsymbol{Q}_{\alpha,\beta_1,u}\right)^{a_1}\right]^{b_1}}|_{(a_1,b_1,\beta_1)>0},$$

and

$$\Psi^{\cdot}(v) = \frac{v\left[1-\boldsymbol{Q}_{\alpha,\beta_2,v}\right]^{a_2 b_2}}{\left\{\boldsymbol{Q}_{\alpha,\beta_2,v}^{a_2} + \left[1-\boldsymbol{Q}_{\alpha,\beta_2,v}\right]^{a_2}\right\}^{b_2}}|_{(a_2,b_2,\beta_2)>0}.$$

BvOB-Fr-FGM (Type II) model

Consider the following functional form for both $\Phi(u)$ and $\Psi(v)$ which satisfy all the conditions stated earlier where

$$\Phi(u)|_{(\sigma_1>0)} = u^{\sigma_1}(1-u)^{1-\sigma_1} \text{ and } \Psi(v)|_{(\sigma_2>0)} = v^{\sigma_2}(1-v)^{1-\sigma_2}.$$

The corresponding bivariate copula (henceforth, BvOB-Fr-FGM **(**Type **II)** copula) can be derived from

$$C_{\sigma,\sigma_1,\sigma_2}(u,v) = uv[1 + \sigma u^{\sigma_1} v^{\sigma_2}(1-u)^{1-\sigma_1}(1-v)^{1-\sigma_2}].$$

BvOB-Fr-FGM (Type III) model

Consider the following functional form for both $\Phi(u)$ and $\Psi(v)$ which satisfy all the conditions stated earlier where

$$P(u) = u[log(1+\overline{u})] \text{ and } Q(v) = v[log(1+\overline{v})].$$

In this case, one can also derive a closed form expression for the associated CDF of the BvOB-Fr-FGM (Type **III**) as

$$C_{\nabla}(u,v) = uv\big(1 + \sigma P(u)Q(v)\big).$$

BvOB-Fr-FGM (Type IV) model

Due to Ghosh and Ray (2016) the CDF of the BvOB-Fr-FGM (Type **IV**) model can be derived from

$$C_\sigma(u,v) = uF^{-1}(v) + vF^{-1}(u) - F^{-1}(u)F^{-1}(v),$$

where

$$F^{-1}(u) = \left(-\frac{1}{\alpha_1^{\beta_1}} ln\left\{\frac{\left[1-(1-u)^{\frac{1}{b_1}}\right]^{\frac{1}{a_1}}}{(1-u)^{\frac{1}{a_1 b_1}} + \left[1-(1-u)^{\frac{1}{b_1}}\right]^{\frac{1}{a_1}}}\right\}\right)^{-\frac{1}{\beta_1}},$$

and

$$F^{-1}(v)=\left(-\frac{1}{\alpha_2^{\beta_2}}ln\left\{\frac{\left[1-(1-v)^{\frac{1}{b_2}}\right]^{\frac{1}{a_2}}}{(1-v)^{\frac{1}{a_2 b_2}}+\left[1-(1-v)^{\frac{1}{b_2}}\right]^{\frac{1}{a_2}}}\right\}\right)^{-\frac{1}{\beta_2}}.$$

2.3 Clayton Copula

The Clayton Copula can be considered as

$$C(v_1,v_2)=(v_1^{-\nabla}+v_2^{-\nabla}-1)^{-\frac{1}{\nabla}}|_{\nabla\in[0,\infty]}.$$

Let us assume that $T\sim$ OB-Fr (a_1,b_1,β_1) and $W\sim$ OB-Fr (a_2,b_2,β_2) . Then, setting

$$v_1=v(t)=1-\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right)^{a_1 b_1}}{\left[\boldsymbol{Q}_{\alpha,\beta_1,t_1}^{a_1}+\left(1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right)^{a_1}\right]^{b_1}}|_{\underline{P_1}>0},$$

and

$$v_2=v(w)=1-\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_2,w}\right)^{a_2 b_2}}{\left[\boldsymbol{Q}_{\alpha,\beta_2,w}^{a_2}+\left(1-\boldsymbol{Q}_{\alpha,\beta_2,w}\right)^{a_2}\right]^{b_2}}|_{\underline{P_2}>0},$$

Then, the BvOB-Fr type distribution can be derived as

$$F(t,w)=C(F_{\underline{P_1}}(t),F_{\underline{P_2}}(w))|_{(\alpha=\alpha_1=\alpha_2)}=\begin{bmatrix}\left(1-\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right)^{a_1 b_1}}{\left[\boldsymbol{Q}_{\alpha,\beta_1,t_1}^{a_1}+\left(1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right)^{a_1}\right]^{b_1}}\right)^{-\nabla}\\+\left(1-\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_2,w}\right)^{a_2 b_2}}{\left[\boldsymbol{Q}_{\alpha,\beta_2,w}^{a_2}+\left(1-\boldsymbol{Q}_{\alpha,\beta_2,w}\right)^{a_2}\right]^{b_2}}\right)^{-\nabla}\\-1\end{bmatrix}^{-\frac{1}{\nabla}}.$$

2.4 Renyi's entropy Copula

Consider theorem of Pougaza and Djafari (2011) where

$$R(u,v)=t_2u+t_1v-t_1t_2,$$

then, the associated BvOB-Fr will be

$$R(t_1,t_2)|_{(\alpha=\alpha_1=\alpha_2)}=R(F_{\underline{P_1}}(t),F_{\underline{P_2}}(w))=-t_1t_2$$
$$+t_2\left\{1-\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right)^{a_1 b_1}}{\left[\boldsymbol{Q}_{\alpha,\beta_1,t_1}^{a_1}+\left[1-\boldsymbol{Q}_{\alpha,\beta_1,t_1}\right]^{a_1}\right]^{b_1}}\right\}+t_1\left\{1-\frac{\left(1-\boldsymbol{Q}_{\alpha,\beta_2,t_2}\right)^{a_2 b_2}}{\left[\boldsymbol{Q}_{\alpha,\beta_2,t_2}^{a_2}+\left[1-\boldsymbol{Q}_{\alpha,\beta_2,t_2}\right]^{a_2}\right]^{b_2}}\right\}.$$

By fixing a and b we then have a 5-dimension parameter BvOB-Fr type distribution.

MOB-Fr extension via Clayton Copula. A straightforward $\hbar$ -dimensional extension from the above will be

$$H(v_i)=\left[\sum_{i=1}^{\hbar} v_i^{-\nabla}+1-\hbar\right]^{-\frac{1}{\nabla}},$$

Then, the MOB-Fr extension can expressed as

$$H(\underline{T})|_{(\alpha=\alpha_i)}=\left[\sum_{i=1}^{\hbar}\left(1-\frac{\left[1-exp\left(-\alpha^{\beta_i}t_i^{-\beta_i}\right)\right]^{a_i b_i}}{\left\{exp\left(-a_i t_i^{-\beta_i}\right)+\left[1-exp\left(-t_i^{-\beta_i}\right)\right]^{a_1}\right\}^{b_i}}\right)^{-\nabla}+1-\hbar\right]^{-\frac{1}{\nabla}},$$

where $\underline{T}=t_1,t_2,\cdots,t_\hbar$.

3. Mathematical properties

3.1 Useful representations

Due to Alizadeh et al. (2016), the PDF in (6) can be expressed as

$$f(t) = \sum_{s=0}^{\infty} \vartheta_s \pi_{(1+s)}(t; \alpha, \beta), \tag{8}$$

where

$$\vartheta_s = \frac{ab}{1+s} \sum_{d_1,d_2=0}^{\infty} \sum_{d_3=s}^{\infty} (-1)^{d_2+k+s} \binom{-(1+b)}{d_1} \binom{-[a(1+d_1)+1]}{d_2} \binom{a(1+d_1)+d_2+1}{d_3} \binom{d_3}{s}$$

and $\pi_{(1+s)}(t; \alpha, \beta)$ is the PDF of the F model with scale parameter $\frac{\alpha}{(1+s)^{-\beta^{-1}}}$ and shape parameter β. By integrating Equation (8), the CDF of t becomes

$$F(t) = \sum_{s=0}^{\infty} \vartheta_s \Pi_{(1+s)}(t; \alpha, \beta), \tag{9}$$

where $\Pi_{(1+s)}(t; \alpha, \beta)$ is the CDF of the F distribution with scale parameter $\frac{\alpha}{(1+s)^{-\beta^{-1}}}$ and shape parameter β.

3.2 Moments and incomplete moments

The ρ^{th} ordinary moment of T is given by

$$\mu'_{\rho,T} = E(T^\rho) = \int_{-\infty}^{\infty} t^\rho f(t) dt, \tag{10}$$

then we obtain

$$\mu'_{\rho,T}|_{(\beta>\rho)} = \sum_{s=0}^{\infty} \alpha^\rho \vartheta_s^{(\rho,\beta)} \Gamma\left(1 - \frac{\rho}{\beta}\right),$$

where $\vartheta_s^{(\rho,\beta)} = \vartheta_s (1+s)^{\frac{\rho}{\beta}}$ and $\Gamma(1+\nu)|_{(\nu \in R^+)} = \nu! = \prod_{\hbar=0}^{\nu-1} (\nu - \hbar)$. Setting $\rho = 1,2,3$ and 4 in (10), we have

$$E(T)|_{(\beta>1)} = \mu'_{1,T} = \sum_{s=0}^{\infty} \alpha \vartheta_s^{(1,\beta)} \Gamma\left(1 - \frac{1}{\beta}\right), E(T^2)|_{(\beta>2)} = \mu'_{2,T} = \sum_{s=0}^{\infty} \alpha^2 \vartheta_s^{(2,\beta)} \Gamma\left(1 - \frac{2}{\beta}\right),$$

$$E(T^3)|_{(\beta>3)} = \mu'_{3,T} = \sum_{s=0}^{\infty} \alpha^3 \vartheta_s^{(3,\beta)} \Gamma\left(1 - \frac{3}{\beta}\right),$$

and

$$E(T^4)|_{(\beta>4)} = \mu'_{4,T} = \sum_{s=0}^{\infty} \alpha^4 \vartheta_s^{(4,\beta)} \Gamma\left(1 - \frac{4}{\beta}\right),$$

where $E(T) = \mu'_1$ is the mean of T. The ρ^{th} incomplete moment, say $I_{\rho,T}(\tau)$, of T can be expressed, from (9), as

$$I_{\rho,T}(\tau) = \int_{-\infty}^{\tau} t^\rho f(t) dt = \sum_{s=0}^{\infty} \vartheta_s \int_{-\infty}^{\tau} t^\rho \pi_{(1+s)}(t; \beta) dt$$

then

$$I_{\rho,T}(\tau)|_{(\beta>\rho)} = \sum_{s=0}^{\infty} \alpha^\rho \vartheta_s^{(\rho,\beta)} \gamma\left(1 - \frac{\rho}{\beta}, (1+s)(a/\tau)^\beta\right), \tag{11}$$

where $\gamma(\nu, \rho)$ is the incomplete gamma function.

$$\gamma(\nu,\rho)|_{(\nu \neq 0,-1,-2,\ldots)} = \int_0^\rho t^{\nu-1} exp(-t)\, dt = \frac{1}{\nu} \rho^\nu \{1F_1[\nu; \nu+1; -\rho]\} = \sum_{s=0}^{\infty} \rho^{\nu+s} \frac{(-1)^s}{s!\,(\nu+s)},$$

and $1F_1[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function. The first incomplete moment given by (11) with $\rho = 1$ as

$$I_{1,T}(\tau)|_{(\beta>1)} = \sum_{s=0}^{\infty} \alpha \vartheta_s^{(1,\beta)} \Gamma\left(1 - \frac{1}{\beta}, (1+s)(a/\tau)^\beta\right).$$

3.3 Index of dispersion (Index)

The Index of dispersion IxDis or the variance to mean ratio can derived as

$$\text{IxDis}(T) = \mu_2/\mu'_1.$$

It is a measure used to quantify whether a set of observed occurrences are clustered or dispersed compared to a standard statistical model. So, it indicates whether a certain statistical model is suitable for over (or under) dispersed datasets and is used widely in ecology as a standard measure for measuring clustering (over-dispersion) or repulsion (under-dispersion). Thus, the measure can be used to assess whether observed data can be modeled using a Poisson process. For any real dataset, when the IxDis is less than 1 , the dataset is said to be "under-dispersed", this important condition can relate to occurrence patterns that are more regular than the randomness associated with a Poisson process.

3.4 Numerical analysis

A numerical analysis for some measures including the Mean $[E(T)]$, variance $[\mathrm{Var}(T)]$, skewness $[\mathrm{Skew}(T)]$, kurtosis $[\mathrm{Kur}(T)]$ and IxDis (T) for the OB-Fr model is presented in Table 1 with useful comments. The same analysis for the standard Fr model is given in Table 2. Based on Tables 1 and 2 we note that, the Skew (T) of the OB-Fr distribution can range in the interval (4.5, 1097), whereas the Skew (Y) of the F model varies only in the interval (1.2115, 3.5**)**. Further, the spread for the Kur (T) of the OB-Fr model is ranging from 25.16 to 1354275, whereas the spread for the Kur (T) of the F model only varies from 4.5 to 98.8 with the above parameter values. The IxDis (T) for the OB-Fr model can be only more 1 so it may be used as an "over-dispersed" model. Howevere the IxDis (T) for the Fr model can be only between 0 and 1 so it may be used as an "under-dispersed" model.

Table 1: E(T), Var(T), Skew(T) and Kur(T)of the OB-Fr model.

a	b	α	β	E(T)	Var(T)	Skew(T)	Kur(T)	IxDis(T)
0.00001				0.3749728	18748.37	486.8805	266686	49999.28
0.0001	2	1.5	1.5	3.747167	187336.7	154.0108	26686.1	49994.22
0.001				37.21382	1858581	48.84929	2686.37	49943.30
0.01				344.6265	17033731	15.98947	289.515	49426.64
0.1				812.7468	35212441	10.58003	130.309	43325.23
0.15				384.5686	14749798	15.93518	299.182	38354.14
0.5	0.00001	3	2	0.9996124	49991.05	298.1656	100016.3	50010.43
	0.025			1975.681	93657162	6.524823	49.68264	47405.01
	0.1			3919.6700	17038598	4.544012	25.16457	43469.47
	0.2			3111.3620	12872279	5.249075	33.35269	41371.84
	0.3			1880.6930	74046202	7.000663	58.60794	39371.77
	0.4			1030.2170	37730469	9.792836	114.3752	36623.80
0.01	0.01	0.01	0.01	0.07390783	3688.67	1097.012	1354275	49909.05
		0.1		0.07453191	3719.683	1092.416	1342959	49907.26
		0.2		0.07472321	3729.19	1091.019	1339529	49906.72
		0.5		0.07497858	3741.881	1089.162	1334976	49906.00
		10		0.07583364	3784.372	1083.013	1319955	49903.61
		50		0.07630605	3807.848	1079.66	1311801	49902.30
		100		0.07651238	3818.101	1078.205	1308271	49901.74
		500		0.07699826	3842.246	1074.803	1300032	49900.42
2	2	2	0.001	89.6136	4473301	31.45153	1114.489	49917.67
			0.1	1242.69	4749234	8.732295	90.96676	38217.26
			0.2	81.7061	1528125	43.42091	2399.856	18702.70
			0.5	3.72747	84.5209	1050.910	5681077	22.67513
			0.55	3.36541	32.9263	679.0338	5226987	9.783743
			0.65	2.93612	9.98449	123.4382	1133658	3.400577
			0.7	2.80062	6.72716	47.69959	359970.1	2.402026
			0.725	2.74496	5.68505	30.85292	192929.7	2.071087

Table 2: E(T), Var(T), Skew(T) and Kur(T)of the Fr model.

α	β	E(T)	Var(T)	Skew(T)	Kur(T)	IxDis(T)
0.5	5	0.582115	0.0334404	3.535071	48.0915	0.057446320
	10	0.534314	0.0055656	1.910339	10.9774	0.010416350
	25	0.512366	0.0007351	1.400443	6.85310	0.001434654
	40	0.507532	0.0002750	1.296997	6.23002	0.000541922
	50	0.505974	0.0001736	1.264099	6.04447	0.000343082
	75	0.5039371	7.575362×10^{-05}	1.221375	5.81425	0.000150324
	85	0.5034646	5.87275×10^{-05}	1.211504	5.561893	0.000116647

2.5	10	2.671572	0.1391401	1.910339	10.97857	0.05208173
	30	2.550936	0.0125170	1.353565	6.562312	0.00490683
	50	2.529868	0.0043398	1.264099	6.045233	0.00171541
	75	2.519686	0.0018938	1.221761	5.760403	0.00075161
	100	1.508808	0.0003799	1.216101	4.525062	0.00025185
3.5	7.5	3.828837	0.512812	2.25012	14.9701	0.1339342
5	4.5	5.950756	4.60640	4.23885	98.8016	0.7740870
15	15	15.6475	1.993072	1.60525	8.282494	0.1273732
10	7.5	10.97054	4.47131	2.29491	15.5896	0.4075740
5.5	5.5	6.294259	3.07532	3.10146	32.22113	0.4885905
60	20	61.8872	17.0379	1.47388	7.33349	0.2753061
10	10	10.68629	2.226241	1.91034	10.97857	0.2083269

3.5 Some generating functions (GF)

The moment generating function (MGF) can be derived using (8) as

$$M_T(\tau)=\sum_{s=0}^{\infty}\ \vartheta_s M_{(1+s)}(\tau;\beta),$$

where $M_{(1+s)}(\tau;\alpha,\beta)$ is the MGF of the F model with scale parameter $\frac{\alpha}{(1+s)^{-\beta^{-1}}}$ and shape parameter β, then

$$M_T(\tau)|_{(\beta>\rho)}=\sum_{s=0}^{\infty}\ \sum_{\rho=0}^{\infty}\ \alpha^{\rho}\frac{\tau^{\rho}}{\rho!}\vartheta_s^{(\rho,\beta)}\Gamma\left(1-\frac{\rho}{\beta}\right).$$

The first ρ derivatives of $M_T(\tau)$, with respect to τ at $\tau=0$, yield the first ρ moments about the origin, i.e.,

$$\mu'_{\rho,T}=E(T^r)=\frac{d^{\rho}}{d\tau^{\rho}}M_T(\tau)|_{(\tau=0\text{ and }\rho=1,2,3,\ldots)},$$

The generating function GF (CGF) is the logarithm of the MGF. Thus, r th cumulant, say κ_ρ, can be obtained from

$$\kappa_{\rho,T}=\frac{d^{\rho}}{d\tau^{\rho}}log\left[\sum_{s=0}^{\infty}\ \sum_{\rho=0}^{\infty}\ \alpha^{\rho}\frac{\tau^{\rho}}{\rho!}\vartheta_s^{(\rho,\beta)}\Gamma\left(1-\frac{\rho}{\beta}\right)\right]|_{(\tau=0,\text{ and }\rho=1,2,3,\ldots)}.$$

The 1st cumulant is the mean ($\kappa_1=\mu'_1$), the 2 nd cumulant is the variance, and the 3 rd cumulant is the same as the 3rd central moment $\kappa_3=\mu_3$. But 4 th and higher order cumulants are not equal to central moments, that being said

$$\kappa_{1,T}=\mu'_{1,T},$$

$$\kappa_{2,T}=\mu'_{2,T}-\mu'^{2}_{1,T}=\mu_{2,T},$$

and

$$\kappa_{3,T}=\mu'_{3,T}-3\mu'_{2,T}\mu'_{1,T}+2\mu'^{3}_{1,T}=\mu_{3,T}.$$

In some cases, theoretical treatments of problems in terms of cumulants are simpler than those using moments. In particular, when two or more RVs are statistically independent, the ρ^{th} order cumulant of their sum is equal to the sum of their ρ^{th} order cumulants. Moreover, the cumulants can be also obtained from

$$\kappa_{\rho,T}|_{\rho\geq 1}=\mu'_{\rho,T}-\sum_{m=0}^{\rho-1}\ \binom{\rho-1}{m-1}\mu'_{\rho-m,T}\kappa_{m,T}.$$

3.6 Residual life and reversed residual life functions

The $\hbar^{th}$ moment of the residual life

$$\nu_{\hbar,T}(\tau)=E\left[(T-\tau)^{\hbar}\ |_{T>\tau,\hbar=1,2,\ldots}\right]$$

the $\hbar^{th}$ moment of the residual life of T is given by

$$\nu_{\hbar,T}(\tau)=\frac{1}{1-F(\tau)}\int_{\tau}^{\infty}\ (T-\tau)^{\hbar}dF(T).$$

Therefore,

$$\nu_{\hbar,T}(\tau)=\frac{1}{1-F(\tau)}\sum_{s=0}^{\infty}\alpha^{\hbar}\vartheta_s^{(\hbar,\beta)*}\Gamma\left(1-\frac{\hbar}{\beta},(1+s)(a/\tau)^{\beta}\right)|_{\beta>\hbar},$$

where

$$\vartheta_s^{(\hbar,\beta)*}=\vartheta_s\sum_{\rho=0}^{\hbar}\binom{\hbar}{\rho}(-\tau)^{\hbar},\Gamma(\nu,\rho)|_{\rho>0}=\int_{\rho}^{\infty}t^{\nu-1}\,exp(-t)\,dt$$

and

$$\Gamma(\nu,\rho)=\Gamma(\nu)-\gamma(\nu,\rho).$$

The $\hbar^{th}$ moment of the reversed residual life, say

$$V_{\hbar,T}(\tau)=E\left[(\tau-T)^{\hbar}\;|_{T\le\tau,\tau>0\text{ and }\hbar=1,2,\ldots}\right]$$

uniquely determines $F(T)$. We obtain

$$V_{\hbar,T}(\tau)=\frac{1}{F(\tau)}\int_0^{\tau}(\tau-T)^{\hbar}dF(t).$$

Then, the $\hbar^{th}$ moment of the reversed residual life of T becomes

$$V_{\hbar,T}(\tau)=\frac{1}{F(\tau)}\sum_{s=0}^{\infty}\alpha^{\hbar}\vartheta_s^{(\hbar,\beta)**}\gamma\left(1-\frac{\hbar}{\beta},(1+s)(a/\tau)^{\beta}\right)|_{\beta>\hbar},$$

where

$$\vartheta_s^{(\hbar,\beta)**}=\vartheta_s\sum_{\rho=0}^{\hbar}(-1)^{\rho}\binom{\hbar}{\rho}\tau^{\hbar-\rho}.$$

4. The maximum likelihood estimation (MLE) method

Let $T_1,T_2,\ldots,T_{\hbar}$ be a random sample from size $\hbar$ from the OB-Fr distribution with parameters a,b,α and β. For determining the MLE of $\underline{P}$, we have the log-likelihood function

$$\ell=\ell(\underline{P})=\hbar\,log\left(ab\beta\alpha^{\beta}\right)-(\beta+1)\sum_{i=1}^{\hbar}log(t_i)-ab\sum_{i=1}^{\hbar}\alpha^{\beta}t^{-\beta}$$

$$+2\sum_{i=1}^{\hbar}log\left(exp\left[-ab\alpha^{\beta}t^{-\beta}\right]+\left\{1-exp\left[-b\alpha^{\beta}t^{-\beta}\right]\right\}^{a}\right)+(a-1)\sum_{i=1}^{\hbar}log\left\{1-exp\left[-b\alpha^{\beta}t^{-\beta}\right]\right\}.$$

The components of the score vector is available if needed. Setting $U_a=U_b=U_\alpha=U_\beta=0$ and solving them simultaneously yields the MLEs. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the 3×3 observed information matrix

$$\mathbf{J}(\underline{P})=\{\partial^2\ell/\partial r\partial s\}|_{(r,s=a,b,\alpha,\beta)},$$

5. Graphical assessment

Graphically and using the biases and mean squared errors ($MSEs$), we can perform the simulation experiments to assess the finite sample behavior of the MLEs. The assessment was based on the following algorithm:

i. Generate N =1000 samples of size $n|_{(n=50,100,\ldots,1500)}$ from the OB-Fr distribution using (7).

ii. Compute the MLEs for the N =1000 samples.

iii. Compute the standard errors (SEs) of the MLEs for the 1000 samples.

iv. Compute the biases and mean squared errors given for $\underline{P}=a,b,\alpha,\beta$. We repeated these steps for $n|_{(n=50,100,\ldots,1500)}$ with $a=b=\alpha=\beta=1$, so computing biases $\left(\text{Bias}_{\underline{P}}(n)\right)$, mean squared errors ($MSEs$) $(MSE_h(n))$ for $\underline{P}=a,b,\alpha,\beta$ and $n|_{(n=50,100,\ldots,1500)}$.

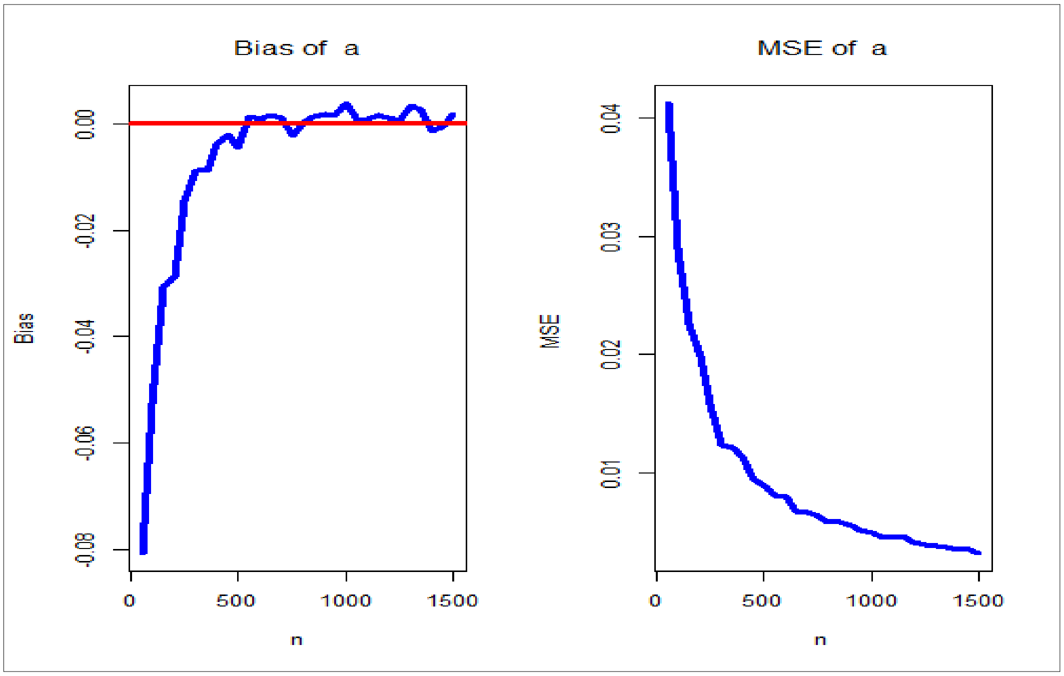


Figure 1: biases and mean squared errors for the parameter *a*.

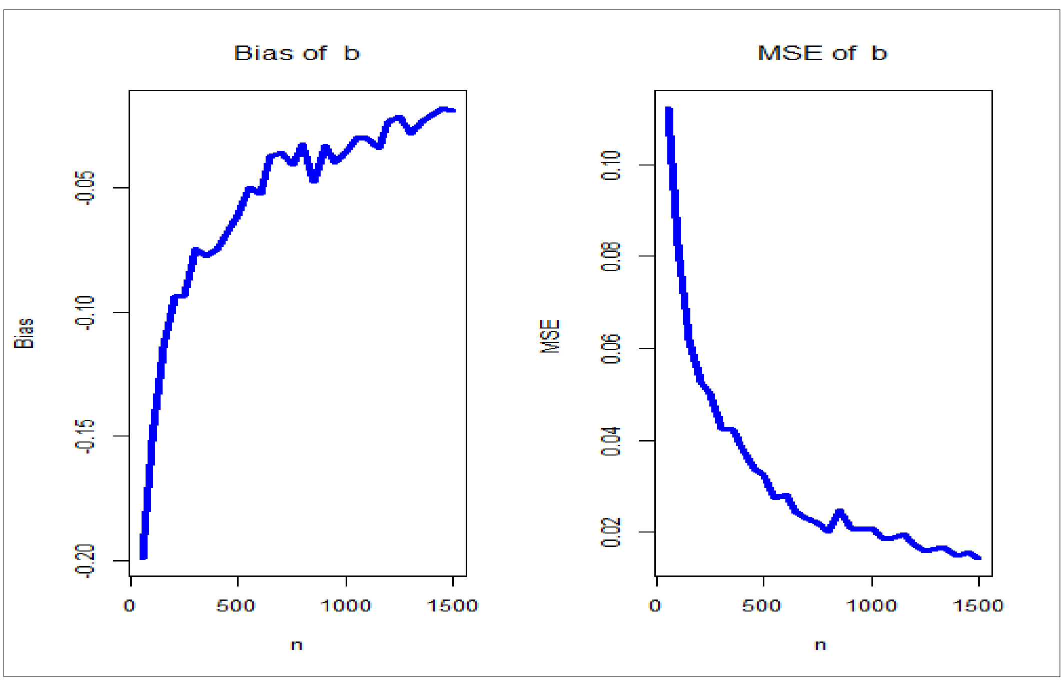


Figure 2: biases and mean squared errors for the parameter *b*.

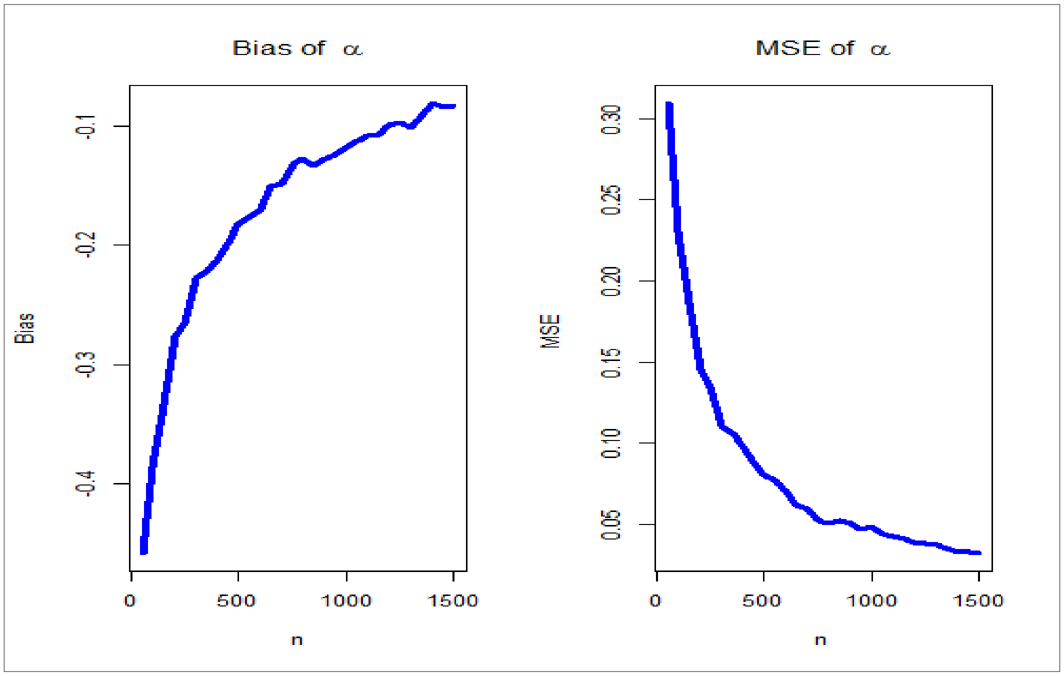


Figure 3: biases and mean squared errors for the parameter ☺

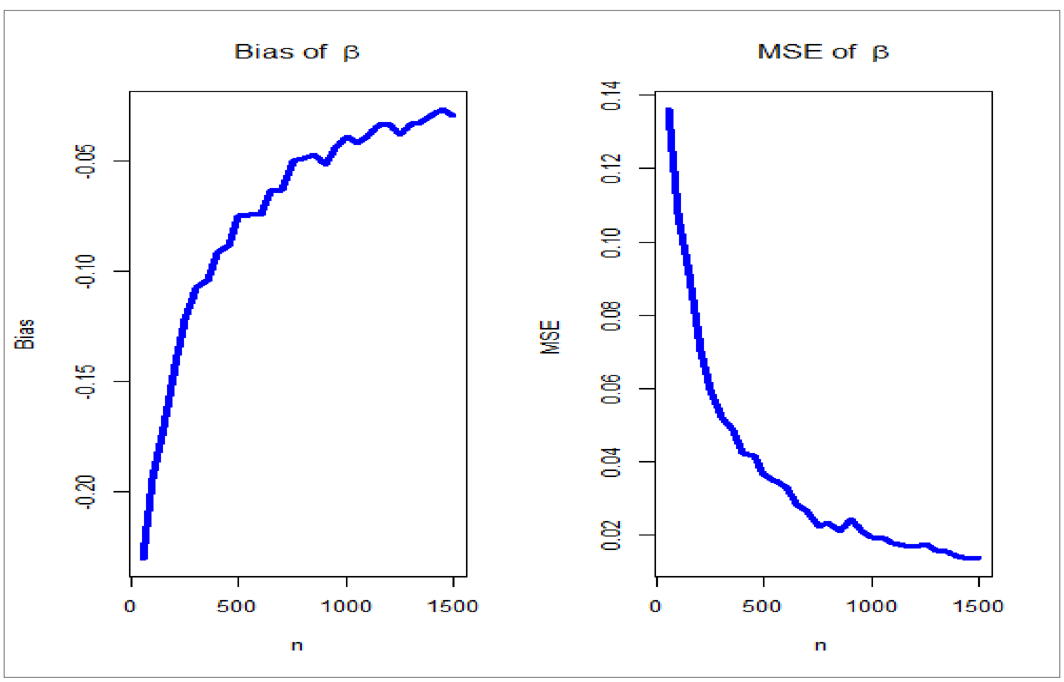


Figure 4: biases and mean squared errors for the parameter ☺

Figures 1, 2, 3 and 4 gives the biases (left panel) and MSEs (right panel) for the parameters α, β, a and b respectively. These Figures (left panels) show how the four biases vary with respect to n and also the right panels show how the four MSEs vary with respect to n . The broken line in Figure 1 corresponds to the biases being 0 . From Figures 1-4, the biases for each parameter are generally negative and decrease to zero as $n \rightarrow \infty$, the MSEs for each parameter decrease to zero as $n \rightarrow \infty$.

6. Modeling uncensored real data for comparing the competitive models

For illustrating the wide applicability of the new OB-Fr model, we consider the Cramér-Von Mises (CVM) statistic, the Anderson-Darling **(**A-D**)** statistic, the Kolmogorov-Smirnov (KS) statistic and its corresponding p-value $(P)v)$. Table 3 below gives the competitive models.

Table 3: The competitive models.

Competitive models (Abbreviation)	Author(s)
Fréchet (Fr)	Fréchet (1927).
Exponentiated-Fr (E-Fr)	Nadarajah and Kotz (2003).
Beta-Fr (B-Fr)	Barreto-Souza et al. (2011).
Marshal-Olkin-Fr (MO-Fr)	Krishna et al. (2013).
Transmuted-Fr (T-Fr)	Mahmoud and Mandouh (2013).
Kumaraswamy-Fr (Kum-Fr)	Mead and Abd-Eltawab (2014).
McDonald-Fr (Mc-Fr)	Shahbaz et al. (2016).
odd log-logistic-IR (OLL-IR)	Yousof et. al (2018a).
odd log-logistic-EFr (OLL-EF)	-
odd log-logistic-EFr (OLL-EIR)	-
Generalized odd log-logistic-IR(GOLL-IR)	-

6.1 Stress data

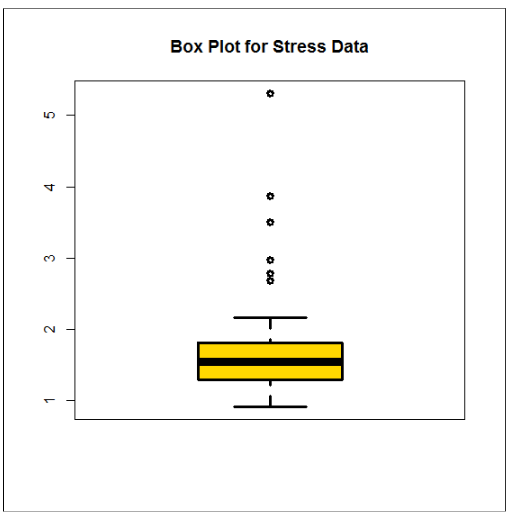
The 1 $s\tau$ data set is an uncensored data set consisting of 100 observations on breaking stress of carbon fibers (in Gba) given by Nichols and Padgett (2006) and these data are used by Mahmoud and Mandouh (2013) to fit the transmuted Fr distribution. Figure 5 gives in its left panel the total time test (TTT) plot (see Aarset (1987) for more details) for data set **I** along with the box plot for discovering the EVs. It indicates that the empirical HRF of data sets **I** is "increasing HRF" and we have six EVs. The statistics CVM, A-D , K-S and $\mathbf{P}v$ for all fitted models are presented in Table 4. The MLEs and corresponding standard errors (SEs) are given in Table 5. From Table 4, the OB-Fr model gives the lowest values the CVM=**0.0664**, A-D=**0.4706**, K-S=**0.0630** and $\mathbf{P}v$ =**0.822** as compared to further Fr models. Therefore, the OB-Fr can be chosen as the best model. Figure 6 gives the estimated (E-PDF) versus the estimated CDF (E-CDF). Figure 6 gives the Probability-Probability (P-P) plot and estimated HRF (E-HRF) for data set **I**. From Figures 6 and 7, we note that the new OB-Fr model provides adequate fits to the empirical functions.

Table 4: CVM, A-D, K-S and P_{v} for data set I.

Model	Goodness of fit criteria			
	CVM	A-D	K-S	$\mathbf{P}v$
OB-Fr	0.0664	0.4706	0.0630	0.822
OLLE-Fr	0.1203	0.9639	0.5561	2.2×10^{-16}
OLLE-IR	0.1553	1.21197	0.65497	2.2×10^{-16}
OLL-IR	0.1553	1.21201	0.6550	2.2×10^{-16}
Fr	0.1090	0.7657	0.0874	0.4282
Kum-Fr	0.0812	0.6217	0.0759	0.6118
Exp-Fr	0.1091	0.7658	0.0874	0.4287
Beta-Fr	0.0809	0.6207	0.0757	0.6147
T-Fr	0.0871	0.6209	0.0782	0.5734
MO-Fr	0.0886	0.6142	0.0763	0.5168
Mc-Fr	0.1333	1.0608	0.0807	0.5332
OLLE-IR	0.1553	1.21197	0.65497	2.2×10^{-16}
OLL-IR	0.1553	1.21201	0.6550	2.2×10^{-16}

Table 5: MLEs and their standard errors (in parentheses) for data set I.

Model	Estimates				
	a	b	c	α	β
OB-Fr	5.1954	0.5990		1.0404	1.232
	(0.000)	(0.000)		(0.000)	(0.000)
OLLExp-Fr	0.1351		3.7216	0.9296	21.319
	(0.011)		(0.0034)	(0.0033)	(0.0034)
OLLE-IR	0.4946		0.067	1.74262	
	(0.04135)		(0.7195)	(9.3007)	
OLL-IR	0.49459			0.45242	
	0.04135			0.03869	
Fr				1.3968	4.3724
				(0.0336)	(0.3278)
Kum-Fr		0.8489	1.6239	1.6341	3.4208
		(16.083)	(0.6979)	(9.049)	(0.7635)
Exp-Fr		0.9395		1.4169	0.9395
		(3.543)		(2.568)	(0.3278)
Beta-Fr		0.7346	1.5830	1.6684	3.5112
		(1.5290)	(0.7132)	(0.7662)	(0.9683)
T-Fr	-0.7166			1.2656	4.7121
	(0.2616)			(0.0579)	(0.3657)
MO-Fr		0.0033		6.2296	1.2419
		(0.0009)		(1.0134)	(0.1181)
Mc-Fr	0.8503	44.423	19.859	0.0203	46.974
	(0.1353)	(25.100)	(6.706)	(0.0060)	(21.871)



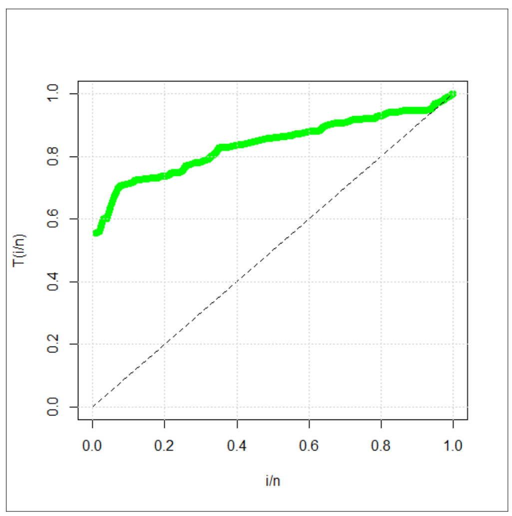


Figure 5: Box plot (left panel) and TTT plot (right panel) for data set **I**.

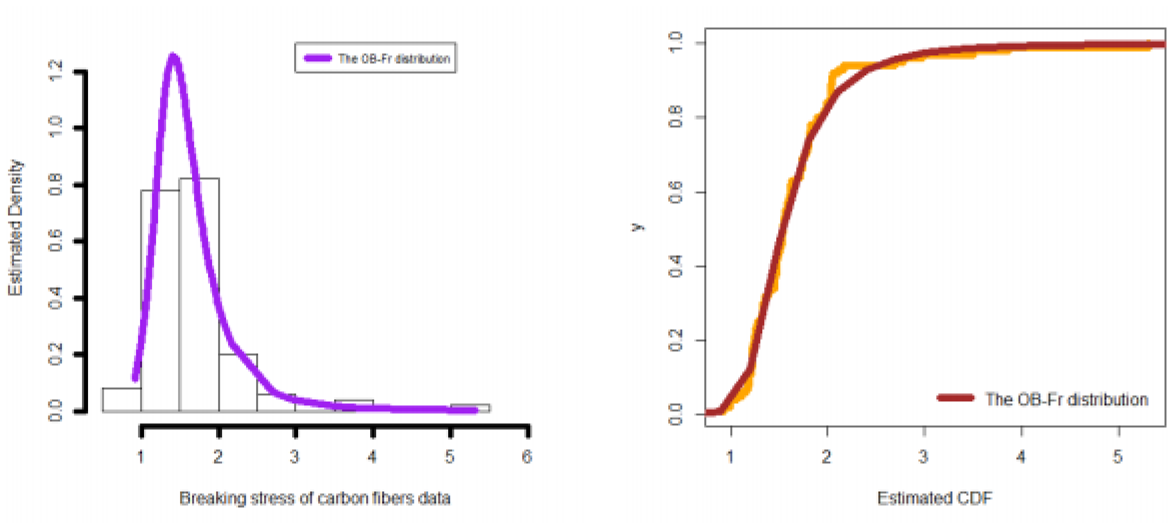


Figure 6: Estimated density, estimated CDF for data set I.

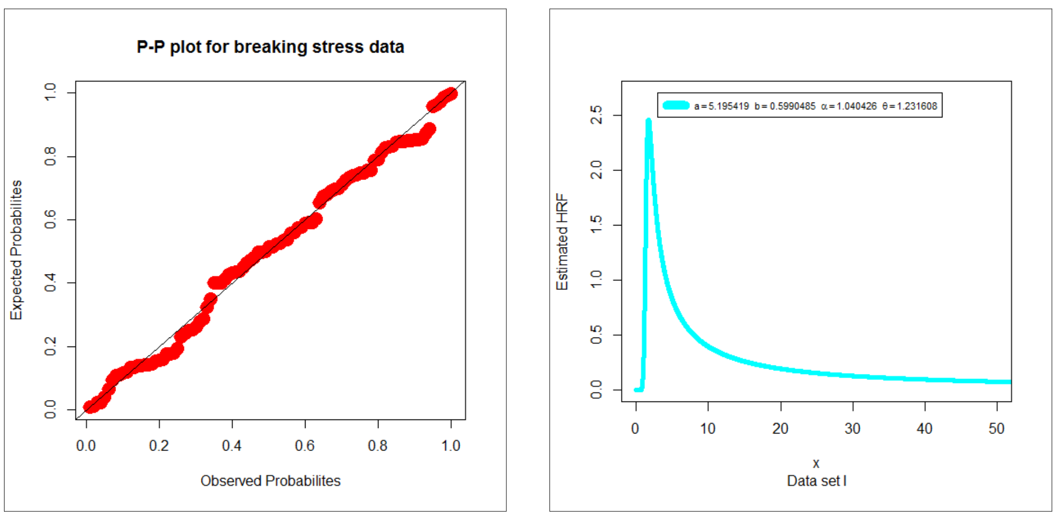


Figure 7: P-P plot and estimated HRF for data set **I**.

5.2 Glass fibers data

The 2 nd data set is generated data to simulate the strengths of glass fibers which was given by Smith and Naylor (1987). Figure 8 gives in its left panel the TTT for data set **II** along with its corresponding box plot for discovering the EVs. It indicates that the empirical HRF of data sets **II** is "increasing HRF" and we have four EVs. The statistics CVM, A-D , K-S and **P** v for all fitted models are presented in Table 6. The MLEs and corresponding SEs are given in Table 7. From Table 6, the OB-Fr model gives the lowest values the CVM=**0.05447**, A-D=**0.3858**, K-S=**0.0797** and **P**v =**0.88270** as compared to further Fr models. Therefore, the OB-Fr can be chosen as the best model. Figure 9 gives the E-PDF versus the E-CDF. Figure 10 gives the P-P plot and E-HRF for data set **II**. based on Figures 9 and 10, we note that the new OB-Fr model provides adequate fits to the empirical functions.

Table 6: CVM, A-D, K-S and P_{v} for data set II.

Model	Goodness of fit criteria			
	CVM	A-D	K-S	**P**v
OB-Fr	0.05447	0.3858	0.0797	0.88270
Fr	0.0707	0.5332	0.0772	0.8185
Exp-Fr	0.0707	0.5332	0.0772	0.8187

OLLE-Fr	0.10487	0.8325	0.55196	6.7×10^{-16}
OLLE-IR	0.1502	1.14697	0.67949	6.7×10^{-16}
OLL-IR	0.15021	1.14697	0.67951	6.7×10^{-16}
Kum-Fr	0.0634	0.4981	0.0715	0.8810
MO-Fr	0.0629	0.4902	0.0813	0.7685
Beta-Fr	0.0640	0.5008	0.0716	0.8804
T-Fr	0.0655	0.4939	0.0735	0.8470
Mc-Fr	0.1161	0.9193	0.0831	0.7455

Table 7: MLEs and their standard errors (in parentheses) for data set II.

Model	Estimates				
	a	b	c	α	β
OB-Fr	6.4757	0.4976		1.0467	1.338
	(0.000)	(0.000)		(0.000)	(0.000)
OLLExp-Fr	0.1449		0.00879	1.2997	24.878
	(0.0129)		(0.000)	(0.000)	(0.000)
OLLE-IR	0.5025		0.0716	1.7048	
	(0.0529)		(1.13062)	(13.47)	
OLL-IR	0.50251			0.45599	
	(0.0529)			(0.0486)	
Fr				1.4108	5.4377
				(0.0344)	(0.5192)
Kum-Fr		0.2855	1.2824	1.9142	4.7731
		(9.1338)	(0.6388)	(12.836)	(1.3134)
Exp-Fr		0.9059		1.4367	5.4379
		(2.764)		(4.324)	(0.5193)
B-Fr		1.2996	1.2649	1.3945	4.7927
		(4.4378)	(0.6640)	(0.9304)	(1.4641)
T-Fr	0.7778			1.5491	4.3139
	(0.2477)			(0.0655)	(0.5849)
MO-Fr		0.0023		5.2383	1.4537
		(0.0004)		(0.8209)	(0.1650)
Mc-Fr		56.227	14.953	0.0073	29.104
		(30.539)	(4.733)	(0.0013)	(11.304)

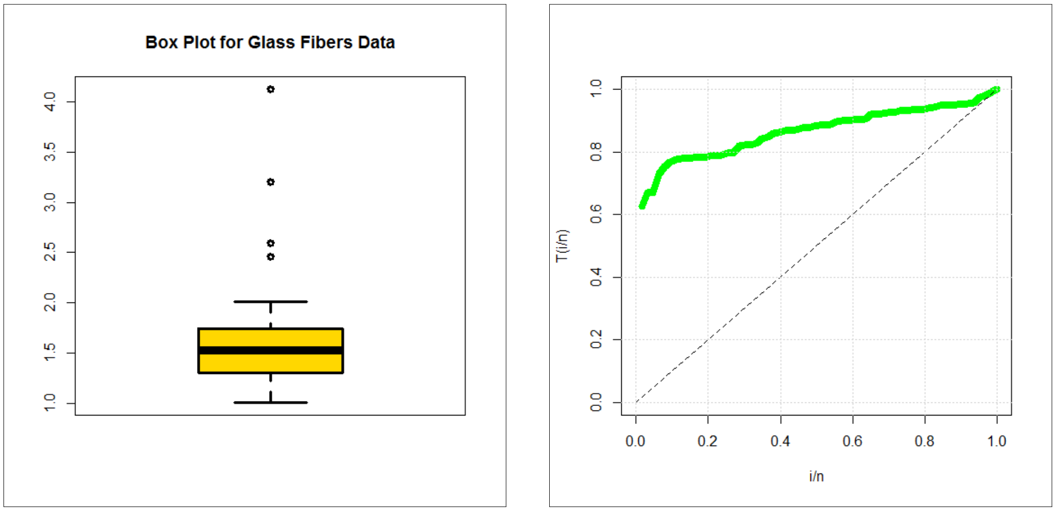


Figure 8: Box plot (left panel) and TTT plot (right panel) for data set **II**.

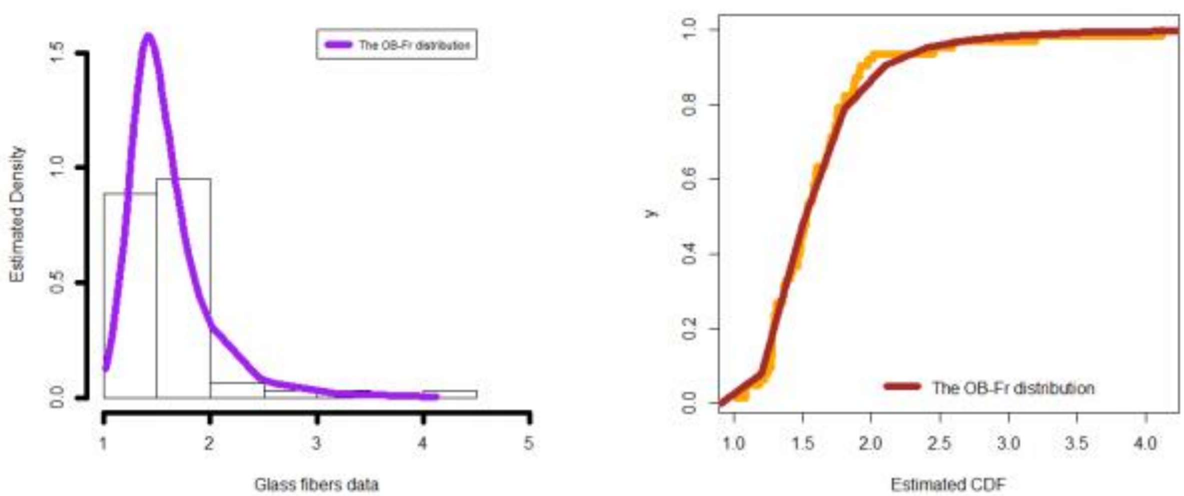


Figure 9: Estimated density, estimated CDF, P-P plot and estimated HRF for data set **II.**

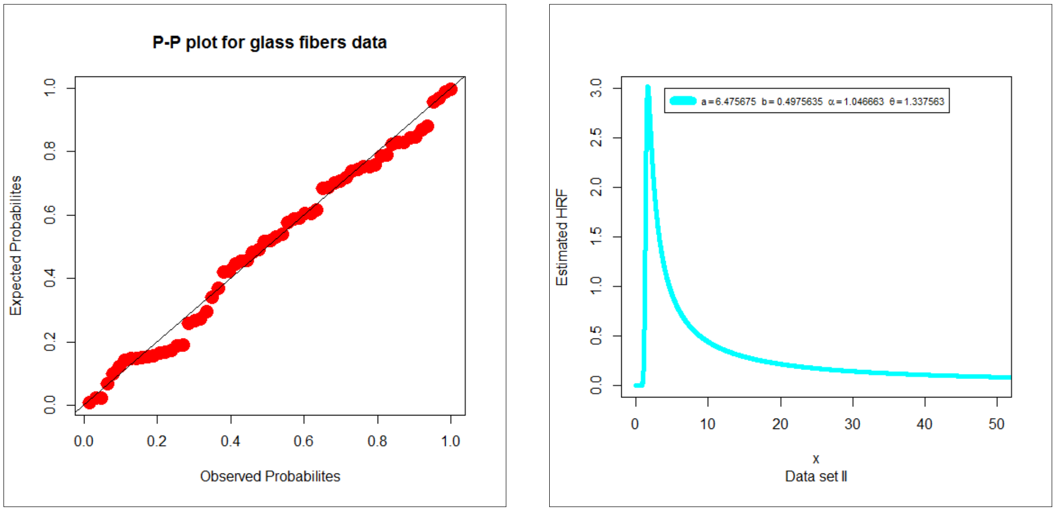


Figure 10: P-P plot and estimated HRF for data set **II**.

6.3 Relief time data

The 3 rd data set (wingo data) represents a complete sample from a clinical trial describe a relief time (in hours) for 50 arthritic patients. Figure 11 gives in its left panel the TTT for data set **III** along with its corresponding box plot for discovering the EVs. Based on Figure 11, the empirical HRF of data sets **III** is "increasing HRF" and we have no EVs. The statistics CVM, A-D , K-S and **P** v for all fitted models are presented in Table 6. The MLEs and corresponding SEs are given in Table 7. From Table 6, the OB-Fr model gives the lowest values the CVM=**0.04903**, A-D=**0.42081**, K-S=**0.09124** and **P**v =**0.7994**. Therefore, the OB-Fr may be chosen as the best model. Figure 12 gives the E-PDF versus the E-CDF. Figure 13 gives the P-P plot and E-HRF for data set **III**. based on Figures 12 and 13, we note that the new OB-Fr model provides adequate fits to the empirical functions.

Table 8: CVM, A-D, K-S and P_{v} for data set III.

Model	Goodness of fit criteria			
	CVM	A-D	K-S	**P**v
OB-Fr	0.0490	0.42081	0.09124	0.7994
GOLLIR	0.1955	1.3498	0.11008	0.5797
OLLExp-Fr	0.1577	1.09876	0.53498	7.4×10^{-13}
Fr	0.3233	2.0301	0.1506	0.2066
Exp-Fr	0.3233	2.0301	0.1506	0.2064
Beta-Fr	0.3611	2.5131	0.1433	0.3601
T-Fr	0.2823	1.8152	0.1370	0.3045

Table 7: MLEs and their standard errors (in parentheses) for data set II.

Model	Estimates				
	a	b	c	α	β
OB-Fr	17.791	6.9955		0.12686	0.17843
	(0.0001)	(4.0354)		(0.000)	(0.000)
GOLLIR	1.961	0.111		1.4123	
	(0.234)	(0.000)		(0.000)	
OLLExp-Fr	0.0669		0.00459	0.3558	32.561
	(0.0076)		(0.0028)	(0.0047)	(0.006)
Fr				0.4859	3.2078
				(0.0227)	(0.3263)

Exp-Fr			0.9047	0.5013	3.2077
			(18.784)	(3.2444)	(0.3263)
Beta-Fr		4.015	1.3349	2.0022	0.87017
		(0.111)	(0.147)	(0.321)	(0.0033)
T-Fr	-0.5816			0.4400	3.4974
	(0.2787)			(0.0290)	(0.3527)

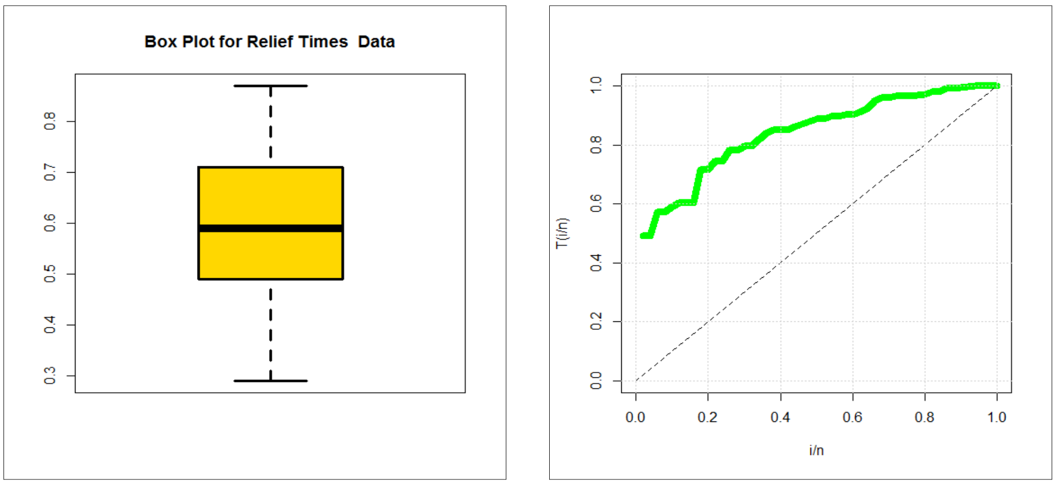


Figure 11: Box plot (left panel) and TTT plot (right panel) for data set **III**.

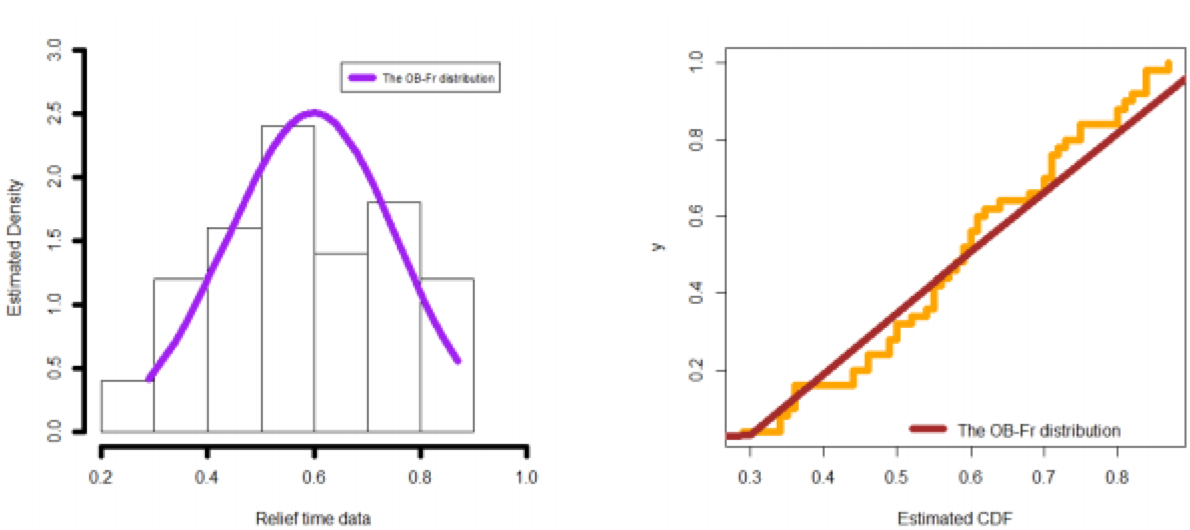


Figure 12: Estimated density, estimated CDF, P-P plot and estimated HRF for data set **III.**

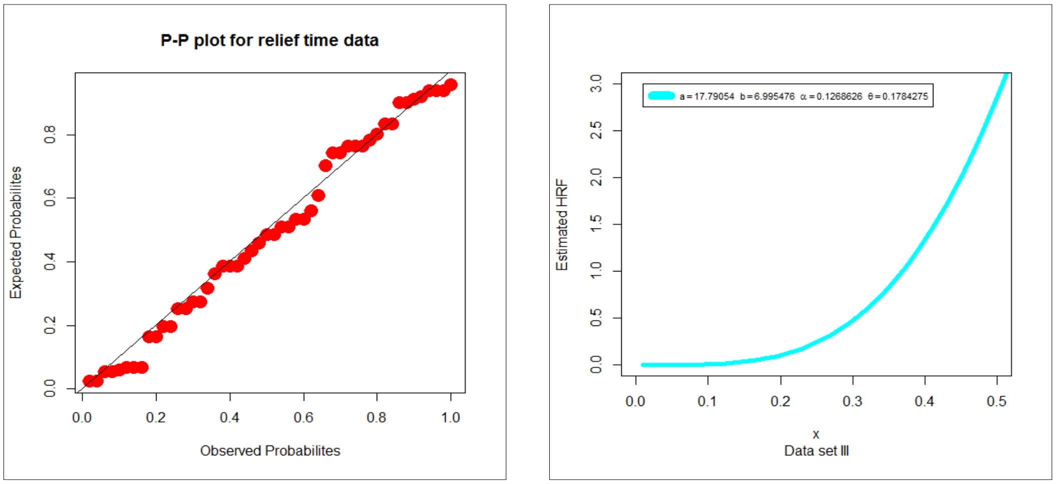


Figure 13: P-P plot and estimated HRF for data set **III**.

7. Concluding remarks

In this article and based on Alizadeh et al. (2016), we expanded the extreme value theory with proposing and studying a new version of the Fréchet model called the generalized Odd-Burr generalized Fréchet model. A straightforward types copula based on Farlie Gumbel Morgenstern family, modified Farlie Gumbel Morgenstern family, Clayton Copula and Renyi's entropy are employed for proposing many bivariate and multivariate type extensions. Some of its mathematical properties such as ordinary moments, incomplete moments, moment generating function, cumulant generating function, residual life and reversed residual life functions are derived. A numerical analysis for some measures including the Mean, variance, skewness, kurtosis and Index of dispersion for the new model are presented with details. The maximum likelihood estimation method is employed to estimate the model parameters. Graphically and using the biases and mean squared errors, we performed the simulation experiments to assess of the finite sample behavior of the maximum likelihood estimations. Three applications are presented for measuring the flexibility and the importance of the new model are presented and also used for comparing the competitive distributions under the uncensored scheme.

As a future potential work, we can modify many new useful goodness-of-fit tests for the right censored distributional validation such as the Nikulin-Rao-Robson goodness-of-fit test statistic, modified Nikulin-Rao-Robson goodness-of-fit test statistic, Bagdonavicius-Nikulin goodness-of-fit test statistic and also modified Bagdonavicius-Nikulin goodness-of-fit test statistic to the new BuXENH model as performed by Ibrahim et al. (2019), Goual et al. (2019, 2020), Mansour et al. (2020a,b,c,d), Yadav et al. (2020) and Goual and Yousof (2020), among others. However, many types of copulas can be used and applied for deriving many new bivariate models based on the new distribution (see Elgohari and Yousof (2020a), Mansour et al. (2020e, f) and El-Morshedy et al. (2021) for more details).

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