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Characterizations of the New Poisson-Weighted Exponential and the Exponentiated Weibull-Geometric Discrete Distributions

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Abstract

Certain characterizations of the new Poisson-weighted exponentiated and the exponentiated Weibull-geometric discrete distributions introduced by Altun(2019) and by Famoye(2019), respectively, are presented here with the intention of completing, in some way, their works.

Key Words: Discrete distributions; Hazard function; Reverse hazard function; Conditional expectation; Characterizations.

1. Introdution

Characterizations of distributions is an important part of distribution theory which has attracted the attention of a good number of researchers in applied sciences, where an investigator is interested to know if their model follows the right distribution. Therefore the investigator relies on conditions under which their model would follow specifically the chosen distribution.
Altun(2019) introduced a new discrete probability model called New Poisson-Weighted Exponential (NPWE) distribution and argued that such a distribution is needed in the case of over-dispersed data sets. Famoye(2019) proposed a discrete distribution called Exponentiated Weibull-Geometric (EWG) which, in part, deals with the under-dispersed or over-dispersed count data. In this short note, we present certain characterizations of NPWE and EWG distributions based on: (i) the conditional expectation of certain function of the random variable for NPWE and EWG and (ii) in terms of the reversed hazard function for EWG. The main goal here is to complete, in some way, the works of Altun and Famoye.
The cumulative distribution function (cdf), $F(x)$, thecorresponding probability mass function (pmf), $f(x)$, the hazard function, $h_F(x)$ and the reversed hazard function $r_F(x)$ of the NPWE distribution are given, respectively, by

$$F(x;\alpha,\theta) = 1 - (1+\alpha+\alpha\theta)^{-x-1}, \quad x = 0,1,\ldots, \tag{1}$$

$$f(x;\alpha,\theta) = \alpha(1+\theta)(1+\alpha+\alpha\theta)^{-x-1}, \quad x = 0,1,\ldots, \tag{2}$$

$$h_F(x) = \alpha(1+\theta), \quad x = 0,1,\ldots, \tag{3}$$

$$r_F(x) = \frac{\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-x-1}}{1-(1+\alpha+\alpha\theta)^{-x-1}}, \quad x = 0,1,\ldots, \tag{4}$$

where α, θ are positive parameters.

The cdf $G(x)$, pmf, $g(x)$, hazard function,$h_G(x)$ and reverse hazard function $r_G(x)$ of the EWG distribution are given, respectively, by

$$G(x;a,c,\theta)=\left(1-\theta^{(x+1)^c}\right)^a,\quad x=0,1,\ldots, \tag{5}$$

$$g(x;a,c,\theta)=\left(1-\theta^{(x+1)^c}\right)^a-\left(1-\theta^{x^c}\right)^a,\quad x=0,1,\ldots, \tag{6}$$

$$h_G(x)=\frac{\left(1-\theta^{(x+1)^c}\right)^a-\left(1-\theta^{x^c}\right)^a}{1-\left(1-\theta^{(x+1)^c}\right)^a},\quad x=0,1,\ldots, \tag{7}$$

$$r_G(x)=1-\left(\frac{1-\theta^{x^c}}{1-\theta^{(x+1)^c}}\right)^a,\quad x=0,1,\ldots, \tag{8}$$

where a,c,θ are positive parameters.

2. Characterization results

Proposition 2.1. Let $X:\Omega\to\mathbb{N}^*=\mathbb{N}\cup\{0\}$ be a random variable. The pmf of X is (2) if and only if

$$E\left\{\left[(1+\alpha+\alpha\theta)^{-X+1}\right]\mid X>k\right\}=\frac{\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-(k-1)}}{\left((1+\alpha+\alpha\theta)^2-1\right)},\quad k\in\mathbb{N}^*. \tag{9}$$

Proof. If X has pmf (2), then the left-hand side of (9) will be

$$\begin{aligned}&(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\left[\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-2x}\right]\right\}\\&=(1+\alpha+\alpha\theta)^{k+1}\alpha(1+\theta)\sum_{x=k+1}^{\infty}(1+\alpha+\alpha\theta)^{-2x}\\&=\frac{\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-(k-1)}}{\left((1+\alpha+\alpha\theta)^2-1\right)},\quad k\in\mathbb{N}^*.\end{aligned}$$

Conversely, if (9) holds, then

$$\begin{aligned}&\sum_{x=k+1}^{\infty}\left\{\left[(1+\alpha+\alpha\theta)^{-x+1}\right]f(x)\right\}=(1-F(k))\frac{\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-(k-1)}}{\left((1+\alpha+\alpha\theta)^2-1\right)}\\&=[1-F(k+1)+f(k+1)]\frac{\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-(k-1)}}{\left((1+\alpha+\alpha\theta)^2-1\right)}.\end{aligned} \tag{10}$$

From (10), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[(1+\alpha+\alpha\theta)^{-x+1}\right]f(x)\right\}=(1-F(k+1))\frac{\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-k}}{(1+\alpha+\alpha\theta)^2-1}. \tag{11}$$

Now, subtracting (11) from (10), we arrive at

$$(1-F(k+1))\left[\left(\frac{\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-(k-1)}}{(1+\alpha+\alpha\theta)^{2}-1}\right)-\left(\frac{\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-k}}{(1+\alpha+\alpha\theta)^{2}-1}\right)\right]$$
$$=\left[(1+\alpha+\alpha\theta)^{-k}-\left(\frac{\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-(k-1)}}{(1+\alpha+\alpha\theta)^{2}-1}\right)\right]\ f(k+1),$$

or

$$(1-F(k+1))\left[\frac{\alpha^{2}(1+\theta)^{2}(1+\alpha+\alpha\theta)^{-k}}{(1+\alpha+\alpha\theta)^{2}-1}\right]=f(k+1)\left[\frac{\alpha(1+\theta)(1+\alpha+\alpha\theta)^{-k}}{(1+\alpha+\alpha\theta)^{2}-1}\right].$$

From the last equality, we have

$$h_F(k+1)=\frac{f(k+1)}{1-F(k+1)}=\alpha(1+\theta),$$

which, in view of (3), implies that X has pmf (2).

Proposition 2.2 Let $X:\Omega\to\mathbb{N}^*=\mathbb{N}\cup\{0\}$ be a random variable. The pmf of X is (6) if and only if

$$E\left\{\left[\left(1-\theta^{(X+1)^c}\right)^{a}+\left(1-\theta^{X^c}\right)^{a}\right]\mid X\ \leq k\right\}=\left(1-\theta^{(k+1)^c}\right)^{a},\quad k\in\mathbb{N}^* \tag{12}$$

Proof. If X has pmf (6), then the left-hand side of (12) will be

$$(G(k))^{-1}\sum_{x=0}^{k}\left\{\left[\left(1-\theta^{(x+1)^c}\right)^{2a}-\left(1-\theta^{x^c}\right)^{2a}\right]\right\}$$
$$=\left(1-\theta^{(k+1)^c}\right)^{-a}\sum_{x=0}^{k}\left\{\left[\left(1-\theta^{(x+1)^c}\right)^{2a}-\left(1-\theta^{x^c}\right)^{2a}\right]\right\}$$
$$=\left(1-\theta^{(k+1)^c}\right)^{-a}\left(1-\theta^{(k+1)^c}\right)^{2a}=\left(1-\theta^{(k+1)^c}\right)^{a},\quad k\in\mathbb{N}^*.$$

Conversely, if (12) holds, then

$$\sum_{x=0}^{k}\left\{\left[\left(1-\theta^{(x+1)^c}\right)^{a}+\left(1-\theta^{x^c}\right)^{a}\right]\ f(x)\right\}=(G(k))\left(1-\theta^{(k+1)^c}\right)^{a}$$
$$=[G(k+1)-g(k+1)]\left(1-\theta^{(k+1)^c}\right)^{a}. \tag{13}$$

From (13), we also have

$$\sum_{x=0}^{k+1}\left\{\left[\left(1-\theta^{(x+1)^c}\right)^{a}+\left(1-\theta^{x^c}\right)^{a}\right]\ f(x)\right\}=(G(k+1))\left(1-\theta^{(k+2)^c}\right)^{a}. \tag{14}$$

Now, subtracting (13) from (14), we arrive at

$$(G(k+1))\left[\left(1-\theta^{(k+2)^c}\right)^{a}-\left(1-\theta^{(k+1)^c}\right)^{a}\right]+\left(1-\theta^{(k+1)^c}\right)^{a}g(k+1)$$
$$=\left[\left(1-\theta^{(k+2)^c}\right)^{a}+\left(1-\theta^{(k+1)^c}\right)^{a}\right]\ f(k+1).$$

or

$$(G(k+1))\left[\left(1-\theta^{(k+2)^c}\right)^{a}-\left(1-\theta^{(k+1)^c}\right)^{a}\right]=g(k+1)\left(1-\theta^{(k+2)^c}\right)^{a}.$$

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From the last equality, we have

$$r_G\,(k+1) = \frac{g\,(k+1)}{G\,(k+1)} = 1 - \left(\frac{\left(1-\theta^{(k+1)^c}\right)}{\left(1-\theta^{(k+2)^c}\right)^a}\right),$$

which, in view of (8), implies that X has pmf (6).

Proposition 2.3. Let $X : \Omega \to \mathbb{N}^* = \mathbb{N} \cup \{0\}$ be a random variable. The pmf of X is (6) if and only if its reverse hazard function, r_G , satisfies the following difference equation

$$r_G\,(k+1) - r_G\,(k) = \left(\frac{1-\theta^{k^c}}{1-\theta^{(k+1)^c}}\right)^a - \left(\frac{1-\theta^{(k+1)^c}}{1-\theta^{(k+2)^c}}\right)^a, \quad k \in \mathbb{N}^*, \tag{15}$$

with the initial condition $r_G\,(0) = 1$.

Proof. Clearly, if X has pmf (6), then (15) holds. Now, if (15) holds, then

$$\sum_{x=0}^{k} \{r_G\,(x+1) - r_G\,(x)\} = \sum_{x=0}^{k} \left\{\left(\frac{1-\theta^{x^c}}{1-\theta^{(x+1)^c}}\right)^a - \left(\frac{1-\theta^{(x+1)^c}}{1-\theta^{(x+2)^c}}\right)^a\right\},$$

or

$$r_G\,(k+1) - r_G\,(0) = -\left(\frac{1-\theta^{(k+1)^c}}{1-\theta^{(k+2)^c}}\right)^a,$$

or, in view of the initial condition $r_F\,(0) = 1$, we have

$$r_G\,(k+1) = 1 - \left(\frac{1-\theta^{(k+1)^c}}{1-\theta^{(k+2)^c}}\right)^a, \quad k \in \mathbb{N}^*,$$

which is the reverse hazard function corresponding to pmf (6).

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