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On The Burr XII-Gamma Distribution: Development, Properties, Characterizations and Applications

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Abstract
We introduce a four-parameter lifetime model with flexible hazard rate called the Burr XII gamma (BXIIG) distribution. We derive the BXIIG distribution from (i) the T-X family technique and (ii) nexus between the exponential and gamma variables. The failure rate function for the BXIIG distribution is flexible as it can accommodate various shapes such as increasing, decreasing, decreasing-increasing, increasing-decreasing-increasing, bathtub and modified bathtub. Its density function can take shapes such as exponential, J, reverse-J, left-skewed, right-skewed and symmetrical. To illustrate the importance of the BXIIG distribution, we establish various mathematical properties such as random number generator, ordinary moments, generating function, conditional moments, density functions of record values, reliability measures and characterizations. We address the maximum likelihood estimation for the parameters. We estimate the adequacy of the estimators via a simulation study. We consider applications to two real data sets to prove empirically the potentiality of the proposed model.

Key Words: Characterizations; Gamma distribution; Maximum Likelihood Estimation; Reliability.

Mathematical Subject Classification: 60E05, 62E10, 62E20

1. Introduction

In the recent decade, many continuous distributions have been introduced in statistical literature. Some of these distributions, however, are not flexible enough for data sets from survival analysis, life testing, reliability, finance, environmental sciences, biometry, hydrology, ecology and geology. Hence, the applications of the generalized models to these fields are clear requisite. Generalization of a distribution is the only way to increase the applicability of a parent distribution. The generalized distributions are derived either by inserting one or more shape parameters or by transform of the parent distribution. So, the generalized distributions will be more suitable than the competing model and sub-models.

The Burr-XII (BXII) distribution among Burr family (Burr, 1943) is widely applied to model insurance data and failure time data. Many generalizations of the BXII distributions are available in literature such as Burr XII modified Weibull (Mdlongwa et al., 2017), Burr XII Uniform (Nasir et al., 2018), Burr XII inverse Rayleigh (Goual and Yousof, 2020), Burr XII-moment exponential (Bhatti et al., 2021) and Burr XII-power Cauchy distribution (Bhatti et al., 2021).

We incorporate gamma distribution into a larger family through an application of the Burr XII cdf. In fact, based on the T-X transform defined by (Alzaatreh et al., 2016), we construct the BXIIG distribution. We aim to propose

BXIIG distribution to obtain flexible density function with various shapes. The BXIIG distribution is important due to (i) it can provide flexible hazard rate and (ii) it can provide better fits than other well-known distributions.

This study is based on the following motivations: (i) to derive the BXIIG model; (ii) to generate distributions with symmetrical, left-skewed, right-skewed, J and reverse-J shaped as well as high kurtosis; (iii) to have monotone and non-monotone failure rate function; (iv) to study numerically descriptive measures for the BXIIG distribution based on parameter values; (v) to derive mathematical properties such as ordinary moments, generating function, conditional moments, density functions of record values, reliability measures and characterizations; (vi) to estimate the precision of the maximum likelihood estimators by means of Monte Carlo simulations; (vii) to reveal the potentiality and utility of the BXIIG model; (viii) to work as the preeminent substitute model to other existing models to explain real data in finance, survival analysis, manufacturing, reliability, life testing and new zones of research; (ix) to deliver better fits than other models and (x) to infer empirically from the goodness of fit statistics (GOFs) and graphical tools.

The contents of the article are structured as follows. Section 2 derives the BXIIG model from (i) the T-X family technique and (ii) linking the exponential and gamma variables. We study basic structural properties, random number generator and sub-models for the BXIIG model. Section 3 presents certain mathematical properties such as ordinary moments, generating function, conditional moments, density functions of record values and reliability measures. Section 4 characterizes the BXIIG distribution. In Section 5, we address the maximum likelihood estimation of the parameters. In Section 6, we evaluate the accuracy of the maximum likelihood estimators (MLEs) via a simulation study. In Section 7, we consider two applications to elucidate the potentiality and utility of the BXIIG model. We test the competency of the new distribution using the goodness of fit statistics. In Section 8, we offer some conclusions.

2. The BXIIG Distribution

We derive the BXIIG distribution from the T-X family technique. We also obtain this model by linking the exponential and gamma variables. We discuss basic structural properties. We highlight the shapes of the density and failure rate functions.

2.1 T-X Family Technique

The cumulative distribution function (cdf) and probability density function (pdf) of the gamma distribution are given, respectively, by

$$G(x)=\gamma_1\left(\kappa,\frac{x}{\theta}\right)=\frac{\gamma\left(\kappa,\frac{x}{\theta}\right)}{\Gamma(\kappa)},\ \kappa,\theta>0, x\geq 0,$$

where $\gamma_1(.,.)$ is the incomplete gamma function ratio and

$$g(x)=\frac{x^{\kappa-1}\,\mathrm{e}^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}.$$

The cumulative hazard rate of the gamma distribution has the form

$$W\left[G(x)\right]=-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right].$$

The cdf of the T-*X* family (Alzaatreh et al., 2016) of distributions has the form

$$F(x)=\int_a^{W\left[G(x;\xi)\right]} r(t)\,dt,\qquad x\in\mathbb{R}, \tag{1}$$

where $r(t)$ is the pdf of the random variable (rv) *T*, where $T \in [a, b]$ for $-\infty \le a < b < \infty$ and $W[G(x;\xi)]$ is a function of the baseline cdf of a rv *X* with the vector parameter ξ, which satisfies the conditions:

i) $W[G(x;\xi)] \in [a, b]$,

ii) $W[G(x;\xi)]$ is differentiable and monotonically non-decreasing and

iii) $\lim_{x\to-\infty} W[G(x;\xi)] \to a$ and $\lim_{x\to\infty} W[G(x;\xi)] \to b$.

The pdf of the T-*X* family can be expressed as

$$f(x) = \left\{\frac{\partial}{\partial x} W[G(x;\xi)]\right\} r\left\{W[G(x;\xi)]\right\}, \qquad x \in \mathbb{R}. \tag{2}$$

We derive the cdf of the BXIIG distribution from the T-X family technique by setting

$$r(t) = \alpha\beta t^{\beta-1}\left(1+t^{\beta}\right)^{-\alpha-1},\ t>0, \alpha>0, \beta>0,$$

and

$$W[G(x)] = -\log\left[1-\gamma_1\left(\kappa, \frac{x}{\theta}\right)\right].$$

The cdf of the BXIIG distribution takes the form

$$F(x) = 1-\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa, \frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{-\alpha}, x \ge 0, \tag{3}$$

where $\alpha, \beta, \theta, \kappa > 0$ are the parameters.

The BXIIG density can be expressed as

$$f(x) = \alpha\beta \frac{x^{\kappa-1} e^{-\frac{x}{\theta}}}{\theta^{\kappa}\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa, \frac{x}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa, \frac{x}{\theta}\right)\right]\right\}^{\beta-1}\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa, \frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{-\alpha-1}, x>0. \tag{4}$$

Hereafter, a rv with pdf (4) is denoted by $X \sim \text{BXIIG}(\alpha, \beta, \theta, \kappa)$. For $\alpha = 1$, the BXIIG distribution reduces to the Log-Log-Gamma (LLG) distribution and, for $\beta = 1$, the BXIIG distribution becomes the Lomax-Gamma (LG) distribution.

2.2 Nexus between the Exponential and Gamma Variable

We derive the BXIIG distribution by linking the exponential and gamma rvs, i.e., $W_1 \sim \exp(1)$ and $W_2 \sim gamma(\alpha, 1)$.

Lemma. (i): If $W_1 \sim \exp(1)$ and $W_2 \sim gamma(\alpha, 1)$, then for $W_1 = \left\{-\log\left[1-\gamma_1\left(\kappa, \frac{X}{\theta}\right)\right]\right\}^{\beta} W_2$, we have that X has density (4).

Proof

If $W_1 \sim \exp(1)$, i.e. $f(w_1) = \mathrm{e}^{-w_1}, w_1 > 0$.

If $W_2 \sim gamma(\alpha,1)$, i.e. $f(w_2) = \frac{w_2^{\alpha-1}\mathrm{e}^{-w_2}}{\Gamma(\alpha)},\ w_2 > 0$.

Then, the joint distribution of the two rvs is $f(w_1, w_2) = \frac{w_2^{\alpha-1}\mathrm{e}^{-w_2}\mathrm{e}^{-w_1}}{\Gamma(\alpha)},\ w_1 > 0,\ w_2 > 0$.

Let $W_1 = \left\{-\log\left[1-\gamma_1\left(\kappa, \frac{X}{\theta}\right)\right]\right\}^{\beta} W_2$.

The joint density of the rvs X and W_2 has the form

$$f(x, w_2) = \frac{w_2^{\alpha-1}\mathrm{e}^{-w_2}\mathrm{e}^{-\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta} w_2}}{\Gamma(\alpha)} \beta \frac{x^{\kappa-1}\mathrm{e}^{-\frac{x}{\theta}}}{\theta^{\kappa}\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1} w_2,\ w_1 > 0,\ w_2 > 0.$$

The marginal density of X takes the form

$$f(x) = \int_0^{\infty} \left\{ \frac{w_2^{\alpha}\,\mathrm{e}^{-\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right) w_2}}{\Gamma(\alpha)} \beta \frac{x^{\kappa-1}\mathrm{e}^{-\frac{x}{\theta}}}{\theta^{\kappa}\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1} \right\} dw_2.$$

After simplification, we obtain (for x>0)

$$f(x) = \alpha\beta \frac{x^{\kappa-1}\mathrm{e}^{-\frac{x}{\theta}}}{\theta^{\kappa}\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1}\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{-\alpha-1},$$

which is the BXIIG density.

2.3 Structural Properties

If $X \sim \mathrm{BXIIG}(\alpha, \beta, \theta, \kappa)$, the survival, failure rate, cumulative failure rate, reverse failure rate, elasticity functions and the Mills ratio of X are given, respectively, by (for x>0)

$$S(x) = \left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{-\alpha},$$

$$h(x) = \frac{\alpha\beta \frac{x^{\kappa-1} e^{-\frac{x}{\theta}}}{\theta^{\kappa}\,\Gamma(\kappa)}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1}}{\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)},$$

$$H(x) = \alpha \ln\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right),$$

$$\mathrm{r}(x) = \frac{d}{dx}\ln\left[1-\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{-\alpha}\right],$$

$$m(x)=\left\{-\frac{d}{dx}\ln\left[\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{-\alpha}\right]\right\}^{-1}$$

and

$$e(x)=\frac{d\ln\left(1-\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{-\alpha}\right)}{d\ln x}.$$

The quantile function of X (for $0<q<1$) follows from

$$\gamma_1\left(\kappa,\frac{x_q}{\theta}\right)=\left(1-\exp\left\{-\left[(1-q)^{-\frac{1}{\alpha}}-1\right]^{\frac{1}{\beta}}\right\}\right),$$

and its random number generator with $Z\sim$ Uniform (0,1) is the solution of the nonlinear equation

$$\gamma_1\left(\kappa,\frac{X}{\theta}\right)=\Gamma(\kappa)\left(1-\exp\left\{-\left[(1-Z)^{-\frac{1}{\alpha}}-1\right]^{\frac{1}{\beta}}\right\}\right).$$

2.4 Shapes of the BXIIG Density and Hazard Rate Functions

We plot the density and failure rate functions of the BXIIG distribution for selected parameter values. The BXIIG density can display numerous shapes such as symmetrical, right-skewed, left-skewed, J, reverse-J and exponential (**Figure 1**). The failure rate function can highlight shapes as modified bathtub, inverted bathtub, decreasing, increasing, increasing-decreasing and decreasing-increasing-decreasing (**Figure 2**). Therefore, the BXIIG distribution is quite flexible and can be applied to numerous data sets.

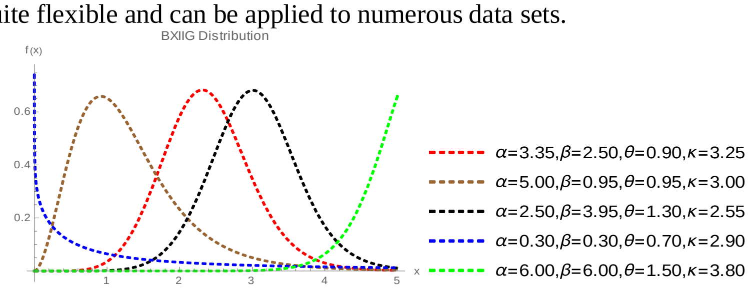

Figure 1: Plots of the BXIIG density

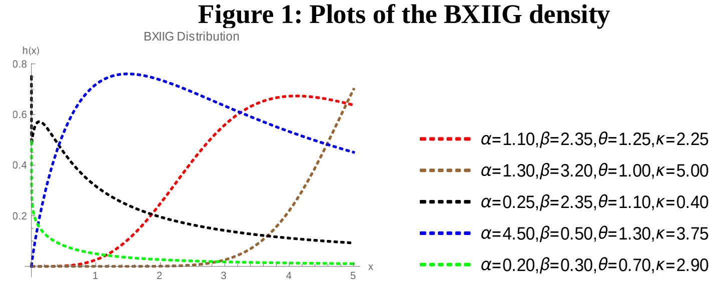

Figure 2: Plots of the BXIIG hazard rate

3. Mathematical Properties

Here, we present certain mathematical and statistical properties such as ordinary moments, generating function, conditional moments, density functions of record values and reliability measure.

3.1 Useful Expansion

In this sub-section, we express the cdf and pdf of the BXIIG distribution in terms of infinite linear combinations of the associated exponentiated-gamma (Exp-G) distributions. Using the following power series that converges everywhere,

$$(1+z)^{-\alpha}=\sum_{n=0}^{\infty}\binom{-\alpha}{n} z^{n},$$

the cdf in Equation (3) can be written as

$$F(x)=-\sum_{n=1}^{\infty}\binom{-\alpha}{n}\underbrace{\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{n\beta}}_{A(x)}. \quad (5)$$

For any real parameter β and $z\in(0,1)$,

$$\left[-\log(1-z)\right]^{\beta}=z^{\beta}+\sum_{\ell=0}^{\infty}a_{\ell}(\beta)\, z^{\ell+\beta+1}, \quad (6)$$

where $a_0(\beta)=\beta/2$, $a_1(\beta)=\beta(3\beta+5)/24$, $a_3(\beta)=\beta\left(\beta^2+5\beta+6\right)/48$, etc. are Stirling polynomials (Flajonet and Odlzko,1990; Flajonet and Sedgewick,2009;Cordeiro et al.,2018).

Applying (6) to the expression $A(x)$ in Equation (5), we obtain

$$\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{n\beta}=\left[\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{n\beta}+\sum_{\ell=0}^{\infty}a_{\ell}(\beta)\left[\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{\ell+n\beta+1}. \quad (7)$$

Applying (7) in Equation (5), the cdf of the BXIIG distribution becomes

$$F(x)=\left\{-\sum_{n=1}^{\infty}\binom{-\alpha}{n}\left[\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{n\beta}-\sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}\binom{-\alpha}{n}a_{\ell}(n\beta)\left[\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{n\beta+\ell+1}\right\}$$

and then

$$F(x)=\sum_{n=1}^{\infty}\nu_n\left[\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{n\beta}+\sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}w_{n,\ell}\left[\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{n\beta+\ell+1}, \quad (8)$$

where $\nu_n=\nu_n(\alpha)=-\binom{-\alpha}{n}$ and $w_{n,\ell}=w_{n,\ell}(\alpha,\beta)=-\binom{-\alpha}{n}a_{\ell}(n\beta)$.

Let $H_{a,\kappa,\theta}(x)=\left[\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{a}$ be the cdf of the Exp-G distribution with power parameter $a>0$. Then,

$$F(x)=\sum_{n=1}^{\infty}\nu_n\, H_{n\beta,\kappa,\theta}(x)+\sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}w_{n,\ell}\, H_{n\beta+\ell+1,\kappa,\theta}(x). \quad (9)$$

Let $h_{a,\kappa,\theta}(x)=\frac{d}{dx}H_{a,\kappa,\theta}(x)$ be the Exp-G density. By differentiating (9), we obtain the linear representation

$$f(x)=\sum_{n=1}^{\infty}\nu_n\, h_{n\beta,\kappa,\theta}(x)+\sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}w_{n,\ell}\, h_{n\beta+\ell,\kappa,\theta}(x), \quad (10)$$

where $h_{n\beta,\kappa,\theta}(x)=\frac{n\beta x^{\kappa-1}e^{-\frac{x}{\theta}}}{\theta^{\kappa}\,\Gamma(\kappa)}\gamma_1\left(\kappa,\frac{x}{\theta}\right)^{n\beta-1}$ and a similar formula for $h_{n\beta+\ell,\kappa,\theta}(x)$ holds.

We have the power series

$$\gamma_1\left(\kappa,\frac{x}{\theta}\right)^{a-1}=\sum_{m=0}^{\infty}s_m\left(a-1\right)\ \gamma_1\left(\kappa,\frac{x}{\theta}\right)^{m}, \tag{11}$$

where $s_m\left(a-1\right)=\sum_{j=m}^{\infty}\left(-1\right)^{j+m}\binom{a-1}{j}\binom{j}{m}$.

Further,

$$\gamma_1\left(\kappa,\frac{x}{\theta}\right)=\frac{1}{\Gamma(\kappa)}\sum_{i=0}^{\infty}\frac{\left(-1\right)^{i}}{i!\left(\kappa+i\right)\theta^{\kappa+i}}\ x^{\kappa+i}=\sum_{i=0}^{\infty}a_i\ x^{\kappa+i},$$

where

$$a_i=a_i\left(\theta,\kappa\right)=\frac{\left(-1\right)^{i}}{\theta^{\kappa+i}\ \Gamma(\kappa)\ i!\left(\kappa+i\right)}.$$

For $m>1$, we can write

$$\gamma_1\left(\kappa,\frac{x}{\theta}\right)^{m}=\sum_{i=0}^{\infty}c_{m,i}\ x^{m\kappa+i}, \tag{12}$$

where the coefficients $c_{m,i}=c_{m,i}\left(\theta,\kappa\right)$ in terms of the a_i's (for $i=1,2,\ldots$) follow from the recurrence relation

$$c_{m,i}=\frac{1}{ia_0}\sum_{p=1}^{i}\left(mp-i+p\right)a_p\,c_{m,i-p},$$

and $c_{m,0}=a_0^{m}$. The coefficient $c_{m,i}$ comes from $c_{m,0},...,c_{m,i-1}$ and hence from $a_0,...,a_i$.

We can write from Equations (11) and (12)

$$\gamma_1\left(\kappa,\frac{x}{\theta}\right)^{a-1}=s_0\left(a-1\right)+\sum_{m=1}^{\infty}s_m\left(a-1\right)\ \sum_{i=0}^{\infty}c_{m,i}x^{m\kappa+i}$$

and then

$$\gamma_1\left(\kappa,\frac{x}{\theta}\right)^{a-1}=s_0\left(a-1\right)+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}d_{m,i}\ x^{m\kappa+i},$$

where $d_{m,i}=d_{m,i}\left(a,\theta,\kappa\right)=s_m\left(a-1\right)c_{m,i}\left(\theta,\kappa\right)$.

Then, the Exp-G density with power parameter a can be expressed as

$$h_{a,\kappa,\theta}\left(x\right)=s_0\left(a-1\right)\frac{a\,x^{\kappa-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}d_{m,i}\ x^{m\kappa+i}\ \frac{a\,x^{\kappa-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}. \tag{13}$$

Next, we introduce the gamma density in (13). The gamma density is

$$\pi\left(x;\theta,\kappa\right)=\frac{x^{\kappa-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)},$$

and then

$$\theta^{\kappa}\ \Gamma(\kappa)\ \pi\left(x;\theta,\kappa\right)=x^{\kappa-1}\,e^{-\frac{x}{\theta}}. \tag{14}$$

We have from (14)

$$h_{a,\kappa,\theta}\left(x\right)=a\,s_0\left(a-1\right)\pi\left(x;\theta,\kappa\right)+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}a\,d_{m,i}\,\frac{\theta^{m\kappa+i}\ \Gamma\left[\left(m+1\right)\kappa+i\right]}{\Gamma(\kappa)}\pi\left(x;\theta,\left(m+1\right)\kappa+i\right).$$

Hence,

$$h_{a,\kappa,\theta}(x)=t_0\ \pi(x;\theta,\kappa)+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}t_{m,i}\ \pi\left(x;\theta,(m+1)\kappa+i\right), \tag{15}$$

where $t_0=t_0(a)=a\,s_0(a-1)$, $t_{m,i}=t_{m,i}(a,\theta,\kappa)=a\,d_{m,i}(a,\theta,\kappa)\,\theta^{m\kappa+i}\ \dfrac{\Gamma\left[(m+1)\kappa+i\right]}{\Gamma(\kappa)}$.

Equation (15) reveals that the Exp-G density is a linear combination of gamma densities.
Finally, combining Equations (10) and (15), we can write

$$\begin{aligned} f(x)=&\sum_{n=1}^{\infty}\nu_n\left[t_0(n\beta)\ \pi(x;\theta,\kappa)+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}t_{m,i}(n\beta,\theta,\kappa)\ \pi\left(x;\theta,(m+1)\kappa+i\right)\right]+\\ &\sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}w_{n,\ell}\left[t_{0,\ell}(n\beta+\ell)\ \pi(x;\theta,\kappa)+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}t_{m,i}(n\beta+\ell,\theta,\kappa)\ \pi\left(x;\theta,(m+1)\kappa+i\right)\right]. \end{aligned} \tag{16}$$

Equation (16) shows that the BXIIG density is a linear combination of gamma densities.

3.2 Moments

The moments are significant tools for statistical analysis in pragmatic sciences. The r[th] ordinary moment of X, say $\mu'_r=E\left(X^r\right)$, can be expressed from (16) as

$$\mu'_r=\int_0^{\infty}x^r\left\{\begin{aligned}&\sum_{n=1}^{\infty}\nu_n\left[n\beta\,s_0(n\beta-1)\frac{x^{\kappa-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}n\beta\ \frac{x^{(m+1)\kappa+i-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}d_{m,i}(n\beta,\theta,\kappa)\right]+\\ &\sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}w_{n,\ell}\left[(n\beta+\ell)s_0(n\beta+\ell-1)\frac{x^{\kappa-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}(n\beta+\ell)\frac{x^{(m+1)\kappa+i-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}d_{m,i}(n\beta+\ell,\theta,\kappa)\right]\end{aligned}\right\}dx.$$

Setting $x/\theta=w$, we can write

$$\mu'_r=\left\{\begin{aligned}&\sum_{n=1}^{\infty}\nu_n\left[n\beta\,s_0(n\beta-1)\frac{\theta^r\Gamma(\kappa+r)}{\Gamma(\kappa)}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}n\beta\frac{\theta^{(m\kappa+i+r)}\Gamma\left[(m+1)\kappa+i+r\right]}{\Gamma(\kappa)}d_{m,i}(n\beta,\theta,\kappa)\right]+\\ &\sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}w_{n,\ell}\left[(n\beta+\ell)s_0(n\beta+\ell-1)\frac{\theta^r\Gamma(\kappa+r)}{\Gamma(\kappa)}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}(n\beta+\ell)\frac{\theta^{(m\kappa+i+r)}\Gamma\left[(m+1)\kappa+i+r\right]}{\Gamma(\kappa)}d_{m,i}(n\beta+\ell,\theta,\kappa)\right]\end{aligned}\right\}$$

The moment generating function (mfg) of X is determined as (for $t\in\mathbb{R}$)

$$\begin{aligned}M_X(t)&=E\left(e^{tX}\right)\\ &=\int_0^{\infty}e^{tx}\left\{\begin{aligned}&\sum_{n=1}^{\infty}\nu_n\left[n\beta\,s_0(n\beta-1)\frac{x^{\kappa-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}n\beta\ \frac{x^{(m+1)\kappa+i-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}d_{m,i}(n\beta,\theta,\kappa)\right]+\\ &\sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}w_{n,\ell}\left[(n\beta+\ell)s_0(n\beta+\ell-1)\frac{x^{\kappa-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}(n\beta+\ell)\frac{x^{(m+1)\kappa+i-1}\,e^{-\frac{x}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}d_{m,i}(n\beta+\ell,\theta,\kappa)\right]\end{aligned}\right\}dx.\end{aligned}$$

Setting $w=\left(\dfrac{1-\theta t}{\theta}\right)x$, we obtain

$$M_X(t)=\left\{\begin{array}{l}\sum_{n=1}^{\infty}\nu_n\left[\frac{n\beta s_0(n\beta-1)}{(1-\theta t)^{\kappa}}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}n\beta\,\theta^{m\kappa+i}\frac{\Gamma[(m+1)\kappa+i]}{(1-\theta t)^{(m+1)\kappa+i}\,\Gamma(\kappa)}d_{m,i}(n\beta,\theta,\kappa)\right]+\\ \sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}w_{n,\ell}\left[\frac{(n\beta+\ell)s_0(n\beta+\ell-1)}{(1-\theta t)^{\kappa}}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}(n\beta+\ell)\theta^{m\kappa+i}\frac{\Gamma[(m+1)\kappa+i]}{(1-\theta t)^{(m+1)\kappa+i}\,\Gamma(\kappa)}d_{m,i}(n\beta+\ell,\theta,\kappa)\right]\end{array}\right\}.$$

The r[th] central moment (μ_r), coefficients of skewness (γ_1) and kurtosis (γ_2) of X are

$$\mu_r=\sum_{\ell=1}^{r}(-1)^{\ell}\binom{r}{\ell}\mu'_{\ell}\,\mu'_{r-\ell},\ \gamma_1=\mu_3\Big/\sqrt[3]{\mu_2}\quad \text{and}\,\beta_2=\mu_4\Big/(\mu_2)^2\ .$$

The numerical values for the mean (μ'_1), median $(\tilde{\mu})$, standard deviation (σ), skewness (γ_1) and kurtosis (γ_2) of the BXIIG distribution for selected values of $\alpha,\beta,\theta,\kappa$ are listed in Table 1.

Table 1: Quantities μ'_1, $\tilde{\mu}$, σ, γ_1 and γ_2 for the BXIIG Distribution

$\alpha,\beta,\theta,\kappa$	μ'_1	$\tilde{\mu}$	σ	γ_1	γ_2
0.5,1,0.5,0.5	2.0183	0.5326	3.2607	2.2702	7.8281
0.5,1.5,0.5,0.5	1.6355	0.4988	2.8029	2.8646	11.7754
0.5,1.5,0.5,2.5	8.2751	2.5001	14.2795	2.8949	11.9938
0.5,1.5,0.5,5	16.5036	4.9938	28.463	2.8946	11.9955
0.5,1.5,0.5,0.5	1.6355	0.4988	2.8029	2.8646	11.7754
0.5,1.05,0.5,0.5	2.0673	0.5439	3.3928	2.3566	8.3275
0.5,1.05,0.5,1.5	6.1842	1.6304	10.1452	2.3580	8.3371
0.5,1.05,1.5,1.5	8.9000	4.1042	11.5929	2.1204	7.2550
0.5,1.5,1.5,1.5	7.6086	3.9038	9.7980	2.5922	10.2654
1,1.5,1.5,1.5	3.8423	2.3550	5.1025	4.5809	32.3353
1.5, 1.5, 1.5, 1.5	2.4751	1.7894	2.6927	5.6776	63.3172
1.5, 0.5, 1.5, 1.5	4.1737	0.9011	8.5247	3.4652	16.3139
4.1,2.2,4.2,0.7	2.3071	2.2844	0.5537	0.3407	3.5272
4.5,2.2,4.2,0.7	2.2601	2.2418	0.5324	0.2847	3.3831
5,2.75,4.2,0.7	2.3570	2.3572	0.4496	0.0528	3.1698
5,2.75,5,0.7	2.8762	2.8798	0.4991	0.0091	3.1700
5,3,4.2,0.7	2.4103	2.4148	0.4261	-0.0135	3.1514
5,5,3,0.9	2.3728	2.3881	0.3192	-0.2564	3.2693
6,4.5,1.5,1.5	1.6337	1.6414	0.3484	-0.0803	3.0426
5,7,5,5	24.113	24.2574	1.8258	-0.4789	3.5713

3.3 Conditional Moments

Life expectancy, mean waiting time and inequality measures can be obtained from incomplete moments.

The r[th] conditional moment $E\left(X^r\middle|X>z\right)$ is $E\left(X^r\middle|X>z\right)=\frac{1}{S(z)}\left[E_{X>z}\left(X^r\right)\right].$

The r[th] lower incomplete moment $E_{X\le z}\left(X^r\right)$ is determined by changing variables

$$E_{X\le z}(X^r)=\left\{\begin{array}{l}\sum_{n=1}^{\infty}\nu_n\left[n\beta s_0(n\beta-1)\theta^r\frac{\gamma(z,\kappa+r)}{\Gamma(\kappa)}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}n\beta\,\theta^{(m\kappa+i+r)}\frac{\gamma[z,(m+1)\kappa+i+r]}{\Gamma(\kappa)}d_{m,i}(n\beta,\theta,\kappa)\right]+\\ \sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}w_{n,\ell}\left[(n\beta+\ell)s_0(n\beta+\ell-1)\theta^r\frac{\gamma(z,\kappa+r)}{\Gamma(\kappa)}+\sum_{m=1}^{\infty}\sum_{i=0}^{\infty}(n\beta+\ell)\theta^{(m\kappa+i+r)}\frac{\gamma[z,(m+1)\kappa+i+r]}{\Gamma(\kappa)}d_{m,i}(n\beta+\ell,\theta,\kappa)\right]\end{array}\right\}.$$

3.4 Record Values distributions

In this sub-section, we consider record values that have wide applications in life testing, industry, hydrology and economics. We investigate the record values densities for the BXIIG distribution.

Definition of Record Value 3.4.1: Based on a sequence $X_i,\ldots,i=1,2...,$ of i.i.d. rvs with cdf F and record times

$$U(1)=1 \text{ and } U(n+1)=\min\left\{j>U(n); X_j>X_{U(n)}\right\},\ n\in N,$$

the rvs $X_{U(n)}$ $(n\in N)$ are called (upper) record values.

The pdf of the i^th^ upper record value $R_i=X_{U(i)}$, with $R_1=X_1$, for the X~BXIIG model, is (for x>0)

$$f_{R_i}(x)=\frac{f(x)}{\Gamma(i)}\left\{-\log\left[S(x)\right]\right\}^{i-1}.$$

Then,

$$f_{R_i}(x)=\frac{\alpha\beta x^{\kappa-1}e^{-\frac{x}{\theta}}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1}\left\{-\log\left[\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{-\alpha}\right]\right\}^{i-1}}{\theta^{\kappa}\ \Gamma(\kappa)\Gamma(i)\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{\alpha+1}}.$$

The joint density of $(R_1,...,R_n)$ is $f_{(R_1,...,R_n)}(x_1,...,x_n)=f(x_n)\prod_{\ell=1}^{n-1}h(x_\ell)$, and then

$$f_{(R_1,...,R_n)}(x_1,...,x_n)=\alpha\beta\frac{x_n^{\kappa-1}e^{-\frac{x_n}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x_n}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_n}{\theta}\right)\right]\right\}^{\beta-1}\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_n}{\theta}\right)\right]\right\}^{\beta}\right)^{-\alpha-1}\times$$

$$\prod_{\ell=1}^{n-1}\alpha\beta\frac{x_\ell^{\kappa-1}e^{-\frac{x_\ell}{\theta}}}{\theta^{\kappa}\ \Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x_\ell}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_\ell}{\theta}\right)\right]\right\}^{\beta-1}\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_\ell}{\theta}\right)\right]\right\}^{\beta}\right)^{-1}.$$

3.5 Stress-strength Reliability

Let X_1 be the strength and X_2 be the stress assuming that X_1 follows the BXIIG distribution $(\alpha_1,\beta,\kappa,\theta)$ and X_2 follows the BXIIG distribution $(\alpha_2,\beta,\kappa,\theta)$. Then, the reliability parameter (Kotz et al., 2003) of the component is computed as

$$\mathrm{R}=\Pr(\mathrm{X}_2<X_1)=\int_0^{\infty}f_{X_1}(x)F_{X_2}(x)dx,$$

and then

$$R=\int_0^\infty \frac{\alpha_1\beta x^{\kappa-1}e^{-\frac{x}{\theta}}}{\theta^\kappa\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1}\times$$
$$\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\right)^{-\alpha_1-1}\left[1-\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\right)^{-\alpha_2}\right]dx.$$

Hence,

$$R=1-\int_0^\infty \frac{\alpha_1\beta x^{\kappa-1}e^{-\frac{x}{\theta}}}{\theta^\kappa\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1}\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\right)^{-\alpha_1-\alpha_2-1}dx$$

and

$$R=\frac{\alpha_2}{(\alpha_1+\alpha_2)},$$

which is independent of the parameters β,κ and θ.

4. Characterizations

In this section, we characterize the BXIIG distribution through: (i) conditional expectation and (ii) truncated moments. We present our characterizations in two subsections.

4.1 Characterization based on Conditional Expectation

Proposition 4.1.1: Let $X:\Omega\to(0,\infty)$ be a continuous rv with cdf $F(x)$.Then, for $\alpha>1$, X has cdf (3) if and only if

$$E\left(\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta \middle| X>t\right)=\frac{1}{\alpha-1}\left(\alpha\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{t}{\theta}\right)\right]\right\}^\beta+1\right). \tag{17}$$

Proof If X has cdf (3), then

$$E\left(\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{X}{\theta}\right)\right]\right\}^\beta \middle| X>t\right)=(1-F(t))^{-1}\int_t^\infty\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta f(x)dx$$
$$=\left\{1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\right\}^\alpha\int_t^\infty\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\alpha\beta\frac{x^{\kappa-1}e^{-\frac{x}{\theta}}}{\theta^\kappa\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{-1}\times$$
$$\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1}\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\right)^{-\alpha-1}dx.$$

Upon integration by parts and simplification, we obtain for $(t>0)$ (17).

Conversely, if (17) holds, then

$$(1-F(t))^{-1}\int_t^\infty\left\{-\log\left[1-\frac{1}{\Gamma(\kappa)}\gamma\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta f(x)dx=\frac{1}{\alpha-1}\left(\alpha\left\{-\log\left[1-\frac{1}{\Gamma(\kappa)}\gamma\left(\kappa,\frac{t}{\theta}\right)\right]\right\}^\beta+1\right),$$

$$\int_t^\infty \left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta f(x)\,dx = \frac{(1-F(t))}{\alpha-1}\left(\alpha\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{t}{\theta}\right)\right]\right\}^\beta+1\right).$$

Differentiating both sides of the above equation with respect to t, we can write

$$-\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{t}{\theta}\right)\right]\right\}^\beta f(t) = \frac{(1-F(t))}{\alpha-1}\frac{\alpha\beta t^{\kappa-1}\,e^{-\frac{t}{\theta}}}{\theta^\kappa\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{t}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{t}{\theta}\right)\right]\right\}^{\beta-1} - \frac{f(t)}{\alpha-1}\left(\alpha\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{t}{\theta}\right)\right]\right\}^\beta+1\right).$$

After simplification and integration, we obtain

$$F(t) = 1-\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{t}{\theta}\right)\right]\right\}^\beta\right)^{-\alpha} \quad \text{for } t \ge 0.$$

4.2 Characterizations based on Truncated Moment of a Function of the Random Variable

In this subsection, we first present a characterization of the BXIIG distribution in terms of a simple relationship between truncated moment of a function of X and another function. This characterization result employs a version of the theorem due to Glänzel (1986); see Theorem G of Appendix A. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 4.2.1 Let $X:\Omega\to(0,\infty)$ be a continuous rv and let $q(x)=\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\right)^{-1}$, $x>0$. The rv X has pdf (4) if and only if the function η defined in Theorem G has the form

$$\eta(x) = \frac{\alpha}{\alpha+1}\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\right)^{-1}, \quad x>0.$$

Proof If X has pdf (4), then for (x>0),

$$[1-F(x)]E[q(X)|X\ge x] = \frac{\alpha}{\alpha+1}\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\right)^{-(\alpha+1)}$$

or

$$E[q(X)|X\ge x] = \frac{\alpha}{\alpha+1}\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\right)^{-1}$$

and

$$\eta(x)-q(x) = -\frac{1}{\alpha+1}\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^\beta\right)^{-1}.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)}{\eta(x)-q(x)}=\frac{\alpha\beta\frac{x^{\kappa-1}e^{-\frac{x}{\theta}}}{\theta^{\kappa}\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1}}{\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)},$$

and

$$s(x)=\ln\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{\alpha},\ x>0,$$

and

$$e^{-s(x)}=\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{-\alpha},\ x>0.$$

In view of Theorem G, X has density (4).

Corollary 4.2.1: Let $X:\Omega\to(0,\infty)$ be a continuous rv. The pdf of X is (4) if and only if there exist functions $\eta(x)$ and $q(x)$ defined in Theorem G satisfying the differential equation

$$\frac{\eta'(x)}{\eta(x)-q(x)}=\frac{\alpha\beta\frac{x^{\kappa-1}e^{-\frac{x}{\theta}}}{\theta^{\kappa}\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1}}{\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)},\ x>0.$$

Remark 4.2.1: The general solution of the differential equation in Corollary 4.2.1is

$$\eta(x)=\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\alpha}\left[-\int\frac{\alpha\beta\frac{x^{\kappa-1}e^{-\frac{x}{\theta}}}{\theta^{\kappa}\,\Gamma(\kappa)}\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta-1}}{\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x}{\theta}\right)\right]\right\}^{\beta}\right)^{\alpha+1}}q(x)\,dx+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in proposition 4.2.1 with D=0. However, it should also be noted that there are other pairs (η,q) satisfying conditions of Theorem G.

5. Maximum Likelihood Estimation

Here, we adopt maximum likelihood estimation technique for estimating the BXIIG parameters. Let $\xi=(\alpha,\beta,\theta,\kappa)^{T}$ be the unknown parameter vector. The log-likelihood function $\ell(\xi)$ for the BXIIG distribution is

$$\ell=n\ln(\alpha)+n\ln(\beta)-n\ln\left[\Gamma(\kappa)\right]-n\kappa\ln(\theta)+(\kappa-1)\sum_{i=1}^{n}\ln x_i-\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)-\sum_{i=1}^{n}\ln\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]+$$
$$(\beta-1)\sum_{i=1}^{n}\ln\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}-(\alpha+1)\sum_{i=1}^{n}\ln\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}^{\beta}\right). \quad (18)$$

We can obtain the MLEs of the parameters α,β,θ and κ of the BXIIG distribution by solving the nonlinear equations $\frac{\partial\ell}{\partial\alpha}=0$, $\frac{\partial\ell}{\partial\beta}=0$, $\frac{\partial\ell}{\partial\theta}=0$ and $\frac{\partial\ell}{\partial\kappa}=0$ either directly or using quasi-Newton procedures in computer softwares such as R, SAS, Ox, MATHEMATICA, MATLAB and MAPLE. They are given by

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\ln\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}^{\beta}\right)=0\,,$$

$$\frac{\delta\ell}{\delta\beta}=\frac{n}{\beta}-\sum_{i=1}^{n}\ln\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}-(\alpha+1)\sum_{i=1}^{n}\frac{\ln\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}}{\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}^{-\beta}\right)}=0\,,$$

$$\frac{\partial\ell}{\partial\kappa}=-n\psi(k)-n\ln(\theta)+\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}\frac{\gamma_1'\left(\kappa,\frac{x_i}{\theta}\right)_{\kappa}}{1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)}+(\beta-1)\sum_{i=1}^{n}\frac{\gamma_1'\left(\kappa,\frac{x_i}{\theta}\right)_{\kappa}\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]^{-1}}{\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}}$$

$$-(\alpha+1)\beta\sum_{i=1}^{n}\frac{\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}^{\beta-1}\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]^{-1}\gamma_1'\left(\kappa,\frac{x_i}{\theta}\right)_{\kappa}}{\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}^{\beta}\right)}=0,$$

$$\frac{\partial\ell}{\partial\theta}=-\frac{n\kappa}{\theta}+\sum_{i=1}^{n}\left(\frac{x_i}{\theta^2}\right)+\sum_{i=1}^{n}\frac{\gamma_1'\left(\kappa,\frac{x_i}{\theta}\right)_{\theta}}{\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]}+(\beta-1)\sum_{i=1}^{n}\frac{\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]^{-1}\gamma_1'\left(\kappa,\frac{x_i}{\theta}\right)_{\theta}}{\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}}$$

$$-(\alpha+1)\beta\sum_{i=1}^{n}\frac{\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]^{-1}\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}^{\beta-1}\gamma_1'\left(\kappa,\frac{x_i}{\theta}\right)_{\theta}}{\left(1+\left\{-\log\left[1-\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\right]\right\}^{\beta}\right)}=0,$$

where $\psi(k)=\frac{\Gamma(\kappa)'}{\Gamma(\kappa)}$, $\partial\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\Big/\partial\kappa=\gamma_1'\left(\kappa,\frac{x_i}{\theta}\right)_{\kappa}$ and $\partial\gamma_1\left(\kappa,\frac{x_i}{\theta}\right)\Big/\partial\theta=\gamma_1'\left(\kappa,\frac{x_i}{\theta}\right)_{\theta}$.

6. Simulation Study

In this section, we perform a simulation study to verify the accuracy of the MLEs of the parameters of the BXIIG distribution. The random number generation is executed by inverting its cdf. The MLEs, say $\left(\hat{\alpha}_i,\hat{\beta}_i,\hat{\theta}_i,\hat{\kappa}_i\right)$, for i=1,2,...,N, have been obtained using the CG routine in R software.

The simulation study is based on graphical results. We generate N=1,000 samples of sizes n=20, 25,…,850 from the BXIIG distribution and consider the true values for α,β,θ and κ as 6,3,1 and 2, respectively.. We also determine the means, biases and mean square errors (MSEs) of the MLEs. The biases and MSEs are calculated by (for $h=\alpha,\beta,\theta,\kappa$)

$Bias_h = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{h}_i - h\right)$ and $MSE_h = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{h}_i - h\right)^2$.

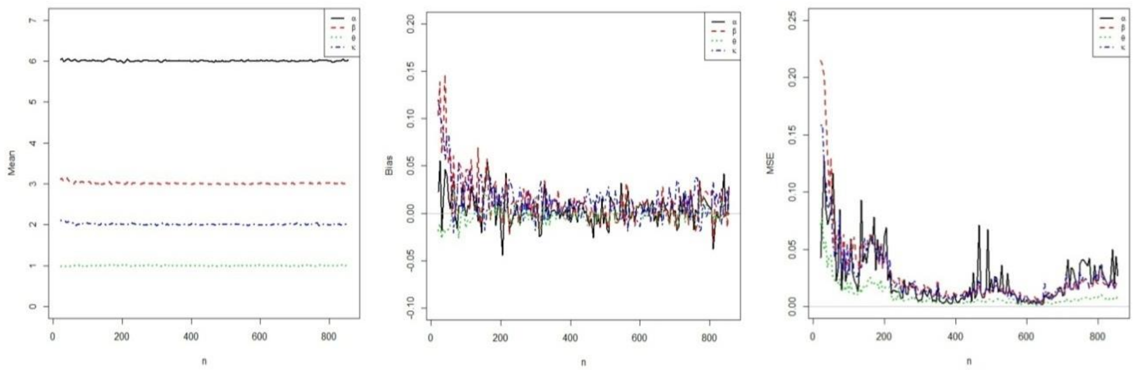

Figure: 3. Empirical means (left), biases (center) and MSEs (right) of the BXIIG model

The results are displayed by Figure 3, which reveal that the empirical means tend to the true parameter values and that the biases and MSEs decrease when the sample size increases. However, the MSE of the estimate $\hat{\alpha}$ can be larger than those of the other parameters.

7. Data Analysis

We consider applications to two real data sets to verify the flexibility, utility and potentiality of the BXIIG model. We compare the BXIIG distribution with some members of the Kumaraswamy gamma (KwG) model (Cordeiro and de Castro, 2011), beta gamma (BG) model (Kong et al., 2007), generalized odd log logistic gamma (GOLLG) model (Haghbin et al., 2017) and Topp Leone gamma (TLG) model (Rezai et al., 2017) for two data sets. For selection of the best distribution, we compute the maximized log-likelihood values $\hat{\ell}$, Akaike Information Criteria (AIC), Kolmogorov-Smirnov (KS), Cramer von Mises (W^*) and Anderson-Darling (A^*) goodness of-fit statistics for all models. The statistics $A^*$and $W^*$are discussed by Chen and Balakrishnan (1995). In general, the best model has the smallest values of the AIC, KS, $A^*$and W^* statistics and the largest value of $\hat{\ell}$.

All computations of the MLEs are performed via the maxLik routine and all statistics are calculated by the goodness of fit test routine in the R software.

Data Set I: Survival Times

Firstly, we consider the guinea pig data set consisting of survival times (in days) of guinea pigs injected with different amount of tubercle bacilli (Bjerkedal, 1960). Recently, this data set has been analyzed by Gupta et al. (1997) and Korkmaz (2017). The data are 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

A descriptive summary for the survival times provide the following values: 72 (sample size), 12 (minimum), 376 (maximum), 70 (median), 99.81944 (mean), 81.11795 (standard deviation), 81.26468 (coefficient of variation), 1.796245 (coefficient of skewness) and 5.614438 (coefficient of kurtosis). The TTT (total time on test) plot for survival times is concave [**Figure 4(left)**] which indicates increasing shaped hazard rate. The boxplot for survival times is positively skewed [**Figure 4(right)**]. So, the BXIIG distribution is suitable to model these survival times.

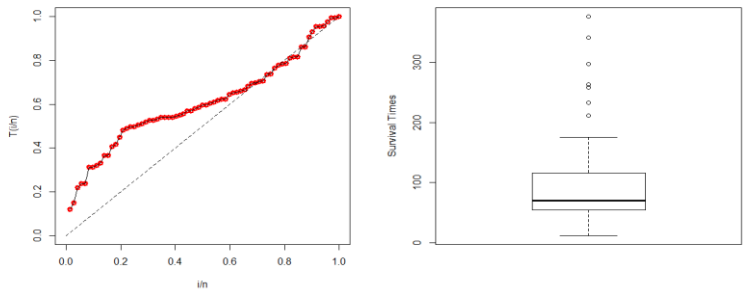

Figure 4: TTT plot (left) and Boxplot (right) for survival times

For this data set, we compare the BXIIG model with the models described before under the above criteria.

Table 2 lists the MLEs, their standard errors (SEs), the maximized log likelihood $\hat{\ell}=\ell\left(\hat{\xi}\right)$ from (18) and the goodness-of-fits statistics from the fitted models. The figures in this table show that the BXIIG model could be chosen as the best distribution among the fitted models since it has the lowest values of the statistics AIC, KS, $A^*$and W^* and the highest $\hat{\ell}$ value.

Table 2. MLEs, SEs of the estimates (in parentheses), $\hat{\ell}$ and the goodness-of-fits statistics for the data set I ([.] denotes p-values of the KS statistics)

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\kappa}$	$-\hat{\ell}$	AIC	KS	A^*	W^*
BXIIG	0.5096 (0.1712)	10.0811 (2.2902)	754.1970 (4.2166)	0.2017 (0.1298)	389.1349	786.2699	0.0824 [0.7122]	0.4674	0.0759
GOLLG	2.5046 (0.2524)	8.2390 (0.0614)	3748.6477 (5.0959)	0.6735 (0.0153)	391.8592	791.7185	0.0975 [0.5002]	1.1244	0.1670
TLG	9.3052 (1.2344)	2.7930 (0.2262)	2861.7113 (42.0107)	0.1652 (0.0001)	389.9928	787.9857	0.0962 [0.5175]	0.7149	0.1266
KwG	26.8317 (0.1162)	0.3484 (0.0436)	28.4782 (0.0246)	0.2143 (0.0243)	390.4703	788.9405	0.1010 [0.4547]	0.9755	0.1833
BG	175.7332 (4.2133)	0.3351 (0.0413)	28.3102 (0.0100)	0.0295 (0.0050)	390.3043	788.6085	0.0993 [0.4761]	0.9369	0.1758

The plots of the fitted density, estimated cdf and P-P plot of the BXIIG distribution are displayed in Figure 5. These plots show that the BXIIG model provides a good fit to these data. Hence, the fitted new distribution successfully captures the kurtosis of the data.

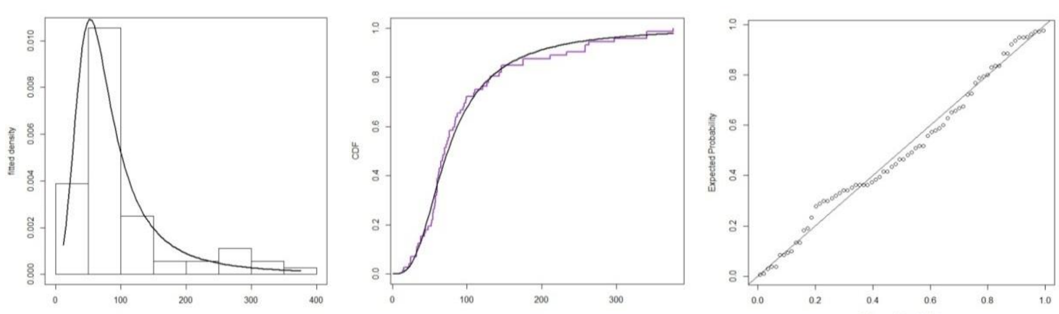

Figure 5. The fitted density (left), estimated cdf (center) and P-P plot (right) of the BXIIG model for the data set I

Data set II: Remission Times

Secondly, the real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients (Lee and Wang, 2003): 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. This data set has been analyzed by Lemonte and Cordeiro (2013).

A descriptive summary for the remission times provide the following values: 128 (sample size), 0.08 (minimum), 79.05 (maximum), 6.395 (median), 9.365625 (mean), 10.50833 (standard deviation), 112.201 (coefficient of variation), 3.286569 (coefficient of skewness) and 18.48308 (coefficient of kurtosis). The TTT plot for the remission times is first concave and then convex [**Figure 6(left)**], which indicates unimodal shaped hazard rate. The boxplot for survival times is positively skewed [**Figure 6(right)**]. So, the BXIIG distribution is suitable to model remission times.

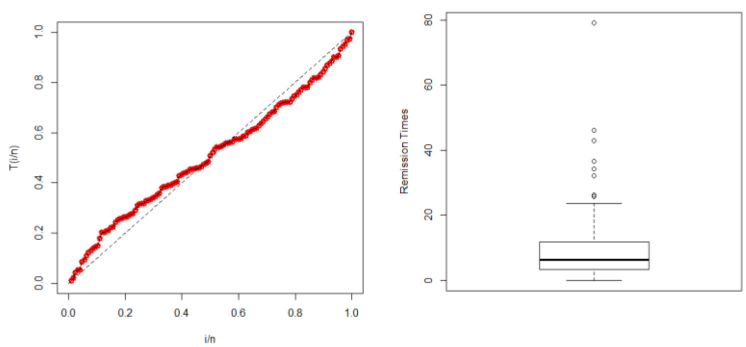

Figure 6: TTT plot (left) and Boxplot (right) for remission times

Table 3 lists the MLEs, their SEs, $\hat{\ell}$ and the goodness-of-fits statistics from the fitted models. The figures in this table reveal that the BXIIG distribution could be chosen as the best model among the fitted models since it has the lowest values of the statistics AIC, KS, $A^*$and W^* and the highest $\hat{\ell}$ value.

Table 3. MLEs, SEs of the estimates (in parentheses), $\hat{\ell}$ and the goodness-of-fits statistics for the data set II ([.] denotes p-values of the KS statistics)

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\kappa}$	$-\hat{\ell}$	AIC	KS	A^*	W^*
BXIIG	1.1869 (0.5290)	3.4449 (1.1625)	35.7153 (12.1125)	0.3279 (0.1066)	409.5026	827.3305	0.0327 [0.9991]	0.1006	0.0152
GOLLG	3.9165 (2.2084)	0.9724 (0.9719)	191.3240 (6.7628)	0.2230 (0.2101)	409.5177	827.3605	0.0341 [0.9984]	0.1169	0.0177
TLG	1.2393 (0.4225)	8.8195 (1.3029)	142329.2 (4.1943)	0.0768 (0.0020)	409.6781	827.6814	0.0355 [0.9969]	0.1255	0.0186
KwG	4.8198 (2.6031)	0.6061 (0.0765)	6.3907 (0.0774)	0.2897 (0.1495)	411.5361	831.0722	0.0580 [0.7820]	0.4341	0.0736
BG	85.7694 (0.6168)	12.8314 (0.8810)	412.9971 (14.5122)	0.0374 (0.0035)	411.2123	830.4245	0.0492 [0.9160]	0.3466	0.0522

The plots of the fitted density, estimated cdf and P-P plot of the BXIIG model are displayed in Figure 7. These plots indicate that the BXIIG model provides a good fit to these data. Hence, the fitted new distribution successfully captures the kurtosis of the data.

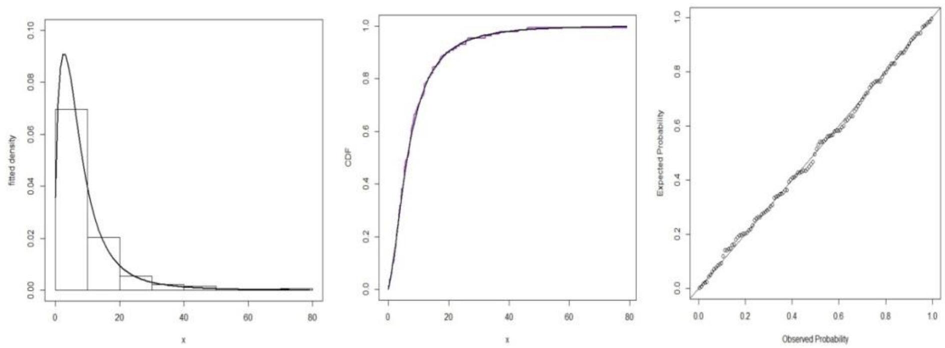

Figure 7. The fitted density (left), estimated cdf (center) and P-P plot (right) of the BXIIG model for the data set II

8. Concluding Remarks

We propose the BXII-Gamma (BXIIG) distribution from (i) the T-X family technique and (ii) link between the exponential and gamma random variables. The BXIIG density highlights various shapes as J, reverse-J, left-skewed, right-skewed and symmetrical shapes. Its hazard rate function has various shapes such as increasing, decreasing, decreasing-increasing, increasing-decreasing-increasing, bathtub and modified bathtub. We study some of its mathematical properties such as random number generator, ordinary moments, generating function, conditional moments, density functions of record values, reliability measure and characterizations. We address the maximum likelihood estimation for the BXIIG parameters. We evaluate the precision of the maximum likelihood estimators via a simulation study. We consider two applications to survival times of guinea pigs and remission times of bladder patients to illustrate the potentiality of the new model. We compute the goodness of fit measures for testing the acceptability of the BXIIG distribution. The potentiality of the BXIIG model illustrates that it is flexible, competitive and parsimonious to other existing distributions to fit lifetime data. Therefore it should be included in the distribution theory to facilitate the researchers. Further, as perspective of future projects, we may consider several intensive subjects (i) statistical inferences using different sampling schemes such as simple random sampling (SRS) and rank set sampling (RSS); (ii) reliability analysis using SRS and RSS; (iii) Bayesian estimation of the BXIIG parameters via SRS and RSS under different loss functions; (iv) Bayesian estimation of the BXIIG parameters via complete and censored samples under different loss functions and (v) the study of the complexity of the BXIIG via Bayesian methods.

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Appendix A

Theorem G. Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a given probability space and let $\mathrm{H}=[a_1,a_2]$ be an interval with $a_1<a_2$ ($a_1=-\infty, a_2=\infty$). Let $X:\Omega\to[a_1,a_2]$ be a continuous random variable with distribution function F and Let $g(x)$ be a real function defined on $\mathrm{H}=[a_1,a_2]$ such that $E[g(X)|\mathrm{X}\geq \mathrm{x}]=h(x)$ for $x\in H$ is defined with some real function $h(x)$ should be in simple form. Assume that $g(x)\varepsilon C([a_1,a_2])$, $h(x)\varepsilon C^2([a_1,a_2])$ and F is twofold continuously differentiable and strictly monotone function on the set $[a_1,a_2]$.We conclude, assuming that the equation $g(x)=h(x)$ has no real solution in the inside of $[a_1,a_2]$.Then F is obtained from the functions $g(x)$ and $h(x)$ as $F(x)=\int_a^x k\left|\frac{h'(t)}{h(t)-g(t)}\right|\exp(-s(t))dt$, where $s(t)$ is the solution of equation $s'(t)=\frac{h'(t)}{h(t)-g(t)}$ and k is a constant, chosen to make $\int_{a_1}^{a_2} dF=1$.