Pak.j.stat.oper.res. Vol.16 No. 4 2020 pp 661-674 DOI: http://dx.doi.org/10.18187/pjsor.v16i4.3442

Pakistan Journal of Statistics and Operation Research

A New Probability Distribution Family Arising from Truncated Power Lomax Distribution with Application to Weibull Model

Hassan A. S[1]., Sabry M. A. H. [2*], Elsehetry A. M[2].

* Corresponding Author:

[1] Faculty of Graduate Studies for Statistical Research: Giza, Egypt, dr.amalelmoslamy@gmail.com
[2] Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research, (FGSSR), Cairo University, Egypt, mohusss@cu.du.eg

Abstract

The truncated distributions have been widely studied, mainly in life-testing and reliability analysis. In this paper, we introduce a new right truncated generator related to power Lomax distribution, referred to right truncated power Lomax--G family. The proposed family is a generalization of recently [0, 1] truncated Lomax-G family. Statistical properties like; moments, moment generating function, probability weighted moments, quantile function, mean deviation, order statistics and Rényi entropy are derived. Five new sub-models from the truncated family are presented. Maximum likelihood estimation is investigated and simulation issues are discussed for truncated power Lomax Weibull model as particular case from the family. The flexibility of the truncated power Lomax Weibull is assessed by applying it to a real data set. The application indicates that the truncated power Lomax Weibull distribution model can give better fits than other well-known lifetime distributions.

Key Words: Power Lomax distribution; Order statistics; Truncated, Maximum likelihood method, Weibull Distribution.

Mathematical Subject Classification: 60E05 **Probability distributions: general theory**

1. Introduction

Numerous classical distributions have been extensively used over the past decades for modeling data in several areas such as engineering, actuarial, environmental and medical sciences, biological studies, demography, economics, finance and insurance. However, in many applied areas such as lifetime analysis, finance and insurance, there is a clear need for more flexible forms of these distributions to model specific types of real data. For that reason, several methods for generating new families of distributions have been studied. Some of the well-known generators are the beta-G by Eugene *et al.* (2002), Kumaraswamy-G by Cordeiro and de Castro (2011), exponentiated generalized-G by Cordeiro *et al.* (2013), Transformed-Transformer (T-X) by Alzaatreh *et al.* (2013), Weibull-G by Bourguignon *et al.* (2014), exponentiated half-logistic-G by Cordeiro *et al.* (2014a), Lomax-G by Cordeiro *et al.* (2014b) odd generalized exponential by Tahir *et al.* (2015), the beta odd log-logistic generalized by Cordeiro *et al.* (2016), exponentiated Weibull-G by Hassan and Elgarhy (2016a), Kumaraswamy Weibull-G by Hassan and Elgarhy (2016b), additive Weibull-G by Hassan and Hemeda (2016), exponentiated extended-G by Elgarhy *et al.* (2017), Type II half logistic-G by Hassan *et al.* (2017a), generalized additive Weibull-G by Hassan *et al.* (2017b), The Lomax-R{Y} family by Mansoor *et al.* (2017), odd Frechet-G by Haq and Elgarhy (2018), inverse Weibull-G by Hassan and Nassr (2018), power Lindley-G by Hassan and Nassar (2019), and Type II generalized Topp-Leone –G (Hassan *et al*., 2019) among others.

An important model is the Lomax (or Pareto II) that has been suggested by Lomax (1954) for modeling lifetime data. It has been widely applied in some areas, such as, analysis of income and wealth data, modeling business failure data, biological sciences, model firm size and queuing problems (see for example Harris (1968), and Atkinson and Harrison (1978)). Also, it has been used in reliability and life testing problems in engineering (see Hassan and Al-Ghamdi (2009) and Hassan *et al.* (2016)). Extended forms of Lomax distribution have been provided by several authors. Our interest here with one of the newly extended forms of Lomax distribution which is power Lomax (PL) distribution. PL distribution has been introduced by Rady *et al.* (2016) by using power transformation of a Lomax distribution. The PL distribution is more flexible and accommodates both inverted bathtub and decreasing hazard rate. The cumulative distribution function (cdf) and probability density function (pdf) of PL distribution are given respectively, by

$$G(u) = 1 - \lambda^{\alpha}\left(\lambda + u^{\beta}\right)^{-\alpha}, \tag{1}$$

and,

$$g(u) \;=\; \alpha\beta\lambda^{\alpha}\, u^{\beta-1}\left(\lambda + u^{\beta}\right)^{-(\alpha+1)}, \quad u > 0\,, \tag{2}$$

where, $\alpha, \beta > 0$ are two shape parameters and $\lambda > 0$ is a scale parameter.

A truncated distribution is defined as a conditional distribution that results from restricting the domain of the statistical distribution. Hence, truncated distributions are used in cases where occurrences are limited to values which lie above or below a given threshold or within a specified range. If occurrences are limited to values which lie below a given threshold, the lower (left) truncated distribution is obtained. Similarly, if occurrences are limited to values which lie above a given threshold, the upper (right) truncated distribution arises. In the literature, several truncated distributions have been provided by several authors. Recently, Abid and Abdulrazak (2017) introduced [0, 1] truncated Fréchet-G of distributions by using the [0, 1] truncated Fréchet distribution as a generator. Hassan *et al*. (2020 a) proposed [0,1] truncated Lomax-G family.

The main aim of this paper is to introduce and study a new truncated family of probability distributions depending on the [0, 1] truncated PL distribution as a generator instead of unbounded random variables. We call the new family as the truncated PL generated (TPL- G). The proposed family is a generalization to truncated Lomax-G family proposed by Hassen et al. (2020 b). We hope that the truncated family yields a better fit in more practical situations.

2. [0, 1] Truncated Power Lomax Distribution

Based on Equation 1, let $\lambda = 1$, truncated power Lomax (TPL) distribution are defined (see Hassan *et al*., 2020 a) defined) as follows

$$r(t) \;=\; \frac{g(t)}{G(1)-G(0)} = \frac{\alpha\beta t^{\beta-1}\left(1+t^{\beta}\right)^{-(\alpha+1)}}{1-2^{-\alpha}}\;, \tag{3}$$

and

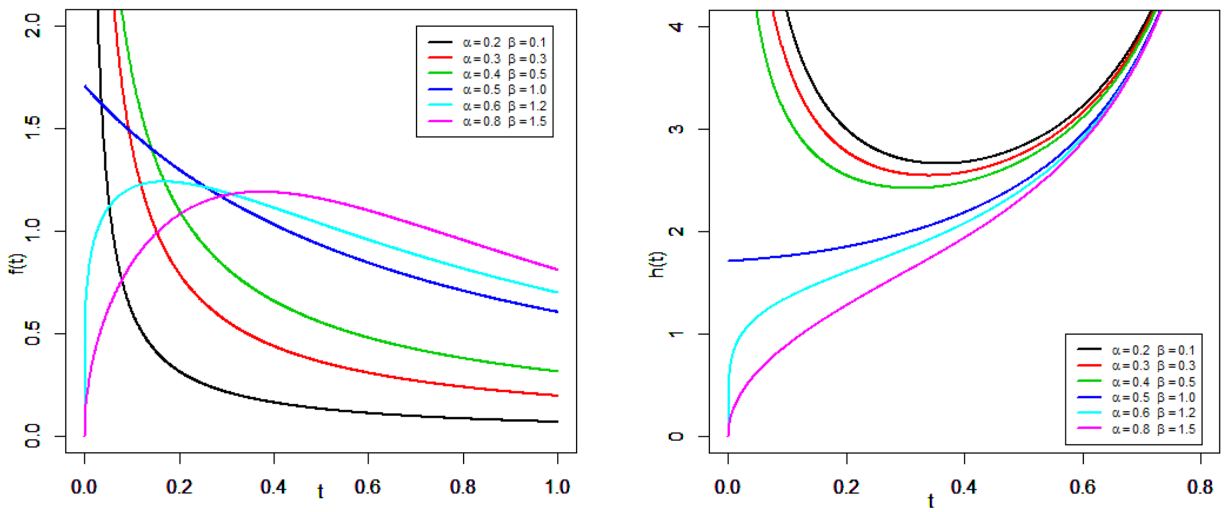
$$R(t) = \frac{\int_0^t g(t)dt}{G(1)-G(0)} = \frac{G(t)-G(0)}{G(1)-G(0)} = \frac{1-(1+t^{\beta})^{-\alpha}}{1-2^{-\alpha}}, \tag{4}$$

respectively, where $\alpha, \beta > 0$ and $0 < t < 1$. The survival, hazard rate function (hrf), reversed hazard rate function and cumulative hazard rate function are given, respectively, by

$$\bar{R}(t) = 1 - R(t) = \frac{\left(1+t^{\beta}\right)^{-\alpha} - 2^{-\alpha}}{1-2^{-\alpha}} \quad , h(t) = \frac{r(t)}{\bar{R}(t)} = \frac{\alpha\beta t^{\beta-1}\left(1+t^{\beta}\right)^{-(\alpha+1)}}{\left(1+t^{\beta}\right)^{-\alpha} - 2^{-\alpha}} \quad ,$$

$$\tau(t) = \frac{r(t)}{R(t)} = \frac{\alpha\beta t^{\beta-1}\left(1+t^{\beta}\right)^{-(\alpha+1)}}{1-(1+t^{\beta})^{-\alpha}} \quad , \text{and } H(t) = -\,ln\left(\bar{R}(t)\right) = -\,ln\left(\frac{\left(1+t^{\beta}\right)^{-\alpha} - 2^{-\alpha}}{1-2^{-\alpha}}\right).$$

Figure 1 gives a variety of possible shapes of the pdf and the hrf of [0,1] TPL distribution for some selected values of parameters. It can be detected from Figure 1 that the pdf shape can be right skewed, reversed J-shape, and unimodel. Also, the shape of the hrf of the [0,1] TPL distribution could be increasing or decreasing, and U-shaped.



(a) **(b)**

Figure 1. (a) pdf and (b) hrf of the [0, 1] TPL distribution for different values of parameters

3. Truncated Power Lomax –G Family

In this section, a new truncated family of distributions is introduced based on [0, 1] truncated power Lomax distribution. The density and distribution function of [0, 1] TPL –G family is defined. Further, the pdf and cdf of TPL-G family are defined by taking the pdf (3) as a generator in the T-X family proposed by Alzaatreh *et al.* (2013).

A random variable X is said to be distributed as TPL-G, denoted by $X \sim \mathrm{TPL-G}$, if it has cdf and pdf as follows

$$F(x) = \int_0^{G(x;\xi)} \frac{\alpha\beta t^{\beta-1}(1+t^{\beta})^{-(\alpha+1)}}{1-2^{-\alpha}} dt = A\left(1-(1+G(x;\xi)^{\beta})^{-\alpha}\right), \tag{5}$$

and

$$f(x) = A\alpha\beta g(x;\xi)G(x;\xi)^{\beta-1}(1+G(x;\xi)^{\beta})^{-\alpha-1}, \tag{6}$$

where $x > 0$ and $\alpha, \beta > 0$, $A = \frac{1}{1-2^{-\alpha}}$ and ξ is the set of parameters of the $G(.)$ distribution. The survival function, $\bar{F}(x)$, and hazard rate function, $h(x)$, are, respectively, given by

$$\bar{F}(x) = 1 - A\left(1-(1+G(x;\xi)^{\beta})^{-\alpha}\right),$$

and

$$h(x) = \frac{A\alpha\beta g(x;\xi)G(x;\xi)^{\beta-1}(1+G(x;\xi)^{\beta})^{-\alpha-1}}{1 - A(1-(1+G(x;\xi)^{\beta})^{-\alpha})}.$$

4. Statistical Properties of the TPL- G family

In this section, some statistical properties of the $\mathrm{TPL-G}$ family of distributions are investigated.

4.1 Quantile function

Let X denotes a random variable has the cdf (5), the quantile function, say $Q_{(u)}$ of X is given by

$$Q_{(u)} = G^{-1}\left\{\sqrt[\beta]{[1-(1-2^{-\alpha})\,u]^{\frac{-1}{\alpha}} - 1}\right\},$$

where, $0 < u < 1$ and $G^{-1}(.)$ is the inverse cumulative distribution function of $G(.)$.

4.2 Useful representation

Some representations of the cdf and pdf for $\mathrm{TPL-G}$ family of distributions will be presented.

It is well-known that, if $\alpha > 0$ and $|Z| < 1$ the generalized binomial theorem is written as follows

$$(1+Z)^{-\alpha} = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha+i-1}{i} Z^i. \tag{7}$$

Then, by applying the binomial theorem (7) in (6), the pdf of $\mathrm{TPL-G}$ distribution where α is real becomes

$$f(x) = \sum_{i=0}^{\infty}\eta_i\, g(x)G(x)^{\beta(i+1)-1}, \tag{8}$$

where, $\eta_i = A\alpha\beta\,(-1)^i\binom{\alpha+i}{i}$.

Further, an expansion for $[F(x)]^h$ is derived, for h is integer. Again, the binomial expansion is worked out.

$$[F(x)]^h = \sum_{k=0}^{\infty} S_k\; G(x)^{\beta k}, \tag{9}$$

where,

$$S_k = A^h \sum_{j=0}^{h} (-1)^{j+k} \binom{h}{j} \binom{\alpha j + k - 1}{k}.$$

4.3 Probability weighted moments

Class of moments, called the probability-weighted moments (PWMs), has been proposed by Greenwood *et al.* (1979). This class is used to derive estimators of the parameters and quantiles of distributions expressible in inverse form. For a random variable X the PWMs, denoted by $\tau_{r,h}$ can be calculated through the following relation

$$\tau_{r,h} = E(X^r F(x)^h) = \int_{-\infty}^{\infty} x^r f(x)\, F(x)^h dx. \quad (10)$$

The PWM of TPL − G is obtained by inserting (8) and (9) into (10) as follows

$$\tau_{r,h} = \sum_{i,k=0}^{\infty} \eta_i\, S_k \int_{-\infty}^{\infty} x^r\, g(x;\xi)\, G(x;\xi)^{\beta(i+k+1)-1}\, dx.$$

Then,

$$\tau_{r,h} = \sum_{i,k=0}^{\infty} \eta_i\, S_k\, \tau_{r,\beta(i+k+1)-1},$$

where,

$$\Upsilon_{r,\beta(i+k+1)-1} = \int_{-\infty}^{\infty} x^r g(x;\xi) G(x;\xi)^{\beta(i+k+1)-1}\, dx.$$

4.4 Moments and Moment Generating function

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the r^{th} moment for the TPL − G family. If X has the pdf (8), then r^{th} moment is obtained as follows

$$\acute{\mu}_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx = \sum_{i=0}^{\infty} \eta_i \int_{-\infty}^{\infty} x^r\, g(x;\xi) G(x;\xi)^{\beta(i+1)-1}\, dx.$$

Then,

$$\acute{\mu}_r = \sum_{i=0}^{\infty} \eta_i\ \tau_{r,\beta(i+1)-1}.$$

For a random variable X it is known that, the moment generating function is defined as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \acute{\mu}_r = \sum_{i,r=0}^{\infty} \frac{t^r}{r!}\, \eta_i\, \tau_{r,\beta(i+1)-1}.$$

4.5 The mean deviation

In statistics, mean deviation about the mean and mean deviation about the median measure the amount of scattering in a population. For random variable X with pdf $f(x)$, cdf $F(x)$, the mean deviation about the mean(μ) and mean deviation about the median (M), are defined by

$$\delta_1 = 2\mu\, F(\mu) - 2T(\mu) \text{ and } \delta_2 = \mu - 2T(M),$$

respectively, where,

$$T(q) = \int_{-\infty}^{q} x\, f(x)\ dx,$$

which is the first incomplete moment.

4.6 Order statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let $X_1, X_2, \dots, X_n$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function $F(x)$. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the corresponding ordered random sample from a population of size n. According to David (1981), the pdf of the k^{th} order statistic, is defined as

$$f_{x_{(k)}}(x) = \frac{1}{B(k,n-k+1)} [F(x)]^{k-1} f(x) [1 - F(x)]^{n-k}, \qquad 0 < x_{(k)} < \infty. \quad (11)$$

By employing the binomial expansion in (11), then the pdf of k^{th} order statistic takes the following form

$$f_{x_{(k)}}(x) = \frac{f(x)}{B(k,n-k+1)} \sum_{m=0}^{n-k} (-1)^m \binom{n-k}{m} F(x)^{m+k-1}, \quad (12)$$

$B(.,.)$ stands for beta function. The pdf of the k^{th} order statistic for TPL − G family is derived by substituting (8) and (9) in (12), replacing h with $m + k - 1$

$$f_{x_{(k)}}(x) = \frac{g(x;\xi)}{B(k,n-k+1)} \sum_{m=0}^{n-k} \sum_{i,k=0}^{\infty} C^* G(x;\xi)^{\beta(i+k+1)-1},$$

where, $C^* = (-1)^m \binom{n-k}{m} \eta_i S_k$, $g(.)$ and $G(.)$ are the pdf and cdf of the TPL − G family, respectively.
Further, the r$^{\text{th}}$ moment of k^{th}order statistics for TPL − G family is defined by:

$$E\left(X^r{}_{(k)}\right) = \int_{-\infty}^{\infty} x^r f_{x_{(k)}}(x)\, dx. \tag{13}$$

By substituting (12) in (13), leads to

$$E\left(X^r{}_{(k)}\right) = \frac{1}{B(k,n-k+1)} \sum_{m=0}^{n-k} \sum_{i,j,k,l=0}^{\infty} C^* \tau_{r,\beta(i+k+1)-1}.$$

4.7 Rényi entropy

An entropy is a concept encountered in physics and engineering. It is a measure of variation or uncertainty of a random variable *X* (see Rényi (1961)). The Rényi entropy of *X* with pdf *f* (*x*) is defined by

$$I_\delta(X) = \frac{1}{1-\delta} log \int_{-\infty}^{\infty} f(x)^\delta\, dx, \quad \delta > 0 \quad and \quad \delta \neq 1.$$

Now, we consider the generalized binomial theorem (7), then the pdf $f(x)^\delta$ can be expressed as follows:

$$f(x)^\delta = \sum_{i=0}^{\infty} C_i\, g(x;\xi)^\delta G(x;\xi)^{\beta(i+\delta)-\delta},$$

where, $C_i = (-1)^i (A\alpha\beta)^\delta \binom{\delta(\alpha+1)+i-1}{i}$. Therefore, the Rényi entropy of TPL − G family of distributions is given by

$$I_\delta(X) = \frac{1}{1-\delta} log \sum_{i=0}^{\infty} C_i \int_{-\infty}^{\infty} g(x;\xi)^\delta G(x;\xi)^{\beta(i+\delta)-\delta}\, dx.$$

5. Sub-Models

This section is devoted to discuss and describe five sub-models of the TPL-G family, namely; TPL -uniform, TPL - Weibull, TPL -Ferchet, TPL -Kumaraswamy and TPL –Toppe-Leone.

5.1 TPL -Uniform Distribution

For $g(x;\theta) = \frac{1}{\theta}$, $0 < x < \theta$, and $G(x;\theta) = \frac{x}{\theta}$ the pdf of TPL -uniform (TPLU) is derived from (6) as the following

$$f(x) = \frac{A\alpha\beta(x)^{\beta-1} \left(1 + \left(\frac{x}{\theta}\right)^\beta\right)^{-\alpha-1}}{\theta^\beta}, \quad 0 < x < \theta.$$

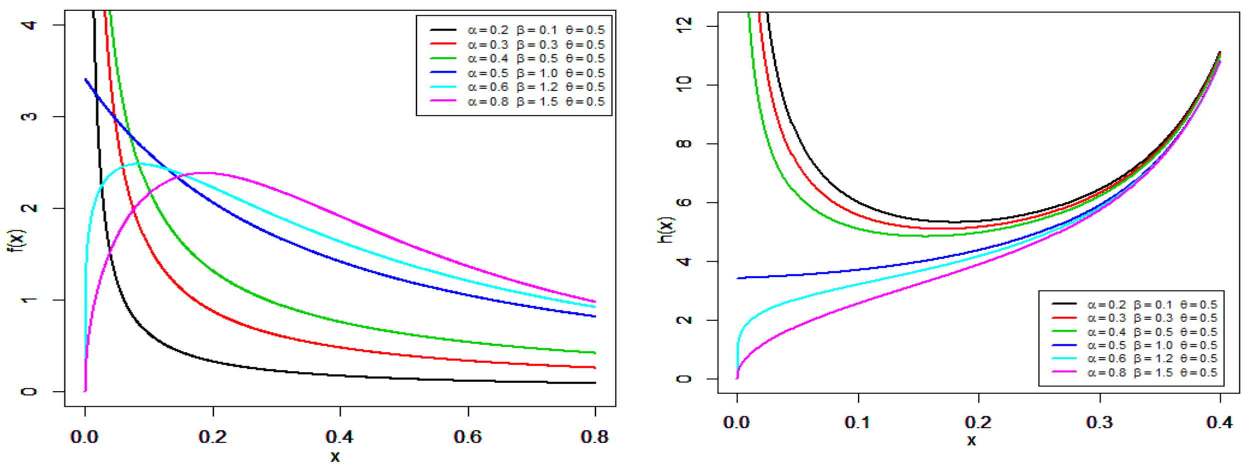
The corresponding cdf takes the following form

$$F(x) = A\left(1 - \left(1 + \left(\frac{x}{\theta}\right)^\beta\right)^{-\alpha}\right).$$

The hrf of TPLU is given by

$$h(x) = \frac{A\alpha\beta(x)^{\beta-1} \left(1 + \left(\frac{x}{\theta}\right)^\beta\right)^{-\alpha-1}}{\theta^\beta \left[1 - A\left(1 - \left(1 + \left(\frac{x}{\theta}\right)^\beta\right)^{-\alpha}\right)\right]}.$$

Plots of pdf and hrf for the TPLU are displayed in Figure 2.



(a) **(b)**

Figure 2: (a) pdf and (b) hrf of the TPLU distribution for different values of parameters

From Figure 2 it appears that the shape of the distribution depends heavily on parameter values.

5.2 TPL -Weibull Distribution

Let us consider the Weibull distribution with $g(x;a,b) = abx^{b-1}e^{-ax^b}$, $x,a,b > 0$ and $G(x;a,b) = 1 - e^{-ax^b}$, we obtain the TPL -Weibull (TPLW) density function as follows

$$f(x) = A\alpha\beta abx^{b-1}e^{-ax^b}\left(1-e^{-ax^b}\right)^{\beta-1}\left(1+\left(1-e^{-ax^b}\right)^{\beta}\right)^{-\alpha-1}, x,a,b > 0.$$

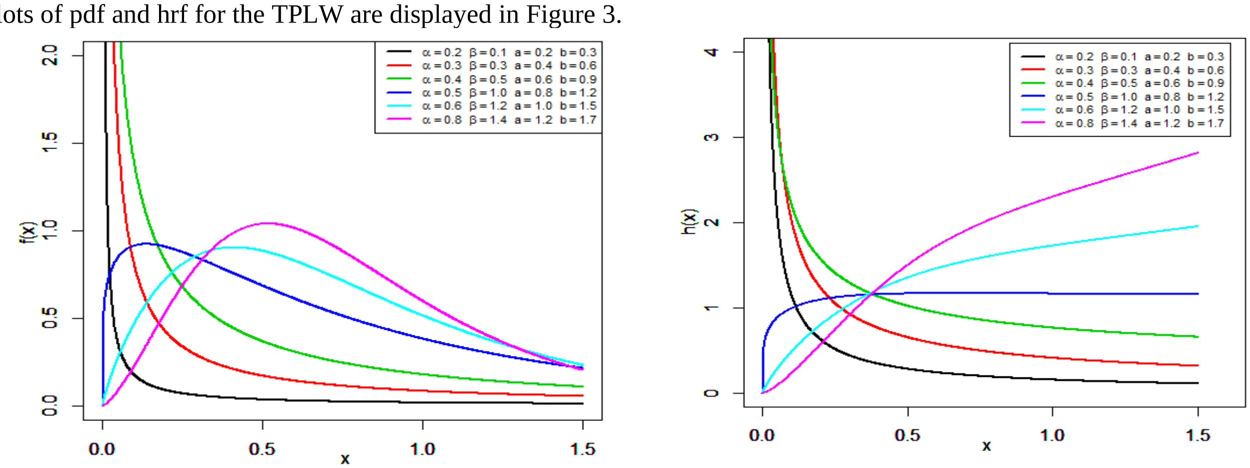
The cdf and hrf of the TPLW distribution are given, respectively, by

$$F(x) = A\left(1-\left(1+\left(1-e^{-ax^b}\right)^{\beta}\right)^{-\alpha}\right),$$

and,

$$h(x) = \frac{A\alpha\beta abx^{b-1}e^{-ax^b}\left(1-e^{-ax^b}\right)^{\beta-1}\left(1+\left(1-e^{-ax^b}\right)^{\beta}\right)^{-\alpha-1}}{1-A\left(1-\left(1+\left(1-e^{-ax^b}\right)^{\beta}\right)^{-\alpha}\right)}.$$

Plots of pdf and hrf for the TPLW are displayed in Figure 3.



(a) **(b)**

Figure 3: (a) pdf and (b) hrf of the TPLW distribution for different values of parameters

5.3 TPL -Frechet Distribution

We consider the Frèchet distribution with $g(x;\mu,\delta)=\delta\mu^{\delta}x^{-\delta-1}e^{-\left(\frac{\mu}{x}\right)^{\delta}}, x,\mu,\delta>0$ and $G(x;\mu,\delta)=1-e^{-\left(\frac{\mu}{x}\right)^{\delta}}$, we obtain the TPL -Frechet (TPLF) density function as follows

$$f(x)=A\alpha\beta\delta\mu^{\delta}x^{-\delta-1}e^{-\left(\frac{\mu}{x}\right)^{\delta}}\left(1-e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{\beta-1}\left(1+\left(1-e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{\beta}\right)^{-\alpha-1}, x,\mu,\delta>0.$$

The cdf and hrf of the TPLF distribution are given, respectively, by

$$F(x)=A\left(1-\left(1+\left(1-e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{\beta}\right)^{-\alpha}\right),$$

and

$$h(x)=\frac{A\alpha\beta\delta\mu^{\delta}x^{-\delta-1}e^{-\left(\frac{\mu}{x}\right)^{\delta}}\left(1-e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{\beta-1}\left(1+\left(1-e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{\beta}\right)^{-\alpha-1}}{1-A\left(1-\left(1+\left(1-e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{\beta}\right)^{-\alpha}\right)}.$$

Plots of pdf and hazard rate function for the TPLF are displayed in Figure 4.

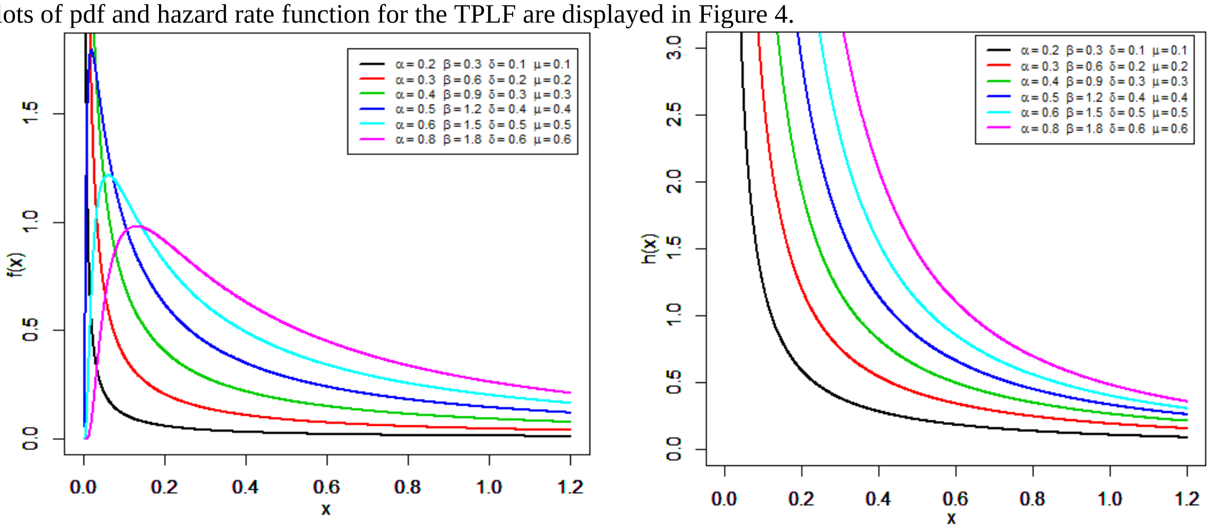


Figure 4: (a) pdf and (b) hrf of the TPLF distribution for different values of parameters

5.4 TPL -Kumaraswamy Distribution

For $g(x;k,s)=ks\,x^{k-1}(1-x^{k})^{s-1}, 0<x<1$, $k,s>0$ and $G(x;k,s)=1-(1-x^{k})^{s}$, we obtain the TPL -Kumaraswamy (TPLK) density function as follows

$$f(x)=A\alpha\beta ksx^{k-1}(1-x^{k})^{s-1}\ [1-(1-x^{k})^{s}]^{\beta-1}\times\{1+[1-(1-x^{k})^{s}]^{\beta}\}^{-\alpha-1}, 0<x<1.$$

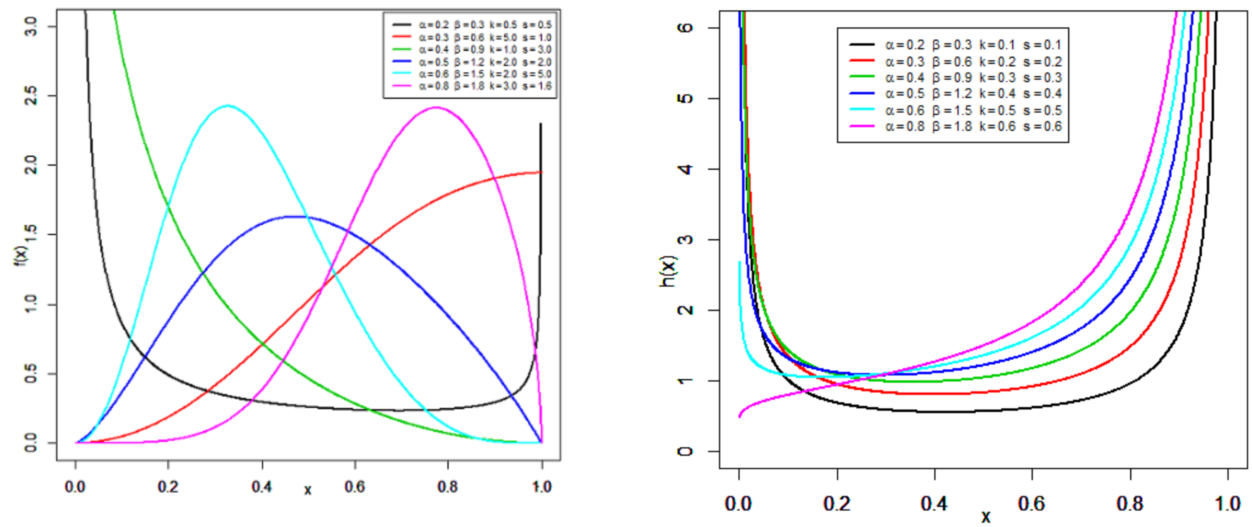
The cdf and hrf of the TPLK distribution are given, respectively, by

$$F(x)=A\left(1-\left(1+[1-(1-x^{k})^{s}]^{\beta}\right)^{-\alpha}\right),$$

and

$$h(x)=\frac{A\alpha\beta ksx^{k-1}(1-x^{k})^{s-1}\ [1-(1-x^{k})^{s}]^{\beta-1}\{1+[1-(1-x^{k})^{s}]^{\beta}\}^{-\alpha-1}}{1-A(1-(1+[1-(1-x^{k})^{s}]^{\beta})^{-\alpha})}.$$

Plots of pdf and hrf for the TPLK are displayed in Figure 5.



(a) (b)

Figure 5:(a) pdf and (b) hrf of the TPLK distribution for different values of parameters

5.5 TPL –Toppe Leone (TPLTL) Distribution

For $g(x;q) = 2q\,x^{q-1}(1-x)(2-x)^{q-1}, 0<x<1 \quad , q>0$ and $G(x;q) = x^q(2-x)^q$, we obtain the TPL – Toppe Leone (TPLTL) density function as follows

$$f(x) = 2A\alpha\beta q x^{q\beta-1}(1-x)(2-x)^{q\beta-1}\ \left(1+x^{q\beta}(2-x)^{q\beta}\right)^{-\alpha}, 0<x<1 \quad , q>0.$$

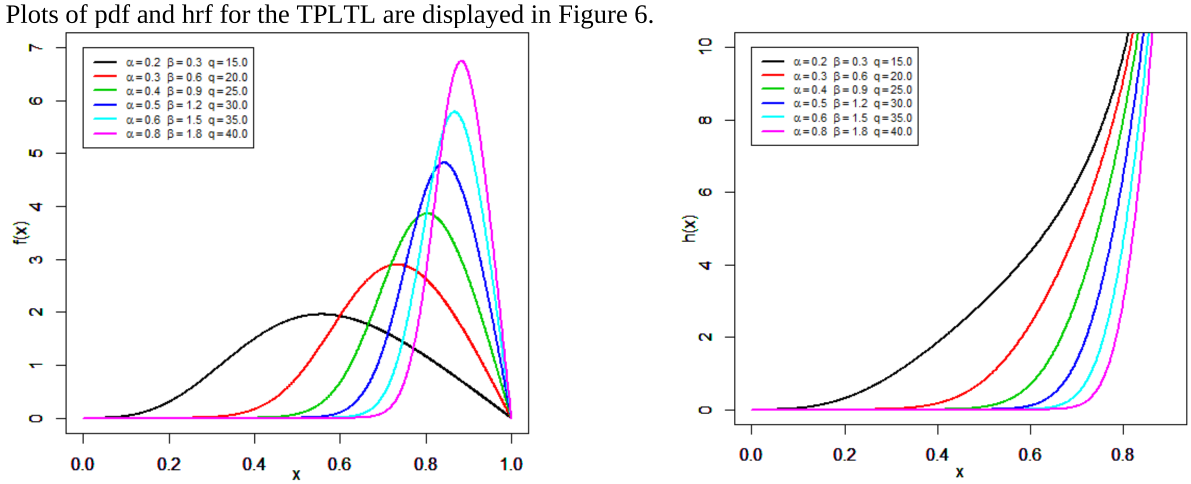
The cdf and hrf of the TPLTL distribution are given, respectively, by

$$F(x) = A\left(1-\left(1+x^{q\beta}(2-x)^{q\beta}\right)^{-\alpha}\right),$$

and

$$h(x) = \frac{2A\alpha\beta q x^{q\beta-1}(1-x)(2-x)^{q\beta-1}\ \left(1+x^{q\beta}(2-x)^{q\beta}\right)^{-\alpha}}{1-A(1-(1+x^{q\beta}(2-x)^{q\beta})^{-\alpha})}.$$

Plots of pdf and hrf for the TPLTL are displayed in Figure 6.



(a) (b)

Figure 6. (a) pdf and (b) hrf of the TPLTL distribution for different values of parameters

6. Maximum Likelihood Method to Estimation

This section deals with the maximum likelihood estimators of the unknown parameters for the TPL-G family of distributions on the basis of complete samples. Let $X_1, X_2, \ldots, X_n$ be the observed values from the TPL-G family with set of parameter $\Phi = (\alpha, \beta, \xi)^T$. The log-likelihood function for parameter vector $\Phi = (\alpha, \beta, \xi)^T$ is obtained as follows

$$\ln L(\Phi) = n\ln\alpha + n\ln\beta - n\ln(1-2^{-\alpha}) + \sum_{i=1}^{n}\ln g(x_i;\xi) + (\beta-1)\sum_{i=1}^{n}\ln G(x_i;\xi) - (\alpha+1)\sum_{i=1}^{n}\ln\left[1+G(x_i;\xi)^{\beta}\right].$$

The partial derivatives of the log-likelihood function with respect to α, β and ξ components of the score vector $U_L = (U_\alpha, U_\beta, U_\xi)^T$ can be obtained as follows

$$U_\alpha = \frac{n}{\alpha} - \frac{n\,2^{-\alpha}\ln 2}{1-2^{-\alpha}} - \sum_{i=1}^{n}\ln\left[1+G(x_i;\xi)^{\beta}\right],$$

$$U_\beta = \frac{n}{\beta} + \sum_{i=1}^{n}\ln G(x_i;\xi) - (\alpha+1)\sum_{i=1}^{n}\frac{G(x_i;\xi)^{\beta}\ln G(x_i;\xi)}{1+G(x_i;\xi)^{\beta}},$$

and

$$U_\xi = \sum_{i=1}^{n}\frac{g'(x_i;\xi)}{g(x_i;\xi)} + (\beta-1)\sum_{i=1}^{n}\frac{G'(x_i;\xi)}{G(x_i;\xi)} - \beta(\alpha+1)\sum_{i=1}^{n}\frac{G'(x_i;\xi)\,G(x_i;\xi)^{\beta-1}}{1+G(x_i;\xi)^{\beta}},$$

where, $g'(x_i;\xi) = \partial g(x_i;\xi)/\partial\xi$ and $G'(x_i,\xi) = \partial G(x_i;\xi)/\partial\xi$. Setting $U_\alpha, U_\beta, and\ U_\xi$ equal to zero and solving these equations simultaneously yield the maximum likelihood estimators $\hat{\Phi} = (\hat{\alpha}, \hat{\beta}, \hat{\xi})^T$ of $\Phi = (\alpha, \beta, \xi)^T$. Unfortunately these equations cannot be solved analytically and numerical iterative methods could be used to solve.

7. Simulation Study

In this section, the performance of the maximum likelihood estimators is assessed in terms of the sample size n. A numerical evaluation is carried out to examine the performance of maximum likelihood estimators for TPLW model (as particular case from the family). The evaluation of estimates is performed based on the biases and the empirical mean square errors (MSEs). The simulation is made using the MATHEMATICA package and the numerical steps are listed as follows:

Step 1: A random sample $X_1, X_2, \ldots X_n$ of sizes; n =10, 20, 30, 50,75 and 100 are considered, these random samples are generated from the TPLW distribution by using inversion method.

Step 2: Six sets of the parameters are considered as Set1 [$\alpha = 0.5, \beta = 0.5, a = 0.5, b = 0.5$], Set2 [$\alpha = 0.5, \beta = 0.5, a = 0.5, b = 1.5$], Set3 [$\alpha = 0.5, \beta = 0.5, a = 0.5, b = 2.0$], Set4 [$\alpha = 1.5, \beta = 0.5, a = 0.5, b = 0.5$], Set5 [$\alpha = 2.0, \beta = 0.5, a = 0.5, b = 0.5$], and Set6 [$\alpha = 1.5, \beta = 0.5, a = 0.5, b = 1.5$]. The maximum likelihood estimate (MLE) of TPLW model is evaluated for each parameters value and for each sample size.

Step 3: Repeat this process 3000 times and then obtain the means, biases and MSEs of the MLE for different values of model parameters at each sample size. Empirical results are reported in Tables 1-3. We can detect from these tables that the estimates are quite stable and are close to the true value of the parameters as the sample sizes increase.

Table 1: MLE, Bias and MSE of Model Parameters for Set 1 and Set 2

n	Parameters	Set 1: $\alpha = 0.5, \beta = 0.5, a = 0.5, b = 0.5$			Set 2: $\alpha = 0.5, \beta = 0.5, a = 0.5, b = 1.5$		
		MLE	Bias	MSE	MLE	Bias	MSE
10	α	0.50013	0.00013	0.00025	0.50059	0.00059	0.00025
	β	0.58751	0.08751	0.07775	0.57699	0.07699	0.06661
	a	0.51592	0.01592	0.00875	0.51870	0.01870	0.00906
	b	0.61550	0.11550	0.14135	1.82395	0.32395	0.96733
20	α	0.50029	0.00029	0.00013	0.50027	0.00027	0.00012
	β	0.53747	0.03747	0.02558	0.53530	0.03530	0.02231
	a	0.50931	0.00931	0.00475	0.51009	0.01009	0.00446
	b	0.54514	0.04514	0.03058	1.65493	0.15493	0.31111
30	α	0.49996	-0.00004	0.00008	0.50002	0.00002	0.00008
	β	0.52537	0.02537	0.01366	0.52548	0.02548	0.01407

	a	0.50554	0.00554	0.00302	0.50600	0.00600	0.00306
	b	0.53427	0.03427	0.01991	1.59985	0.09985	0.17032
50	α	0.50018	0.00017	0.00005	0.50015	0.00015	0.00005
	β	0.51336	0.01336	0.00748	0.51287	0.01287	0.00693
	a	0.50443	0.00443	0.00186	0.50320	0.00320	0.00183
	b	0.52023	0.02022	0.01077	1.54944	0.04944	0.09067
75	α	0.50003	0.00003	0.00003	0.50019	0.00019	0.00003
	β	0.50990	0.00990	0.00497	0.50854	0.00854	0.00493
	a	0.50280	0.00280	0.00127	0.50302	0.00302	0.00135
	b	0.51380	0.01380	0.00672	1.53698	0.03698	0.06242
100	α	0.50012	0.00012	0.00002	0.50008	0.00008	0.00003
	β	0.50552	0.00552	0.00350	0.50710	0.00710	0.00362
	a	0.50220	0.00220	0.00091	0.50199	0.00199	0.00095
	b	0.50794	0.00794	0.00446	1.52819	0.02819	0.04420

Table 2: MLE, Bias and MSE of Model Parameters for Set 3 and Set 4

n	Parameters	Set 3: $\alpha = 0.5, \beta = 0.5, a = 0.5, b = 2.0$			Set 4: $\alpha = 1.5, \beta = 0.5, a = 0.5, b = 0.5$		
		MLE	Bias	MSE	MLE	Bias	MSE
10	α	0.50056	0.00056	0.00025	1.51148	0.01148	0.01935
	β	0.57704	0.07704	0.06617	0.60320	0.10320	0.11383
	a	0.51867	0.01867	0.00886	0.51342	0.01342	0.00688
	b	2.42679	0.42679	1.74762	0.61597	0.11597	0.14514
20	α	0.50048	0.00048	0.00012	1.50696	0.00696	0.00985
	β	0.53158	0.03158	0.02233	0.54673	0.04673	0.03267
	a	0.51064	0.01064	0.00454	0.50832	0.00832	0.00368
	b	2.19139	0.19139	0.56633	0.55332	0.05332	0.03763
30	α	0.50023	0.00023	0.00008	1.50717	0.00717	0.00616
	β	0.52471	0.02471	0.01383	0.52280	0.02280	0.01605
	a	0.50699	0.00699	0.00302	0.50609	0.00609	0.00235
	b	2.13850	0.13850	0.30946	0.52797	0.02797	0.01894
50	α	0.49998	-0.00002	0.00005	1.50341	0.00341	0.00386
	β	0.51448	0.01448	0.00755	0.51501	0.01501	0.00956
	a	0.50276	0.00276	0.00188	0.50330	0.00330	0.00149
	b	2.07118	0.07118	0.16556	0.51742	0.01742	0.01086
75	α	0.49997	-0.00003	0.00003	1.50109	0.00109	0.00263
	β	0.50966	0.00966	0.00483	0.51287	0.01287	0.00618

	a	0.50178	0.00178	0.00128	0.50213	0.00213	0.00101
	b	2.04795	0.04795	0.10956	0.51487	0.01487	0.00686
100	α	0.50000	0.00001	0.00002	1.50145	0.00145	0.00193
	β	0.50743	0.00743	0.00347	0.50802	0.00802	0.00425
	a	0.50177	0.00177	0.00092	0.50120	0.00120	0.00078
	b	2.04063	0.04063	0.07781	0.50833	0.00833	0.00498

Table 3: MLE, Bias and MSE of Model Parameters for Set 5 and Set 6

n	Parameters	Set5: $\alpha = 2.0, \beta = 0.5, a = 0.5, b = 0.5$			Set6: $\alpha = 1.5, \beta = 0.5, a = 0.5, b = 1.5$		
		MLE	Bias	MSE	MLE	Bias	MSE
10	α	2.02690	0.02690	0.06281	1.5114	0.01143	0.01942
	β	0.60687	0.10687	0.90418	0.6046	0.10467	0.11833
	a	0.51150	0.01150	0.00666	0.5126	0.01269	0.00692
	b	0.61663	0.11663	0.22513	1.8382	0.33823	1.25968
20	α	2.01666	0.01666	0.03020	1.5065	0.00652	0.00998
	β	0.54769	0.04769	0.03915	0.5366	0.03669	0.03114
	a	0.50688	0.00688	0.00334	0.5063	0.00631	0.00370
	b	0.54819	0.04819	0.03885	1.6099	0.10990	0.26747
30	α	2.01104	0.01104	0.02046	1.5007	0.00077	0.00640
	β	0.53232	0.03232	0.02420	0.5350	0.03507	0.01997
	a	0.50513	0.00513	0.00228	0.5033	0.00339	0.00242
	b	0.53194	0.03194	0.02221	1.6102	0.11027	0.19135
50	α	2.00299	0.00299	0.01158	1.5023	0.00234	0.00391
	β	0.51893	0.01893	0.01042	0.5176	0.01760	0.00978
	a	0.50147	0.00147	0.00137	0.5032	0.00323	0.00153
	b	0.51754	0.01754	0.01050	1.5612	0.06124	0.10012
75	α	2.00460	0.00460	0.00790	1.5023	0.00231	0.00259
	β	0.51118	0.01118	0.00639	0.5098	0.00984	0.00600
	a	0.50202	0.00202	0.00093	0.5027	0.00278	0.00101
	b	0.51172	0.01172	0.00663	1.5372	0.03724	0.05851
100	α	2.00379	0.00379	0.00585	1.5019	0.00199	0.00190
	β	0.50870	0.00870	0.00482	0.5059	0.00591	0.00431
	a	0.50169	0.00169	0.00068	0.5016	0.00168	0.00075
	b	0.50949	0.00949	0.00501	1.5205	0.02059	0.04312

8. Applications to real data

In this section, real data set are analyzed to illustrate the merit of TPLW distribution compared to some models; namely, Type I half logistic Weibull (TIHLW) (Cordeiro *et al.* (2015)), Weibull Weibull (WW) (Abouelmagd *et al.* (2017)), Weibull exponential (WE) (Oguntunde *et al.* (2015)), and beta Weibull (BW) (Lee *et al.* (2007)) dsitributions.

We obtain the MLE and their corresponding standard errors (in parentheses) of the model parameters. To compare the distribution models, we consider criteria like; minus two of log-likelihood function (-2lnL), Akaike information criterion (AIC), the correct Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan Quinn information criterion (HQIC), the Kolmogorov-Smirnov test (K-S) and p-value. However, the better distribution corresponds to the smaller values of -2 lnL, AIC, CAIC, BIC, HQIC, K-S criteria and largest p-value. Furthermore, we plot the histogram for each data set and the estimated pdf of the TPLW, TIHLW, WE, BW and WW models. Moreover, the plots of empirical cdf of the data set and estimated pdf of TPLW, TIHLW, WE, BW and WW models are displayed in Figure 7.

The data set have been obtained from Hinkley (1977) and represents thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data are as follows: 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Table 4 gives the MLE of parameters of the TPLW and their standard error (SE). The values of the log-likelihood functions, AIC, CAIC, BIC, HQIC, K-S and p-values are presented in Table 5.

Table 4: MLE and SE of the model parameters for data set

Model	MLEs and SE				
TPLW $(\alpha, \beta, a, \mathrm{b})$	48.865 (143.009)	26.911 (138.04)	1.767 (4.393)	0.217 (0.508)	
TIHLW(α, β, λ)	0.626 (0.124)	1.532 (0.293)	0.889 (0.1791)	-	-
WE(α, β, λ)	-	-	35.218 (0.26269)	1.69 (0.234)	0.06 (0.044)
BW (a, b, α, β)	25.851 (1.533)	15.276 (0.787)	0.884 (0.201)	0.335 (0.027)	-
WW$(\alpha, \beta, \lambda, \gamma)$	39.853 (0.414)	3.154 (0.518)	0.196 (0.102)	0.5 (0.072)	-

Table 5: The values of -2LnL, AIC, BIC, CAIC, HQIC, K-S and p-value for data set

Model	-2LnL	AIC	CAIC	BIC	HQIC	K-S	p-value
TPLW	76.209	84.209	85.809	82.118	86.002	0.0581	0.9999
TIHLW	106.63	112.63	112.211	113.562	115.358	0.069	0.9988
WE	112.10	118.10	119.031	116.539	119.452	0.0753	0.996
BW	149.89	157.89	157.326	159.497	161.522	0.0795	0.9913
WW	138.19	146.19	145.623	147.794	149.819	0.0754	0.9955

We find that the TPLW distribution with four-parameter provides a better fit than some new models. It has the smallest K-S, AIC, CAIC, BIC and HQIC values among those considered here. Plots of the fitted densities and the histogram are given in Figures 7.

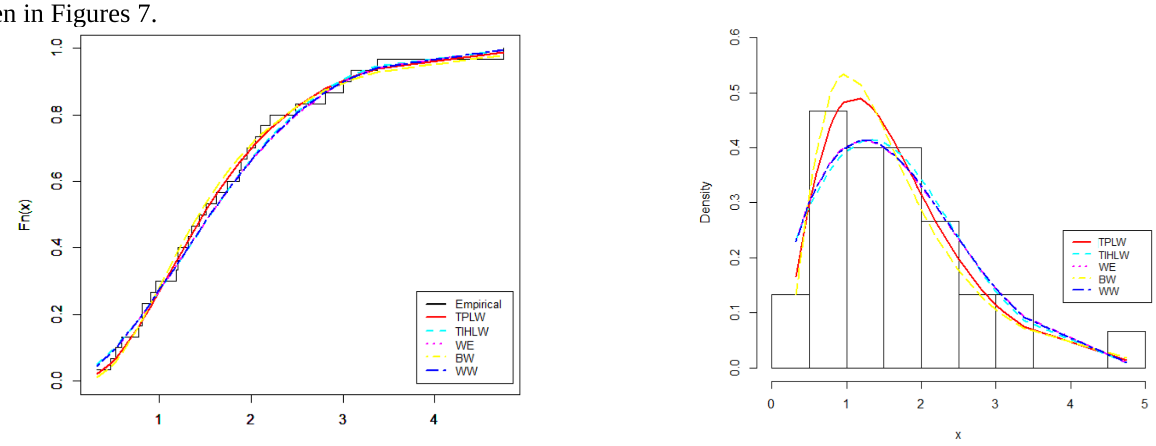


Figure 7. Estimated cumulative and estimated densities for all models for data set

9. Summary and Conclusion

In this paper, a new truncated family of probability distributions called the right truncated power Lomax-G family is introduced. The new family is considered as the generalization of truncated Lomax-G family presented by Hassan et al. (2020 b). Several structural properties of the family, such as, linear representations for the density function and cumulative distribution function, expressions for the ordinary moments, moment generating function and order statistics are investigated. Special sub-models are presented. A simulation study is carried out for one particular case, to assess the finite sample behavior of the maximum likelihood estimates. It can be detected that the estimates are quite stable and are close to the true value of the parameters as the sample sizes increase. An application to a real life data shows that the right truncated power Lomax Weibull distribution is a strong and better competitor for the Type I half logistic Weibull distribution, Weibull Weibull distribution, Weibull exponential distribution, and the beta Weibull distribution.

Acknowledgments

The authors would like to thank the editor and the anonymous referees, for their valuable and very constructive comments, which have greatly improved the contents of the paper.

References

1. Abid, S.H., & Abdulrazak, R.K. (2017). [0,1] truncated Fréchet-G generator of distributions. Applied Mathematics, 7, 51-66
2. Abouelmagd, T.H.M., Saeed Al-Mualim, Elgarhy, M., Ahmed, Z. Afify & Munir Ahmad (2017). Properties of the four–parameter Weibull distribution and its Applications. Pakistan Journal of Statistics, 33, 449-466
3. Alzaatreh, A., Lee, C. & Famoye, F. (2013). A new method for generating families of continuous distributions. Metron, 71, 63-79
4. Atkinson, A., & Harrison, A. (1978). Distribution of Personal Wealth in Britain. Cambridge University Press, Cambridge
5. Bourguignon, M., Silva, R.B. & Cordeiro, G. M. (2014). The Weibull– G family of probability distributions. Journal of Data Science, 12, 53-68
6. Cordeiro, G.M. & de Castro, M. (2011). A new family of generalized distribution. Journal of Statistical Computations and Simulation, 81, 883-898.
7. Cordeiro, G.M., Alizadeh, M. & Marinho, E. P. R. D. (2015). The type I half-logistic family of distributions. Journal of Statistical Computation and Simulation, 86, 707-728.
8. Cordeiro, G.M., Alizadeh, M. & Ortega, E. M. M. (2014 a). The exponentiated half-logistic family of distributions: properties and applications. Journal of Probability and Statistics, Article ID 864396, 21 pages. http://dx.doi.org/10.1155/2014/864396
9. Cordeiro, G.M., Ortega, E.M.M. & da Cunha, D.C.C. (2013). The exponentiated generalized class of distributions. Journal of Data Science, 11, 1-27
10. Cordeiro, G. M., Ortega, E.M.M, Popovic, B.V. & Pescim, R.R. (2014 b). The Lomax generator of distributions: properties, minification process and regression model. Applied Mathematics and Computation, 247, 465-486
11. Cordeiro, G.M., Alizadeh, M. Tahir, M.H. Mansoor , M. Bourguignonk, M. & Hamedani, G.G. (2016). The beta odd log-logistic generalized family of distributions. Hacettepe Journal of Mathematics and Statistics, 45, 1175 – 1202
12. David, H.A. (1981). Order Statistics. Second edition. Wiley, New York.
13. Elgarhy, M., Haq, M. & Ozel, G. (2017). A new exponentiated extended family of distributions with Applications. Gazi University Journal of Science, 30, 101-115
14. Eugene, N., Lee, C. & Famoye, F. (2002). The beta-normal distribution and its applications. Communications in Statistics & Theory and Methods, 31, 497-512
15. Greenwood, J. A., Landwehr, J. M., & N.C. Matalas, (1979). Probability weighted moments: definitions and relations of parameters of several distributions expressible in inverse form. Water Resources Research, 15, 1049-1054

16. Haq, M. & Elgarhy, M. (2018). The odd Frèchet-G family of probability distributions. Journal of Statistics Applications & Probability, 7, 185-201
17. Harris, C.M. (1968). The Pareto distribution as a queue service discipline. Operations Research, 16 (2), 307–313
18. Hassan A.S. & Al-Ghamdi, A. (2009) Optimum step stress accelerated life testing for Lomax distribution. Journal of Applied Sciences Research, 5, 2153–2164.
19. Hassan, A.S. & Elgarhy, M. (2016 a). Kumaraswamy Weibull-generated family of distributions with applications. Advances and Applications in Statistics, 48, 205-239
20. Hassan, A.S. & Elgarhy, M. (2016 b). A New Family of Exponentiated Weibull-Generated Distributions. International Journal of Mathematics & its Applications, 4, 135-148
21. Hassan, A.S., & Hemeda, S.E. (2016). The additive Weibull-G family of probability distributions. International Journals of Mathematics & Its Applications, 4, 151-164
22. Hassan, A. S., & Nassr, S. G. (2018). The inverse Weibull generator of distributions: properties and applications. Journal of Data Sciences, 16(4), 732-742.
23. Hassan, A.S. & Nassar, S.G. (2019). Power Lindley-G family. Annals of Data Science, 6(2), 189–210.
24. Hassan, A.S., M.S. Assar & A. Shelbaia (2016). Optimum Step-Stress Accelerated Life Test Plan for Lomax Distribution with an Adaptive Type-II Progressive Hybrid Censoring. Journal of Advances in Mathematics and Computer Science. 13(2), 1-19.
25. Hassan, A.S., Elgarhy, M., & Shakil, M. (2017a). Type II half logistic family of distributions with applications. Pakistan Journal of Statistics &Operation Research, 13, 245-264
26. Hassan, A.S., Elgarhy, M., & Ahmad, Z. (2019). Type II Generalized Topp–Leone Family of Distributions: Properties and Applications. Journal of Data Science, 17(4), 638-659
27. Hassan, A.S., Sabry, M., A. & Elsehetry, A. M. (2020 a). Truncated power Lomax distribution with application to flood data. Journal of Statistics Applications & Probability, 9(2), 347-359
28. Hassan, A.S., Sabry, M., A. & Elsehetry, A. M. (2020 b). A new family of upper-truncated distributions: properties and estimation. Thailand Statistician, 18(2): 196-214
29. Hassan, A.S., Hemeda, S.E., Maiti, S.S. & Pramanik, S. (2017 b). The generalized additive Weibull-G family of probability distributions. International Journal of Statistics and Probability, 6, 65-83
30. Hinkley, D. (1977). On quick choice of power transformations. Journal of the Royal Statistical Society, Series (c), Applied Statistics, 26, 67–69
31. Lee, C., Famoye, F., & Olumolade, O. (2007). Beta-Weibull distribution: some properties and applications to censored data. *Journal of Modern Applied Statistical Methods*, 6, 173-186
32. Lomax, K. S. (1954). Business failures: another example of the analysis of failure data. Journal of the American Statistical Association, 49, 847-852
33. Mansoor, M., Tahir, M.H., Cordeiro, G.M., Alzaatreh, A., & Zubair, M. (2017). A new family of distributions to analyze lifetime data. Journal of Statistical Theory and Applications, 16, 490-507
34. Oguntunde, P.E., Balogun, O.S., Okagbue, H.I. & Bishop, S.A. (2015). The Weibull-Exponential distribution: Its properties and applications. Journal of Applied Sciences,15, 1305-1311
35. Rady, E.H.A., Hassanein, W.A., & Elhaddad, T.A. (2016). The power Lomax distribution with an application to bladder cancer data. SpringerPlus, 5, 1-22
36. Renyi A. (1961). On measures of entropy and information. In: Proceedings of the 4th Fourth Berkeley Symposium on Mathematical Statistics and Probability, 547-561. University of California Press, Berkeley.
37. Tahir, M.H., Cordeiro, G.M., Alizadeh, M., Mansoor, M., Zubair, M. & Hamedani, G.G., (2015). The odd generalized exponential family of distributions with applications. Journal of Statistical Distributions and Applications, 2, 1-28