Pak.j.stat.oper.res. Vol.18 No. 3 2022 pp 537-553 DOI: http://dx.doi.org/10.18187/pjsor.v18i3.3420

Pakistan Journal of Statistics and Operation Research

Expanding the Nadarajah Haghighi Model: Characterizations, Properties and Application

Mohamed Ibrahim[1], G.G. Hamedani[2],
Nadeem Shafique Butt[3] and Haitham M. Yousof [4*]

* Corresponding Author

[1] Department of Applied, Mathematical and Actuarial Statistics, Faculty of Commerce, Damietta University, Damietta, Egypt. mohamed_ibrahim@du.edu.eg
[2] Department of Mathematics, Statistics and Computer Science, Marquette University, USA. gholamhoss.hamedani@marquette.edu
[3] Department of Family and Community Medicine King Abdul Aziz University, Jeddah, Kingdom of Saudi Arabia. nshafique@kau.edu.sa
[4] Department of Statistics, Mathematics and Insurance, Benha University, Egypt. haitham.yousof@fcom.bu.edu.eg

Abstract

A new three-parameter Nadarajah Haghighi model is introduced and studied. The new density has various shapes such as the right skewed, left skewed and symmetric and its corresponding hazard rate shapes can be increasing, decreasing, bathtub, upside down and constant. Characterization results are obtained based on two truncated moments and in terms of the hazard function. An example is presented for illustrating the importance of the new model.

Key Words: Characterizations; Modeling; Nadarajah Haghighi Model; Hazard Rate Function.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1.Introduction and motivation

In recent years, it has become quite usual to employ modern continuous distribution generators instead of traditional ones. In the statistical literature, the process of enlarging a class of distributions by including additional shape parameter(s) is well-known. Modeling and lifespan data analysis are essential in many practical fields, including, among others, health, engineering, and finance. Different forms of survival data have been modelled using a variety of lifetime distributions. The produced distribution has a significant impact on the effectiveness of the techniques employed in a statistical study. Additionally, without selecting the appropriate probability distribution, statistical modelling of the phenomena, applications, or the validity of data are all impossible (the mathematical form of the model). As a result, significant effort has been made to investigate novel statistical approaches. Additionally, the difficulties associated with computing special functions in the new expanded families may be readily overcome because to the computational and analytical capabilities found in programming languages like R, Maple, and Mathematica.

Numerous statisticians are inspired to create new expanded models by these tools. However, several significant issues concerning actual data still exist and do not fit into any of the conventional statistical models. The ability to describe both monotonic and non-monotonic failure rates, even while the baseline failure rate may be monotonic, is what drives generalized distributions for lifetime data modelling. The additional shape parameter(s) is to induce skewness and to change the tail weights. Moreover, by expanding popular families of lifetime distributions, several classes of distributions have been created, and their varied properties have been examined. Because of their adaptable characteristics, generalized families of distributions have drawn the attention of theoretical and applied statisticians.

Among the parametric distributions, the exponential (Exp) model is perhaps the most widely applied model in several fields. The Exp model has "constant" hazard rate function (HRF). A new generalization of the Exp distribution as an alternative to the gamma, Weibull and exponentiated exponential (EExp) distributions was recently proposed and studied by Nadarajah and Haghighi (2011). The cumulative distribution function (CDF) of the Nadarajah and Haghighi (NH) model is given by

$$G_{a,b}(y) = 1 - exp[1 - (1 + by)^a]\,|_{(y>0)}.$$

For $a = 1$, the NH model reduces to Exp the model. Lemonte (2013) proposed and studied a new three-parameter exponential-type model called the generalized Nadarajah-Haghighi (GNH). Ortega et al. (2015) investigated the Gamma-NH (GamNH) with a new regression model. Lemonte et al. (2016) proposed the Marshall-Olkin-NH (MONH). Yousof and Korkmaz (2017) proposed and studied the Topp-Leone Nadarajah-Haghighi (TLNH) model bsed on the Topp-Leone family (Rezaei et al. (2017). Based on the odd Lindley family (Silva et al. (2017)), Yousof et al. (2017) derived and studied the odd Lindley Nadarajah-Haghighi (OLNH) distribution. Using beta family, Dias et al. (2018) introduced and studied the beta-NH (BNH). Ibrahim (2019) studied the odd log-logistic NH (OLLNH) and Proportional reversed hazard rate (PRHRNH) models with its corresponding statistical properties and different methods of estimation. Recently, Nascimento et al. (2019) employed the $G_{a,b}(y)$ for building and new system of NH densities based on the odd ratio (OR) function $O_{\underline{\psi}}(y) = G_{a,b}(y)/[1 - G_{a,b}(y)]$. Following Lemonte (2013), the CDF of the GNH model is given by

$$G_{\theta,a,b}(y) = \left(1 - \tau_{y;a,b}\right)^{\theta}|_{(y>0)}, \tag{1}$$

where

$$\tau_{y;a,b} = exp[1 - (1 + by)^a],$$

and the corresponding probability density function (PDF) is

$$g_{\theta,a,b}(y) = \theta ab(1 + by)^{a-1}\tau_{y;a,b}\left(1 - \tau_{y;a,b}\right)^{\theta-1}|_{(y>0)}, \tag{2}$$

where the parameters $\theta > 0$ and $a > 0$ control the shape of the distribution and $b > 0$ is the scale parameter. When $\theta = 1$, the GNH model reduces to the NH model (see Nadarajah and Haghighi (2011)). When $a = 1$, we have the generalized exponential (GExp) model (see Gupta and Kundu (1999)). When $\theta = a = 1$, we have the standard Exp model. On the other hand, Silva et al. (2017) presented a new class of distributions called the odd Lindley G (OL-G) family. The PDF and CDF of the OL-G family of distributions are given by

$$f_{\underline{\psi}}(y) = \frac{1}{2}\frac{g_{\underline{\psi}}(y)}{\overline{G}^{3}_{\underline{\psi}}(y)}\,exp\left[-\frac{G_{\underline{\psi}}(y)}{\overline{G}_{\underline{\psi}}(y)}\right]|_{(y\in R)}, \tag{3}$$

and

$$F_{\underline{\psi}}(y) = 1 - \frac{1 + \overline{G}_{\underline{\psi}}(y)}{2\overline{G}_{\underline{\psi}}(y)}\,exp\left[-\frac{G_{\underline{\psi}}(y)}{\overline{G}_{\underline{\psi}}(y)}\right]|_{(y\in R)}, \tag{4}$$

respectively, where $G_{\underline{\psi}}(y)$ is the CDFof any baseline model, $\overline{G}_{\underline{\psi}}(y) = 1 - G_{\underline{\psi}}(y)$ is the survival function (SF) of any baseline model, $g_{\underline{\psi}}(y) = \frac{d}{dy}G_{\underline{\psi}}(y)$ is the PDFof the baseline model. To this end, we use (1), (2) and (3) to obtain the three-parameter odd Lindley generalized Nadarajah-Haghighi (OLGNH) PDF as

$$f_{\theta,a,b}(y) = \frac{\theta ab}{2}(1 + by)^{a-1}\frac{\tau_{y;a,b}\left(1 - \tau_{y;a,b}\right)^{\theta-1}}{\left[1 - \left(1 - \tau_{y;a,b}\right)^{\theta}\right]^{3}}\,exp\left[-\frac{\left(1 - \tau_{y;a,b}\right)^{\theta}}{1 - \left(1 - \tau_{y;a,b}\right)^{\theta}}\right]. \tag{5}$$

The corresponding CDF is given by

$$F_{\theta,a,b}(y) = 1 - \frac{1 + \left[1 - \left(1 - \tau_{y;a,b}\right)^{\theta}\right]}{2\left[1 - \left(1 - \tau_{y;a,b}\right)^{\theta}\right]}\,exp\left[-\frac{\left(1 - \tau_{y;a,b}\right)^{\theta}}{1 - \left(1 - \tau_{y;a,b}\right)^{\theta}}\right]. \tag{6}$$

The new CDF in (4) can be used for presenting a new discrete G family for modeling the count data (see Aboraya et al. (2020), Chesneau et al. (2021), Ibrahim et al. (2021) and Yousof et al. (2021) for more details). The OLGNH density function can be expressed as an infinite mixture of GNH PDF as follows

$$f(y) = \sum_{\kappa_1,\kappa_2=0}^{\infty} c_{\kappa_1,\kappa_2}\,\pi_{\theta^*,a,b}(y)|_{(\theta^*=(\kappa_1+\kappa_2+1)\theta)}, \tag{7}$$

where

$$c_{\kappa_1,\kappa_2}=\frac{(-1)^{\kappa_2}\Gamma(\kappa_1+\kappa_2+3)}{2(\kappa_1+\kappa_2+1)\kappa_1!\,\kappa_2!\,\Gamma(\kappa_2+3)},$$

and

$$\pi_{\theta^*,a,b}(y)=\theta^* ab\tau_{y;a,b}(1+by)^{a-1}\left[1-\tau_{y;a,b}\right]^{\theta^*-1}$$

represents the GNH density with power parameter $\theta^*>0$. The CDF of OLGNH model can be given by integrating (7) as

$$F(y)=\sum_{\kappa_1,\kappa_2=0}^{\infty} c_{\kappa_1,\kappa_2}\,\Pi_{\kappa_1+\kappa_2+1}(y), \tag{8}$$

where

$$\Pi_{\theta^*,a,b}(y)=\left[1-\tau_{y;a,b}\right]^{\theta^*}$$

is the CDF of the GNH model with power parameter $\theta^*>0$. Table 1 gives some sub-models from the OLGNH model. In this paper, we present some characterizations of the OLGNH model in terms of a simple relationship between two two truncated moments (TTMs) and based on HRF, this characterization results are stable in the sense of weak convergence. The PDF of the OLGNH distribution has various useful shapes such as the right skewed, left skewed and symmetric. The HRF of the OLGNH model produces flexible HRF shapes such as "increasing", "decreasing", "bathtub (**U**-HRF)", "upside down" and "constant". Some of its properties such as moments, incomplete moments, moment generating function (MGF), moments of residual life and reversed residual life are mathematically derived. A numerical analysis for the variance, skewness and kurtosis measures is presented. The uncensored exceedances of flood peaks data are employed for comparing the combative model.

Table 1: Some sub-models from the OLGNH model.

θ	a	b	Model	CDF	Author
1			OLNH	$1-\frac{1-\tau_{y;a,b}}{2\tau_{y;a,b}}exp\left(-\left\{\left[1-\tau_{y;a,b}\right]^{-1}-1\right\}^{-1}\right)$	Yousof et al. (2017)
	1		OLGExp	$1-\frac{\left[1-\tau_{y,b}\right]^{\theta}}{2\left\{1-\left[1-\tau_{y;b}\right]^{\theta}\right\}}exp\left(-\left\{\left[1-\tau_{y;b}\right]^{-\theta}-1\right\}^{-1}\right)$	Yousof et al. (2017)
1	1		OLExp	$1-\frac{1-\tau_{y,b}}{2\tau_{y;b}}exp\left(-\left\{\left[1-\tau_{y;b}\right]^{-1}-1\right\}^{-1}\right)$	Yousof et al. (2017)

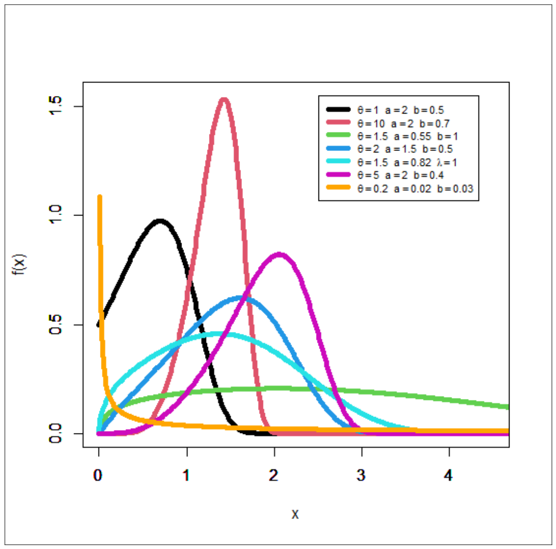


Figure 1: Plots of the OLGNH PDF for some parameter values.

 DOI: http://dx.doi.org/10.18187/pjsor.v18i3.3420

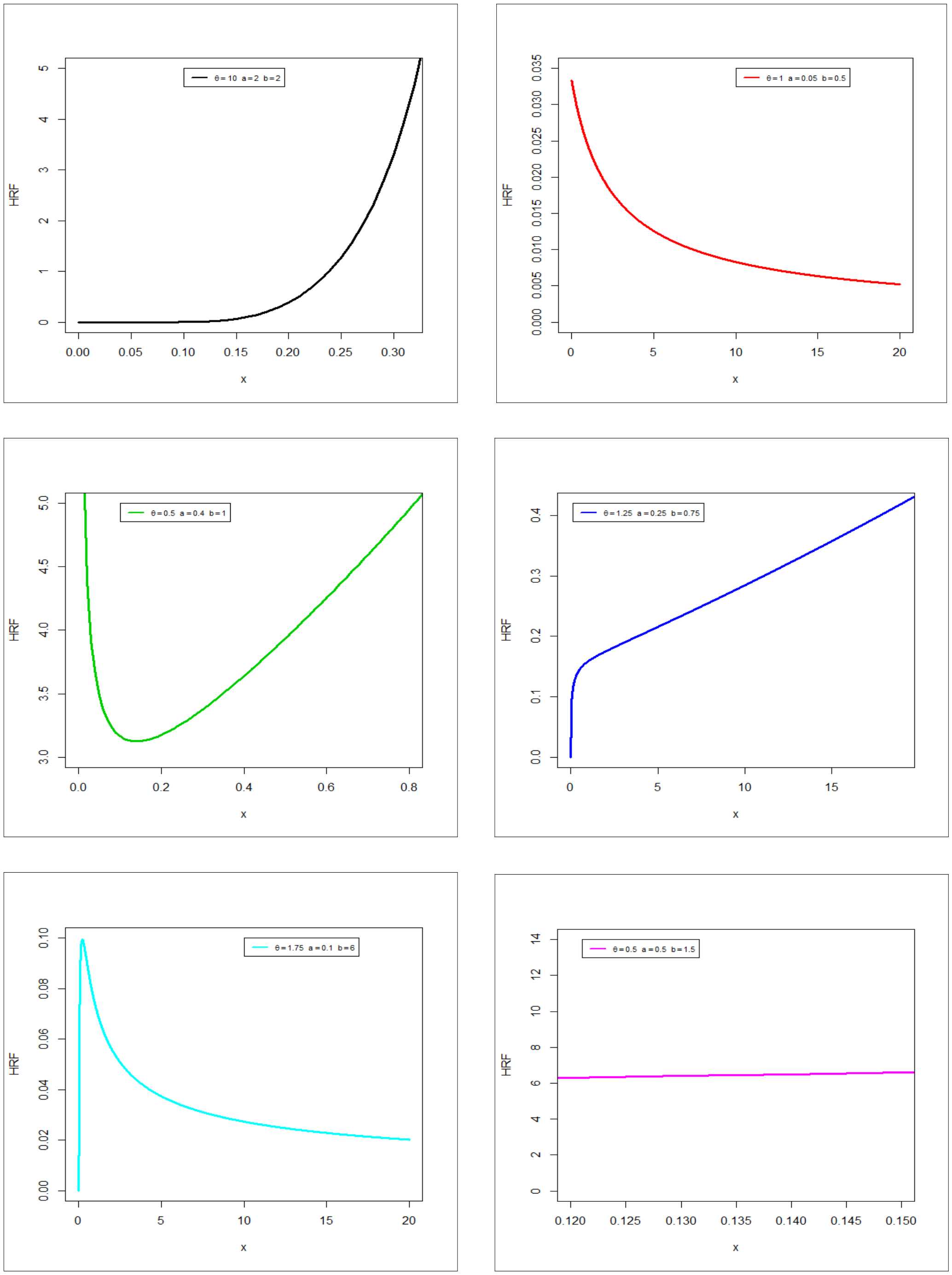


Figure 2: Plots of the OLGNH HRF for some parameter values.

Figure 1 shows that the OLGNH distribution has various PDF shapes such as the right skewed, left skewed and symmetric. Figure 2 shows that the OLGNH model produces flexible HRF shapes such as "increasing $(\theta = 10, a = 2, b = 2)$", "decreasing $(\theta = 1, a = 0.05, b = 0.5)$", "bathtub (**U**-HRF) $(\theta = 0.5, a = 0.4, b = 1)$", "upside down $(\theta = 1.25, a = 0.1, b = 6)$" and "constant $(\theta = 0.5, a = 0.5, b = 1.5)$". These plots indicate that the OLGNH model is very useful in fitting different data sets with various shapes.

2.Characterizing the OLGNH model

2.1Based on two truncated moments (TTMs)

We present some characterizations of the OLGNH model in terms of a simple relationship between two TTMs. The first result employs the theorem according to Glänzel (1987), see Theorem 2.1.1 below. The result holds also when the interval H is not closed, it could be also applied when the CDF F does not have a certain closed form. Due to Glänzel (1990), this characterization results are stable in the sense of weak convergence.

Theorem 2.1.1.
Let (Ω, F, P) be a given probability space and let $H=[d_H, e]$ be an interval for some $d_H < e$ $(d_H=-\infty, e=+\infty$ might as well be allowed). Let $Y:\Omega \rightarrow H$ be a continuous RV with the CDF F and let $\nabla_{(1)}$ and $\nabla_{(1)}$ be two real functions defined on H such that $E[\nabla_{(2)}(Y) \mid Y \geq y] = E[\nabla_{(1)}(Y) \mid Y \geq y]\zeta(Y)|_{(y \in H)}$, is defined with some real function ζ. Let $\nabla_{(1)}, \nabla_{(2)} \in c^1(H)$, $\zeta \in c^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that $\zeta \nabla_{(1)} = \nabla_{(2)}$ has no real solution in the interior of H. Then F is uniquely determined by the functions $\nabla_{(1)}, \nabla_{(2)}$ and ζ, particularly

$$F(y)=\int_a^y c\left|\frac{\zeta'(v)}{\zeta(v)\nabla_{(1)}(v)-\nabla_{(2)}(v)}\right| exp\left(-s(v)\right)\, dv\,,$$

where the function s is a solution of the differential equation (Diff-E) $s'=\frac{\zeta' \nabla_{(1)}}{\zeta \nabla_{(1)}-\nabla_{(2)}}$ and c is the normalization constant, such that $\int_H \ dF=1$.

Remark 2.1.1. The goal in Theorem 2.1.1 is to have $\zeta(y)$ as simple as possible.
Proposition 2.1.1. Let $Y : \Omega \rightarrow (0,\infty)$ be a continuous RV and let

$$\nabla_{(1)}(Y)=\frac{\left(\left[1-\left(1-\tau_{y;a,b}\right)^{\theta}\right]^3 exp\left\{\left[\left(1-\tau_{y;a,b}\right)^{-\theta}-1\right]^{-1}\right\}\right)}{\left[1-\tau_{y;a,b}\right]^{\theta-1}}$$

and $\nabla_{(2)}(Y)=\nabla_{(1)}(Y)\tau_{y;a,b}|_{(y>0)}$. The RV Y has PDF (5) if and only if the function ζ defined in Theorem 2.1.1 has the form $\zeta(y)=\frac{1}{2}\tau_{y;a,b}|_{(y>0)}$.

Proof. Let Y be a RV with PDF (5), then $\left(1-F(y)\right)E\left[\nabla_{(1)}(Y) \mid_{(Y\geq y)}\right]=\frac{\theta}{2}\tau_{y;a,b}|_{(y>0)}$, and

$$\left(1-F(y)\right)E\left[\nabla_{(2)}(Y) \mid_{(Y\geq y)}\right]=\frac{\theta}{4}exp[2-2(1+by)^a]\,|_{(y>0)},$$

and finally, $\zeta(y)\nabla_{(1)}(Y)-\nabla_{(2)}(Y)=-\frac{1}{2}\nabla_{(1)}(Y)\tau_{y;a,b}<0\,|_{(y>0)}$. Conversely, if ζ is given as above, then

$$s'(y)=\frac{\zeta'(y)\nabla_{(1)}(Y)}{\zeta(y)\nabla_{(1)}(Y)-\nabla_{(2)}(Y)}=ab(1+by)^{a-1}|_{(y>0)},$$

and hence $s(y)=(1+by)^a-1\,|_{(y>0)}$. Now, in view of Theorem 2.1.1, Y has density (5).

Corollary 2.1.1.
Let $Y : \Omega \rightarrow (0,\infty)$ be a continuous RV and let $\nabla_{(1)}(Y)$ be as in Proposition 2.1.1. The PDF of Y is (5) if and only if there exist functions ∇_2 and ζ defined in Theorem 2.1.1 satisfying the Diff-E

$$\frac{\zeta'(y)\nabla_{(1)}(Y)}{\zeta(y)\nabla_{(1)}(Y)-\nabla_{(2)}(Y)}=ab(1+by)^{a-1}|_{(y>0)}.$$

Corollary A.1.2.
A general solution of the Diff-E in Corollary 2.1.1 is

$$\xi(y)=exp[-1+(1+by)^a]\left[-\int \quad ab(1+by)^{a-1}\tau_{y;a,b}\left(\nabla_{(1)}(Y)\right)^{-1}\nabla_{(2)}(Y)+D\right],$$

where D is a constant. Note that a set of functions satisfying the above Diff-E is given in Proposition 2.1.1 with $D=0$. However, it should be also noted that there are other triplets $(\nabla_1, \nabla_2, \zeta)$ satisfying the conditions of Theorem 2.1.1.

2.2 Based on HRF

It is well known that the HRF, $\hbar_F$, of a twice differentiable distribution function, F, satisfies the first order Diff-E

$$\frac{f'}{f}=\frac{\hbar_F'}{\hbar_F}-\hbar_F.$$

For many univariate continuous distributions, this is the only characterization available in terms of the HRF. The following proposition establishes a characterization of OLGNH distribution in terms of the HRF, which is not of the above trivial form.

Proposition 2.2.1.

Let $Y\ :\ \Omega\rightarrow(0,\infty)$ be a continuous RV. The PDF of Y is (5) if and only if its HRF $\hbar_F$ satisfies the Diff-E

$$\begin{bmatrix}\hbar_F'(y)\\+ab(1+by)^{a-1}\hbar_F\end{bmatrix}=\frac{\theta ab\tau_{y;a,b}}{\left(\frac{d}{dy}\left\{\frac{(1+by)^{a-1}\left[1-\tau_{y;a,b}\right]^{\theta-1}}{\left\{1+\left[1-\tau_{y;a,b}\right]^{\theta}\right\}\left\{1-\left[1-\tau_{y;a,b}\right]^{\theta}\right\}^{2}}\right\}\right)^{-1}}\big|_{(y>0)}.$$

Proof. Is straightforward and hence omitted.

3. Properties

3.1 Moments and moment generating function (MGF)

The r[th] moment of Y, say μ_r', follows from (7) as

$$\mu_r'=E(Y^r)=\sum_{\kappa_1,\kappa_2,\zeta_1=0}^{\infty}\sum_{\zeta_2=0}^{r}c_{\kappa_1,\kappa_{22}}\,V_{\zeta_1,\zeta_2}^{[\theta^*,r]}\Gamma\left(\frac{\zeta_2}{a}+1,1+\zeta_1\right), \tag{9}$$

where $V_{\zeta_1,\zeta_2}^{[\theta^*,r]}=\theta^*b^{-r}(-1)^{r+\zeta_1-\zeta_2}(1+\zeta_1)^{-\left(\frac{\zeta_2}{a}+1\right)}exp(1+\zeta_1)\binom{\theta^*-1}{\zeta_1}\binom{r}{\zeta_2}$. Or

$$\mu_r'=E(Y^r)=\sum_{\kappa_1,\kappa_2=0}^{\infty}\sum_{\zeta_1=0}^{\theta^*-1}\sum_{\zeta_2=0}^{r}c_{\kappa_1,\kappa_2}\,V_{\zeta_1,\zeta_2}^{[\theta^*,r]}\Gamma\left(\frac{\zeta_2}{a}+1,1+\zeta_1\right)\big|_{(\theta^*>0\text{ and integer})}$$

The variance (Var(Y)), skewness (Ske(Y)) and kurtosis (Ku(Y)) measures can be calculated from the ordinary moments using well-known relationships. Table 2 give a numerical analysis for the $E(Y)$, Var(Y), Ske(Y) and Ku(Y) for the OLGNH distribution. Based on Table 2 we note that: The skewness of the OLGNH distribution is always positive. The kurtosis of the OLGNH distribution can be only more than three.

The parameter b has a fixed effect on the Ske(Y) and Ku(Y) for all different values of all other parameter, when θ =0.5 and a =0.35, Ske(Y) = 0.7405238 and Ku (Y) =3.306488 for any value of the parameter b, when θ =2 and a =0.15, Ske(Y) =3.608101 and Ku (Y) =23.82384 for any value of the parameter b. The mean of the OLGNH model increases as θ increases. The mean of the OLGNH model decreases as a increases. The mean of the OLGNH model decreases as b increases. Based on Tables 2 and 3 we note that, the skewness of the OLGNH distribution can range in the interval (0.7, 11.7), whereas the skewness of the GNH distribution varies only in the interval (0.435, 3.17). Further, the spread for the OLGNH kurtosis is ranging from 3.306 to 256.4, whereas the spread for the GNH kurtosis only varies from 3.37 to 9 with the above parameter values.

Table 2: E(Y), Var(Y), Ske(Y) and Ku(Y) of the OLGNH distribution.

θ	a	b	E(Y)	Var(Y)	Ske(Y)	Ku(Y)
0.05	0.2	4	0.003295	0.000212	11.71085	256.3675
0.10			0.037353	0.0103854	7.055484	94.4865
0.50			2.107714	11.96853	3.863399	27.19396
1			8.35206	134.0818	2.989657	16.71037
2			27.73286	1056.389	2.340927	11.00752
5			106.3427	10203.5	1.745389	7.166646
10			251.9827	42840.98	1.429758	5.632246
20			536.2111	1487620	1.191500	4.692096
50			1274.686	613750.4	0.957295	3.944527
75			1797.298	1074415	0.8747941	3.721898
100			2264.638	1564916	0.8223643	3.591269
2	0.15	2.5	267.9287	180084.6	3.608101	23.82384

	0.20		44.37258	2704.355	2.340927	11.00752
	0.25		15.53627	219.533	1.711201	6.913076
	0.30		7.768556	40.60619	1.327297	5.09611
	0.35		4.72495	11.94158	1.065177	4.135207
5	0.35	0.50	55.21963	1035.263	0.7405238	3.306488
		1	27.60981	258.8157	0.7405238	3.306488
		2	13.80491	64.70392	0.7405238	3.306488
2	0.15	1	669.8218	1125529	3.608101	23.82384
		2	334.9109	281382.3	3.608101	23.82384
		3	223.2739	125058.8	3.608101	23.82384
		5	133.9644	45021.16	3.608101	23.82384
		10	66.98218	11255.29	3.608101	23.82384

Table 3: E(Y), Var(Y), Ske(Y) and Ku(Y) of the GNH distribution.

θ	a	b	E(Y)	Var(Y)	Ske(Y)	Ku(Y)
1	1	1	1	1	2	9
2			1.5	1.25	1.609969	7.08
5			2.283333	1.463611	1.339221	6.025973
10			2.928968	1.549768	1.241416	5.703086
20			3.597740	1.596163	1.190993	5.548813
50			4.499205	1.625133	1.160248	5.458834
75			4.901356	1.631689	1.153366	5.439116
100			5.187378	1.634984	1.149918	5.429296
200			5.878031	1.639947	1.144738	5.414611
500			6.792823	1.642936	1.141626	5.405815
5	0.5	5	2.248778	4.20089400	3.176333	22.94556
	1		0.4566667	0.05854444	1.339221	6.025973
	2		0.1568573	0.00398623	0.7435537	3.776470
	3		0.09323104	0.00117149	0.5688394	3.366604
	5		0.0512124	0.00030503	0.435122	6.044819
1.5	2	0.1	4.75017	10.469720	1.04087	4.228012
		0.5	0.950034	0.4187887	1.04087	4.228012
		1	0.475017	0.1046971	1.04087	4.228012
		5	0.09500339	0.0041879	1.04087	4.228012
2.5	2.5	4	0.1137808	0.0034247	0.7489036	3.576191
2	2	3	0.1823958	0.01180120	0.9293953	4.007148
1	2	4	0.0947340	0.00615846	1.253913	4.772774
4	2	1	0.7265553	0.1023400	0.7698508	3.792370
0.5	2.5	1	0.1828302	0.04319620	1.664298	6.044604
1.5	2.5	5	0.0719500	0.00222103	0.923067	3.827275

The MGF $M_Y(t) = E(e^{tY})$ of Y . Clearly, the first one can be derived using (7) as

$$M_Y(t) = \sum_{\kappa_1,\kappa_2,\zeta_1,r=0}^{\infty} \sum_{\zeta_2=0}^{r} c_{\kappa_1,\kappa_2} \frac{t^r}{r!} V_{\zeta_1,\zeta_2}^{[\theta^*,r]} \Gamma\left(\frac{\zeta_2}{a} + 1, 1 + \zeta_1\right).$$

Or

$$M_Y(t) = \sum_{\kappa_1,\kappa_2,r=0}^{\infty} \sum_{\zeta_1=0}^{\theta^*-1} \sum_{\zeta_2=0}^{r} c_{\kappa_1,\kappa_2} \frac{t^r}{r!} V_{\zeta_1,\zeta_2}^{[\theta^*,r]} \Gamma\left(\frac{\zeta_2}{a} + 1, 1 + \zeta_1\right) |_{(\theta^*>0 \text{ and integer})}.$$

3.2 Incomplete moments

The r th incomplete moment, say $I_{r,t}$, of Y can be expressed using (7) as

$$I_{r,t}=\int_{-\infty}^{t} y^r f(y)dy=\sum_{\kappa_1,\kappa_2,\zeta_1=0}^{\infty}\sum_{\zeta_2=0}^{r} c_{\kappa_1,\kappa_2}\, V_{\zeta_1,\zeta_2}^{[\theta^*,r]}\begin{bmatrix}\Gamma\left(\frac{\zeta_2}{a}+1,1+\zeta_1\right)\\ -\Gamma\left(\frac{\zeta_2}{a}+1,(1+\zeta_1)(1+bt)^a\right)\end{bmatrix}. \quad (10)$$

Or

$$I_{r,t}=\sum_{\kappa_1,\kappa_2=0}^{\infty}\sum_{\zeta_1=0}^{\theta^*-1}\sum_{\zeta_2=0}^{r} c_{\kappa_1,\kappa_2}\, V_{\zeta_1,\zeta_2}^{[\theta^*,r]}\begin{bmatrix}\Gamma\left(\frac{\zeta_2}{a}+1,1+\zeta_1\right)\\ -\Gamma\left(\frac{\zeta_2}{a}+1,(1+\zeta_1)(1+bt)^a\right)\end{bmatrix}\Big|_{(\theta^*>0\text{ and integer})}.$$

3.3 Residual life (RL) and reversed residual life (RRL)

The r[th] moment of the RL, say $z_r(t)=E[(Y-t)^r]\,|_{(Y>t\text{ and r}=1,2,\ldots)}$. The r th moment of the RL of Y is given as $z_r(t)=\frac{1}{1-F(t)}\int_t^{\infty}(Y-t)^r\,dF(y)$. Then

$$z_r(t)=\frac{1}{1-F(t)}\sum_{\kappa_1,\kappa_2,\zeta_1=0}^{\infty}\sum_{\zeta_2=0}^{r} c_{\kappa_1,\kappa_2}^{(1)}\, V_{\zeta_1,\zeta_2}^{[\theta^*,r]}\Gamma\left(\frac{\zeta_2}{a}+1,1+\zeta_1\right),$$

where $c_{\kappa_1,\kappa_2}^{(1)}=c_{\kappa_1,\kappa_2}\sum_{h=0}^{n}\binom{n}{h}(-t)^{n-h}$. Or

$$z_r(t)=\frac{1}{1-F(t)}\sum_{\kappa_1,\kappa_2=0}^{\infty}\sum_{\zeta_1=0}^{\theta^*-1}\sum_{\zeta_2=0}^{r} c_{\kappa_1,\kappa_2}^{(1)}\, V_{\zeta_1,\zeta_2}^{[\theta^*,r]}\Gamma\left(\frac{\zeta_2}{a}+1,1+\zeta_1\right)\Big|_{(\theta^*>0\text{ and integer})}.$$

The r th moment of the RRL, say $Z_r(t)=E[(t-Y)^r]\,|_{(Y\le t,t>0\text{ and r}=1,2,\ldots)}$. Then $Z_r(t)=\frac{1}{F(t)}\int_0^{t}(t-Y)^r\,dF(y)$. Then, the r th moment of the RRL of y becomes

$$Z_r(t)=\frac{1}{F(t)}\sum_{\kappa_1,\kappa_2,\zeta_1=0}^{\infty}\sum_{\zeta_2=0}^{r} c_{\kappa_1,\kappa_2}^{(2)}\, V_{\zeta_1,\zeta_2}^{[\theta^*,r]}\begin{bmatrix}\Gamma\left(\frac{\zeta_2}{a}+1,1+\zeta_1\right)\\ -\Gamma\left(\frac{\zeta_2}{a}+1,(1+\zeta_1)(1+bt)^a\right)\end{bmatrix},$$

where

$c_{\kappa_1,\kappa_2}^{(2)}=c_{\kappa_1,\kappa_2}\sum_{h=0}^{n}(-1)^h\binom{n}{h}t^{n-h}$. Or

$$Z_r(t)=\frac{1}{F(t)}\sum_{\kappa_1,\kappa_2=0}^{\infty}\sum_{\zeta_1=0}^{\theta^*-1}\sum_{\zeta_2=0}^{r} c_{\kappa_1,\kappa_2}^{(2)}\, V_{\zeta_1,\zeta_2}^{[\theta^*,r]}\begin{bmatrix}\Gamma\left(\frac{\zeta_2}{a}+1,1+\zeta_1\right)\\ -\Gamma\left(\frac{\zeta_2}{a}+1,(1+\zeta_1)(1+bt)^a\right)\end{bmatrix}\Big|_{(\theta^*>0\text{ and integer})}.$$

3.4 Order statistics

Suppose $Y_1,Y_2,\ldots,Y_n$ is a random sample (RS) from an OLGNH model. Let $Y_{i\,:\,n}$ denote the i[th] order statistic. The PDF of $Y_{i\,:\,n}$ can be expressed as

$$f_{i\,:\,n}(y)=\frac{f(y)}{B(i,n-i+1)}F(y)^{i-1}[1-F(y)]^{n-i}. \quad (11)$$

We can write the density function of $Y_{i\,:\,n}$ in (11) as

$$f_{i\,:\,n}(y)=\sum_{\kappa_1,p=0}^{\infty}\sum_{\zeta_1=0}^{\kappa_2+n-i} \upsilon_{\zeta_1,\kappa_1,p}\,\pi_{(\zeta_1+\kappa_1+p)\theta}(y), \quad (12)$$

where

$$\upsilon_{\zeta_1,\kappa_1,p}=\sum_{\kappa_2=0}^{n-1}\frac{(-1)^{\kappa_2+\kappa_1}}{B(i,n-i+1)\kappa_1!\,[(\zeta_1+\kappa_1+p)\theta+1]}\binom{(\zeta_1+\kappa_1+p)\theta}{\zeta_1+\kappa_1}\binom{\kappa_2+n-i}{\zeta_1}\binom{i-1}{\kappa_2}.$$

The p th moment of $Y_{i\,:\,n}$ is given by

$$E(Y_{i\ :\ n}^{p})=\sum_{\kappa_1,p,w=0}^{\infty}\sum_{\zeta_1=0}^{\kappa_2+n-i}\sum_{l=0}^{r}\upsilon_{\zeta_1,\kappa_1,p}\,V_{\zeta_1,i}^{[(\zeta_1+\kappa_1+p)\theta,r]}\Gamma\left(\frac{l}{a}+1,1+w\right). \tag{13}$$

Or

$$E(Y_{i\ :\ n}^{p})=\sum_{\kappa_1,p=0}^{\infty}\sum_{\zeta_1=0}^{\kappa_2+n-i}\sum_{l=0}^{r}\sum_{w=0}^{(\zeta_1+\kappa_1+p)\theta-1}\upsilon_{\zeta_1,\kappa_1,p}\,V_{\zeta_1,i}^{[(\zeta_1+\kappa_1+p)\theta,r]}\Gamma\left(\frac{l}{a}+1,1+w\right)|_{((\zeta_1+\kappa_1+p)\theta>0\text{ and integer})}.$$

4. Maximum likelihood estimation (MLE)

Maximum likelihood estimation (MLE) is a statistical technique for estimating the parameters of a probability distribution that has been assumed given some observed data. This is accomplished by maximizing a likelihood function to make the observed data as probable as possible given the assumed statistical model. The maximum likelihood estimate is the location in the parameter space where the likelihood function is maximized. Maximum likelihood is a popular approach for making statistical inferences since its rationale is clear and adaptable. The derivative test for figuring out maxima can be used if the probability function is differentiable. The ordinary least squares estimator, for illustration, maximizes the likelihood of the linear regression model, letting the first-order requirements of the likelihood function to be explicitly solved in some circumstances. However, in the majority of cases, it will be essential to use numerical techniques to determine the probability function's maximum. MLE is typically comparable to maximum a posteriori estimates under a uniform prior distribution on the parameters from the viewpoint of Bayesian inference. MLE is a specific example of an extremum estimator in frequentist inference, with likelihood as the objective function. Let $Y_1,\dots,Y_n$ be a RS from this distribution with parameter vector $\underline{\Psi}=(\theta,a,b)^T$. The log-likelihood function for $\underline{\Psi}$, say $\ell(\underline{\Psi})$, is given by

$$\begin{aligned}\ell(\underline{\Psi})=n\log\left(\frac{1}{2}\right)+n\log(\theta)+n\log(a)+n\log(b)+(a-1)\sum_{i=0}^{n}\log(1+by_i)\\+\sum_{i=0}^{n}[1-(1+by_i)^a]+(\theta-1)\sum_{i=0}^{n}\log\left(1-\tau_{(y_i;a,b)}\right)\\-3\sum_{i=0}^{n}\log\left[1-\tau_{(y_i;a,b)}(\theta)\right]-\sum_{i=0}^{n}\frac{\tau_{(y_i;a,b)}(\theta)}{1-\tau_{(y_i;a,b)}(\theta)},\end{aligned} \tag{14}$$

where $\tau_{(y_i;a,b)}(\theta)=\left[1-\tau_{(y_i;a,b)}\right]^{\theta}$ and $\tau_{(y_i;a,b)}=exp[1-(1+by_i)^a]$. Equation (14) can be maximized by using the different programs like R, SAS or by solving the nonlinear equations obtained by differentiating (14). The score vector elements are easily to be derived as

$$\frac{\partial\ell(\underline{\Psi})}{\partial\theta}=\frac{n}{\theta}+\sum_{i=0}^{n}\log\left[1-\tau_{(y_i;a,b)}\right]+3\sum_{i=0}^{n}\frac{\tau_{(y_i;a,b)}(\theta)\log\left[1-\tau_{(y_i;a,b)}\right]}{1-\left[1-\tau_{(y_i;a,b)}\right]^{\theta}}-\sum_{i=0}^{n}\frac{\tau_{(y_i;a,b)}(\theta)\log\left[1-\tau_{(y_i;a,b)}\right]}{\left\{1-\tau_{(y_i;a,b)}(\theta)\right\}^2},$$

$$\frac{\partial\ell(\underline{\Psi})}{\partial a}=\frac{n}{a}+\sum_{i=0}^{n}\log(1+by_i)-\sum_{i=0}^{n}(1+by_i)^a\log(1+by_i)+(\theta-1)\sum_{i=0}^{n}\frac{(1+by_i)^a\tau_{(y_i;a,b)}\log(1+by_i)}{1-\tau_{(y_i;a,b)}}$$

$$\begin{aligned}\frac{\partial\ell(\underline{\Psi})}{\partial b}=\frac{n}{b}+(a-1)\sum_{i=0}^{n}\frac{y_i}{1+by_i}-a\sum_{i=0}^{n}y_i(1+by_i)^{a-1}+(\theta-1)\sum_{i=0}^{n}\frac{ay_i(1+by_i)^{a-1}\tau_{(y_i;a,b)}}{1-\tau_{(y_i;a,b)}}\\+3\sum_{i=0}^{n}\frac{a\theta y_i(1+by_i)^{a-1}\tau_{(y_i;a,b)}\tau_{(y_i;a,b)}(\theta)}{1-\tau_{(y_i;a,b)}(\theta)}-\sum_{i=0}^{n}\frac{a\theta y_i(1+by_i)^{a-1}\tau_{(y_i;a,b)}\tau_{(y_i;a,b)}(\theta)}{\left\{1-\tau_{(y_i;a,b)}(\theta)\right\}^2}.\end{aligned}$$

Setting $\frac{\partial\ell(\Psi)}{\partial\theta}=0$ and $\frac{\partial\ell(\Psi)}{\partial a}=0$ and solving them simultaneously using any software like "R" yields the MLEs for the model parameters. The Newton-Raphson algorithms are employed for the numerically solving in such cases. As usual, under regularity conditions, the properties of consistency and asymptotic normality are satisfied. Especially, the asymptotic distribution behind the MLEs is multivariate normal, with mean and covariance matrix derived to the inverse of the expected Fisher covariance matrix. This asymptotic distribution is useful to construct confidence intervals (CIs), confidence regions, and various kinds of likelihood test.

5. Data analysis

In this section, we propose an application to a real data set based to show the importance and flexibility of the OLGNH model. The OLGNH is compared with the MONH, GamNH, exponentiated Weibull NH (New), OLNH, GNH, PRHRNH, OLLNH and BNH distributions. The selection of the best model is applied using the estimated log-likelihood, Kolmogorov-Smirnov (K-S), Akaike Information Criterion (AIC), Consistent Akaike IC (CAIC), Bayesian IC (BIC), and Hannan-Quinn IC (HQIC). Consider the data of Choulakian and Stephens (2001), the data consist of 72 exceedances for the years 1958-1984, rounded to one decimal place, the data are: 1.70, 2.2, 14.40, 1.40, 18.7, 8.5, 25.50, 11.6, 14.10, 22.1, 1.1, 2.50, 14.40, 1.7, 37.6, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 0.6, 2.2, 39.0, 0.3, 15.0, 1.1, 0.40, 20.6, 5.30, 0.7, 1.90, 13.0, 12.0, 9.3, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 25.5, 3.40, 11.90, 21.50, 27.6, 36.40, 2.7, 64.0, 1.5, 2.50, 27.4, 1.0, 27.10, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0. Some other useful real-life data set can be used (see Merovci et al. (2017 and 2020)).

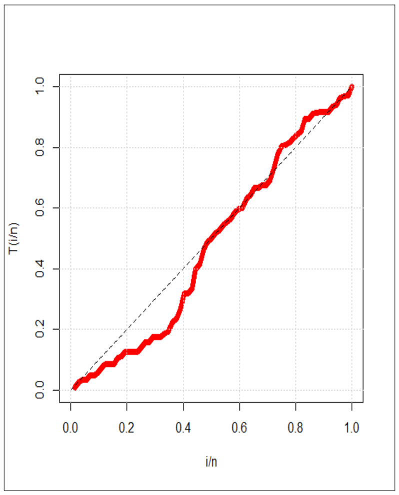
The total time on test (TTT) plot (Aarset (1987)) is useful for exploring the HRF of the used data set. The TTT plot, the quantile-quantile (Q-Q) plot and the box plot for the exceedances of flood peaks data is given in Figure 3. Figure 4 give the estimated PDF, estimated CDF and estimated HRF plots for the OLGNH for the exceedances of flood peaks data. Figure 5 give the P-P and Kaplan-Meier survival plots of the OLGNH for the exceedances of flood peaks data. Table 4 give the estimates of the competitive models fitted to the Choulakian and Stephens data. Table 5 give statistics of the competitive models fitted to the Choulakian and Stephens data. The results displayed in Table 4 show that the OLGNH distribution has the lowest AIC, CAIC, BIC, HQIC and has the biggest estimated log-likelihood among all the fitted models. So, it could be chosen as the best model under these criteria among all the fitted models. So, the new model may be chosen as the best model under these criteria. Yousof et al. (2017) introduced the OLNH and studied its properties, the model of Yousof et al. (2017) has three parameters one of them is a scale parameter. However, the new OLGNH model has three shape parameter, and its flexibility is better that the OLNH model as shown in Table 5. Based on Figures 4 and 5, the OLGNH model provides a closer fit to the empirical PDF and CDF. Also, from Figure 3, we have a bathtub-shaped (**U**-shape) HRF for the exceedances of flood peaks data, which are in accordance with TTT plot.

Table 4: Estimates of the competitive models fitted to the Choulakian and Stephens data.

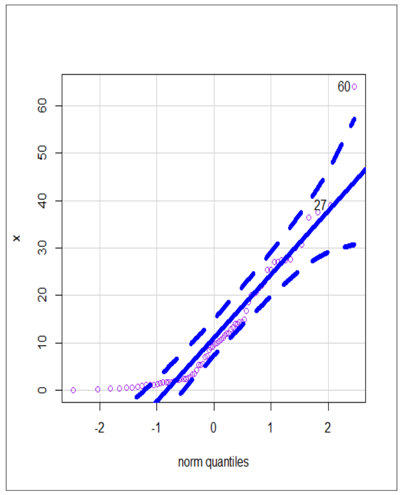
Model	Estimates (SD)			
Exp(b)	0.082			
	(0.01)			
NH(a,b)	0.841	0.1094		
	(0.259)	(0.059)		
RNH(a,b)	0.125	6.28		
	(0.012)	(2.919)		
OLLNH(θ,a,b)	0.777	1.501	0.051	
	(0.105)	(0.685)	(0.033)	
OLNH(θ,a,b)	0.7293	0.2519	1.8065	
	(0.6059)	(0.052)	(3.355)	
PRHRNH(θ,a,b)	0.364	1.714	0.031	
	(0.068)	(1.191)	(0.031)	
GamNH(θ,a,b)	0.7286	1.9299	0.0242	
	(0.1385)	(1.7591)	(0.0312)	
MONH(θ,a,b)	23.77	0.0011	0.2660	
	(5.5053)	(0.0003)	(0.0895)	
GNH(θ,a,b)	0.7289	1.7126	0.0309	
	(0.1404)	(1.2607)	(0.0330)	
BNH(θ,β,a,b)	0.8381	316.0285	0.6396	0.0003
	(0.1215)	(4.2194)	(0.8227)	(0.0004)
EWNH(θ,β,a,b)	2.7591	0.3989	0.4732	0.6129
	(1.742)	(0.167)	(0.158)	(0.959)
OLGNH(θ,a,b)	**2.5565**	**0.2009**	**13.1264**	
	(3.3153)	**(0.06599)**	**(49.51049)**	

Table 5: Statistics of the competitive models fitted to the Choulakian and Stephens data.

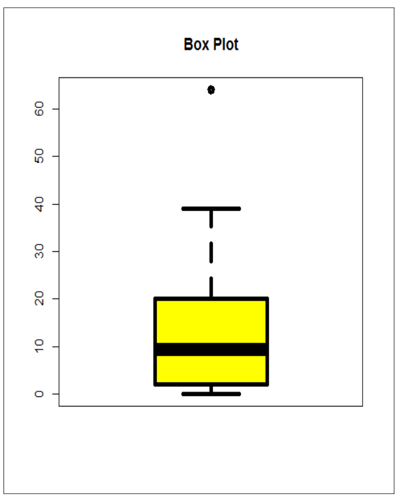
Model	loglike	AI_C	CAI_C	BI_C	HQI_C
OLGNH	**-250.276**	**506.552**	**506.905**	**513.381**	**509.271**
OLLNH	-250.412	506.824	507.188	513.655	509.542
RNH	-251.722	507.445	507.624	513.990	509.744
NH	-251.987	507.974	508.153	515.532	509.795
OLNH	-250.589	507.183	507.535	514.014	509.962
PRHRNH	-300.832	607.666	608.024	614.494	610.385
GamNH	-250.917	507.834	508.187	514.663	510.555
MONH	-251.087	508.175	508.532	515.005	510.894
EWNH	-250.032	508.064	508.666	517.171	511.690
GNH	-250.925	507.849	508.202	514.679	510.571
BNH	-251.356	510.713	511.315	519.822	514.344
Exp	-252.128	506.256	506.313	513.533	507.162



TTT plot



Q-Q plot



Box plot

Figure 3: TTT plot of the exceedances of flood peaks data.

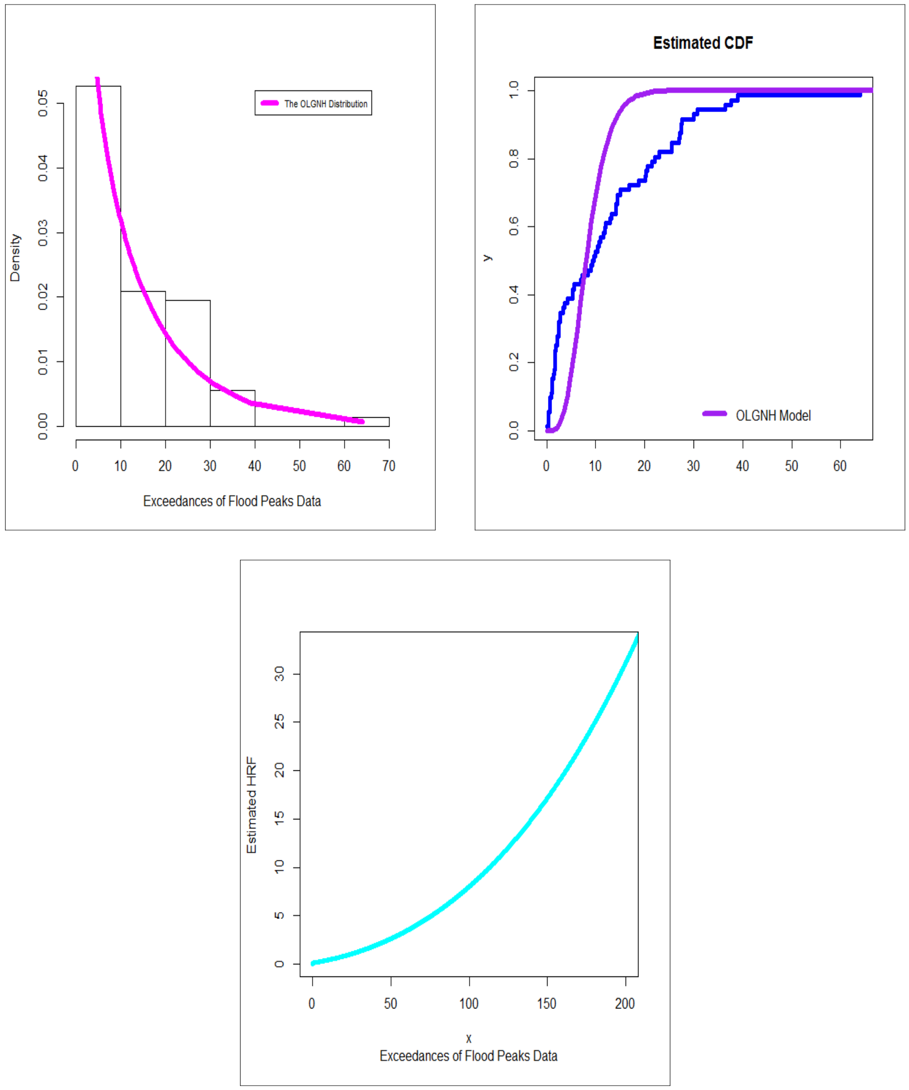


Figure 4: EPDF, ECDF and EHRF plots.

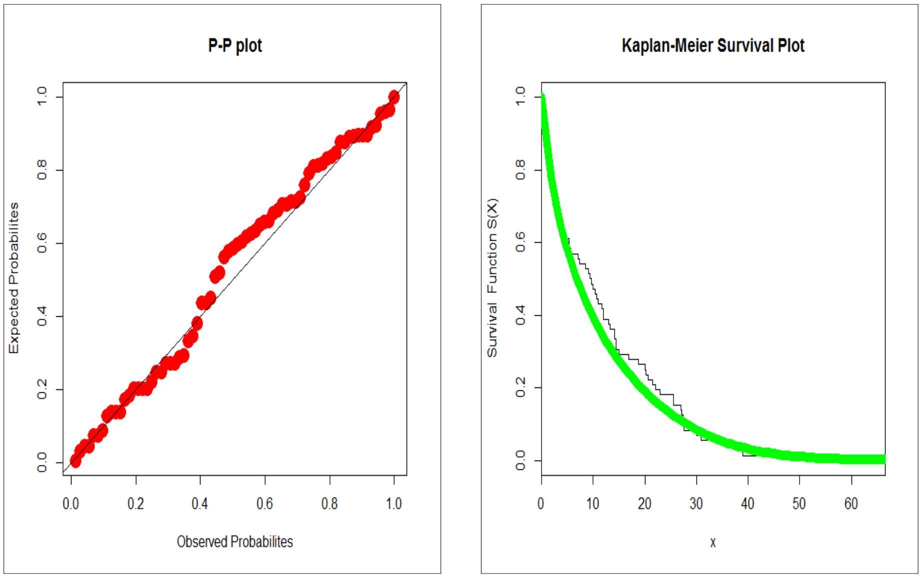


Figure 5: P-P and Kaplan-Meier survival plots.

6. Conclusions

In this paper, a new three-parameter version of the Nadarajah Haghighi model is introduced and studied. Some of its properties such as moments, incomplete moments, moment generating function (MGF), moment of residual life and reversed residual life are mathematically derived. The new density has various shapes such as the right skewed, left skewed and symmetric and its corresponding hazard rate shapes can be "increasing", "decreasing", "bathtub", "upside down" and "constant". Characterization results are obtained based on two TTMs and in terms of the HRF. Uncensored and Censored validation via a modified chi-squared goodness-of-fit test is presented under the new model. The modified chi-squared test is based on the Nikulin-Rao-Robson statistic. A numerical analysis for the variance, skewness and kurtosis measures is presented. The uncensored exceedances of flood peaks data are employed for comparing the combative model. As a future work, we will consider many new useful goodness-of-fit tests for right censored validation such as the Nikulin-Rao-Robson goodness-of-fit test and Bagdonavičius-Nikulin goodness-of-fit test as performed by Ibrahim et al. (2019), Goual et al. (2019, 2020), Mansour et al. (2020a-f), Yadav et al. (2020), Goual and Yousof (2020), Aidi et al. (2021) and Ibrahim et al. (2022), among others. Some useful reliability studies based on multicomponent stress-strength and the remained stress-strength concepts can be presented (Rasekhi et al. (2020), Saber et al. (2022a,b), Saber and Yousof (2022)). For modelling of the bivariate real data sets, we shall derive some new bivariate odd Lindley generalized Nadarajah-Haghighi type distribution using "Farlie-Gumbel-Morgenstern copula" (FGMC), modified,"Clayton copula ", "Renyi's entropy copula (REC) and "Ali-Mikhail-Haq copula (AMHC)" (see also Shehata and Yousof (2021a,b), Elgohari and Yousof (2020a,b), Al-babtain et al. (2020a,b), Elgohari and Yousof (2021), Alizadeh et al. (2020a,b), Elgohari et al. (2021), Aryal and Yousof (2017), Aryal et al. (2017), Ali et al. (2021a,b), Shehata et al. (2021), Hamedani et al. (2022) and Shehata et al. (2022)). Some new acceptance sampling plans based on the complementary geometric Weibull-G family or based on some special members can be presented in separate article (see Ahmed and Yousof (2022) and Ahmed et al. (2022)). Some recent studies on estimating the survival rates and risk claim-size data can be found in Shrahili et al. (2021) and Mohamed et al. (2022a,b,c).

References

1. Aarset, M. V. (1987). How to identify bathtub hazard rate. IEEE Transactions on Reliability, 36, 106-108.
2. Aboraya, M., M. Yousof, H. M., Hamedani, G. G. and Ibrahim, M. (2020). A new family of discrete distributions with mathematical properties, characterizations, Bayesian and non-Bayesian estimation methods. Mathematics, 8, 1648.
3. Ahmed, B. and Yousof, H. M. (2022). A New Group Acceptance Sampling Plans based on Percentiles for the Weibull Fréchet Model. Statistics, Optimization & Information Computing, forthcoming

4. Ahmed, B., Ali, M. M. and Yousof, H. M. (2022). A Novel G Family for Single Acceptance Sampling Plan with Application in Quality and Risk Decisions, Annals of Data Science, forthcoming.
5. Aidi, K., Butt, N. S., Ali, M. M., Ibrahim, M., Yousof, H. M. and Shehata, W. A. M. (2021). A Modified Chi-square Type Test Statistic for the Double Burr X Model with Applications to Right Censored Medical and Reliability Data. Pakistan Journal of Statistics and Operation Research, 17(3), 615-623.
6. Al-babtain, A. A., Elbatal, I. and Yousof, H. M. (2020a). A new flexible three-parameter model: properties, Clayton copula, and modeling real data. Symmetry, 12, 440.
7. Al-Babtain, A. A., Elbatal, I. and Yousof, H. M. (2020b). A new three parameter Fréchet model with mathematical properties and applications. Journal of Taibah University for Science, 14, 265-278.
8. Ali, M. M., Ibrahim, M. and Yousof, H. M. (2021a). Expanding the Burr X model: properties, copula, real data modeling and different methods of estimation. Optimal Decision Making in Operations Research & Statistics: Methodologies and Applications, VOL 1, 26-49.
9. Ali, M. M., Yousof, H. M. and Ibrahim, M. (2021b). A new version of the generalized Rayleigh distribution with copula, properties, applications and different methods of estimation. Optimal Decision Making in Operations Research & Statistics: Methodologies and Applications, VOL 1, 1-25.
10. Alizadeh, M., Jamal, F., Yousof, H. M., Khanahmadi, M. and Hamedani, G. G. (2020a). Flexible Weibull generated family of distributions: characterizations, mathematical properties and applications. University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics, 82, 145-150.
11. Alizadeh, M., Rasekhi, M., Yousof, H. M., Ramires, T. G. and Hamedani G. G. (2018). Extended exponentiated Nadarajah-Haghighi model: mathematical properties, characterizations and applications. Studia Scientiarum Mathematicarum Hungarica, 55, 498-522.
12. Alizadeh, M., Yousof, H. M., Jahanshahi, S. M. A., Najibi, S. M. and Hamedani, G. G. (2020b). The transmuted odd log-logistic-G family of distributions. Journal of Statistics and Management Systems, 23, 1-27.
13. Aryal, G. R. and Yousof, H. M. (2017). The exponentiated generalized-G Poisson family of distributions. Economic Quality Control, 32, 1-17.
14. Aryal, G. R., Ortega, E. M., Hamedani, G. G. and Yousof, H. M. (2017). The Topp-Leone generated Weibull distribution: regression model, characterizations and applications. International Journal of Statistics and Probability, 6, 126-141.
15. Chesneau, C., Yousof, H. M., Hamedani, G. and Ibrahim, M. (2022). A New One-parameter Discrete Distribution: The Discrete Inverse BurrDistribution: Characterizations, Properties, Applications, Bayesian and Non-Bayesian Estimations. Statistics, Optimization & Information Computing, forthcoming.
16. Dias, C. R. B., Alizadeh, M. and Cordeiro, G. M. (2018). The beta Nadarajah-Haghighi distribution. Hacettepe Journal of Mathematics and Statistics, 47, 1302 13203.
17. Elgohari, H. and Yousof, H. M. (2020a). A Generalization of Lomax Distribution with Properties, Copula and Real Data Applications. Pakistan Journal of Statistics and Operation Research, 16(4): 697-711.
18. Elgohari, H. and Yousof, H. M. (2021). A New Extreme Value Model with Different Copula, Statistical Properties and Applications. Pakistan Journal of Statistics and Operation Research, 17(4): 1015-1035.
19. Elgohari, H. and Yousof, H. M. (2020b). New Extension of Weibull Distribution: Copula, Mathematical Properties and Data Modeling. Statistics, Optimization & Information Computing, 8(4): 972-993.
20. Elgohari, H., Ibrahim, M. and Yousof, H. M. (2021). A New Probability Distribution for Modeling Failure and Service Times: Properties, Copulas and Various Estimation Methods. Statistics, Optimization & Information Computing, 8(3): 555-586.
21. Goual, H., Yousof, H. M. and Ali, M. M. (2019). Validation of the odd Lindley exponentiated exponential by a modified goodness of fit test with applications to censored and complete data. Pakistan Journal of Statistics and Operation Research, 15(3), 745-771.
22. Goual, H. and Yousof, H. M. (2020). Validation of Burr XII inverse Rayleigh model via a modified chi-squared goodness-of-fit test. Journal of Applied Statistics, 47(3), 393-423.
23. Goual, H., Yousof, H. M. and Ali, M. M. (2020). Lomax inverse Weibull model: properties, applications, and a modified Chi-squared goodness-of-fit test for validation. Journal of Nonlinear Sciences & Applications, 13(6), 330-353.
24. Hamedani, G. G., Korkmaz, M. Ç., Butt, N. S. and Yousof H. M. (2022). The Type II Quasi Lambert G Family of Probability Distributions, Pakistan Journal of Statistics and Operation Research, forthcoming.
25. Ibrahim, M., Aidi, K., Ali, M. M. and Yousof, H. M. (2022). A Novel Test Statistic for Right Censored Validity under a new Chen extension with Applications in Reliability and Medicine. Annals of Data Science, forthcoming.
26. Ibrahim, M., Ali, M. M. and Yousof, H. M. (2021). The discrete analogue of the Weibull G family: properties, different applications, Bayesian and non-Bayesian estimation methods. Annals of Data Science, forthcoming.

27. Ibrahim, M., Yadav, A. S., Yousof, H. M., Goual, H. and Hamedani, G. G. (2019). A new extension of Lindley distribution: modified validation test, characterizations and different methods of estimation. Communications for Statistical Applications and Methods, 26(5), 473-495.
28. Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, 75-84.
29. Glänzel, W. (1990). Some consequences of a characterization theorem based on truncated moments, Statistics: A Journal of Theoretical and Applied Statistics, 21 (4), 613-618.
30. Goual, H., Yousof, H. M. and Ali, M. M. (2019). Validation of the odd Lindley exponentiated exponential by a modified goodness of fit test with applications to censored and complete data. Pak. J. Stat. Oper. Res. 15(3), 745-771.
31. Goual, H. and Yousof, H. M. (2019). Validation of Burr XII inverse Rayleigh model via a modified chi-squared goodness-of-fit test. Journal of Applied Statistics, 47(1), 1-32.
32. Gupta, R.D. and Kundu, D. (1999). Generalized exponential distributions. Aust. N. Z. J. Stat. 41, 173-188.
33. Ibrahim, M., Goual, H., Alotaibi, R., Ali, M.M. Rezk, H. R. and Yousof, H. M. (2020). Censored and uncensored validation for the double Burr XII model using a new Nikulin-Rao-Robson goodness-of-fit test with Bayesian and non-Bayesian estimation, Journal of Taibah University for Science, forthcoming.
34. Lemonte, A. J. (2013). A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Computational Statistics & Data Analysis, 62, 149-170.
35. Lemonte, A. J., Cordeiro, G. M. and Moreno-Arenas, G. (2016). A new useful three-parameter extension of the exponential distribution. Statistics, 50(2), 312-337.
36. Mansour, M. M., Ibrahim, M., Aidi, K., Shafique Butt, N., Ali, M. M., Yousof, H. M. and Hamed, M. S. (2020a). A New Log-Logistic Lifetime Model with Mathematical Properties, Copula, Modified Goodness-of-Fit Test for Validation and Real Data Modeling. Mathematics, 8(9), 1508.
37. Mansour, M. M., Butt, N. S., Ansari, S. I., Yousof, H. M., Ali, M. M. and Ibrahim, M. (2020b). A new exponentiated Weibull distribution's extension: copula, mathematical properties and applications. Contributions to Mathematics, 1 (2020) 57–66. DOI: 10.47443/cm.2020.0018
38. Mansour, M., Korkmaz, M. C., Ali, M. M., Yousof, H. M., Ansari, S. I., & Ibrahim, M. (2020c). A generalization of the exponentiated Weibull model with properties, Copula and application. Eurasian Bulletin of Mathematics, 3(2), 84-102.
39. Mansour, M., Rasekhi, M., Ibrahim, M., Aidi, K., Yousof, H. M. and Elrazik, E. A. (2020d). A New Parametric Life Distribution with Modified Bagdonavičius–Nikulin Goodness-of-Fit Test for Censored Validation, Properties, Applications, and Different Estimation Methods. Entropy, 22(5), 592.
40. Mansour, M., Yousof, H. M., Shehata, W. A. and Ibrahim, M. (2020e). A new two parameter Burr XII distribution: properties, copula, different estimation methods and modeling acute bone cancer data. Journal of Nonlinear Science and Applications, 13(5), 223-238.
41. Mansour, M. M., Butt, N. S., Yousof, H. M., Ansari, S. I. and Ibrahim, M. (2020f). A Generalization of Reciprocal Exponential Model: Clayton Copula, Statistical Properties and Modeling Skewed and Symmetric Real Data Sets. Pakistan Journal of Statistics and Operation Research, 16(2), 373-386.
42. Merovci, F., Alizadeh, M., Yousof, H. M. and Hamedani, G. G. (2017). The exponentiated transmuted-G family of distributions: theory and applications. Communications in Statistics-Theory and Methods, 46(21), 10800-10822.
43. Merovci, F., Yousof, H. M. and Hamedani, G. G. (2020). The Poisson Topp Leone Generator of Distributions for Lifetime Data: Theory, Characterizations and Applications. Pakistan Journal of Statistics and Operation Research, 16(2), 343-355.
44. Mohamed, H. S., Cordeiro, G. M., and Yousof, H. M. (2022a). The synthetic autoregressive model for the insurance claims payment data: modeling and future prediction. Optimization & Information Computing, forthcoming.
45. Mohamed, H. S., Cordeiro, G. M., Minkah, R., Yousof, H. M. and Ibrahim, M. (2022b). A size-of-loss model for the negatively skewed insurance claims data: applications, risk analysis using different methods and statistical forecasting. Journal of Applied Statistics, forthcoming.
46. Mohamed, H. S., Ali, M. M. and Yousof, H. M. (2022c). The Lindley Gompertz Model for Estimating the Survival Rates: Properties and Applications in Insurance, Annals of Data Science, forthcoming.
47. Nadarajah, S.and Haghighi, F. (2011). An extension of the exponential distribution. Statistics. 45, 543-558.
48. Nascimento, A. D. C., Silva, K. F., Cordeiro, G. M., Alizadeh, M. and Yousof, H. M. (2019). The odd Nadarajah-Haghighi family of distributions: properties and applications. Studia Scientiarum Mathematicarum Hungarica, 56(2), 1-26.
49. Ortega, E. M., Lemonte, A. J., Silva, G. O. and Cordeiro, G. M. (2015). New flexible models generated by gamma random variables for lifetime modeling. Journal of Applied Statistics, 42(10), 2159-2179.

50. Rao, K. C., Robson, D. S. (1974). A Chi-square statistic for goodness-of-fit tests within the exponential family. Communication in Statistics, 3, 1139-1153
51. Rasekhi, M., Saber, M. M. and Yousof, H. M. (2020). Bayesian and classical inference of reliability in multicomponent stress-strength under the generalized logistic model. Communications in Statistics-Theory and Methods, 50(21), 5114-5125.
52. Rezaei, S., Sadr, B. B., Alizadeh, M. and Nadarajah, S. (2017). Topp-Leone generated family of distributions: Properties and applications. Communications in Statistics: Theory and Methods 46(6), 2893-2909.
53. Saber, M. M. and Yousof, H. M. (2022). Bayesian and Classical Inference for Generalized Stress-strength Parameter under Generalized Logistic Distribution, Statistics, Optimization & Information Computing, forthcoming.
54. Saber, M. M. Marwa M. Mohie El-Din and Yousof, H. M. (2022a). Reliability estimation for the remained stress-strength model under the generalized exponential lifetime distribution, Journal of Probability and Statistics, 20a21, 1-10
55. Saber, M. M., Rasekhi, M. and Yousof, H. M. (2022b). Generalized Stress-Strength and Generalized Multicomponent Stress-Strength Models, Statistics, Optimization & Information Computing, forthcoming.
56. Silva, F. S., Percontini, A., de Brito, E., Ramos, M. W., Venancio, R. and Cordeiro, G. M. (2017). The Odd Lindley-G Family of Distributions. Austrian Journal of Statistics, 46(1), 65-87.
57. Shehata, W. A. M. and Yousof, H. M. (2021a). The four-parameter exponentiated Weibull model with Copula, properties and real data modeling. Pakistan Journal of Statistics and Operation Research, 17(3), 649-667.
58. Shehata, W. A. M. and Yousof, H. M. (2021b). A novel two-parameter Nadarajah-Haghighi extension: properties, copulas, modeling real data and different estimation methods. Statistics, Optimization & Information Computing, forthcoming
59. Shehata, W. A. M., Butt, N. S., Yousof, H. and Aboraya, M. (2022). A New Lifetime Parametric Model for the Survival and Relief Times with Copulas and Properties. Pakistan Journal of Statistics and Operation Research, 18(1), 249-272.
60. Shehata, W. A. M., Yousof, H. M. and Aboraya, M. (2021). A Novel Generator of Continuous Probability Distributions for the Asymmetric Left-skewed Bimodal Real-life Data with Properties and Copulas. Pakistan Journal of Statistics and Operation Research, 17(4), 943-961.
61. Shrahili, M.; Elbatal, I. and Yousof, H. M. Asymmetric Density for Risk Claim-Size Data: Prediction and Bimodal Data Applications. Symmetry 2021, 13, 2357.
62. Yadav, A. S., Goual, H., Alotaibi, R. M., H. Rezk, Ali, M. M., Yousof, H. M. (2020). Validation of the Topp-Leone-Lomax model via a modified Nikulin-Rao-Robson goodness-of-fit test with different methods of estimation. Symmetry, 12(1), 1-26. doi: 10.3390/sym1201005
63. Yousof, H. M., Chesneau, C., Hamedani, G. and Ibrahim, M. (2021). A New Discrete Distribution: Properties, Characterizations, Modeling Real Count Data, Bayesian and Non-Bayesian Estimations. Statistica, 81(2), 135-162.
64. Yousof, H. M. and Korkmaz, M. C. (2017). Topp-Leone Nadarajah-Haghighi distribution: mathematical properties and applications, International Journal of Applied Mathematics. Journal of Statisticians: Statistics and Actuarial Sciences, 2, 119-128.
65. Yousof, H. M., Korkmaz, M. C. Hamedani G. G. (2017). The odd Lindley Nadarajah-Haghighi distribution. J. Math. Comput. Sci., 7, 864-882.