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Exponentiated Exponential - Exponentiatd Weibull Linear Mixed Distribution

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Abstract

The present article deals with the linear mixing of exponentiated Exponential and exponentiated Weibull distributions. The proposed model is named as exponentiated Exponential-Exponentiated Weibull linear mixed distribution. Several existing distributions arise as special cases of the proposed model. Statistical properties such as shape of the density and the distribution function, moments, generating functions and reliability are studied. An empirical study is presented for mean, variance, coefficient of skewness, and coefficient of kurtosis. The method of maximum likelihood is used for estimation of the parameters. Applicability of the proposed model in the field of reliability and medical sciences accommodate its validity.

Key Words: Exponentiated Exponential; Exponentiated Weibull; Empirical Study; Inference; Maximum Likelihood.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

Various researchers have done a considerable work for the finite mixture of distributions. Their motivation in mixing of the distributions is to provide a model which has potential to deal with different causes of failures. Several works are available regarding mixture models. Jewell (1982) has proposed a mixture of exponential distributions. Nassar and Mahmoud (1985) have studied characterization of mixture of exponential distributions. Nassar (1988) has obtained two properties which characterize mixtures of exponential distributions. Titterington et al. (1985) have discussed finite mixture models in detail. Lindsay (1995) has worked on mixture models. Gharib (1995) has presented two characterizations of the of a gamma mixture distribution. Gharib (1996) has derived characterizations of the exponential distribution via mixing distributions. Rider (1961) has estimated the parameters of the mixture of two exponential distributions using method of moments. Al-Hussaini (1999) has developed the mixture of two exponential distributions in context of Bayesian analysis. Bartoszewicz (2002) has derived the Laplace transform order relations for the mixtures using results on stochastic orders. Jaheen (2005) has used the maximum likelihood and Bayes methods for estimating the parameters of the finite mixture of two exponential distributions based on record statistics. Radhakrishna et al. (1992) have given estimation of parameters in a two-component mixture generalized gamma distribution. Al-Hussaini et al. (2000) have introduced finite mixture of two-component Gompertz distribution. Sultan

et al. (2007) have developed a mixture of two inverse Weibull distributions. Mubarak (2011) has constructed mixture of two Frechet distributions. Bakoban (2010) has suggested a mixture of exponential and exponentiated gamma distributions. Abu-Zinadah (2010) has studied mixture of exponentiated Pareto and exponential distributions. Shawky and Bakoban (2009) have defined mixture of two component exponentiated gamma distribution. Badr and Shawky (2014) have suggested a mixture of exponentiated Frechet distribution. Nair and Abdul (2010) have provided a mixture of exponential distributions. Soliman (2006) has consider the finite mixture of Rayleigh distribution.
Gupta and Kundu (2001) have introduced a new generalized exponential distribution which is named as exponentiated exponential (EE) distribution. The generalized exponential distribution is the particular case of Gompertz-Verhulst distribution. The distribution function (CDF) of the EE distribution is given as

$$F_{EE}(y) = (1 - e^{-y\beta_1})^{\alpha_1}, \quad y > 0, \;\; \alpha_1, \beta_1 > 0, \tag{1}$$

where α_1 is shape parameter and β_1 is scale parameter. The density function (PDF) of EE distribution is

$$f_{EE}(y) = \alpha_1 \beta_1 e^{-y\beta_1} \left(1 - e^{-y\beta_1}\right)^{\alpha_1 - 1} \quad y > 0, \;\; \alpha_1, \beta_1 > 0. \tag{2}$$

If the shape parameter is equal to 1, then EE distribution reduces to classical exponential distribution. Mudholkar and Srivastava (1993) have proposed exponentiated Weibull (EW) distribution. The CDF of EW distribution is provided as

$$F_{EW}(y) = (1 - e^{-(y\beta_2)^{\lambda}})^{\alpha_2}, \quad y > 0, \;\; \alpha_2, \beta_2, \lambda > 0, \tag{3}$$

where λ and α_2 are shape parameters and β_2 is scale parameter. The PDF of EW distribution is

$$f_{EW}(y) = \alpha_2 \beta_2^{\lambda} \lambda y^{\lambda - 1} e^{-(\beta_2 y)^{\lambda}} \left(1 - e^{-(\beta_2 y)^{\lambda}}\right)^{\alpha_2 - 1}, \quad y > 0, \;\; \alpha_2, \beta_2, \lambda > 0. \tag{4}$$

In modeling real data, the common failure time distributions need not be the same but can be a mixture of different lifetime distributions. Each of these distinct lifetime distributions can represent a different type of cause of failure for the population. In this article, we have presented a finite mixture of exponentiated exponential and exponentiated Weibull distributions, named as *"exponentiated exponential-exponentiated Weibull Linear Mixed Distribution* (*EE-EW LMD*)". Several characteristics of the proposed model are studied. The proposed model consists of five parameters which makes it more flexible and adaptable to handle the complex situation of the modeling phenomena.
The article layout follows. In Section 2, we have introduced the new model and have discussed its properties. In Section 3, we have conducted an empirical study. In Section 4, estimation of the model parameters have been done. In Section 5, we have illustrated the applicability of the proposed model. In Section 6, some concluding remarks are stated.

2. Exponentiated Exponential-Exponentiated Weibull Linear Mixed Distribution and its Properties

In this section, we will derive the distribution function (CDF) and density function (PDF) of the EE-EW LMD.

2.1. The Model

A random variable X is said to have a linear mixed distribution if its CDF is given as

$$F(y) = \sum_{i=1}^{k} p_i F_i(t), \tag{5}$$

where each F_i is a CDF and $p_1, ...p_k$ are the mixing proportions, which are non-negative and sum to one. The CDF of EE-EW LMD is obtained by using the CDF's of exponentiated-exponential and exponentiated-Weibull distributions in the above equation and is given as

$$F_{EE-EW}(y) = p_1(1 - e^{-y\beta_1})^{\alpha_1} + p_2(1 - e^{-(y\beta_2)^{\lambda}})^{\alpha_2}, \;\; y > 0, \; \alpha_1, \alpha_2, \beta_1, \beta_2, \lambda > 0, \; p_1 + p_2 = 1. \tag{6}$$

The density function of EE-EW LMD is

$$f_{EE-EW}(y) = p_1\alpha_1\beta_1 e^{-y\beta_1}\left(1-e^{-y\beta_1}\right)^{\alpha_1-1} + p_2\alpha_2\beta_2^{\lambda}\lambda y^{\lambda-1}e^{-(\beta_2 y)^{\lambda}}\left(1-e^{-(\beta_2 y)^{\lambda}}\right)^{\alpha_2-1}. \quad (7)$$

For simplicity, we use $p_1 = p$ and $p_2 = 1-p$. The proposed model reduces to exponentiated exponential-exponentiated exponential linear mixed distribution for $\lambda = 1$. For $\alpha_1 = 1$, it reduces to exponential-exponentiated Weibull linear mixed distribution. For $\alpha_2 = 1$, is converts into exponentiated exponential-Weibull linear mixed distribution. For $\alpha_1 = \alpha_2 = 1$, it transforms into exponential-Weibull linear mixed distribution. For $p = 0$, it turns into exponentiated Weibull distribution. For $p = 1$, it reduces to exponentiated Exponential distribution. We use $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ for simplicity.

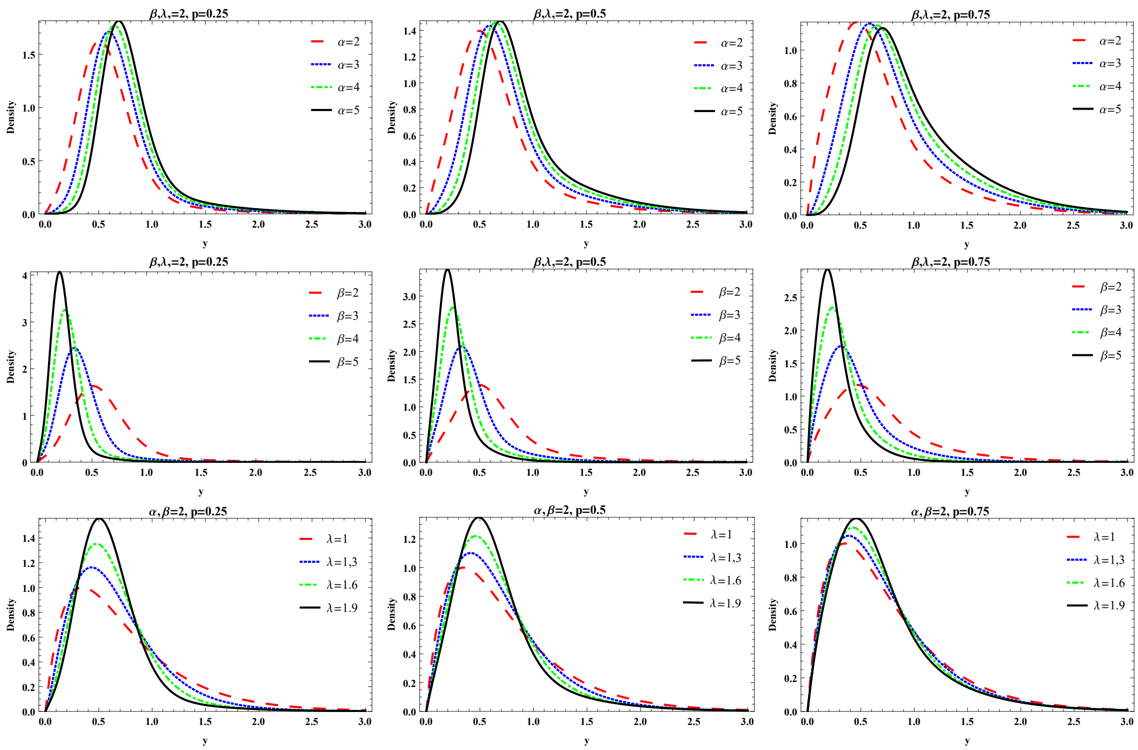


Figure 1: Plots of Density for Different Values of the Parameters

2.2. Shape of CDF and PDF

The distribution function of the EE-EW LMD distribution is given in (6) as

$$F_{EE-EW}(y) = p(1-e^{-y\beta_1})^{\alpha_1} + (1-p)(1-e^{-(y\beta_2)^{\lambda}})^{\alpha_2}.$$

Using following identity given by Prudnikov et al. (1986)

$$(1+x)^a = \sum_{j=0}^{\infty}\frac{\Gamma(a+1)}{j!(a+1-j)}x^j,$$

the CDF of the EE-EW LMD distribution is written as

$$F(x) = p\sum_{j=0}^{\infty}(-1)^j\frac{(\alpha_1+1)}{j!\Gamma(\alpha_1+1-j)}(e^{-y\beta_1})^j + (1-p)\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(\alpha_2+1)}{j!\Gamma(\alpha_2+1-j)}(e^{-(y\beta_2)^{\lambda}})^j. \quad (8)$$

Further, the PDF of the EE-EW LMD is written as

$$f_{EE-EW}(y) = p\beta_1 \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(\alpha_1+1)}{j!\Gamma(\alpha_1-j)}(e^{-y\beta_1})^{j+1} + (1-p)\lambda\beta_2^{\lambda}y^{\lambda-1}\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(\alpha_2+1)}{j!\Gamma(\alpha_2-j)}(e^{-(y\beta_2)^{\lambda}})^{j+1}. \tag{9}$$

It can be seen that the density function is weighted sum of exponential and Weibull density functions.

2.3. Moments

Moments are very useful in studying several properties of the distribution. Moments are used to obtain the mean, variance, coefficient of skewness and coefficient of kurtosis for any distribution. The k-th moment of the EE-EW LMD is obtained by using (9) as

$$\mu_k' = pA^*\Gamma(k+1) + (1-p)B^*\Gamma\left(\frac{k}{\lambda}+1\right),$$

where

$$A^* = \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(\alpha_1+1)}{j!\Gamma(\alpha_1-j)}\frac{1}{\beta_1^k}\left(\frac{1}{j+1}\right)^{k+1}$$
$$B^* = \sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(\alpha_2+1)}{j!\Gamma(\alpha_2-j)}\frac{1}{\beta_2^k}\left(\frac{1}{j+1}\right)^{\frac{k}{\lambda}+1},$$

and $k>0$. The mean of the EE-EW LMD is obtained by using $k=1$ in above equation and is given as

$$\begin{aligned}\mu_k' =& p\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(\alpha_1+1)}{j!\Gamma(\alpha_1-j)}\frac{1}{\beta_1}\left(\frac{1}{j+1}\right)^{2}\Gamma(2)\\ &+(1-p)\sum_{j=0}^{\infty}(-1)^j \frac{\Gamma(\alpha_2+1)}{j!\Gamma(\alpha_2-j)}\frac{1}{\beta_2}\left(\frac{1}{j+1}\right)^{\frac{1}{\lambda}+1}\Gamma\left(\frac{1}{\lambda}+1\right).\end{aligned}$$

The coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) can be obtained as

$$CV = \sqrt{\frac{\mu_2}{\mu_1'}}.$$

$$CS = \frac{\mu_3}{\mu_2^{3/2}}.$$

$$CK = \left(\frac{\mu_4}{\mu_2^2}\right) - 3.$$

These coefficients can be computed for different values of the parameters.

2.4. Moment Generating Function

The moment generating function (mgf) of a distribution is obtained as

$$M_y(t) = \int_{-\infty}^{\infty} e^{ty} f(y)dy.$$

Using the relation $e^{ty} = \sum_{m=0}^{\infty} \frac{t!y^m}{m!}$, the mgf of the EE-EW LMD is

$$M_y(t) = pa^*\Gamma(m+1) + (1-p)b^*\Gamma(\frac{m}{\lambda}+1),$$

where

$$a^* = \sum_{j,m=0}^{\infty} (-1)^j \frac{t!\Gamma(\alpha_1+1)}{j!m!\Gamma(\alpha_1-j)} \frac{1}{\beta_1^m} \left(\frac{1}{j+1}\right)^{m+1}$$
$$b^* = \sum_{j,m=0}^{\infty} (-1)^j \frac{t!\Gamma(\alpha_2+1)}{j!m!\Gamma(\alpha_2-j)} \frac{1}{\beta_2^m} \left(\frac{1}{j+1}\right)^{\frac{m}{\lambda}+1}.$$

Moment generating function can be used to obtain moments of the distribution.

2.5. Reliability and Hazard Function

The reliability function (rf) and hazard rate function (hrf) of the mixture model are defined as

$$R(y) = pR_1(y) + (1-p)R_2(y)$$
$$h(y) = \frac{f(y)}{R(y)}.$$

So the rf of the EE-EW LMD is given as

$$R_{EE-EW}(y) = p(1-(1-e^{-y\beta_1})^{\alpha_1}) + (1-p)(1-(1-e^{-(y\beta_2)^\lambda})^{\alpha_2}).$$

The hrf is given as

$$h_{EE-EW}(y) = \frac{p\alpha_1\beta_1 e^{-y\beta_1}\left(1-e^{-y\beta_1}\right)^{\alpha_1-1} + (1-p)\alpha_2\beta_2^\lambda \lambda y^{\lambda-1} e^{-(\beta_2 y)^\lambda}\left(1-e^{-(\beta_2 y)^\lambda}\right)^{\alpha_2-1}}{p(1-(1-e^{-y\beta_1})^{\alpha_1}) + (1-p)(1-(1-e^{-(y\beta_2)^\lambda})^{\alpha_2})}$$

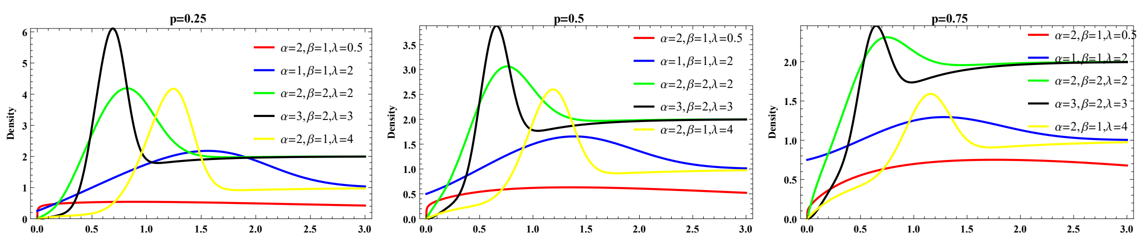


Figure 2: Plots of Hazard Rate Function for Different Values of the Parameters

3. Empirical Study

In this section, we have performed an empirical study for the mean, variance, coefficient of skewness and coefficient of kurtosis. The results of this empirical study are given in the following tables.

From Table (1), it is observed that different combination of the parameters provide different values for the mean. This table shows that for fixed values of α and λ, as the values of β increases the mean of the proposed model decreases. For the fixed values of α and β, as λ increases the mean of the proposed model also increases. For the fixed values of λ and β, as α increases the mean of the proposed model also increases.
From Table (2), it is observed that different combination of the parameters provide different values for the variance. From this table, we can see that for fixed values of α and λ, as the values of β increases the variance of the proposed model decreases. For fixed values of α and β, as λ increases the variance of the proposed model decreases. For fixed

Table 1: Mean of EE-EW Distribution for Different Values of Parameters

		p=0.25			p=0.5			p=0.75		
			λ			λ			λ	
α	β	2	3	4	2	3	4	2	3	4
1	1	0.9147	0.9197	0.9298	0.9431	0.9465	0.9532	0.9716	0.9732	0.9766
1	2	0.4573	0.4599	0.4649	0.4716	0.4732	0.4766	0.4858	0.4866	0.4883
1	3	0.3049	0.3066	0.3099	0.3144	0.3155	0.3177	0.3239	0.3244	0.3255
2	1	1.2343	1.1829	1.1630	1.3229	1.2886	1.2753	1.4114	1.3943	1.3877
2	2	0.6172	0.5915	0.5815	0.6614	0.6443	0.6377	0.7057	0.6972	0.6938
2	3	0.4114	0.3943	0.3877	0.4410	0.4295	0.4251	0.4705	0.4648	0.4626
3	1	1.4261	1.3372	1.2993	1.5619	1.5026	1.4773	1.6976	1.6680	1.6553
3	2	0.7131	0.6686	0.6497	0.7809	0.7513	0.7387	0.8488	0.8340	0.8277
3	3	0.4754	0.4457	0.4331	0.5206	0.5009	0.4924	0.5659	0.5560	0.5518

Table 2: Variance of EE-EW Distribution for Different Values of Parameters

		p=0.25			p=0.5			p=0.75		
			λ			λ			λ	
α	β	2	3	4	2	3	4	2	3	4
1	1	0.4134	0.3311	0.3001	0.6105	0.5555	0.5345	0.8061	0.7785	0.7678
1	2	0.1033	0.0828	0.0750	0.1526	0.1389	0.1336	0.2015	0.1946	0.1920
1	3	0.0459	0.0368	0.0333	0.0678	0.0617	0.0594	0.0896	0.0865	0.0853
2	1	0.4764	0.4033	0.3819	0.7499	0.7079	0.6965	1.0078	0.9901	0.9859
2	2	0.1191	0.1008	0.0955	0.1875	0.1770	0.1741	0.2520	0.2475	0.2465
2	3	0.0529	0.0448	0.0424	0.0833	0.0787	0.0774	0.1120	0.1100	0.1095
3	1	0.5218	0.4696	0.4600	0.8384	0.8214	0.8238	1.1182	1.1186	1.1241
3	2	0.1304	0.1174	0.1150	0.2096	0.2054	0.2059	0.2795	0.2797	0.2810
3	3	0.0580	0.0522	0.0511	0.0932	0.0913	0.0915	0.1242	0.1243	0.1249

values of λ and β, as α increases the variance of the proposed model increases.
Table (3) provides values of coefficient of skewness (CS). It can be seen from this table that for fixed values of α and λ, the increasing values of β demonstrate the increasing behavior of CS. For fixed values of α and β, the CS increases as the values of λ increases. For fixed values of β and λ, one can see that the values of CS decreases as α increases.
Table (4) provides values of coefficient of kurtosis (CK). For the fixed values of α and λ, the increasing values of β demonstrate the mixed behavior for CK. For the fixed values of α and β, the CK increases as the values of λ increases. For the fixed values of β and λ, one can see that the values of CS are mixed as α increases.

4. Inference

In this section we have discussed the maximum likelihood estimation of the parameters of EE-EW LMD. For this let $y_1, y_2, ..., y_n$ be a random sample from $EEEW_(x, \alpha_1, \beta_1, \alpha_2, \beta_2, \lambda)$ distribution then their likelihood function is written as

$$L\left(f(y_i)\right) = \prod_{i=1}^{n} f(y_i).$$

Table 3: Coefficient of Skewness EE-EW distribution for Different Values of Parameters

		p=0.25			p=0.5			p=0.75		
		λ			λ			λ		
α	β	2	3	4	2	3	4	2	3	4
1	1	0.5806	0.6028	0.6105	0.5896	0.5978	0.6005	0.5932	0.5961	0.5969
1	2	0.7071	0.6935	0.6854	-0.6547	-0.4006	-0.2811	0.4631	0.3683	0.3269
1	3	0.5722	0.5798	0.5816	0.6245	0.6275	0.6277	0.6771	0.6773	0.6769
2	1	0.5459	0.5637	0.5726	0.5727	0.5802	0.5834	0.5833	0.5860	0.5871
2	2	0.0383	0.4540	0.5061	0.6281	0.5205	0.4796	0.6776	0.6861	0.6885
2	3	0.5536	0.5713	0.5797	0.6456	0.6544	0.6589	0.7071	0.7071	0.7070
3	1	0.5417	0.5591	0.5684	0.5718	0.5792	0.5826	0.5837	0.5864	0.5875
3	2	-0.2139	-0.7059	-0.6944	0.6822	0.7005	0.7038	0.5890	0.5999	0.6033
3	3	0.5561	0.5821	0.5953	0.6849	0.6937	0.6981	0.5833	0.5673	0.5541

Table 4: Coefficient of Kurtosis EE-EW Distribution for Different Values of Parameters

		p=0.25			p=0.5			p=0.75		
		λ			λ			λ		
α	β	2	3	4	2	3	4	2	3	4
1	1	2.1895	2.2093	2.2161	2.2068	2.2139	2.2161	2.2130	2.2154	2.2162
1	2	1.0114	1.0786	1.1543	2.2274	2.1961	2.1790	2.2075	2.2158	2.2188
1	3	2.1946	2.2061	2.2089	2.2697	2.2713	2.2716	2.0540	2.0623	2.0683
2	1	2.1832	2.2012	2.2078	2.2054	2.2120	2.2142	2.2134	2.2156	2.2164
2	2	2.1411	1.9712	1.9204	2.1990	2.2152	2.2189	2.1287	2.1351	2.1367
2	3	2.1974	2.2210	2.2312	2.1262	2.1012	2.0852	1.1071	1.0991	1.0922
3	1	2.1851	2.2025	2.2090	2.2078	2.2141	2.2163	2.2159	2.2180	2.2188
3	2	2.2553	2.2591	2.2540	2.1391	2.1552	2.1588	2.0907	2.0960	2.0972
3	3	2.2205	2.2432	2.2507	1.5154	1.4346	1.3803	1.4656	1.4853	1.5056

The log-likelihood function for EE-EW LMD is written as

$$\begin{aligned}\ell =& n\ln p + n\ln\alpha_1 + n\ln\beta_1 + (1-\alpha_1)\sum_{i=1}^{n}\ln(1-e^{-y_i\beta_1}) - \beta_1\sum_{i=1}^{n} y_i \\ &+ \ln q + n\ln\alpha_2 + n\ln\lambda + n\lambda\ln\beta_2 + (\lambda-1)\sum_{i=1}^{n}\ln y_i \\ &+ (\alpha_2-1)\sum_{i=1}^{n}\ln[1-e^{-(y_i\beta_2)^\lambda}] - \sum_{i=1}^{n}(y_i\beta_2)^\lambda. \end{aligned} \tag{10}$$

Differentiating w.r.t $\alpha_1, \alpha_2, \beta_1, \beta_2, \lambda$ and equating resulting derivatives to zero, the likelihood equations are

$$\frac{n}{\alpha_1} + \sum_{i=1}^{n}\ln(1-e^{-y_i\beta_1}) = 0 \tag{11}$$

$$\frac{n}{\alpha_2} + \sum_{i=1}^{n} ln[1-e^{-(y_i\beta_2)^\lambda}] = 0 \tag{12}$$

$$\frac{n}{\beta_1} + (\alpha_1)\sum_{i=1}^{n}\frac{y_i e^{-y_i\beta_1}}{1-e^{-y_i\beta_1}} - \sum_{i=1}^{n} y_i = 0, \tag{13}$$

$$\frac{n\lambda}{\beta_2}+(1-\alpha_2)\lambda\beta_2^{\lambda-1}\sum_{i=1}^{n}\frac{e^{-(y_i\beta_2)^\lambda}}{1-e^{-(y_i\beta_2)^\lambda}}y_i^\lambda-\lambda\beta_2^{\lambda-1}\sum_{i=1}^{n}y_i^\lambda=0 \tag{14}$$

$$\frac{n}{\lambda}+n\ln\beta_2+\sum_{i=1}^{n}\ln y_i+(\alpha_2-1)\lambda\sum_{i=1}^{n}\frac{e^{-(y_i\beta_2)^\lambda}}{1-e^{-(y_i\beta_2)^\lambda}}y_i^\lambda\ln(y_i\beta_2)-\beta_2^{\lambda^1}\sum_{i=1}^{n}y_i^\lambda\ln(y_i\beta_2)=0. \tag{15}$$

The maximum likelihood estimators $\hat{\alpha_1},\hat{\alpha_2},\hat{\beta_1},\hat{\beta_2}$ and $\hat{\gamma}$ are obtained by solving (11)-(15) numerically or by using nlm() function in R. The asymptotic normality of maximum likelihood estimates can be used to obtain the asymptotic confidence intervals for the unknown parameters. This results is stated as

$$\sqrt{n}(\hat{\theta}-\theta)\longrightarrow N_5(0,I^{-1}(\theta)), \tag{16}$$

where

$$I(\theta)=p\alpha_1\beta_1\left(1-e^{-x\beta_1}\right)^{\alpha_1-1}e^{-x\beta_1}+(1-p)\alpha_2\lambda\beta_2^\lambda x^{\lambda-1}\left(1-e^{(x\beta_2)^\lambda}\right)^{\alpha_2-1}e^{(x\beta_2)^\lambda}. \tag{17}$$

The entries of Fisher information matrix are given as

$$I(\theta)=-\frac{1}{n}\begin{bmatrix} E(\frac{\partial^2 I}{\partial\alpha_1^2}) & E(\frac{\partial^2 I}{\partial\alpha_1\partial\beta_1}) & E(\frac{\partial^2 (I)}{\partial\alpha_1\partial\alpha_2}) & E(\frac{\partial^2 I}{\partial\alpha_1\partial\beta_2}) & E(\frac{\partial^2 I}{\partial\alpha_1\lambda}) \\ E(\frac{\partial^2 I}{\partial\beta_1\alpha_1}) & E(\frac{\partial^2 I}{\partial\beta_1^2}) & E(\frac{\partial^2 I}{\partial\beta_1\partial\alpha_2}) & E(\frac{\partial^2 I}{\partial\beta_1\partial\beta_2}) & E(\frac{\partial^2 I}{\partial\beta_1\partial\lambda}) \\ E(\frac{\partial^2 I}{\partial\alpha_2\partial\alpha_1}) & E(\frac{\partial^2 I}{\partial\alpha_2\partial\beta_1}) & E(\frac{\partial^2 I}{\partial\alpha_2^2}) & E(\frac{\partial^2 I}{\partial\alpha_2\partial\beta_2}) & E(\frac{\partial^2 I}{\partial\alpha_2\partial\lambda}) \\ E(\frac{\partial^2 I}{\partial\beta_2\partial\alpha_1}) & E(\frac{\partial^2 I}{\partial\beta_2\partial\beta_1}) & E(\frac{\partial^2 I}{\partial\beta_2\partial\alpha_2}) & E(\frac{\partial^2 I}{\partial\beta_2^2}) & E(\frac{\partial^2 I}{\partial\beta_2\partial\lambda}) \\ (\frac{\partial^2 I}{\partial\lambda\partial\alpha_1}) & (\frac{\partial^2 I}{\partial\lambda\partial\beta_1}) & (\frac{\partial^2 I}{\partial\lambda\partial\alpha_2}) & (\frac{\partial^2 I}{\partial\lambda\partial\beta_2}) & E(\frac{\partial^2 I}{\partial\lambda^2}) \end{bmatrix}$$

Inverting the above matrix, the asymptotic variances and covariances of the ML estimators for $\alpha_1,\alpha_2,\beta_1\beta_2$ and λ can be obtained. Using above, approximate $100(1-\lambda)\%$ confidence intervals for $\alpha_1,\alpha_2,\beta_1\beta_2$ and λ are obtained as:

$$\hat{\alpha_1}\pm Z_{\frac{\beta_j}{2}}\sqrt{E(\frac{\partial^2 I}{\partial\alpha_1^2})},\ \hat{\alpha_2}\pm Z_{\frac{\beta}{2}}\sqrt{E(\frac{\partial^2 I}{\partial\alpha_2^2})},\ \hat{\beta_1}\pm Z_{\frac{\beta}{2}}\sqrt{E(\frac{\partial^2 I}{\partial\beta_1^2})},\ \hat{\beta_2}\pm Z_{\frac{\beta}{2}}\sqrt{E(\frac{\partial^2 I}{\partial\beta_2^2})},$$
$$\hat{\lambda}\pm Z_{\frac{\beta}{2}}\sqrt{E(\frac{\partial^2 I}{\partial\lambda^2})}. \tag{18}$$

In the following we have given a real data application of the proposed distribution.

5. Application

In this section, we have discussed real data applications of the EE-EW LMD. We have used data sets from the field of reliability and medical sciences. Data sets are given by Bethea (1995) and Birnbaum and Saunders (1969) and are given in the appendix. The results of estimated parameters, -2 Log-likelihood (-2ll), AIC (Akaike information criterion), and BIC (Bayesian Information criterion) are obtained. For computational and graphical analysis, we have used software $mathematica$. The results are compared with following well known distributions.

1. Burr XII distribution (B-XII) by Rodriguez (1977)

$$f(y) = \frac{\alpha\gamma y^{\alpha\gamma-1}}{(1+y^{\alpha})^{\gamma+1}}$$

2. Poisson Lomax (PL) distribution by Al-Zahrani and Sagor (2014)

$$f(y) = \frac{\alpha\beta\lambda(1+\beta y)^{-(\alpha+1)}e^{-\lambda(1+\beta y)^{-\alpha}}}{1-e^{-\lambda}}$$

3. Inverse Weibull (IW) distribution given by Hanook et al. (2013)

$$f(y) = \alpha y^{-(\alpha+1)}e^{-y^{-\alpha}} \tag{19}$$

4. Burr type II (B-II) distribution by Rodriguez (1977)

$$f(y) = \frac{\alpha(1+e^{-y})^{-(\alpha+1)}}{e^{y}}$$

5. A new Generalized Weighted Weibull (GWW) distribution by Abbas et al. (2019)

$$f(y) = 2\beta\alpha\gamma(1+\lambda^{\gamma})y^{\gamma-1}e^{-2\alpha y^{\gamma}(1+\lambda^{\gamma})}[1-e^{-2\alpha y^{\gamma}(1+\lambda^{\gamma})}]^{\beta-1} \tag{20}$$

6. Rayleigh (R) distribution by Abdi and Kaveh (1998)

$$f(y) = 2\alpha y e^{-\alpha y^2}\left(1-e^{-\alpha y^2}\right)$$

Results of the descriptive statistics, estimated model parameters, and goodness of fit criterion are given in tables 5-7.

Table 5: Descriptive Statistics for the Data Sets

Data	Max	Min	Mean	Median	Variance	S.D	Skewness	Kurtosis
I	53	0.9	14.675	10.75	186.697	13.667	1.3487	4.2799
II	147	5	68.34	67	502.631	22.4194	0.409	4.2524

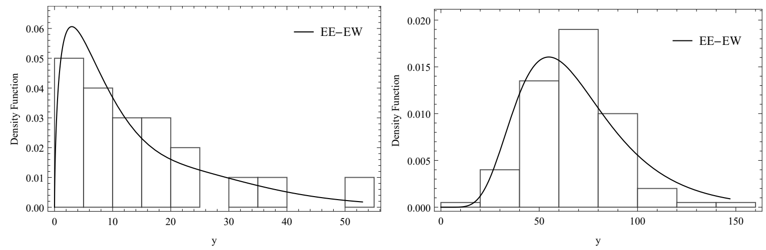
Table 6: Estimated Parameters of EEEW distribution

Data	Model			Parameters		
I	EEEW	α_1=2.0946	β_1=0.1585	α_2=17.2106	β_2 =0.0972	λ=0.7892
	GWEx	α=0.653534	β=81.3931	γ=0.2167	λ=3.1883	
	PL	α=0.9901	β =4.4292	λ =24.4338		
	BXII	α =0.082	β =5.4724			
	BII	α =28.2605				
	IW	α =0.5838				
	R	α =0.0041				
II	EEEW	α_1 =4777.44	β_1 =1218.62	α_2 =9.67	β_2 =0.04	λ =1319.79
	GWEx	α =1.14	β =83997.70	λ =0.15	γ =59.63	
	PL	α =1.81	β =0.35	λ =201.30		
	BXII	α =0.03	β =7.98			
	BII	α =14882.50				
	IW	α =0.32				
	R	α =0.0002				

The results indicate that the proposed model is a better fit to both the considering data sets as compared with the other distributions because it has smaller values of -2ℓ, AIC, and BIC. The graphical representation of the density plot also demonstrates the adequacy of the EE-EW LMD distribution.

Table 7: Estimated Values of -2ℓ, AIC, and BIC of EEEW distribution

Data Set I							
	EEEW	GWEx	PL	BXII	BII	IW	R
-2ℓ	146.7	163.029	152.108	175.492	494.757	149.195	230.946
AIC	154.7	167.029	158.108	177.492	496.757	157.195	232.946
BIC	158.683	169.021	161.095	178.488	497.752	161.178	233.942
Data Set II							
-2ℓ	927.843	1316.61	1018.44	1378.96	11946.3	1006.36	1049.64
AIC	935.843	1320.61	1024.44	1380.96	11948.3	1014.36	1051.64
BIC	939.826	1322.6	1027.42	1381.96	11949.3	1018.34	1052.64



Data I Data II

Figure 3: Plots of Density for EEEW distribution

6. Conclusion

In this article, we have proposed a mixture of exponentiated exponential distribution and exponentiated Weibull distribution. The proposed model is named as exponentiated Exponential-exponentiated Weibull linear mixed distribution. Several characteristics of the new model are studied. An empirical study of model parameters through mean, variance, coefficient of skewness, and coefficient of kurtosis are discussed. Method of maximum likelihood is used for estimation of the model parameters. Two real-life data sets are used to test the competency of the new model. We have found that the proposed model is more compatible and flexible as compared with the competing models.

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Appendix

Data I 0.9, 1.5, 2.3, 3.2, 3.9, 5.0, 6.2, 7.5, 8.3, 10.4, 11.1, 12.6, 15.0, 16.3, 19.3, 22.6, 24.8, 31.5, 38.1 , 53.0.

Data II 5, 25, 31, 32, 34, 35, 38, 39, 39, 40, 42, 43, 43, 43, 44, 44, 47, 47, 48, 49, 49, 49, 51, 54, 55, 55, 55, 56, 56, 56, 58, 59, 59, 59, 59, 59, 63, 63, 64, 64, 65, 65, 65, 66, 66, 66, 66, 66, 67, 67, 67, 68, 69, 69, 69, 69, 71, 71, 72, 73, 73, 73, 74, 74, 76, 76, 77, 77, 77, 77, 77, 77, 79, 79, 80, 81, 83, 84, 84, 86, 86, 87, 90, 91, 92, 92, 92, 92, 93, 94, 97, 98, 98, 99, 101, 103, 105, 109, 136, 147.s