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# Marshall–Olkin Alpha Power Lomax Distribution: Estimation Methods, Applications on Physics and Economics

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**Abstract**

The aim of this article is to introduce and study a new extension of Lomax distribution with a four-parameter named as the Marshall–Olkin alpha power Lomax (MOAPL) distribution. Some statistical properties of this distribution are discussed as quantile, median, linear representation, non-central moments, and moment generating function. Maximum likelihood estimation (MLE), maximum product spacing (MPS), and Least Square (LS) method for the MOAPL distribution parameters are discussed. The problem of this article is to describe real-life phenomena by using statistical distributions. For this reason, the theory of statistical distribution and generating new distributions are of great interest. Many authors studied and generated new distributions from old ones. A numerical study using real data analysis and Monte-Carlo simulation is performed to compare different methods of estimation. The superiority of the new model over some well-known distributions is illustrated by physics and economics real data sets. The MOAPL model can produce better fits than some well-known distributions as Marshall–Olkin Lomax, alpha power Lomax, Lomax distribution, Marshall–Olkin alpha power exponential, Kumaraswamy-generalized Lomax, exponentiated Lomax, and power Lomax.

**Key Words:** Marshall–Olkin alpha power; Lomax distribution; maximum likelihood estimation; maximum product spacing; least square method and data analysis.

**Mathematical Subject Classification:** 60E05, 62E15

## 1. Introduction

The Lomax distribution has been introduced by Lomax (1954). It is an important model for lifetime analysis and business failure data, moreover, it has been widely applied in a variety of contexts, known as Lomax or Pareto type II distribution. The probability distribution of Lomax is a heavy-tail and often used in business, economics, and actuarial modeling. Many authors have discussed more applications by using Lomax distribution. For examples, see Chahkandi and Ganjali (2009), Hassan and Al-Ghamdi (2009), Abd-Elfattah and Alharbey (2010), Ashour et al. (2011), Nasiri and Hosseini (2012), Helu et al. (2015), Hassan et al. (2016) and El-Sherpieny et al. (2020), all of them used Lomax distribution for different applications. The CDF and pdf of Lomax distribution with parameters $\beta$ and $\lambda$ are given respectively, as

$$F(x;\beta,\lambda) = 1-\left(1+\frac{x}{\lambda}\right)^{-\beta}; \quad x \geq 0, \beta, \lambda > 0, \tag{1.1}$$

$$f(x;\beta,\lambda)=\frac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(1+\beta)}. \tag{1.2}$$

Modified and extended versions of the Lomax distribution have been studied by many authors as Rao et al. (2009) when introduced the Marshall-Olkin extended Lomax (MOL) distribution. Gupta et al. (2010) discussed estimating the reliability R=P(X>Y) in the Marshall–Olkin extended Lomax distribution. Al-Zahrani and Sagor (2014) introduced Poisson Lomax distribution. El-Bassiouny et al. (2015) introduced the exponential Lomax distribution. Cordeiro et al. (2015) introduced gamma Lomax distribution. Kilany (2016) introduced the weighted Lomax distribution. Rady et al. (2016) introduced power Lomax distribution. Oguntunde et al. (2017) introduced Gompertz Lomax distribution. Nwezza and Ugwuowo (2020) introduced the Marshall-Olkin Gumbel-Lomax distribution (MOGL) distribution.

Marshall-Olkin family has been introduced by Marshall and Olkin (1997) to construct new extended distributions. The MO extended distribution offers a wide range of behavior than the basic distribution from which they are derived. For more examples, see Ghitany (2005), Ghitany et al. (2007), Alice and Jose (2005), Okasha and Kayid (2016), and Ahmad and Almetwally (2020). On the other hand, the alpha power (AP) family has been proposed by Mahdavi and Kundu (2017) which construct more extended distributions. A lot of work in distributions based on AP transformation had been done, for example, Nassar et al. (2017), Elbatal et al. (2018), Dey et al. (2019 a, b), Hassan et al. (2019), and Basheer (2019). Recently, The Marshall Olkin alpha power (MOAP) family has been proposed by Nassar et al. (2019) to get new distributions. The cdf of MOAP is given by

$$F(x)=\begin{cases}\dfrac{\alpha^{G(x)}-1}{(\alpha-1)\left(\theta+\dfrac{(1-\theta)}{(\alpha-1)}(\alpha^{G(x)}-1)\right)} & if\ \alpha>0,\alpha\neq 1,\theta>0\\ G(x) & if\ \alpha=1\end{cases}, \tag{1.3}$$

The pdf of MOAP can be expressed as

$$f(x)=\begin{cases}\dfrac{\theta\ln(\alpha)}{\alpha-1}\dfrac{g(x)\alpha^{G(x)}}{\left(\theta+\dfrac{(1-\theta)}{(\alpha-1)}(\alpha^{G(x)}-1)\right)^2} & if\ \alpha>0,\alpha\neq 1,\theta>0\\ g(x) & if\ \alpha=1\end{cases} \tag{1.4}$$

The aim of this paper is to clarify two things. Firstly, propose and study a new lifetime distribution called Marshall Olkin alpha power Lomax (MOAPL) distribution based on the MOAP family. Some statistical properties of the MOAPL distribution are provided. Secondly, parameters estimation for the MOAPL distribution is discussed by using MLE, MPS, and LS methods. To evaluate the performance of the estimators, an extensive simulation study is carried out. Our MOAPL model as well as some other well-known distributions are illustrated by three real data sets. The MOAPL model can produce better fits than some well-known distributions.

The paper is organized as follows; in section 2, we introduce the description and notation of MOAPL distribution, while in section 3 the statistical properties of MOAPL are discussed. In section 4 we discuss the parameter estimation of MOAPL distribution. In section 5, the Monte-Carlo simulation study is presented to compare the performance of the estimation of the parameters for different methods. In section 6, applications of three real data sets are studied. Lastly in section 7, we discuss the results and conclusions of the current study.

## 2. Model Description and Notation

In this section, we will introduce the MOAPL distribution and some of its sub-models.

### 2.1. MOAPL Distribution

The MOAP family and Lomax distribution have been used to generate MOAPL distribution. It is represented by the random variable $X\sim MOAPL(\alpha,\beta,\theta,\lambda)$. By using Equations (1.3, 1.4, 1.1, and 1.2), the pdf of MOAPL distribution is given as:

$$f(x,\Omega)=\begin{cases}\dfrac{\theta\beta\ln(\alpha)}{\lambda(\alpha-1)}\dfrac{\left(1+\frac{x}{\lambda}\right)^{-(1+\beta)}\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}}{\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}-1\right)\right)^{2}} & if\ \alpha>0,\alpha\neq 1,\theta>0\\ \dfrac{\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(1+\beta)} & if\ \alpha=1\end{cases}\tag{2.1}$$

where $\Omega=(\alpha,\beta,\theta,\lambda)$, $\Omega>\mathbf{0}$. The cdf of MOAPL distribution takes this form

$$F(x,\Omega)=\begin{cases}\dfrac{\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}-1}{(\alpha-1)\left(\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}-1\right)\right)} & if\ \alpha>0,\alpha\neq 1,\theta>0\\ 1-\left(1+\frac{x}{\lambda}\right)^{-\beta} & if\ \alpha=1\end{cases}\tag{2.2}$$

Fig. 1 display plots of the pdf Equation (2.1) of the MOAPL distribution for some parameters values as follows

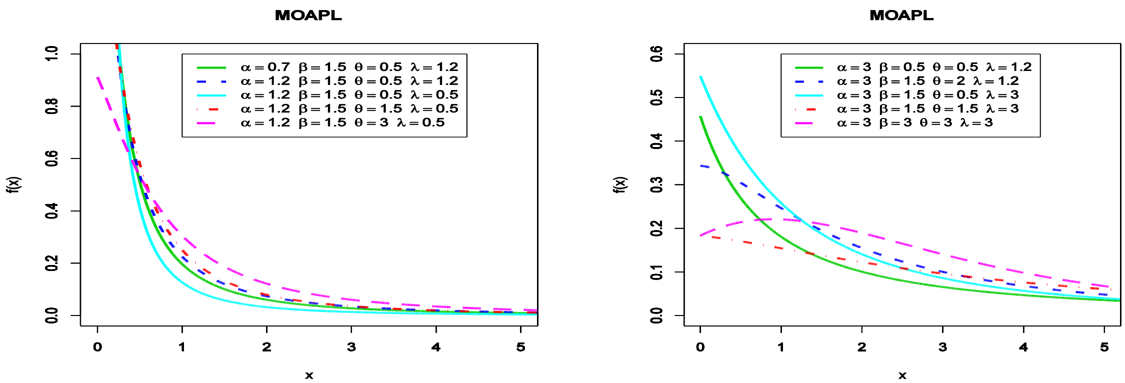


Fig. 1. Plots of the pdf of the MOAPL distribution with Some Values of Parameters.

### 2.2. Sub-Models from MOAPL

Many sub-models can be derived from MOAPL distribution as will be shown in Table 1:

**Table 1:** Sub-Models from MOAPL distribution.

| Models | $\alpha$ | $\beta$ | $\theta$ | $\lambda$ |
|---|---|---|---|---|
| Marshall Olkin alpha power Pareto (MOAPP) distribution (new) | $\alpha$ | $\beta$ | $\theta$ | 1 |
| Alpha power Lomax (APL) distribution (new) | $\alpha$ | $\beta$ | 1 | $\lambda$ |
| Marshall Olkin Lomax (MOL) distribution [Rao et al. (2009)] | 1 | $\beta$ | θ | λ |
| Alpha power Pareto (APP) distribution (new) | $\alpha$ | $\beta$ | 1 | 1 |
| Marshall Olkin Pareto (MOP) distribution [Bdair and Ahmad (2019)] | 1 | $\beta$ | θ | 1 |
| Lomax distribution | 1 | $\beta$ | 1 | λ |
| Pareto distribution | 1 | $\beta$ | 1 | 1 |

### 2.3. Reliability functions of MOAPL distribution

The survival function of MOAPL distribution is given by

$$S(x,\Omega)=\begin{cases}\dfrac{\theta(\alpha-1)-\theta\left(\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}-1\right)}{(\alpha-1)\theta+(1-\theta)\left(\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}-1\right)} & if\ \alpha>0,\alpha\neq 1\\ \left(1+\frac{x}{\lambda}\right)^{-\beta} & if\ \alpha=1\end{cases}\tag{2.3}$$

The hazard function of a lifetime random variable $X$ with MOAPL distribution is given by

$$h(x,\Omega)=\begin{cases}\dfrac{\beta\ln(\alpha)\left(1+\frac{x}{\lambda}\right)^{-(1+\beta)}\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}}{\lambda(\alpha-1)-\lambda\left(\alpha^{1-\left(1+\frac{x}{\lambda}\right)^{-\beta}}-1\right)} & if\ \alpha>0,\alpha\neq1\\ 1-\left(1+\dfrac{x}{\lambda}\right)^{-\beta} & if\ \alpha=1\end{cases} \tag{2.4}$$

Figs. 2 display plots of the hazard function Equation (2.4) of the MOAPL distribution for some values of the parameters as follows

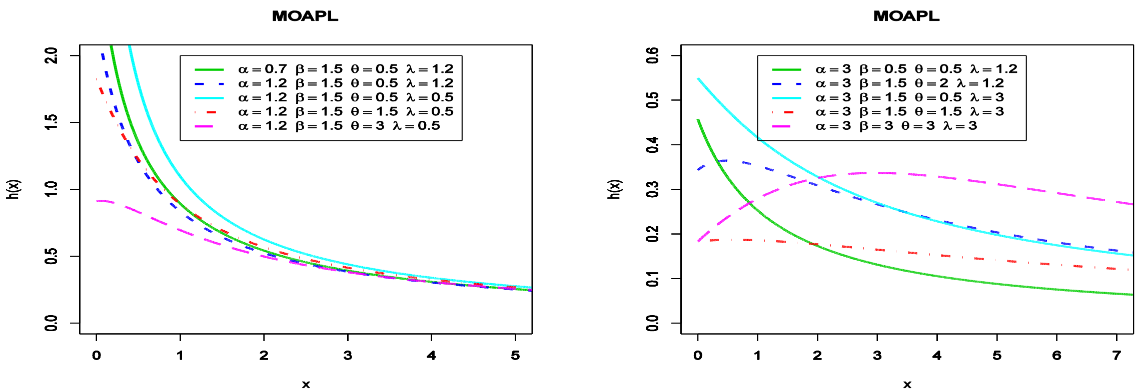


Fig. 2. Plots of the hazard of the MOAPL with Some Values of the Parameters.

## 3. Statistical Properties

In this section, we obtain some statistical properties of the MOAPL distribution such as quantile, median, linear representation, non-central moments, and moment generating function.

### 3.1. Quantile Function

By inverting Equation (2.2), we have the quantile function of the MOAPL distribution as following

$$x_q=\begin{cases}\lambda\left(\left[1-\dfrac{1}{\ln(\alpha)}\ln\left(1+\dfrac{\theta q\alpha-q+1}{1-q(1-\theta)}\right)\right]^{-1/\beta}-1\right) & if\ \alpha>0,\alpha\neq1\\ \lambda\left([1-q]^{-1/\beta}-1\right) & if\ \alpha=1\end{cases} \tag{3.1}$$

where $0<q<1$. From Equation (3.1), we can obtain the median (M) or the second quartile of MOAPL distribution when $q=0.5$ as follows

$$M=\begin{cases}\lambda\left(\left[1-\dfrac{1}{\ln(\alpha)}\ln\left(\dfrac{2+\theta(\alpha+1)}{1+\theta}\right)\right]^{-1/\beta}-1\right) & if\ \alpha>0,\alpha\neq1\\ \lambda\left([2]^{1/\beta}-1\right) & if\ \alpha=1\end{cases} \tag{3.2}$$

We can obtain the first and third quartiles ($q_1$ *and* $q_3$) of MOAPL distribution when $q=0.25$ and $q=0.75$ respectively, as follows

$$q_1=\begin{cases}\lambda\left(\left[1-\dfrac{1}{\ln(\alpha)}\ln\left(\dfrac{\theta\alpha+\theta}{3+\theta}\right)\right]^{-1/\beta}-1\right) & if\ \alpha>0,\alpha\neq1\\ \lambda\left(\left[\dfrac{3}{4}\right]^{-1/\beta}-1\right) & if\ \alpha=1\end{cases} \tag{3.3}$$

and

$$q_3 = \begin{cases} \lambda\left(\left[1-\frac{1}{\ln(\alpha)}\ln\left(\frac{3\theta\alpha+3\theta+2}{1+3\theta}\right)\right]^{-1/\beta}-1\right) & if\ \alpha>0, \alpha\neq 1 \\ \lambda\left(\left[\frac{1}{4}\right]^{-1/\beta}-1\right) & if\ \alpha=1 \end{cases} \quad (3.4)$$

From Equations (3.2, 3.3, and 3.4) first, second and third quartiles of MOAPL distribution, we can obtain the Galton skewness (Sk), also known as Bowley's skewness that defined as

$$Sk = \frac{q_3 - 2M + q_1}{q_3 - q_1}, \quad (3.5)$$

and also kurtosis (Ku) measure which given as

$$Ku = \frac{x_{0.875} - x_{0.625} - x_{0.375} + x_{0.125}}{q_3 - q_1} \quad (3.6)$$

### 3.2. Linear Representation

We provide a useful linear representation for the MOAPL density. Nassar et al. (2019) derived a mixture representation of the MOAP family density as follows,

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \omega_m h_{j+1}(x), h_{j+1}(x) = (\beta+1)g(x)G^{\beta}(x)$$

denotes the exponentiated-G (exp-G) PDF with power parameter β > 0, and the coefficient $\omega_m$ depending on α and θ takes the form

$$\omega_m = \begin{cases} (-1)^j(k+1)\theta(1-\theta)^k \binom{k}{j}\alpha^{k-j}\frac{(\log(\alpha))^{m+1}(j+1)^m}{(\alpha-1)^{k+1}(m+1)!}; & 0<\theta<1 \\ (-1)^j(k+1)\left(1-\frac{1}{\theta}\right)^k \binom{k}{j}\frac{(\log(\alpha))^{m+1}(k-j+1)^m}{\theta(\alpha-1)^{k+1}(m+1)!}; & \theta>1 \end{cases}$$

Using the pdf and cdf of the Lomax distribution Equation (1.1 and 1.2), the MOAPL distribution can be rewritten as

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{\infty} \omega_m \frac{(j+1)}{(1+l)}\frac{\beta(1+l)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\beta(1+l)-1}, \quad (3.7)$$

where $\frac{\beta(1+l)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-(1+\beta)-\beta l}$ is distributed as Lomax with $(\beta(1+l), \lambda)$.

### 3.3. Non-central moments and moment generating function

The r-th non-central moment of X form MOAPL distribution by using linear representation Equation (3.7) can be rewritten as

$$\mu_r' = E(X^r) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}\sum_{l=0}^{\infty} \omega_m \frac{(j+1)}{(1+l)}\frac{\lambda^r\Gamma(\beta(1+l)-r)\Gamma(1+r)}{\Gamma(\beta(1+l))}, \quad (3.8)$$

The mean, variance, Skewness and Kurtosis can be obtained from (3.8).
The moment generating function (mgf) is as follows:

$$M(t) = E(e^{tX}) = \sum_{h=0}^{\infty}\frac{t^h}{h!}E(X^h)$$

where $E(X^h)$ is obtained from equation (3.8).

The first quartile, median, third quartile, skewness, kurtosis, mean and variance of MOAPL distribution with different values of the parameters are computed using the R program and displayed in Table 2. The results indicate that, the first and third quartiles, median, skewness, kurtosis, mean, and variance are increasing whenever any one of the MOAPL distortion parameters had increased while the remaining parameters are fixed values. But $\lambda$ is not effective for skewness and kurtosis measures.

**Table 2:** The First and Third Quartile, median, skewness, kurtosis, Mean and Variance of the MOAPL distribution with Different Values of the Parameters

| $\alpha$ | $\beta$ | $\theta$ | $\lambda$ | Q1 | median | Q3 | SK | kt | mean | Variance |
|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 0.75 | 0.75 | 0.0774 | 0.2134 | 0.5374 | 7.5219 | 85.0827 | 0.5340 | 1.3113 |
| | | | 2 | 0.2063 | 0.5690 | 1.4330 | 7.5219 | 85.0827 | 1.4239 | 9.3249 |
| | 2 | 2 | 0.75 | 0.1921 | 0.4893 | 1.1151 | 7.0429 | 76.8682 | 1.0310 | 3.7820 |
| | | | 2 | 0.5123 | 1.3048 | 2.9735 | 7.0429 | 76.8682 | 2.7493 | 26.8939 |
| | 2 | 0.75 | 0.75 | 0.1237 | 0.3265 | 0.7777 | 7.2601 | 80.5402 | 0.7413 | 2.1983 |
| | | | 2 | 0.3297 | 0.8706 | 2.0739 | 7.2601 | 80.5402 | 1.9767 | 15.6324 |
| | 2 | 2 | 0.75 | 0.2955 | 0.7124 | 1.5437 | 6.8870 | 74.3251 | 1.3982 | 6.2254 |
| | | | 2 | 0.7881 | 1.8996 | 4.1166 | 6.8870 | 74.3251 | 3.7286 | 44.2697 |
| 5 | 2 | 2 | 0.75 | 0.4266 | 0.9585 | 1.9812 | 6.8008 | 72.9662 | 1.7778 | 9.1506 |
| | | | 2 | 1.1376 | 2.5559 | 5.2833 | 6.8008 | 72.9662 | 4.7407 | 65.0712 |
| | 2 | 5 | 0.75 | 0.8432 | 1.7652 | 3.4511 | 6.6416 | 70.4575 | 3.0505 | 23.4505 |
| | | | 2 | 2.2485 | 4.7073 | 9.2031 | 6.6416 | 70.4575 | 8.1347 | 166.7589 |
| | 5 | 2 | 0.75 | 0.1480 | 0.2925 | 0.5077 | 2.4201 | 12.7668 | 0.3832 | 0.1245 |
| | | | 2 | 0.3947 | 0.7800 | 1.3539 | 2.4201 | 12.7668 | 1.0220 | 0.8852 |
| | 5 | 5 | 0.75 | 0.2638 | 0.4669 | 0.7441 | 2.1299 | 10.9980 | 0.5664 | 0.2027 |
| | | | 2 | 0.7034 | 1.2452 | 1.9843 | 2.1299 | 10.9980 | 1.5105 | 1.4416 |

## 4. Parameter Estimation

In this section, the parameter estimation for the MOAPL distribution using MLE, MPS, and least squares (LS) estimation methods in the presence of complete sample will be discussed in detail.

### 4.1. MLE method

The log-likelihood function of MOAPL distribution, is given by:

$$l(\Omega) = n\,ln\left(\frac{\theta\beta\ln(\alpha)}{\lambda(\alpha-1)}\right) - (1+\beta)\sum_{i=1}^{n} ln\left(1+\frac{x_i}{\lambda}\right) + ln(\alpha)\sum_{i=1}^{n}\left(1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right) + 2\sum_{i=1}^{n} ln\left[\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}} - 1\right)\right]. \quad (4.1)$$

Equation (4.1) can be maximized directly by using the R package by an optim function, to solve the non-linear likelihood equations obtained by differentiating Equation (4.1) with respect to $\Omega$ and equating to zero. The non-linear likelihood equations are given as

$$\frac{\partial l(\Omega)}{\partial\alpha} = \frac{\frac{n}{\alpha} - \frac{n\ln(\alpha)}{(\alpha-1)}}{\ln(\alpha)} + \frac{1}{\alpha}\sum_{i=1}^{n}\left(1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\right) + 2(\theta-1)\sum_{i=1}^{n}\frac{\frac{\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}}-1}{(\alpha-1)^2} + \frac{\left(1+\frac{x_i}{\lambda}\right)^{-\beta}-1}{\alpha^{\left(1+\frac{x_i}{\lambda}\right)^{-\beta}}(\alpha-1)}}{\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}}-1\right)},$$

$$\frac{\partial l(\Omega)}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n} ln\left(1+\frac{x_i}{\lambda}\right) + ln(\alpha)\sum_{i=1}^{n}\left(\left(1+\frac{x_i}{\lambda}\right)^{-\beta}\ln\left(1+\frac{x_i}{\lambda}\right)\right) + 2\frac{(1-\theta)}{(\alpha-1)}\sum_{i=1}^{n}\mathfrak{H}(x_i,\Omega),$$

$$\frac{\partial l(\Omega)}{\partial\theta} = \frac{n}{\theta} + 2\sum_{i=1}^{n}\frac{1-\left(\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}}-1\right)/(\alpha-1)}{\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}}-1\right)},$$

and

$$\frac{\partial l(\Omega)}{\partial \lambda} = \frac{-n}{\lambda} + \sum_{i=1}^{n} \frac{(1+\beta)x_i}{\lambda(\lambda + x_i)} - ln(\alpha)\beta \sum_{i=1}^{n} \left(1 + \frac{x_i}{\lambda}\right)^{-\beta-1} \frac{x_i}{\lambda^2} + 2\frac{(1-\theta)\beta \ln(\alpha)}{\lambda^2(\alpha-1)} \sum_{i=1}^{n} \wp(x_i, \Omega).$$

**Where** $\mathfrak{H}(x_i, \Omega) = \frac{\alpha^{-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}} \ln\left(1+\frac{x_i}{\lambda}\right)}{\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}} - 1\right)}$ and $\wp(x_i, \Omega) = \frac{\alpha^{-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}} \left(1+\frac{x_i}{\lambda}\right)^{-\beta-1} x_i}{\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}} - 1\right)}$.

### 4.2. MPS Method

MPS method is used to estimate the parameters of continuous univariate models as an alternative to the MLE method. The uniform spacings of a random sample $x_1 < \cdots < x_n$ of size $n$ from the MOAPL distribution can be defined by

$$D_i(\Omega) = F(x_i, \Omega) - F(x_{i-1}, \Omega); i = 1,2, \ldots, n+1$$

where $D_i$ denotes to the uniform spacings and $\sum_{i=1}^{n+1} D_i = 1$. The MPS estimators can be obtained by maximizing $G(\Omega) = \frac{1}{n+1}\sum_{i=1}^{n+1} \ln(D_i(\Omega))$.

To more information of MPS method, see Cheng and Amin (1983), Almetwally and Almongy (2019b,a) and Almongy Almetwally (2020).

The natural logarithm of the product spacing function for the MPS of MOAPL distribution is given by

$$\ln G(\Omega) = \frac{1}{n+1}\left( \left[\ln\left(\mathcal{H}(x_n, \Omega)\right) - \ln\left(\varphi(x_n, \Omega)\right)\right] + \ln(\mathscr{b}(x_1, \alpha, \beta, \lambda) - 1) - \ln\left(\varphi(x_1, \Omega)\right) + \sum_{i=2}^{n} \ln\left(\frac{\mathscr{b}(x_i, \alpha, \beta, \lambda)}{\varphi(x_i, \Omega)} - \frac{\mathscr{b}(x_{i-1}, \alpha, \beta, \lambda)}{\varphi(x_{i-1}, \Omega)}\right)\right) \quad (4.2)$$

where $\varphi(x_i, \Omega) = (\alpha - 1)\theta + (1-\theta)\left(\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}} - 1\right)$, $\mathscr{b}(x_i, \alpha, \beta, \lambda) = \alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}}$

$\mathcal{H}(x_i, \Omega) = \theta(\alpha - 1) - \theta\left(\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}} - 1\right)$,

The partial derivatives of MPS for Equation (4.2) with respect to the unknown parameters cannot be solved explicitly, so numerical methods like the conjugate gradient's algorithms can be used to calculate the MPS of Ω.

### 4.3. Least Squares Method

Swain et al. (1988) presented the LS method for parameter estimation of a distribution. It dependent on the observed sample $x_1 < \cdots < x_n$ from $n$ ordered a random sample of any distribution with CDF, where $F(.)$ denotes the CDF, hence we get $E\left(F(x_i)\right) = \frac{i}{(n+1)}$. The LS method can be written as follows, $P(\Omega) = \sum_{i=1}^{n}\left(F(x_i; \Omega) - \frac{i}{(n+1)}\right)^2$. The LS method of MOAPL distribution can be written as follows

$$P(\Omega) = \sum_{i=1}^{n}\left(\frac{\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}} - 1}{(\alpha-1)\left(\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-\left(1+\frac{x_i}{\lambda}\right)^{-\beta}} - 1\right)\right)} - \frac{i}{(n+1)}\right)^2. \quad (4.3)$$

After differentiating equation (4.3) with respect to parameters Ω and then equating them to zero, numerical methods like the optimization algorithm can be used to calculate the LS estimator of Ω.

## 5. Simulation Study

In this section; a Monte Carlo simulation is done to estimate the parameters based on a complete sample by using MLE, MPS, and LS methods. Using the “bbmle" package through the R program and using the following:

Simulation algorithm: Monte Carlo experiments were carried out based on 10000 random samples for following data generated from MOAPL distribution by using the quantile Equation (3.1), where $x$ is distributed as MOAPL distribution for different parameters $\Omega = (\alpha, \beta, \theta, \lambda)$ with different actual values of the parameter and for different samples sizes $n = 50, 100$ *and* $200$. We can find the parameter estimation by using Equations (4.1, 4.2, and 4.3) and a mle2 function in “bbmle" package through the R program. We could define the best method as which minimizes the Bias and mean squared error (MSE) of the estimator.

The following conclusions can be drawn from Tables (3-5):

1. All the estimates reveal the property of consistency, i.e., the Bias and MSE decrease when $n$ increase.
2. The LS estimates have more relative efficiency than MLE and MPS for most parameters of MOAPL distribution.

**Table 3:** MLE, MPS, and LS estimation methods with different values of parameters when $n = 50$

| actual value | | | | estimate | MLE | | MPS | | LS | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $\beta$ | $\theta$ | $\lambda$ | | Bias | MSE | Bias | MSE | Bias | MSE |
| 3 | 1.3 | 0.5 | 0.5 | $\hat{\alpha}$ | -0.0598 | 0.2626 | 0.0373 | 0.0526 | 0.0969 | 0.0126 |
| | | | | $\hat{\beta}$ | 0.4292 | 1.1444 | -0.0655 | 0.2407 | 0.0774 | 0.0133 |
| | | | | $\hat{\theta}$ | 0.2004 | 0.3249 | 0.0781 | 0.1780 | 0.0420 | 0.0040 |
| | | | | $\hat{\lambda}$ | 0.4354 | 0.9808 | 0.0832 | 0.1346 | 0.0064 | 0.0039 |
| | | | 1.5 | $\hat{\alpha}$ | -0.0923 | 0.9529 | 0.0306 | 0.0711 | 0.0805 | 0.0104 |
| | | | | $\hat{\beta}$ | 0.2966 | 0.7823 | -0.1037 | 0.1829 | 0.0708 | 0.0129 |
| | | | | $\hat{\theta}$ | 0.3488 | 0.6372 | 0.0319 | 0.1192 | 0.0264 | 0.0083 |
| | | | | $\hat{\lambda}$ | 0.4801 | 2.8616 | -0.0142 | 0.1521 | 0.0503 | 0.0114 |
| | | | 3 | $\hat{\alpha}$ | -0.0737 | 1.4473 | 0.0253 | 0.0651 | 0.0839 | 0.0097 |
| | | | | $\hat{\beta}$ | 0.1973 | 0.3840 | -0.1091 | 0.1731 | 0.0576 | 0.0128 |
| | | | | $\hat{\theta}$ | 0.3120 | 0.5640 | -0.0093 | 0.0801 | 0.0245 | 0.0073 |
| | | | | $\hat{\lambda}$ | 0.3072 | 2.8251 | -0.0082 | 0.0648 | 0.0599 | 0.0121 |
| | | 2 | 0.5 | $\hat{\alpha}$ | -0.1177 | 1.1392 | 0.0094 | 0.1011 | 0.0997 | 0.0137 |
| | | | | $\hat{\beta}$ | 0.3502 | 0.8473 | -0.0353 | 0.1075 | 0.0503 | 0.0089 |
| | | | | $\hat{\theta}$ | -0.0376 | 0.7530 | 0.0045 | 0.0962 | 0.0547 | 0.0114 |
| | | | | $\hat{\lambda}$ | 0.7980 | 3.6709 | 0.0141 | 0.0901 | 0.0241 | 0.0048 |
| | | | 1.5 | $\hat{\alpha}$ | -0.0028 | 1.7885 | 0.0085 | 0.1029 | 0.0917 | 0.0114 |
| | | | | $\hat{\beta}$ | 0.1844 | 0.2742 | -0.0563 | 0.0851 | 0.0414 | 0.0105 |
| | | | | $\hat{\theta}$ | 0.2330 | 0.8575 | -0.0209 | 0.1592 | 0.0632 | 0.0112 |
| | | | | $\hat{\lambda}$ | 0.7937 | 4.5178 | -0.0548 | 0.2349 | 0.0396 | 0.0173 |
| | | | 3 | $\hat{\alpha}$ | 0.0707 | 1.7543 | -0.0023 | 0.1138 | 0.0949 | 0.0127 |
| | | | | $\hat{\beta}$ | 0.1099 | 0.1329 | -0.0680 | 0.0767 | 0.0313 | 0.0123 |
| | | | | $\hat{\theta}$ | 0.3462 | 0.8592 | -0.0953 | 0.3745 | 0.0161 | 0.0246 |
| | | | | $\hat{\lambda}$ | 0.4887 | 2.9177 | -0.0428 | 0.2058 | 0.0842 | 0.0137 |
| | 0.5 | 2 | 0.5 | $\hat{\alpha}$ | 0.0383 | 0.0157 | -0.0032 | 0.0090 | 0.0861 | 0.0114 |
| | | | | $\hat{\beta}$ | 0.0057 | 0.0089 | -0.0416 | 0.0104 | 0.0247 | 0.0023 |
| | | | | $\hat{\theta}$ | -0.0363 | 0.5879 | -0.2116 | 0.5799 | 0.0436 | 0.0117 |
| | | | | $\hat{\lambda}$ | 0.2466 | 0.3544 | 0.1774 | 0.2783 | 0.0650 | 0.0127 |
| | | | 1.5 | $\hat{\alpha}$ | 0.0872 | 0.0476 | 0.0202 | 0.0229 | 0.0904 | 0.0117 |
| | | | | $\hat{\beta}$ | 0.0163 | 0.0092 | -0.0332 | 0.0095 | 0.0159 | 0.0024 |
| | | | | $\hat{\theta}$ | 0.1298 | 0.4827 | -0.1203 | 0.4590 | 0.0274 | 0.0201 |
| | | | | $\hat{\lambda}$ | 0.2033 | 0.5025 | 0.0928 | 0.4642 | 0.0955 | 0.0132 |
| | | | 3 | $\hat{\alpha}$ | 0.1091 | 0.1101 | 0.0044 | 0.0637 | 0.1018 | 0.0138 |
| | | | | $\hat{\beta}$ | 0.0184 | 0.0097 | -0.0314 | 0.0097 | 0.0127 | 0.0024 |
| | | | | $\hat{\theta}$ | 0.2051 | 0.6438 | -0.1167 | 0.4937 | 0.0436 | 0.0208 |
| | | | | $\hat{\lambda}$ | 0.1400 | 0.5368 | 0.0077 | 0.4008 | 0.0796 | 0.0123 |

**Table 4:** MLE, MPS, and LS estimation methods with different values of parameters when $n = 100$

| actual value | | | | estimate | MLE | | MPS | | LS | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $\beta$ | $\theta$ | $\lambda$ | | Bias | MSE | Bias | MSE | Bias | MSE |
| 3 | 1.3 | 0.5 | 0.5 | $\hat{\alpha}$ | -0.0361 | 0.1159 | 0.0320 | 0.0299 | 0.0837 | 0.0104 |
| | | | | $\hat{\beta}$ | 0.2205 | 0.3471 | -0.0421 | 0.1109 | 0.0755 | 0.0109 |
| | | | | $\hat{\theta}$ | 0.1198 | 0.1982 | 0.0691 | 0.1150 | 0.0354 | 0.0027 |
| | | | | $\hat{\lambda}$ | 0.2867 | 0.5715 | 0.0397 | 0.0836 | -0.0011 | 0.0019 |
| | | | 1.5 | $\hat{\alpha}$ | -0.0644 | 0.4434 | 0.0282 | 0.0374 | 0.0797 | 0.0100 |
| | | | | $\hat{\beta}$ | 0.1352 | 0.2076 | -0.0651 | 0.0871 | 0.0705 | 0.0095 |
| | | | | $\hat{\theta}$ | 0.2077 | 0.2949 | 0.0164 | 0.0579 | 0.0128 | 0.0039 |
| | | | | $\hat{\lambda}$ | 0.2121 | 1.1271 | -0.0363 | 0.0853 | 0.0494 | 0.0084 |
| | | | 3 | $\hat{\alpha}$ | -0.0934 | 0.8233 | 0.0243 | 0.0344 | 0.0829 | 0.0096 |
| | | | | $\hat{\beta}$ | 0.1005 | 0.1344 | -0.0662 | 0.0855 | 0.0537 | 0.0085 |
| | | | | $\hat{\theta}$ | 0.1571 | 0.1950 | -0.0152 | 0.0359 | 0.0071 | 0.0023 |
| | | | | $\hat{\lambda}$ | 0.1356 | 1.2850 | -0.0218 | 0.0330 | 0.0669 | 0.0099 |
| | | 2 | 0.5 | $\hat{\alpha}$ | -0.1250 | 0.5604 | 0.0070 | 0.0567 | 0.0991 | 0.0133 |
| | | | | $\hat{\beta}$ | 0.1585 | 0.2007 | -0.0237 | 0.0484 | 0.0455 | 0.0054 |
| | | | | $\hat{\theta}$ | -0.0905 | 0.4205 | 0.0061 | 0.0528 | 0.0483 | 0.0109 |
| | | | | $\hat{\lambda}$ | 0.3768 | 1.0308 | -0.0040 | 0.0376 | 0.0108 | 0.0017 |
| | | | 1.5 | $\hat{\alpha}$ | -0.0487 | 1.0194 | 0.0053 | 0.0533 | 0.0907 | 0.0107 |
| | | | | $\hat{\beta}$ | 0.0933 | 0.0895 | -0.0329 | 0.0417 | 0.0425 | 0.0054 |
| | | | | $\hat{\theta}$ | 0.1147 | 0.3993 | -0.0153 | 0.0754 | 0.0573 | 0.0085 |
| | | | | $\hat{\lambda}$ | 0.4143 | 1.7032 | -0.0529 | 0.1207 | 0.0203 | 0.0106 |
| | | | 3 | $\hat{\alpha}$ | 0.1162 | 1.3501 | -0.0048 | 0.0558 | 0.0899 | 0.0121 |
| | | | | $\hat{\beta}$ | 0.0581 | 0.0551 | -0.0380 | 0.0385 | 0.0342 | 0.0057 |
| | | | | $\hat{\theta}$ | 0.1987 | 0.3584 | -0.0696 | 0.1934 | -0.0045 | 0.0185 |
| | | | | $\hat{\lambda}$ | 0.1999 | 1.3046 | -0.0447 | 0.1061 | 0.0792 | 0.0099 |
| | 0.5 | 2 | 0.5 | $\hat{\alpha}$ | 0.0213 | 0.0096 | -0.0111 | 0.0051 | 0.0886 | 0.0109 |
| | | | | $\hat{\beta}$ | 0.0015 | 0.0049 | -0.0225 | 0.0049 | 0.0243 | 0.0014 |
| | | | | $\hat{\theta}$ | -0.0212 | 0.5600 | -0.1582 | 0.3466 | 0.0349 | 0.0101 |
| | | | | $\hat{\lambda}$ | 0.1591 | 0.1995 | 0.0948 | 0.1151 | 0.0491 | 0.0078 |
| | | | 1.5 | $\hat{\alpha}$ | 0.0804 | 0.0359 | 0.0061 | 0.0105 | 0.0898 | 0.0115 |
| | | | | $\hat{\beta}$ | 0.0041 | 0.0046 | -0.0201 | 0.0044 | 0.0167 | 0.0011 |
| | | | | $\hat{\theta}$ | 0.0590 | 0.5124 | -0.1014 | 0.2486 | 0.0118 | 0.0143 |
| | | | | $\hat{\lambda}$ | 0.2120 | 0.6210 | 0.0571 | 0.2242 | 0.0853 | 0.0108 |
| | | | 3 | $\hat{\alpha}$ | 0.0805 | 0.0801 | -0.0059 | 0.0299 | 0.1026 | 0.0138 |
| | | | | $\hat{\beta}$ | 0.0106 | 0.0050 | -0.0170 | 0.0049 | 0.0140 | 0.0011 |
| | | | | $\hat{\theta}$ | 0.1509 | 0.5733 | -0.0910 | 0.2664 | 0.0291 | 0.0140 |
| | | | | $\hat{\lambda}$ | 0.0856 | 0.6126 | 0.0020 | 0.1820 | 0.0659 | 0.0089 |

**Table 5:** MLE, MPS, and LS estimation methods with different values of parameters when $n = 200$

| actual value | | | | estimate | MLE | | MPS | | LS | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | $\beta$ | $\theta$ | $\lambda$ | | Bias | MSE | Bias | MSE | Bias | MSE |
| 3 | 1.3 | 0.5 | 0.5 | $\hat{\alpha}$ | -0.0099 | 0.0637 | 0.0268 | 0.0198 | 0.0780 | 0.0092 |
| | | | | $\hat{\beta}$ | 0.1222 | 0.1140 | -0.0199 | 0.0527 | 0.0602 | 0.0063 |
| | | | | $\hat{\theta}$ | 0.0824 | 0.1310 | 0.0619 | 0.0766 | 0.0252 | 0.0015 |
| | | | | $\hat{\lambda}$ | 0.1978 | 0.2798 | 0.0222 | 0.0588 | 0.0006 | 0.0010 |
| | | | 1.5 | $\hat{\alpha}$ | -0.0310 | 0.3020 | 0.0220 | 0.0228 | 0.0820 | 0.0100 |
| | | | | $\hat{\beta}$ | 0.0713 | 0.0693 | -0.0359 | 0.0415 | 0.0580 | 0.0057 |
| | | | | $\hat{\theta}$ | 0.1408 | 0.1725 | 0.0141 | 0.0277 | 0.0062 | 0.0011 |
| | | | | $\hat{\lambda}$ | 0.1236 | 0.6104 | -0.0378 | 0.0551 | 0.0510 | 0.0065 |
| | | | 3 | $\hat{\alpha}$ | -0.0429 | 0.5061 | 0.0203 | 0.0219 | 0.0829 | 0.0096 |
| | | | | $\hat{\beta}$ | 0.0546 | 0.0527 | -0.0363 | 0.0413 | 0.0420 | 0.0042 |
| | | | | $\hat{\theta}$ | 0.0840 | 0.0728 | -0.0093 | 0.0165 | 0.0016 | 0.0010 |
| | | | | $\hat{\lambda}$ | 0.0510 | 0.6254 | -0.0208 | 0.0193 | 0.0745 | 0.0099 |
| | | 2 | 0.5 | $\hat{\alpha}$ | -0.1023 | 0.4632 | 0.0092 | 0.0453 | 0.0899 | 0.0112 |
| | | | | $\hat{\beta}$ | 0.1241 | 0.1425 | -0.0130 | 0.0350 | 0.0363 | 0.0027 |
| | | | | $\hat{\theta}$ | -0.0771 | 0.3562 | 0.0102 | 0.0423 | 0.0515 | 0.0101 |
| | | | | $\hat{\lambda}$ | 0.3026 | 0.8433 | -0.0009 | 0.0270 | 0.0061 | 0.0007 |
| | | | 1.5 | $\hat{\alpha}$ | -0.1072 | 0.6704 | 0.0026 | 0.0321 | 0.0878 | 0.0102 |
| | | | | $\hat{\beta}$ | 0.0549 | 0.0336 | -0.0176 | 0.0205 | 0.0343 | 0.0028 |
| | | | | $\hat{\theta}$ | 0.0360 | 0.2422 | -0.0036 | 0.0390 | 0.0566 | 0.0072 |
| | | | | $\hat{\lambda}$ | 0.2911 | 0.7819 | -0.0308 | 0.0592 | 0.0117 | 0.0058 |
| | | | 3 | $\hat{\alpha}$ | -0.0025 | 0.6363 | -0.0037 | 0.0327 | 0.0917 | 0.0117 |
| | | | | $\hat{\beta}$ | 0.0328 | 0.0230 | -0.0197 | 0.0190 | 0.0281 | 0.0028 |
| | | | | $\hat{\theta}$ | 0.1256 | 0.1871 | -0.0335 | 0.0935 | -0.0052 | 0.0123 |
| | | | | $\hat{\lambda}$ | 0.1136 | 0.5400 | -0.0276 | 0.0526 | 0.0760 | 0.0084 |
| | 0.5 | 2 | 0.5 | $\hat{\alpha}$ | 0.0102 | 0.0037 | -0.0053 | 0.0027 | 0.0851 | 0.0108 |
| | | | | $\hat{\beta}$ | 0.0025 | 0.0021 | -0.0110 | 0.0022 | 0.0193 | 0.0008 |
| | | | | $\hat{\theta}$ | -0.0102 | 0.2290 | -0.0798 | 0.1759 | 0.0392 | 0.0086 |
| | | | | $\hat{\lambda}$ | 0.0723 | 0.0750 | 0.0472 | 0.0545 | 0.0346 | 0.0041 |
| | | | 1.5 | $\hat{\alpha}$ | 0.0271 | 0.0073 | 0.0036 | 0.0052 | 0.0882 | 0.0114 |
| | | | | $\hat{\beta}$ | 0.0057 | 0.0023 | -0.0088 | 0.0022 | 0.0138 | 0.0006 |
| | | | | $\hat{\theta}$ | 0.0340 | 0.1712 | -0.0513 | 0.1294 | 0.0074 | 0.0091 |
| | | | | $\hat{\lambda}$ | 0.0750 | 0.1712 | 0.0308 | 0.1281 | 0.0815 | 0.0097 |
| | | | 3 | $\hat{\alpha}$ | 0.0306 | 0.0248 | -0.0044 | 0.0163 | 0.0967 | 0.0129 |
| | | | | $\hat{\beta}$ | 0.0070 | 0.0024 | -0.0085 | 0.0024 | 0.0109 | 0.0005 |
| | | | | $\hat{\theta}$ | 0.0635 | 0.1645 | -0.0481 | 0.1240 | 0.0221 | 0.0096 |
| | | | | $\hat{\lambda}$ | 0.0377 | 0.1593 | -0.0015 | 0.0919 | 0.0630 | 0.0077 |

## 6. Application of Real Data Analysis

This section is devoted to illustrate the potentiality of the MOAPL distribution for three real data sets. MOAPL distribution is compared with other competitive models, namely: the Marshall–Olkin Lomax (MOL), alpha power Lomax (APL), Lomax distribution, Marshall–Olkin alpha power exponential (MOAPE), Kumaraswamy-generalized Lomax (KGL) [Shams (2019)], exponentiated Lomax (EL) [Abdul-Moniem and Abdel-Hameed (2012)] and power Lomax (PL).

### 6.1. Physical data:

The first data set introduced by Birnbaum and Saunders (1969). The data refers to the actual fatigue data on the fatigue life of 6061-T6 aluminum coupons and consists of 101 observations with maximum stress per cycle 21,000 psi. The data is as follows: 370, 706, 716, 746, 785, 797, 844, 855, 858, 886, 886, 930, 960 ,988, 990, 1000, 1010, 1016, 1018, 1020, 1055, 1085, 1102, 1102, 1108, 1115 ,1120, 1134, 1140, 1199, 1200, 1200, 1203 ,1222, 1235, 1238 ,1252, 1258, 1262, 1269,1270, 1290, 1293, 1300, 1310, 1313, 1315, 1330, 1355, 1390, 1416, 1419, 1420, 1420, 1450, 1452, 1475, 1478, 1481, 1485, 1502, 1505, 1513, 1522, 1522, 1530, 1540, 1560, 1567, 1578, 1594, 1602, 1604, 1608, 1630, 1642, 1674, 1730, 1750, 1750,1763, 1768, 1781, 1782, 1792, 1820, 1868, 1881, 1890, 1893, 1895, 1910, 1923, 1940 ,1945, 2023, 2100, 2130, 2215, 2268 and 2440.

Table 6. MLE, K–S Distance and P-values with Different Models for the Physical Data Set.

| | $\alpha$ | $\beta$ | $\theta$ | $\lambda$ | D | P-Value |
|---|---|---|---|---|---|---|
| Lomax | - | 0.781 | 536.145 | - | 0.471 | 0.000 |
| MOAPL | 313.565 | 10.128 | 717.650 | 1074.195 | 0.052 | 0.951 |
| APL | 3296.489 | 12.168 | - | 5809.975 | 0.173 | 0.005 |
| MOL | - | 8.181 | 797.585 | 1057.447 | 0.094 | 0.329 |
| MOAPE | 27.709 | 0.003 | 27.729 | - | 0.091 | 0.367 |
| KGL | 65.428 | 0.967 | 45.320 | 81.269 | 0.060 | 0.866 |
| EL | 122.025 | 4.399 | - | 586.421 | 0.119 | 0.115 |
| PL | 1.817 | 0.802 | - | 464.274 | 0.459 | 0.000 |

In table 6, the MOAPL model has the highest p-value and the lowest distance (D) of Kolmogorov Smirnov (K-S) value when compare with all other models used here to fit the current physical data. Figure 3 shows the fit empirical, histogram, QQ-plot and PP-plot as follows

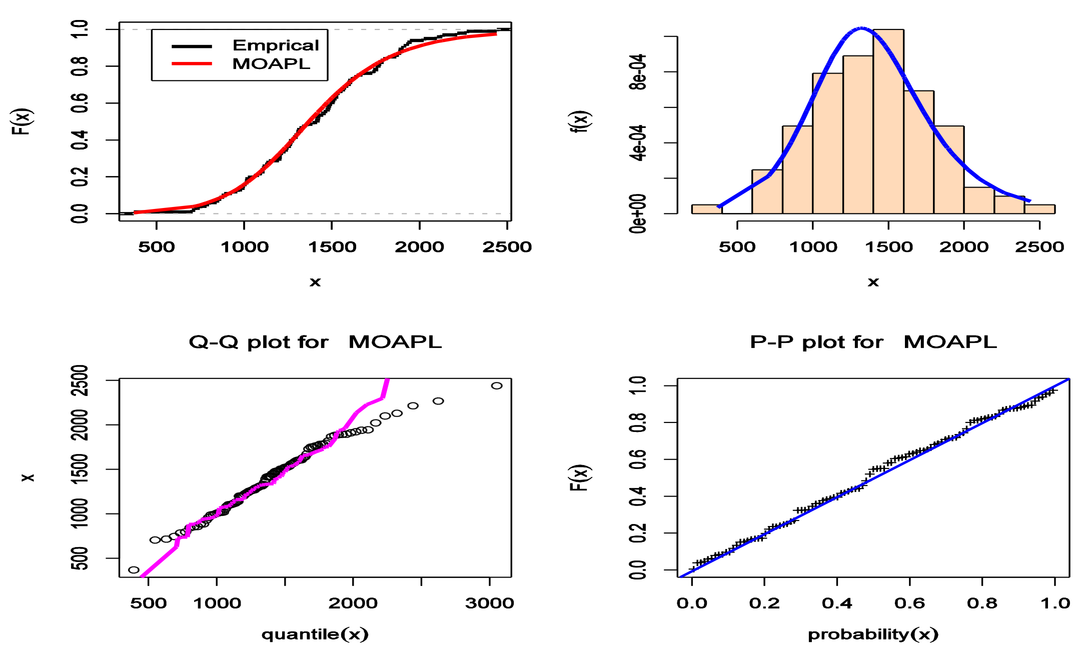

Figure 3. Cumulative function and empirical cdf, histogram and the Fitted MOAPL distribution, Q-Q plot, and P-P plot for the MOAPL distribution for the physical data set.

Table 7. Log-likelihood and different criteria with different models for the physical data set.

| | *ll* | AIC | CAIC | BIC | HQIC |
|---|---|---|---|---|---|
| Lomax | 886.965 | 1777.931 | 1778.053 | 1783.161 | 1780.048 |
| MOAPL | 747.751 | 1503.502 | 1503.919 | 1513.962 | 1507.737 |
| APL | 771.559 | 1549.117 | 1549.364 | 1556.962 | 1552.293 |
| MOL | 752.796 | 1511.592 | 1511.840 | 1519.438 | 1514.769 |
| MOAPE | 751.184 | 1508.369 | 1508.616 | 1516.214 | 1511.545 |
| KGL | 747.937 | 1503.874 | 1504.290 | 1514.334 | 1508.108 |
| EL | 763.880 | 1533.761 | 1534.008 | 1541.606 | 1536.937 |
| PL | 878.261 | 1762.522 | 1762.770 | 1770.368 | 1765.698 |

Table 7 shows that the new model (MOAPL) fits the data better than the MOAPE, MOL, APL, PL, KGL, EL, and Lomax models based on these different criteria as the Akaike information criterion (AIC), correct Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan–Quinn information criterion (HQIC) values.

### 6.2. Carbon Fibres Data:

Nichols and Padgett (2006) discussed data set on breaking stress of carbon fibres (in Gba). The data are recorded as follows 3.70,2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19,3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83,1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00,1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65. Hassan and Abd-Allah (2018) discussed this data to fit exponentiated Weibull-Lomax distribution where the AIC is 292.624, CAIC is 293.249, BIC is 305.65 and HQIC is 297.896.

Table 8. MLE, K–S Distance and P-values with different models for the carbon fibers data Set.

| | $\alpha$ | $\beta$ | $\theta$ | $\lambda$ | D | P-Value |
|---|---|---|---|---|---|---|
| Lomax | - | 38.60224 | 99.47269 | - | 0.3236 | 0.000 |
| MOAPL | 645.4590 | 2024.721 | 8.0386 | 1293.6335 | 0.0669 | 0.7606 |
| APL | 652.314 | 180.3115 | - | 193.6307 | 0.1097 | 0.1803 |
| MOL | - | 1203.7696 | 17.6117 | 1046.675 | 0.1142 | 0.1473 |
| KGL | 5.997914 | 3.2863 | 6.0538 | 5.7720 | 0.0968 | 0.3057 |
| EL | 8.1596 | 42.3372 | - | 39.69147 | 0.11078 | 0.1717 |
| PL | 2.26854 | 3.21081 | - | 50.16789 | 0.10213 | 0.2479 |

In table 8, the MOAPL model has the highest p-value and the lowest distance of K-S value when compared with all other models used here to fit the current carbon fibers data. Figure 4 shows the fit empirical, histogram, QQ-plot and PP-plot as follows

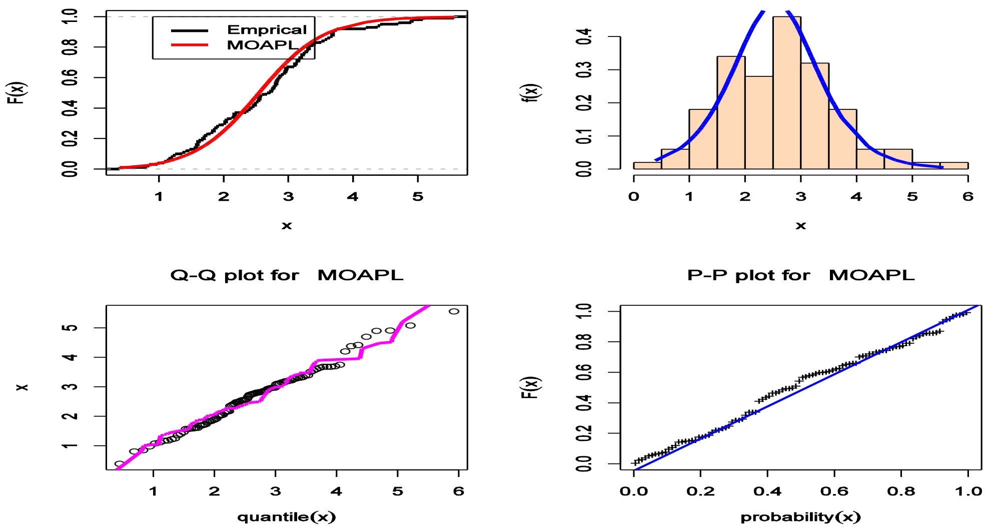


Figure 4. Cumulative function and empirical cdf, histogram and the Fitted MOAPL distribution, Q-Q plot, and P-P plot for the MOAPL distribution for the carbon fibers data set.

Table 9. Log-likelihood and Different Criteria with Different Models for the Carbon Fibres Data Set.

| | *ll* | AIC | CAIC | BIC | HQIC |
|---|---|---|---|---|---|
| Lomax | 197.4766 | 398.9531 | 399.0769 | 404.1635 | 401.0619 |
| MOAPL | 141.3647 | 290.7294 | 291.1504 | 301.15 | 294.648 |
| APL | 148.099 | 302.1981 | 302.4481 | 310.0136 | 305.3611 |
| MOL | 148.958 | 303.9161 | 304.1661 | 311.7316 | 307.0792 |
| KGL | 143.5193 | 295.0762 | 295.4973 | 305.4969 | 299.2937 |
| EL | 146.9003 | 299.8006 | 300.0506 | 307.6162 | 302.9637 |

Table 9 shows that the new model (MOAPL) fits the data better than the MOL, APL, PL, KGL, EL, and Lomax models based on different criteria as the AIC, CAIC, BIC, and HQIC values.

### 6.3. Economic data:

Almetwally et al. (2019) discussed data set of 31 observation for GDP growth (% per year) of Egypt. The data are recorded as follows 10.01132, 3.756100, 9.907171, 7.401136, 6.091518, 6.602036, 2.646586, 2.519411, 7.930073, 4.972375, 5.701749, 1.078837, 4.431994, 2.900787, 3.973172, 4.642467, 4.988731, 5.491124, 4.036373, 6.105463, 5.367998, 3.535252, 2.370460, 3.192285, 4.089940, 4.478960, 6.853908, 7.090271, 7.157617, 4.673845, 5.145106.

Table 10. MLE, K–S Distance, and P-values with different models for the economic data Set.

| | $\alpha$ | $\beta$ | $\theta$ | $\lambda$ | D | P-Value |
|---|---|---|---|---|---|---|
| Lomax | - | 17.81213 | 88.27927 | - | 0.34398 | 0.00088 |
| MOAPL | 741.9532 | 226.47981 | 6.1994 | 293.7377 | 0.04508 | 0.997 |
| APL | 686.3087 | 140.1373 | - | 291.6849 | 0.11603 | 0.7553 |
| MOL | - | 14.74783 | 29.29223 | 18.02297 | 0.12096 | 0.7099 |
| KGL | 6.940726 | 2.305151 | 5.920281 | 6.309118 | 0.073515 | 0.9916 |
| EL | 8.159856 | 26.46506 | - | 47.50951 | 0.096371 | 0.9092 |
| PL | 1.905924 | 2.811477 | - | 165.1069 | 0.12905 | 0.6337 |

In table 10, the MOAPL model has the highest p-value and the lowest distance of K-S value when compared with all other models used here to fit the current economic data. Figure 5 shows the fit empirical, histogram, QQ-plot and PP-plot as follows

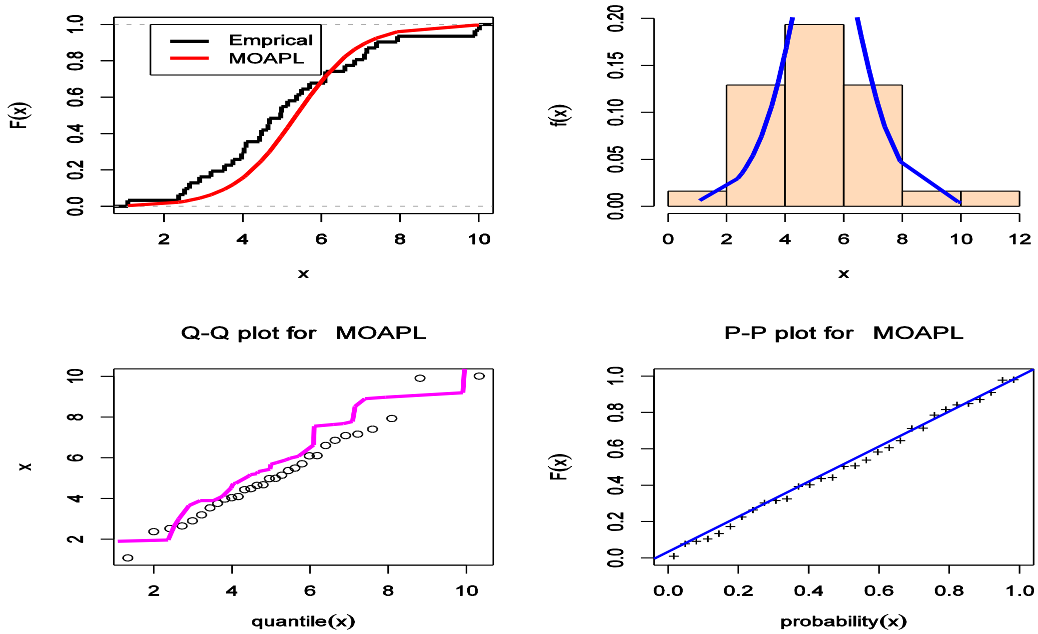


Figure 5. Cumulative function and empirical cdf, histogram and the Fitted MOAPL distribution, Q-Q plot, and P-P plot for the MOAPL distribution for the economic data set.

Table 11. Log-likelihood and Different Criteria with different models for the economic data Set.

| | ll | AIC | CAIC | BIC | HQIC |
|---|---|---|---|---|---|
| Lomax | 82.44368 | 168.8874 | 169.3159 | 171.7553 | 169.8223 |
| MOAPL | 65.55508 | 139.1102 | 140.6486 | 144.8461 | 140.9799 |
| APL | 67.08035 | 140.1607 | 141.0496 | 144.4627 | 141.563 |
| MOL | 67.22326 | 140.4616 | 141.3505 | 144.7636 | 141.864 |
| KGL | 66.25209 | 140.5061 | 142.0446 | 146.242 | 142.3759 |
| EL | 66.99368 | 139.9874 | 140.8763 | 144.2893 | 141.3897 |

Table 9 shows that the MOAPL fits the economic data better than the MOL, APL, PL, KGL, EL, and Lomax models based on these different criteria as the AIC, CAIC, BIC, and HQIC values.

## 7. Conclusion

In this paper, we propose a new four-parameter model, called the Marshall-Olkin alpha power Lomax (MOAPL) distribution, which is a new extension of the Lomax distribution. The MOAPL distribution is motivated by the wide utilization of the Lomax model in life testing and provides more flexibility to analyze lifetime data. Some structural properties of the MOAPL distribution are provided as quantile, median, linear representation, non-central moments, and moment generating function. We provide some applications of MOAPL distribution in the context of statistics. The parameter estimation of MOAPL distribution is derived by MLE, MPS, and LS. The methods of estimation are employed to estimate the model parameters and simulation results are provided to assess the model performance. Three real-life data proposed model provides a consistently better fit than the MOL, APL, PL, KGL, EL, and Lomax distributions.

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