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Fuzzy solution of fully fuzzy multi-objective linear fractional programming problems

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Abstract

In this study, we present a novel method for solving fully fuzzy multi-objective linear fractional programming problems without transforming to equivalent crisp problems. First, we calculate the fuzzy optimal value for each fractional objective function and then we convert the fully fuzzy multi-objective linear fractional programming problem to a single objective fuzzy linear fractional programming problem and find its fuzzy optimal solution which inturn yields a fuzzy Pareto optimal solution for the given fully fuzzy multi-objective linear fractional programming problem. To demonstrate the proposed strategy, a numerical example is provided.

Key Words: Fuzzy Multi-objective linear fractional programming; triangular fuzzy numbers; parametric form; fuzzy arithmetic; fuzzy ranking; Pareto optimal solution.

Mathematical Subject Classification: 03E72,90C32, 90C70.

1. Introduction

Multi-objective linear fractional programming problems(MOLFPP) occurs when there are several ratios of quantities to be optimized simultaneously. It has a variety of important uses, including calculating malting efficiency in DNA and health care, in corporate factory planning (supply/demond ratios, manufacturing/worker), in transport planning (seat/vehicle, speed/distance), in election commission planning (people/areas, booth/ officer) and in educational planning (school/teacher, result output/teacher).

Bellman and Zadeh (Zadeh, 1965) introduced the method of decision making in fuzzy environment. Guzel and Sivri (N.Guzel and M.Sivri, 2020) proposed multiple efficient solution to the MOLFPP. Nuran Guzel(Guzel, 2013) introduced a new solution procedure to solve MOLFPP by reducing it to linear programming problem(LPP). Farhana Akond Pramy (Pramy, 2020) introduced graded mean integration method to solve the fuzzy multi objective linear fractional programming problem (FMOLFPP). Surapati Pramanik et.al (Surapati Pramanik and Mandal, 2018) convert each objective function with fuzzy parameters into crisp objective functions using α- cut of the fuzzy numbers. Duran Toksari (Toksari, 2008) solving MOLFPP using Taylor's series method. Loganathan and Ganesan (Loganathan and Ganesan, 2019) presented a solution for fully fuzzy linear fractional programming problem(FFLFPP) with a new approach. Suvasis Nayak et.al (Nayak and Ojha, 2019) proposed a solution approach to MPLFPP using parametric functions. El-Saeed Ammar et.al (Ammar and Muamer, 2016) developed an approach for solving MOLFPP with fuzzy rough coefficients. Adem (Cevikel, 2008) tackling FMOLFPP, using Q-Taylor series approach(FMOLFPP). Srikumar Acharya et.al (Srikumar Acharya and Mishra, 2019) solving a multi-probabilistic fractional programming problem

with two parameters using the Cauchy distribution. Hamiden et.al (Khalifa and Ammar, 2020)investigated a MOLFPP that involves probabilistic parameters on the right-hand side of constraints. Moslem Ganji et.al (Ganji and Saraj, 2019) solved MOLFPP under uncertainty via robust optimization approach. Stanojevi et.al (B. Stanojevi, 2020)proposed a pair of piece-wise linear membership functions to solve FMOLFPP.
In this paper, we consider FFMOLFPP that included triangular fuzzy numbers and proposed a new method for its solution. The proposed method facilitate the decision maker to solve FFMOLFPP without transforming it to an equivalent crisp problem. The remainder of this study is laid out as follows: In Section 2, we recall the notion of fuzzy numbers, ranking and arithmetic operations on the parametric form of fuzzy numbers. In section 3, the general FFMOLFPP, its mathematical formulation and relevant theorems are presented. Also a new algorithm is proposed for the solution of FFMOLFPP without transforming to equivalent crisp problem. In Section 4, a numerical example is given to illustrate the efficacy of the proposed method. Section 5 provides the conclusion part.

2. Preliminaries

Definition 2.1. *A fuzzy set $\tilde{a}$ defined on the set of real numbers R is said to be a fuzzy number, if its membership function $\tilde{a} : R \rightarrow [0,1]$ has the following characteristics:*

- *$\tilde{a}$ is convex, (i.e.) $\tilde{a}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\tilde{a}(x_1), \tilde{a}(x_2)\}, \lambda \in [0,1]$, for all $x_1, x_2 \in R$.*
- *$\tilde{a}$ is normal, (i.e.) there exists an $x \in R$ such that $\tilde{a}(x) = 1$.*
- *$\tilde{a}$ is piecewise continuous.*

We use $F(R)$ to denote the set of all fuzzy numbers defined on R.

Definition 2.1. *A triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3) \in F(R)$ can also be represented as a pair $\tilde{a} = (\underline{a}, \overline{a})$ of functions $\underline{a}(h), \overline{a}(h)$ for $0 \leq h \leq 1$, which satisfies the following requirements:*

- *$\underline{a}(h)$ is a bounded monotonic increasing left continuous function.*
- *$\overline{a}(h)$ is a bounded monotonic decreasing left continuous function.*
- *$\underline{a}(h) \leq \overline{a}(r), 0 \leq h \leq 1$.*

An arbitrary triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$ can also be written as $\tilde{a} = (a_0, a_*, a^*)$, where $a_0 = \left(\frac{\underline{a}(1) + \overline{a}(1)}{2}\right)$ is the location index number, $a_* = (a_0 - \underline{a})$ and $a^* = (\overline{a} - a_0)$ are the left and the right fuzziness index functions respectively.

2.1. Ranking of Triangular Fuzzy Numbers

Several approaches for the ranking of fuzzy numbers have been proposed in the literature. An efficient approach for comparing the fuzzy numbers is by the use of a ranking function based on their graded means. We define the magnitude of the triangular fuzzy number $\tilde{a}$ by $R(\tilde{a}) = \left(\frac{a^* + 4a_0 - a_*}{4}\right) = \left(\frac{\underline{a} + \overline{a} + a_0}{4}\right)$.
For any two triangular fuzzy numbers $\tilde{a} = (a_1, a_2, a_3) = (a_0, a_*, a^*)$ and $\tilde{b} = (b_1, b_2, b_3) = (b_0, b_*, b^*)$ in $F(R)$ we have

- $\tilde{a} \succeq \tilde{b}$ if and only if $R(\tilde{a}) \geq R(\tilde{b})$
- $\tilde{a} \preceq \tilde{b}$ if and only if $R(\tilde{a}) \leq R(\tilde{b})$
- $\tilde{a} \approx \tilde{b}$ if and only if $R(\tilde{a}) = R(\tilde{b})$

2.2. Arithmetic Operations of Triangular Fuzzy Numbers

Ming Ma et. al. (Ming Ma and Kandel, 2017) have proposed a new fuzzy arithmetic based upon both location index and fuzziness index functions. The location index number is taken in the ordinary arithmetic, whereas the fuzziness index functions are considered to follow the lattice rule which is least upper bound in the lattice L. i.e. for $u, v \in L$, We define $u \vee v = \max\{u, v\}$ and $u \wedge v = \min\{u, v\}$. The arithmetic operations on any two triangular fuzzy number $\tilde{u} = (u_1, u_2, u_3), \tilde{v} = (v_1, v_2, v_3)$ are defined by

$$\tilde{u} \star \tilde{v} = (u_0, u_*, u^*) \star (v_0, v_*, v^*) = ((u_0 \star v_0), \max\{u_*, v_*\}, \max\{u^*, v^*\}),$$

where $\star = \{+, -, \times, \div\}$.

Note: Division is possible only when the location index number of the denominator fuzzy number is non-zero.

3. Fully Fuzzy Multi-Objective Linear Fractional Programming Problem (FFMOLFPP)

3.1. Fully fuzzy linear fractional programming (FFLFPP)

A general FFLFPP is given by

$$\begin{aligned} \max \tilde{z} =& \frac{\sum \tilde{c_j}\tilde{y_j} + \tilde{\alpha_j}}{\sum \tilde{d_j}\tilde{y_j} + \tilde{\beta_j}} \\ \text{subject to } & \sum_{i=1}^{n} \tilde{a}_{ij}\tilde{y}_j \preceq \tilde{b}_i, i = 1, 2, ...m \\ \text{and } \quad & \tilde{y}_j \succeq \tilde{0}, \text{ for all } j = 1, 2, ...n. \end{aligned} \tag{1}$$

In matrix form, the general FFLFPP is expressed as

$$\max \tilde{z} = \frac{\tilde{\mathbf{c}}\tilde{\mathbf{y}} + \tilde{\alpha}}{\tilde{\mathbf{d}}\tilde{\mathbf{y}} + \tilde{\beta}} \quad \text{subject to } \tilde{A}\tilde{\mathbf{y}} \preceq \tilde{\mathbf{b}} \quad \text{and} \quad \tilde{\mathbf{y}} \succeq \tilde{\mathbf{0}}.$$

where $\tilde{A} = (\tilde{a}_{ij})_{(m\times n)}$, $\tilde{\mathbf{y}} = (\tilde{y}_1, \tilde{y}_2, ..., \tilde{y}_n)$, $\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, ..., \tilde{b}_m)$, $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2, ..., \tilde{c}_n)$, $\tilde{\mathbf{d}} = (\tilde{d}_1, \tilde{d}_2, ..., \tilde{d}_n)$, $\tilde{\alpha}, \tilde{\beta} \in F(R)$.

3.2. Fully Fuzzy Multi-Objective Linear Fractional Programming Problem (FFMOLFPP)

A general FFMOLFPP is given by

$$\begin{aligned} \mathbf{P}_I : \max \tilde{\mathbf{z}}(\tilde{y}) =& [\tilde{z_1}(\tilde{y}), \tilde{z_2}(\tilde{y}), ..., \tilde{z_k}(\tilde{y})] \\ \text{subject to } & \tilde{A}\tilde{\mathbf{y}} \preceq \tilde{\mathbf{b}} \\ \text{and } \quad & \tilde{\mathbf{y}} \succeq \tilde{\mathbf{0}}, \end{aligned} \tag{2}$$

where $\tilde{z}_i(\tilde{y})=\dfrac{\tilde{\mathbf{c}}_i\tilde{\mathbf{y}} + \tilde{\alpha}_i}{\tilde{\mathbf{d}}_i\tilde{\mathbf{y}} + \tilde{\beta}_i}, i = 1, 2, ...k$; $\tilde{\mathbf{c}}_i, \tilde{\mathbf{d}}_i$ are n-dimensional fuzzy vectors and $\tilde{\alpha_i}, \tilde{\beta_i} \in F(R)$.

Based on Guzel's (N.Guzel and M.Sivri, 2020) procedure, the FFMOLFPP (2) is transformed into an equivalent non-fractional multi objective FFLPP as

$$\begin{aligned} \mathbf{P}_{II} : & \max_{\tilde{y}\in S}\{N_i(\tilde{y}) - \tilde{z}_i^*(D_i(\tilde{y}))\} \\ \text{subject to } & \tilde{A}\tilde{\mathbf{y}} \preceq \tilde{\mathbf{b}} \\ \text{and } \quad & \tilde{\mathbf{y}} \succeq \tilde{\mathbf{0}}. \end{aligned} \tag{3}$$

Also the FFMOLFPP (2) is transformed into an equivalent non-fractional single objective FFLPP as

$$\begin{aligned} \mathbf{P}_{III} : \max \tilde{z} = &\left\{ \sum_{i=1}^{k} [N_i(\tilde{y}) - \tilde{z}_i^*(D_i(\tilde{y}))]/\tilde{y} \in S \right\} \\ \text{subject to } &\tilde{A}\tilde{\mathbf{y}} \preceq \tilde{\mathbf{b}} \\ \text{and } \quad &\tilde{\mathbf{y}} \succeq \tilde{\mathbf{0}}. \end{aligned} \tag{4}$$

where $\tilde{\mathbf{y}}_i^*, z_i^*$ are maximum point and the value of each objective function, that is $\tilde{z}_i^* = \tilde{z}_i(\tilde{\mathbf{y}}_i^*)$= $\max \left\{ \dfrac{N_i(\tilde{y})}{D_i(\tilde{y})}/\tilde{y} \in S \right\}$, $i = 1, 2, ...k$.

Definition 3.1. *Let $\tilde{S}$ be the set of all fuzzy feasible solutions of the FFMOLFPP (2). A feasible solution $\tilde{\mathbf{y}}^*$ is said to be fuzzy Pareto optimum solution of the FFMOLFPP (2), if there does not exist another feasible solution $\tilde{\mathbf{y}} \in \tilde{S}$ such that $\tilde{z}_i(\tilde{\mathbf{y}}) \succeq \tilde{z}_i(\tilde{\mathbf{y}}^*)$ for all i and $\tilde{z}_j(\tilde{\mathbf{y}}) \succ \tilde{z}_j(\tilde{\mathbf{y}}^*)$ atleast one j.*

Theorem 3.1. *For any fully fuzzy linear fractional programming problem (1), if $\tilde{z}^* \approx \dfrac{N(\tilde{\mathbf{y}^*})}{D(\tilde{\mathbf{y}^*})} \approx \max \left\{ \dfrac{N(\tilde{\mathbf{y}})}{D(\tilde{\mathbf{y}})}/\tilde{\mathbf{y}} \in S \right\}$, then $F(\tilde{z}^*, \tilde{\mathbf{y}^*}) \approx \max \{N(\tilde{\mathbf{y}}) - \tilde{z}^* D(\tilde{\mathbf{y}})/\tilde{\mathbf{y}} \in S\} \approx \tilde{0}$.*

Theorem 3.2. *Let $\tilde{\mathbf{y}^*}$ be a fuzzy pareto optimal solution of $\mathbf{P}_I$ if and only if $\tilde{\mathbf{y}^*}$ is also a fuzzy pareto optimal solution of $\mathbf{P}_{II}$.*

Proof: Let $G_i(\tilde{y}) = \dfrac{N_i(\tilde{y})}{D_i(\tilde{y})}$ and $F_i(\tilde{y}) = \{N_i(\tilde{y}) - \gamma_i^* D_i(\tilde{y})\}$ for i=1,2,...k. Assume that $\tilde{\mathbf{y}^*}$ is a fuzzy pareto optimal solution of $\mathbf{P}_I$. Suppose that $\tilde{\mathbf{y}^*}$ is not a fuzzy pareto optimal solution to $\mathbf{P}_{II}$. Then by definition, there exists an $\tilde{\mathbf{y}} \in S$ such that $F_i(\tilde{\mathbf{y}}) \succeq F_i(\tilde{\mathbf{y}^*})$ for all $i = 1, 2, ..., k$ and $F_i(\tilde{\mathbf{y}}) \succ F_i(\tilde{\mathbf{y}^*})$ for at least one $j = 1, 2, ..., k$.

$\Rightarrow N_i(\tilde{\mathbf{y}}) - \gamma_i^* D_i(\tilde{\mathbf{y}}) \succeq N_i(\tilde{\mathbf{y}^*}) - \gamma_i^* D_i(\tilde{\mathbf{y}^*})$ for all i and $N_j(\tilde{\mathbf{y}}) - \gamma_j^* D_j(\tilde{\mathbf{y}}) \succ N_j(\tilde{\mathbf{y}^*}) - \gamma_j^* D_j(\tilde{\mathbf{y}^*})$ for at least one $j = 1, 2, ..., k$.
Since $\gamma_i^* \approx \dfrac{N_j(\tilde{\mathbf{y}^*})}{D_j(\tilde{\mathbf{y}^*})}$, we have $N_j(\tilde{\mathbf{y}^*}) - \gamma_j^* D_j(\tilde{\mathbf{y}^*}) \approx \tilde{0}$.

$\Rightarrow N_i(\tilde{\mathbf{y}}) - \gamma_i^* D_i(\tilde{\mathbf{y}}) \succeq \tilde{0}$ for all i and $N_j(\tilde{\mathbf{y}}) - \gamma_j^* D_j(\tilde{\mathbf{y}}) \succ \tilde{0}$ for at least one $j = 1, 2, ..., k$.

$\Rightarrow \dfrac{N_i(\tilde{\mathbf{y}})}{D_i(\tilde{\mathbf{y}})} \succeq \dfrac{N_i(\tilde{\mathbf{y}^*})}{D_i(\tilde{\mathbf{y}^*})}$ for all i and $\dfrac{N_j(\tilde{\mathbf{y}})}{D_j(\tilde{\mathbf{y}})} \succ \dfrac{N_j(\tilde{\mathbf{y}^*})}{D_j(\tilde{\mathbf{y}^*})}$ for at least one $j = 1, 2, ..., k$.
This contradicts the fact that $\tilde{\mathbf{y}^*}$ is a fuzzy pareto optimal solution of $\mathbf{P}_I$. Hence $\tilde{\mathbf{y}^*}$ is also a fuzzy pareto optimal solution of $\mathbf{P}_{II}$.
Conversely, let $\tilde{\mathbf{y}^*}$ is a fuzzy pareto optimal solution of $\mathbf{P}_{II}$. Suppose that $\tilde{y^*}$ is not a fuzzy pareto optimal for $\mathbf{P}_I$. Then there exists an $\tilde{\mathbf{y}} \in S$ such that $G_i(\tilde{\mathbf{y}}) \succeq G_i(\tilde{\mathbf{y}^*})$ for all $i = 1, 2, ..., k$ and $G_i(\tilde{\mathbf{y}}) \succ G_i(\tilde{\mathbf{y}^*})$ for at least one $j = 1, 2, ..., k$.
$\Rightarrow \dfrac{N_i(\tilde{\mathbf{y}})}{D_i(\tilde{\mathbf{y}})} \succeq \dfrac{N_i(\tilde{\mathbf{y}^*})}{D_i(\tilde{\mathbf{y}^*})}$ for all i and $\dfrac{N_j(\tilde{\mathbf{y}})}{D_j(\tilde{\mathbf{y}})} \succ \dfrac{N_j(\tilde{\mathbf{y}^*})}{D_j(\tilde{\mathbf{y}^*})}$ for at least one $j = 1, 2, ..., k$.
$\Rightarrow \dfrac{N_i(\tilde{\mathbf{y}})}{D_i(\tilde{\mathbf{y}})} \succeq \gamma_i^*$ for all i and $\dfrac{N_j(\tilde{\mathbf{y}})}{D_j(\tilde{\mathbf{y}})} \succ \gamma_j^*$ for at least one $j = 1, 2, ..., k$.
$\Rightarrow N_i(\tilde{\mathbf{y}}) \succeq \gamma_i^* D_i(\tilde{\mathbf{y}})$ for all $i = 1, 2, ..., k$ and$N_j(\tilde{\mathbf{y}}) \succ \gamma_j^* D_j(\tilde{\mathbf{y}})$ for at least one $j = 1, 2, ..., k$.
$\Rightarrow N_i(\tilde{\mathbf{y}}) - \gamma_i^* D_i(\tilde{\mathbf{y}}) \succeq \tilde{0}$for all i and $N_j(\tilde{\mathbf{y}}) - \gamma_j^* D_j(\tilde{\mathbf{y}}) \succ \tilde{0}$ for at least one $j = 1, 2, ..., k$.
$\Rightarrow F_i(\tilde{\mathbf{y}}) \succeq \tilde{\mathbf{0}}$ for all i=1,2,...,k. and $F_j(\tilde{\mathbf{y}}) \succ \tilde{\mathbf{0}}$ for at least one $j = 1, 2, ..., k$.
$\Rightarrow F_i(\tilde{\mathbf{y}}) \succeq F_i(\tilde{\mathbf{y}^*})$ for all $i = 1, 2, ..., k$ and $F_j(\tilde{\mathbf{y}}) \succ F_j(\tilde{\mathbf{y}^*})$ for at least one $j = 1, 2, ..., k$, which contradicts the fact that $\tilde{\mathbf{y}^*}$ is a pareto optimal solution of $\mathbf{P}_{II}$. Hence the pareto optimal solution of $\mathbf{P}_{II}$ is also a pareto optimal solution of $\mathbf{P}_I$.

Theorem 3.3. *If $\tilde{\mathbf{y}^*}$ is a fuzzy optimal solution of $\mathbf{P}_{III}$, then $\tilde{\mathbf{y}^*}$ is also a fuzzy pareto optimal solution of $\mathbf{P}_{II}$.*

Proof: Assume that $\tilde{\mathbf{y}^*}$ is a fuzzy optimal solution of $\mathbf{P}_{III}$. Suppose that $\tilde{\mathbf{y}^*}$ is not a fuzzy pareto optimal solution of $\mathbf{P}_{II}$. Then there exists an $\tilde{\mathbf{y}} \in S$ such that $F_i(\tilde{\mathbf{y}}) \succeq F_i(\tilde{\mathbf{y}^*})$ for all i and $F_i(\tilde{\mathbf{y}}) \succ F_i(\tilde{\mathbf{y}^*})$ for at least one $j = 1, 2, ..., k$.

$\Rightarrow N_i(\tilde{\mathbf{y}}) - \gamma_i^* D_i(\tilde{\mathbf{y}}) \succeq N_i(\tilde{\mathbf{y}^*}) - \gamma_i^* D_i(\tilde{\mathbf{y}^*})$ for all i and $N_j(\tilde{\mathbf{y}}) - \gamma_j^* D_j(\tilde{\mathbf{y}}) \succ N_j(\tilde{\mathbf{y}^*}) - \gamma_j^* D_j(\tilde{\mathbf{y}^*})$ for at least one $j = 1, 2, ..., k$.

Since $\gamma_i^* \approx \dfrac{N_j(\tilde{\mathbf{y}^*})}{D_j(\tilde{\mathbf{y}^*})}$, we have $N_j(\tilde{\mathbf{y}^*}) - \gamma_j^* D_j(\tilde{\mathbf{y}^*}) \approx \tilde{0}$.
$\Rightarrow N_i(\tilde{\mathbf{y}}) - \gamma_i^* D_i(\tilde{\mathbf{y}}) \succeq \tilde{0}$ for all i and $N_j(\tilde{\mathbf{y}}) - \gamma_j^* D_j(\tilde{\mathbf{y}}) \succ \tilde{0}$ for at least one $j = 1, 2, ..., k$.

By summing over k, we get

$$\sum_{i=1}^{k}[N_i(\tilde{\mathbf{y}}) - \gamma_i^* D_i(\tilde{\mathbf{y}})] \succeq \tilde{0} \text{ for all } i \text{ and}$$

$$\sum_{j=1}^{j=k}[N_j(\tilde{\mathbf{y}}) - \gamma_j^* D_j(\tilde{\mathbf{y}})] \succ \tilde{0} \text{ for at least one } j = 1, 2, ..., k.$$

$$\Rightarrow \sum_{i=1}^{k}[N_i(\tilde{\mathbf{y}}) - \gamma_i^* D_i(\tilde{\mathbf{y}})] \succeq \sum_{i=1}^{k}[N_i(\tilde{\mathbf{y}^*}) - \gamma_i^* D_i(\tilde{\mathbf{y}^*})] \text{ for all } i \text{ and}$$

$$\sum_{j=1}^{k}[N_j(\tilde{\mathbf{y}}) - \gamma_j^* D_j(\tilde{\mathbf{y}})] \succ \sum_{i=1}^{k}[N_j(\tilde{\mathbf{y}^*}) - \gamma_j^* D_j(\tilde{y^*})] \text{ for at least one } j = 1, 2, ..., k,$$

which contradicts our assumption. Hence $\tilde{\mathbf{y}^*}$ is also a fuzzy pareto optimal solution of $\mathbf{P}_{II}$.

Theorem 3.4. *If $\tilde{\mathbf{y}^*}$ is a fuzzy optimal solution of $\mathbf{P}_{III}$, then $\tilde{\mathbf{y}^*}$ is also a fuzzy pareto optimal solution of $\mathbf{P}_I$.*

Proof: Let $\tilde{\mathbf{y}^*}$ be a fuzzy optimal solution of $\mathbf{P}_{III}$. Suppose that $\tilde{\mathbf{y}^*}$ is not a fuzzy pareto optimal solution of $\mathbf{P}_I$. Then there exists a $\tilde{\mathbf{y}} \in S$ such that
$G_i(\tilde{\mathbf{y}}) \succeq G_i(\tilde{\mathbf{y}^*})$ for all $i = 1, 2, ..., k$ and $G_i(\tilde{\mathbf{y}}) \succ G_i(\tilde{\mathbf{y}^*})$ for at least one $j = 1, 2, ..., k$.
$\Rightarrow \dfrac{N_i(\tilde{\mathbf{y}})}{D_i(\tilde{\mathbf{y}})} \succeq \dfrac{N_i(\tilde{\mathbf{y}^*})}{D_i(\tilde{\mathbf{y}^*})}$ for all i and $\dfrac{N_j(\tilde{\mathbf{y}})}{D_j(\tilde{\mathbf{y}})} \succ \dfrac{N_j(\tilde{\mathbf{y}^*})}{D_j(\tilde{\mathbf{y}^*})}$ for at least one $j = 1, 2, ..., k$.
$\Rightarrow \dfrac{N_i(\tilde{\mathbf{y}})}{D_i(\tilde{\mathbf{y}})} \succeq \gamma_i^*$ for all i and $\dfrac{N_j(\tilde{y})}{D_j(\tilde{y})} \succ \gamma_j^*$ for at least one $j = 1, 2, ..., k$.
$\Rightarrow N_i(\tilde{\mathbf{y}}) - \gamma_i^* D_i(\tilde{\mathbf{y}}) \succeq \tilde{0}$ for all i and $N_j(\tilde{\mathbf{y}}) - \gamma_j^* D_j(\tilde{\mathbf{y}}) \succ \tilde{0}$ for at least one $j = 1, 2, ..., k$.
Summing over k, we get

$$\sum_{i=1}^{i=k}[N_i(\tilde{\mathbf{y}}) - \gamma_i^* D_i(\tilde{\mathbf{y}})] \succeq \tilde{0} \text{ for all } i \text{ and}$$

$$\sum_{j=1}^{j=k}[N_j(\tilde{\mathbf{y}}) - \gamma_j^* D_j(\tilde{\mathbf{y}})] \succ \tilde{0} \text{ for at least one } j = 1, 2, ..., k.$$

$$\Rightarrow \sum_{i=1}^{i=k}[N_i(\tilde{\mathbf{y}}) - \gamma_i^* D_i(\tilde{\mathbf{y}})] \succeq \sum_{i=1}^{i=k}[N_i(\tilde{\mathbf{y}^*}) - \gamma_i^* D_i(\tilde{\mathbf{y}^*})] \text{ for all } i \text{ and}$$

$$\sum_{j=1}^{j=k}[N_j(\tilde{\mathbf{y}}) - \gamma_j^* D_j(\tilde{\mathbf{y}})] \succ \sum_{i=1}^{i=k}[N_j(\tilde{\mathbf{y}^*}) - \gamma_j^* D_j(\tilde{y^*})] \text{ for at least one } j = 1, 2, ..., k.$$

which is a contradiction to our assumption. There fore $\tilde{\mathbf{y}^*}$ is also a fuzzy pareto optimal solution of $\mathbf{P}_I$.

3.3. Algorithm

Step 1: Given a FFMOLFPP with n objective functions.
Step 2: Consider each FFLFPP separately subject to the same set of constraints of the FFMOLFPP.
Step 3: Convert each FFLFPP into an equivalent FFLPP.
Step 4: Solve each FFLPP separately by expressing all its fuzzy decision parameters in their parametric forms using fuzzy version of Simplex algorithm.
Step 5: Obtain the fuzzy optimal value of the objective function of each FFLFPP from the optimal solution of the respective FFLPP.
Step 6: Reduce the given FFMOLFPP with n objective functions to an equivalent single objective FFLPP using the optimal values obtaine in Step 5.
Step 7: Using the Simplex approach, solve the single-objective FFLFPP obtained in step 6.
Step 8: Calculate the fuzzy Pareto optimal solution for the given FFMOLFPP using the fuzzy optimal solution for the single objective FFLFPP.

4. Numerical Example

We consider a FFMOLFPP involving triangular fuzzy numbers.

$$\begin{aligned} \max \tilde{z_1} \approx & \frac{\tilde{2}\tilde{y_1} + \tilde{4}\tilde{y_2} + \tilde{5}}{\tilde{2}\tilde{y_1} + \tilde{6}} \\ \max \tilde{z_2} \approx & \frac{\tilde{1}\tilde{y_1} + \tilde{6}\tilde{y_2} + \tilde{50}}{\tilde{1}\tilde{y_1} + \tilde{1}\tilde{y_2} + \tilde{8}} \\ \text{subject to } & \tilde{2}\tilde{y_1} + \tilde{2}\tilde{y_2} \preceq \tilde{140} \\ & \tilde{y_2} \succeq \tilde{8} \\ & \tilde{y_1} \succeq \tilde{16} \\ \text{and } & \tilde{y_1}, \tilde{y_2} \succeq \tilde{0}. \end{aligned} \tag{5}$$

Solution: Assume that the triangular fuzzy numbers $\tilde{5}, \tilde{1}, \tilde{2}, \tilde{140}, \tilde{8}, \tilde{16}$ and $\tilde{6}$ are of the forms $\tilde{5} = (4, 5, 6), \tilde{1} = (0, 1, 2), \tilde{2} = (0, 2, 4), \tilde{140} = (135, 140, 145), \tilde{8} = (4, 8, 12), \tilde{16} = (13, 16, 19), \tilde{6} = (5, 6, 7)$. By applying the proposed algorithm, we have two FFLFPP. Each FFLFPP is solved separately as follows.

Problem I: Consider the first FFLFPP subject to the constraints of the given FFMOLFPP (5).

$$\begin{aligned} \max \tilde{z_1} \approx & \frac{\tilde{2}\tilde{y_1} + \tilde{4}\tilde{y_2} + \tilde{5}}{\tilde{2}\tilde{y_1} + \tilde{6}} \\ \text{subject to } & \tilde{2}\tilde{y_1} + \tilde{2}\tilde{y_2} \preceq \tilde{140} \\ & \tilde{y_2} \succeq \tilde{8} \\ & \tilde{y_1} \succeq \tilde{16} \\ \text{and } & \tilde{y_1}, \tilde{y_2} \succeq \tilde{0} \end{aligned} \tag{6}$$

Transform this FFLFPP in to an equivalent FFLPP, we have

$$\begin{aligned} \max \tilde{z_1} \approx & \tilde{0.34}\tilde{q_1} + \tilde{4}\tilde{q_2} + \tilde{0.83} \\ \text{subject to } & \tilde{48.7}\tilde{q_1} + \tilde{2}\tilde{q_2} \preceq \tilde{23.3} \\ & \tilde{2.66}\tilde{q_1} + \tilde{q_2} \succeq \tilde{1.33} \\ & \tilde{6.33}\tilde{q_1} \succeq \tilde{2.67} \\ \text{and } & \tilde{q_1}, \tilde{q_2} \succeq \tilde{0}. \end{aligned} \tag{7}$$

The parametric form of this problem is

$$\begin{aligned} \max \tilde{z_1} \approx &(0.34, 0.5-0.5h, 0.5-0.5h)\tilde{q}_1 + (4, 1-h, 1-h)\tilde{q}_2 \\ &+ (0.83, 0.3-0.3r, 0.3-0.3h) \\ \text{subject to } &(48.7, 2-2h, 2-2h)\tilde{q}_1 + (2, 1-h, 1-h)\tilde{q}_2 \preceq (23.3, 4-4h, 4-4h) \\ &(2.66, 3-3h, 3-3h)\tilde{q}_1 + (1, 1-h, 1-h)\tilde{q}_2 \succeq (1.33, 1-h, 1-h) \\ &(6.33, 2-2h, 2-2h)\tilde{q}_1 + (0,0,0)\tilde{q}_2 \succeq (2.67, 3-3h, 3-3h) \\ \text{and } &\tilde{q}_1, \tilde{q}_2 \succeq \tilde{0} \end{aligned} \tag{8}$$

Applying the fuzzy version of simplex method, the fuzzy optimal solution of the FFLPP (7) is $\tilde{q}_1 = (0.42, 4-4h, 4-4h), \tilde{q}_2 = (1.43, 4-4h, 4-4h)$ and hence the optimal solution of the FFLFPP (6) is $\tilde{y}_1 = (15.75, 4-4h, 4-4h), \tilde{y}_2 = (53.63, 4-4h, 4-4h)$ with $\max \tilde{z_1} = (6.69, 4-4h, 4-4h)$.

problem II: Consider the second FFLFPP subject to the constraints of the given FFMOLFPP (5).

$$\begin{aligned} \max \tilde{z_2} \approx &\frac{\tilde{1}\tilde{y_1} + \tilde{6}\tilde{y_2} + \tilde{50}}{\tilde{1}\tilde{y_1} + \tilde{1}\tilde{y_2} + \tilde{8}} \\ \text{subject to } &\tilde{2}\tilde{y_1} + \tilde{2}\tilde{y_2} \preceq \tilde{140} \\ &\tilde{y_2} \succeq \tilde{8} \\ &\tilde{y_1} \succeq \tilde{16} \\ \text{and } \quad &\tilde{y_1}, \tilde{y_2} \succeq \tilde{0} \end{aligned} \tag{9}$$

Transform this FFLFPP into an equivalent FFLPP, we have

$$\begin{aligned} \max \tilde{z_2} \approx &-\tilde{5.25}\tilde{q_1} + \tilde{0.25}\tilde{q_2} + \tilde{6.25} \\ \text{subject to } &\tilde{19.5}\tilde{q_1} + \tilde{19.5}\tilde{q_2} \preceq \tilde{17.5} \\ &\tilde{1}\tilde{q_1} + \tilde{2}\tilde{q_2} \succeq \tilde{1} \\ &\tilde{3}\tilde{q_1} + \tilde{2}\tilde{q_2} \succeq \tilde{2} \\ \text{and } \quad &\tilde{q_1}, \tilde{q_2} \succeq \tilde{0}. \end{aligned} \tag{10}$$

The parametric form of this problem is

$$\begin{aligned} \max \tilde{z_2} = &(-5.25, 1-h, 1-h)\tilde{q}_1 + (0.25, 0.5-0.5h, 0.5-0.5h)\tilde{q}_2 \\ &+ (6.25.25, 1-h, 1-h) \\ \text{subject to } &(19.5, 2-2h, 2-2h)\tilde{q}_1 + (19.5, 2-2h, 2-2h)\tilde{q}_2 \leq (17.5, 3-3h, 3-3h) \\ &(1, 1-h, 1-h)\tilde{q}_1 + (2, 2-2h, 2-2h)\tilde{q}_2 \geq (1, 1-h, 1-h) \\ &(3, 3-3h, 3-3h)\tilde{q}_1 + (2, 2-2h, 2-2h)\tilde{q}_2 \geq (2, 2-2h, 2-2h) \\ \text{and } &\tilde{q}_1, \tilde{q}_2 \geq \tilde{0} \end{aligned} \tag{11}$$

Applying the fuzzy version of simplex method, the fuzzy optimal solution of the FFLPP (10) is $\tilde{q}_1 = (0.2, 3-3h, 3-3h), \tilde{q}_2 = (0.69, 3-3h, 3-3h)$ and hence the fuzzy optimal solution of the FFLFPP (9) is $\tilde{y}_1 = (14.55, 3-3h, 3-3h), \tilde{y}_2 = (50.2, 3-3h, 3-3h)$ with $\max \tilde{z_2} = (5, 3-3h, 3-3h)$.
Now convert the given FFMOLFPP (5) into an equivalent FFLPP, we have

$$\begin{aligned} \max \tilde{z} = &\{[(2, 2-2h, 2-2h)\tilde{y}_1 + (4, 1-h, 1-h)\tilde{y}_2 + (5, 2-2h, 2-2h)] \\ &- (6.69, 4-4h, 4-4h)[(2, 2-2h, 2-2h)\tilde{y}_1 + (6, 4-4h, 4-4h)]\} \\ &+ \{[(1, 1-h, 1-h)\tilde{y}_1 + (6, 4-4h, 4-4h)\tilde{y}_2 + (50, 3-3h, 3-3h)] \\ &- (5, 3-3h, 3-3h)[(1, 1-h, 1-h)\tilde{y}_1 + (1, 1-h, 1-h)\tilde{y}_2 \\ &+ (8, 0.2-0.2h, 0.2-0.2h)]\} \\ &\approx (-15.4, 4-4h, 4-4h)\tilde{y}_1 + (5, 3-3h, 3-3h)\tilde{y}_2 + (25.2, 4-4h, 4-4h) \end{aligned}$$

That is

$$\begin{aligned} \max \tilde{z} \approx & (-15.4, 4-4h, 4-4h)\tilde{y_1} + (5, 3-3h, 3-3h)\tilde{y_2} \\ & + (25.2, 4-4h, 4-4h) \\ \text{subject to } & (2, 2-2h, 2-2h)(\tilde{y_1} + (2, 2-2h, 2-2h)\tilde{y_2} \\ & \preceq (140, 5-5h, 5-5h) \\ & \tilde{y_2} \succeq (8, 4-4h, 4-4h) \\ & \tilde{y_1} \succeq (16, 3-3h, 3-3h) \\ & \tilde{y_1}, \tilde{y_2} \succeq \tilde{0}. \end{aligned} \tag{12}$$

Applying the fuzzy version of Simplex method, the fuzzy optimal solution of the FFLPP (12) is $\tilde{y_1} = (16, 3-3h, 3-3h)$ and $\tilde{y_2} = (54, 5-5h, 5-5h)$.
Hence the fuzzy Pareto optimal solution of the given FFMOLFPP (5) is $\tilde{y_1}$ = (16,3-3h,3-3h) and $\tilde{y_2}$ = (54,5-5h,5-5h) with $\max \tilde{z_1} = (6.658, 5-5h, 5-5h)$ and $\max \tilde{z_2} = (5, 5-5h, 5-5h)$.
For different values of $h \in [0, 1]$, the proposed method gives various fuzzy Pareto optimal solutions for the given FFMOLFPP.

Table 1

Values of h	$\tilde{y_1}$	$\tilde{y_2}$	$\max \tilde{z_1}$	$\max \tilde{z_2}$
$h = 1$	$(16, 16, 16)$	$(54, 54, 54)$	$(6.658, 6.658, 6.658)$	$(5, 5, 5)$
h=0.75	(15.25,16,16.75)	(52.75,54,55.25)	(5.408,6.658,7.908)	(3.75,5,6.25)
h=0.5	(14.5,16,17.5)	(51.5,54,56.5)	(4.158,6.658,9.158)	(2.5,5,7.5)
h=0.25	(13.75,16,18.25)	(50.25,54,57.75)	(2.908,6.658,10.408)	(1.25,5,8.75)
h=0	(13,16,19)	(49,54,59)	(1.658,6.658,11.658)	(0,5,10)

5. Concluding Remarks

We have proposed a new method for tackling fully fuzzy multi-objective linear fractional programming problems with triangular fuzzy numbers. The proposed method enables the decision maker to solve a fully fuzzy multi-objective linear fractional programming problem (FFMOLFPP) without having to convert a crisp version. It also allows the decision maker to select his or her preferred solution. It can be seen that the proposed method produces less ambiguous results when compared to other existing methods.

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