Pak.j.stat.oper.res. Vol.17 No. 2 2021 pp 343-356 DOI: http://dx.doi.org/10.18187/pjsor.v17i2.3360

Pakistan Journal of Statistics and Operation Research

Weighted Power Lomax Distribution and its Length Biased Version: Properties and Estimation Based on Censored Samples

Amal S. Hassan [1], Ehab M. Almetwally [2], Mundher Abdullah Khaleel [3], Heba F. Nagy [4*]

* Corresponding Author

1. Department of Mathematical Statistics, Cairo University, Faculty of Graduate Studies for Statistical Research, Egypt, dr.amalelmoslamy@gmail.com
2 Department of Mathematical Statistics, Cairo University, Faculty of Graduate Studies for Statistical Research, Egypt, ehabxp_2009@hotmail.com
3. Department of Mathematics, Faculty of Computer Science and Mathematics, University of Tikrit, Tikrit, Iraq, mun880088@tu.edu.iq
4. Department of Mathematical Statistics, Cairo University, Faculty of Graduate Studies for Statistical Research, Egypt, heba_nagy_84@yahoo.com

Abstract

In this paper, a weighted version of the power Lomax distribution referred to the weighted power Lomax distribution is introduced. The new distribution comprises the length biased and the area biased of the power Lomax distribution as new models as well as contains an existing model as the length biased Lomax distribution as special model. Essential distributional properties of the weighted power Lomax distribution are studied. Maximum likelihood and maximum product spacing methods are proposed for estimating the population parameters in cases of complete and type-II censored samples. Asymptotic confidence intervals of the model parameters are obtained. A sample generation algorithm along with a Monte Carlo simulation study is provided to demonstrate the pattern of the estimates for different sample sizes. Finally, a real-life data set is analyzed as an illustration and its length biased distribution is compared with some other lifetime distributions.

Key Words: Power Lomax distribution; Maximum product spacing; Approximate confidence intervals; Coverage probability; Monte Carlo simulation

Mathematical Subject Classification: 62F10

1. Introduction

Weighted distributions (**WDs**)are handled in studies associated with reliability, biomedicine, meta-analysis, econometrics, survival analysis, renewal processes, physics, ecology and branching processes which are found in Zelen and Feinleib (1969), Patil and Ord (1976), Patil and Rao (1978), Gupta and Keating (1986), Oluyede (1999) and references therein. In fact, these distributions arise in practice when observations from a sample are recorded with unequal probabilities.

Suppose that T is a nonnegative random variable with the probability density function (**pdf**) $f_w(t)$. The pdf of the weighted random variable T is defined by:

$$f_w(t)=\frac{\omega(t)\,g(t)}{E(\omega(T))},\qquad t>0, \tag{1}$$

where $\omega(t)$ is a nonnegative weight function and $E(\omega(T))>0.$ Different choices of $\omega(t)$ give different WDs, that is, for $\omega(t)=t^s, s>0,$ the pdf in (1) is called as WD of order s. Also, for s=1 or s = 2, the pdf (1) is called as the length-biased (size-biased) and the area-biased distributions, respectively.

The Lomax (**Lo**) distribution is an important model in lifetime analysis. It has been widely applied in a variety of contexts; analysis of income and wealth data (Harris (1968)), modelling business failure data (Atkinson and Harrison (1978)) and, biological sciences (Holland *et al.* (2006)), model firm size and queuing problems (Corbellini *et al.* (2010)) and reliability modelling and life testing (Hassan and Al-Ghamdi (2009) and Hassan *et al.* (2016)).

In recent times, many generalizations and extensions of the Lo distribution have been provided by many authors, including exponentiated Lo distribution (Abdul-Moniem and Abdel-Hameed (2012)), beta Lo distribution, Kumaraswamy Lo distribution and McDonald Lo distribution (Lemonte and Cordeiro (2013)), gamma-Lo distribution (Cordeiro *et al.* (2013)), Weibull Lo (WLo) distribution (Tahir *et al.* (2015)). Gumbel-Lo distribution (Tahir *et al.* (2016)), power Lo distribution (Rady *et al.* (2016)), exponentiated Lomax geometric distribution (Hassan and Abdelghafar (2017)), power Lo Poisson distribution (Hassan and Nassr (2018)), exponentiated Weibull-Lo distribution (Hassan and Abd-Allah (2018)), inverse power Lo distribution (Hassan and Abd-Allah (2019)), inverse exponentiated Lo distribution (Hassan and Mohamed (2019a)), Weibull inverse Lo distribution (Hassan and Mohamed (2019b)), type II half logistic Lo distribution (Hassan *et al.* (2020a), Zubair Lomax (Bantan *et al.* (2020)) and truncated power Lomax (Hassan *et al.* (2020b) among others.

The power Lo (**PLo**) distribution has been proposed by Rady *et al.* (2016) as a new extension of the Lo distribution with an extra shape parameter. The pdf of the PLo distribution with shape parameters *a*, *b* and scale parameter *c* is defined by

$$g(t) = abc^{a} t^{b-1} (c+t^{b})^{-a-1}; \quad t,a,b,c>0. \tag{2}$$

The cumulative distribution function (**cdf**) of the PLo distribution is as follows:

$$G(t) = 1 - c^{a} (c+y^{b})^{-a}. \tag{3}$$

The s^{th} moment corresponding to (2) is given by

$$\mu_s' = a\,c^{\frac{s}{b}} \mathrm{B}\left(a-(s/b), 1+(s/b)\right) = a\,c^{\frac{s}{b}} D_s, \tag{4}$$

where $D_s = \mathrm{B}\left(a-(s/b), 1+(s/b)\right)$, $s = 1,2,3,...$ and B(.,.) is the beta function.

In view of the importance of the PLo distribution as well as the idea of the WD, we introduce a weighted version of the PLo distribution called the weighted PLo (**WPLo**) distribution. The WPLo distribution can (i) be viewed as an alternate model to some new extended forms of the Lo and generalized exponential distributions. (ii) hold both inverted bathtub and decreasing hazard rate and (iii) have wider applications in some areas. We discuss the estimation of the population parameter via maximum likelihood (**ML**) and maximum product spacing (**MPS**) methods in the case of complete and type II censoring (**TIIC**) samples. Application to real data for its length biased version is considered.

The rest of the article is organized as follows. The weighted version for PLo distribution is described in Section 2. Section 3 gives moments and related measures, entropy measure and stochastic ordering for WPLo distribution. Section 4 deals with the point and approximate confidence interval (**CI**) of the model parameters based on the ML and MPS procedures. A simulation study is presented in Section 5. Real data illustration is described in Section 6 for studying the application of the length biased PLo (**LBPLo**) distribution. In the end, concluding remarks are implemented.

2. Weighted Power Lomax Distribution

Here, we obtain the WPLo distribution by considering the weight function $\omega(t) = t^{s}$, and using (2) by substituting them in (1), as appear in the following definition

Definition 1: *A nonnegative continuous random variable, T, is said to follow the WPLo* distribution *with parameters a, b, c and s if its pdf is of the form:*

$$f_w(t) = bc^{a-\frac{s}{b}} D_s^{-1} t^{s+b-1} (c+t^{b})^{-a-1}, \qquad t,a,b,c>0, \quad s = 1,2,... \tag{5}$$

where *a* and *b* are shape parameters and $D_s = \mathrm{B}\left(a-(s/b), 1+(s/b)\right)$, $s = 1,2,3,...$ A random variable with pdf (5) will be denoted by *T* ~ (*a*, *b*, *c*, *s*). The associated cdf of the WPLo distribution is given by:

$$F_w(t) = D_s^{-1} \gamma\left(a-(s/b), 1+(s/b), t^{b}/(c+t^{b})\right), \qquad t,a,b,c>0, \quad s = 1,2,... \tag{6}$$

where $\gamma\left(a-(s/\mathrm{b}), 1+(s/b), t^{b}/(c+t^{b})\right) = \int_0^{t^{b}/(c+t^{b})} (1-y)^{a-(s/b)-1} y^{1+(s/b)-1} dy$, is the incomplete beta function.

Some special sub-models can be obtained from (5) as follows

i. For, *s* =1 in (5), we get LBPLo distribution as a new model.
ii. For, *s* =2 in (5), we get area biased PLo (**ABPLo**) distribution as a new model.
iii. For, *s* =1, *b*=1 in (5), we get length biased Lo (**LBLo**) distribution (Ahmad *et al.* (2016)).
iv. For, *s* =2, *b*=1 in (5), we get area biased Lo (**ABLo**) distribution as a new model.

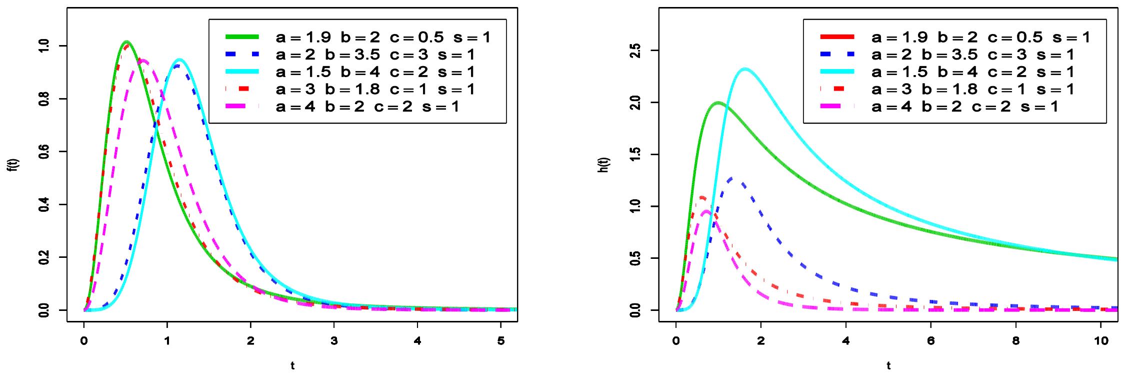
The survival function (**sf**) of the WPLo distribution is then,

$$\bar{F}_w(t)=1-D_s^{-1}\gamma\left(a-(s/b),1+(s/b),t^b/(c+t^b)\right).$$

The hazard rate function (**hrf**) is

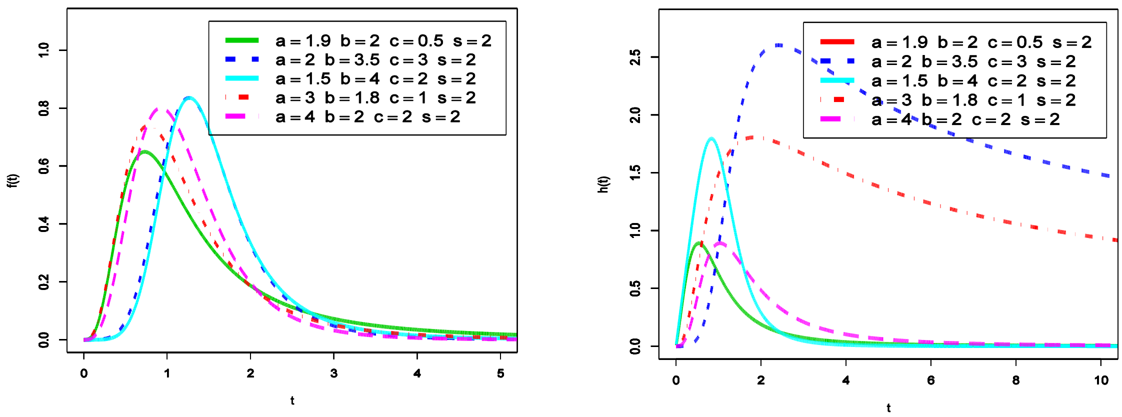
$$h_w(t)=\frac{bc^{a-\frac{s}{b}}D_s^{-1}t^{s+b-1}(c+t^b)^{-a-1}}{1-D_s^{-1}\gamma\left(a-(s/b),1+(s/b),t^b/(c+t^b)\right)}.$$

Fig.1, Fig.2 and Fig.3 show the pdfs and hrfs for WPLo distribution for some choices values of a, b, c and s.



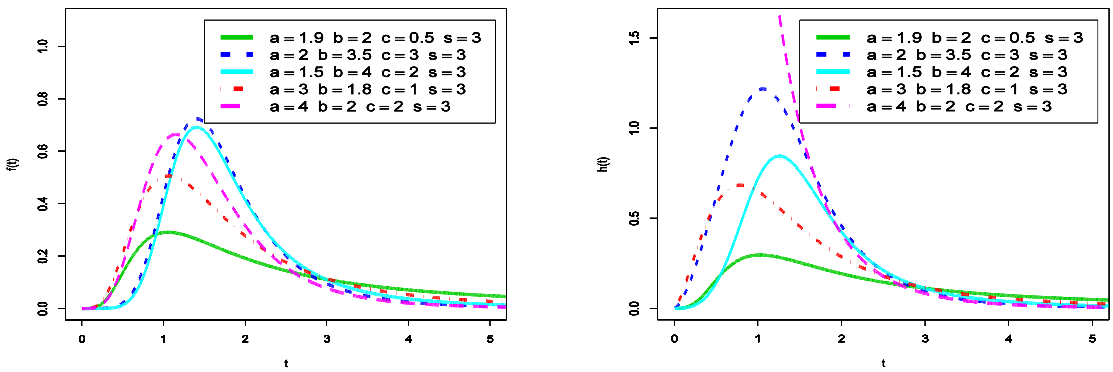
(i) (ii)

Fig. 1: **(i)** The pdf plots and **(ii)** The hrf plots of the LBPLo distribution



(iii) (iv)

Fig. 2: **(iii)** The pdf plots and **(iv)** The hrf plots of the ABPLo distribution



(v) (vi)

Fig. 3: **(v)** The pdf plots and **(vi)** The hrf plots of the WPLo distribution

From the above figures, we conclude that the pdf of the WPLo distribution and their particular cases (LBPLo and ABPLo) can take different shapes according to some values of *a*, *b*, *c* and *s*. Also, their hrf can be increasing; decreasing, reversed J-shaped, and up-side down shapes

3. Main Properties

In this section, some structural properties of the WPLo distribution are discussed.

3.1 Moments and Associated Measures

The moments of a random variable are necessary in statistical analysis, especially in applied works. The r^{th} moment, μ_r', for r =1,2,… of the WPLo distribution is obtained directly by using the pdf (5), hence, we have

$$\mu_r' = c^{\frac{r}{b}} D_s^{-1} B\left\{\frac{r+s}{b}+1,\ a-\left(\frac{r+s}{b}\right)\right\},\ ab-s>r,\quad r=1,2,...$$

Similarly, the r^{th} central moment of a given random variable T, can be defined as

$$\mu_r = E(T-\mu_1')^r = \sum_{i=0}^{r}(-1)^i\binom{r}{i}(\mu_1')^i\,\mu_{r-i}'.$$

The coefficient of skewness (**SK**) and coefficient of kurtosis (**KU**) are defined by

$$SK = \frac{\mu_3}{\mu_2^{3/2}},\quad KU = \frac{\mu_4}{\mu_2^2}.$$

Thus, numerical values of the μ_1', variance (σ^2), coefficient of variation (**CV**), SK and KU of the WPLo distribution for some certain values of parameters are listed in Table 1.

Table 1: Some moments measures of the WPLo distribution

Parameters	μ_1'	σ^2	CV	SK	KU
a=2, *b*=3, *c*=2, *s*=1	1.260	0.382	0.490	2.362	24.949
a=3, *b*=3, *c*=2, *s*=2	1.172	0.213	0.394	1.411	8.562
a=4, *b*=3, *c*=0.5, *s*=1	0.556	0.043	0.373	0.771	4.484
a=4, *b*=3, *c*=0.5, *s*=2	0.633	0.049	0.351	0.899	5.084
a=3, *b*=4, *c*=0.5, *s*=1	0.68	0.043	0.306	0.645	4.351
a=5, *b*=6, *c*=3, *s*=3	0.974	0.029	0.174	0.051	3.242
a=6, *b*=5, *c*=3, *s*=3	0.949	0.035	0.197	0.110	3.197
a=7, *b*=3, *c*=2, *s*=3	0.858	0.062	0.290	0.510	3.633

Table 1 shows that the LBPLo and ABPLo distributions are positively skewed and leptokurtic for selected values of parameters.

Further, a simple formula for the p^{th} incomplete moment of T, say $\upsilon_p(y)=E(T^p\,|T<y)$, is obtained as follows:

$$\upsilon_p(y) = c^{\frac{p}{b}} D_s^{-1}\gamma\left\{\frac{p+s}{b}+1,\ a-\left(\frac{p+s}{b}\right),\frac{y^b}{c+y^b}\right\},\quad p=1,2,...,$$

where $\gamma\left(.,.,y^b/(c+y^b)\right)$ is the incomplete beta function. For p =1, we get the first incomplete moment. The essential applications of the first incomplete moment are the Lorenz and Bonferroni curves.

3.2 Residual and Reversed Life Functions

The residual life plays vital role in many situations like life testing and reliability theory. The m^{th} moment of the residual life (**RL**) is defined by:

$$\varsigma_m(x) = E[(T-x)^m\,|T>x] = \frac{1}{\bar{F}(x)}\int_x^{\infty}(t-x)^m\, f(t)dt.$$

Employing the binomial expansion and pdf; $f_{\text{w}}(t)$, then $\varsigma_m(x)$ can be written as follows:

$$\varsigma_m(x) = \frac{1}{\bar{F}_w(x)}\sum_{j=0}^{m}\binom{m}{j}(-x)^{m-j}\int_x^{\infty} bc^{a-\frac{s}{b}} D_s^{-1} t^{j+s+b-1}\left(c+t^b\right)^{-(a+1)}dt.$$

So, after simplification the m^{th} moment of the RL of the WPLo distribution is obtained as follows:

$$\varsigma_m(x)=\frac{1}{\bar{F}_w(x)}\sum_{j=0}^{m}\binom{m}{j}(-x)^{m-j}c^{\frac{j}{b}}D_s^{-1}\gamma\left(\frac{j+s}{b}+1,a-\left(\frac{j+s}{b}\right),\left(\frac{c}{c+x^b}\right)\right).$$

where $\gamma\left(.,.,\left(c/(c+x^b)\right)\right)$ is the incomplete beta function and $\bar{F}_w(x)$ is the sf of the WPLo distribution. For m =1, we get the mean RL of the WPLo distribution which represents the expected additional life length for a unit which is alive at age ***x*.**

The n^{th} moment of the reversed RL, say $\xi_n(x)=E[(T-x)^n \,|T\le x],$ for $x > 0$ and n = 1, 2,... uniquely determines $F(t)$. Therefore, the n^{th} moment of the reversed RL of T is defined by

$$\xi_n(x)=\frac{1}{F(x)}\int_0^x (t-x)^n \, f(t)dt.$$

Using pdf (5), the n^{th} moment of the reversed RL of the WPLo distribution is as follows

$$\xi_n(x)=\frac{1}{F_w(x)}\sum_{j=0}^{n}\binom{n}{j}(-x)^{n-j}c^{\frac{j}{b}}D_s^{-1}\gamma\left(\frac{j+s}{b}+1,a-\left(\frac{j+s}{b}\right),\frac{x^b}{c+x^b}\right).$$

For n =1, we get the mean waiting time also called the mean reversed RL function which represent the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,x)$.

3.3 Quantile function

The q^{th} quantile; t_q (also called the percentile of order q) of the WPLo distribution can be obtained from (6) as follows

$$F(t_q)=D_s^{-1}\gamma\left(a-(s/b),1+(s/b),t^b/(c+t^b)\right)-q=0. \qquad (7)$$

It is a complex equation so by using iteration technique as a Newton-Raphson we obtain the quartiles. Further, from (7), the values of t_q for $q\sim$ uniform (0, 1) provides the random values generated from the WPLo distribution.

3.4 Rényi Entropy

For a certain random phenomenon under study, it is important to quantify the uncertainty associated with the random variable of interest. One of the most famous measures used to quantify the variability of random variable is the Rényi entropy. It is defined for $\lambda>0$ and $\lambda\neq 1,$ as follows

$$E_\lambda(T)=\frac{1}{1-\lambda}\log\int_{-\infty}^{\infty} f(t)^\lambda \, dt.$$

Using pdf (5), the Rényi entropy of the WPLo distribution can be written as follows

$$E_\lambda(T)=\frac{1}{1-\lambda}\log\int_0^\infty b^\lambda c^{\lambda(a-\frac{s}{b})}D_s^{-\lambda}t^{\lambda(s+b-1)}(c+t^b)^{-\lambda(a+1)}dt.$$

After simplification, the Rényi entropy of the WPLo distribution is given by

$$E_\lambda(T)=\frac{1}{1-\lambda}\log\left\{b^{\lambda-1}c^{\left(\frac{1-\lambda}{b}\right)}D_s^{-\lambda}\mathrm{B}\left[\frac{\lambda(s+b-1)+1}{b},\lambda(a+1)-\left(\frac{\lambda(s+b-1)+1}{b}\right)\right]\right\}.$$

3.5 Stochastic ordering

Shaked and Shanthikumar (2007) mentioned that, for independent random variables T and W with cdfs F_T and F_W respectively, T is said to be smaller than W if the following ordering holds;

- Stochastic order($T \le_{sr} W$) if $F_T(t) \ge F_W(t)$ for all t.
- Likelihood ratio order ($T \le_{lr} W$) if $f_T(t)/f_W(t)$ is decreasing in t.
- Hazard rate order ($T \le_{hr} W$) if $h_T(t) \ge h_W(t)$ for all t.
- Mean residual life order ($X \le_{mrl} Y$) if $m_T(t) \ge m_W(t)$ for all t.

We have the following chain of implications among the various partial orderings discussed above:

$$T \leq_{lr} W \Rightarrow T \leq_{hr} W \Rightarrow T \leq_{mrl} W$$
$$\Downarrow$$
$$T \leq_{sr} W$$

Theorem 1: Let T ~ WPLo distribution (a_1, b_1, c_1, s_1) and W ~ WPLo distribution (a_2, b_2, c_2, s_2) If, $a_1 \geq a_2$, $b_1 \geq b_2, c_1 \geq c_2$, and $s_1 \geq s_2$, then $T \leq_{lr} W$, $T \leq_{hr} W$, $T \leq_{mrl} W$, and $T \leq_{sr} W$.

Proof

It is sufficient to show $f_T(t)/f_W(t)$ is a decreasing function of t; therefore,

$$\frac{d}{dt}\log\frac{f_T(t)}{f_W(t)} = \frac{(s_1+b_1-1)}{t} - \frac{(a_1+1)b_1 t^{b_1-1}}{c_1+t^{b_1}} - \frac{(s_2+b_2-1)}{t} + \frac{(a_2+1)b_2 t^{b_2-1}}{c_2+t^{b_2}}.$$

Now if $a_1 \geq a_2$, $b_1 \geq b_2, c_1 \geq c_2$, and $s_1 \geq s_2$, then $\frac{d}{dt}\log\frac{f_T(t)}{f_W(t)} \leq 0$, which implies that W is stochastically greater than T with respect to likelihood ratio order i.e., $T \leq_{lr} W$. Similarly, we can conclude for $T \leq_{hr} W$, $T \leq_{mrl} W$, and $T \leq_{sr} W$.

4. Parameter Estimation

This section provides the ML and MPS estimators of the population parameters for WPLo distribution via complete and TIIC. Further, the asymptotic CI for model parameters is given.

4.1. ML Estimators

The ML population parameter estimators for the WPLo distribution are derived in case of TIIC and complete samples. Let T_1, T_2,..., T_n be independent and identically WPLo distribution random variables representing the lifetimes of n independent units. In TIIC case, only the first prefixed k $(k \leq n)$ failures, say $T_{(1)}$, $T_{(2)}$,…, $T_{(k)}$ are observed. These failures correspond to the first k order statistics of the random sample T_1, T_2,..., T_n. The log-likelihood function, denoted by $\ln \ell$, for the WPLo distribution based on TIIC, is obtained as follows:

$$\ln\ell = \ln C_1 + k\ln\left(\Gamma(a+1)\right) - k\ln\left(\Gamma(a-s/b)\right) - k\ln\left(\Gamma(1+s/b)\right) + k(a-s/b)\ln c + k\ln b + (s+b-1)\sum_{i=1}^{k}\ln(t_i)$$
$$-(a+1)\sum_{i=1}^{k}\ln(c+t_i^b) + (n-k)\left(\Xi(a,b,c,s)\right),$$

where $C_1 = \frac{n!}{n-k!}$, $\Xi(a,b,c,s) = \ln\left(1 - D_s^{-1}\gamma\left(a-(s/b), 1+(s/b), t^b/(c+t^b)\right)\right)$, and for simplicity we write t_i instead of $t_{(i)}$. The partial derivatives with respect to a, b, c and s are obtained as follows

$$\frac{\partial \ln\ell}{\partial a} = k\Psi(a+1) - k\Psi(a-s/b) + k\ln c - \sum_{i=1}^{k}\ln(c+t_i^b) + \frac{(n-k)\partial\Xi(a,b,c,s)}{\partial a}, \tag{8}$$

$$\frac{\partial \ln\ell}{\partial b} = \frac{ks}{b^2}\Psi(1-s/b) - \frac{ks}{b^2}\Psi(a-s/b) + \frac{ks}{b^2}\ln c + \frac{k}{b} + \sum_{i=1}^{k}\ln(t_i) - \sum_{i=1}^{k}\frac{(a+1)t_i^b\ln t_i}{(c+t_i^b)} + \frac{(n-k)\partial\Xi(a,b,c,s)}{\partial b}, \tag{9}$$

$$\frac{\partial \ln\ell}{\partial c} = \frac{k}{c}(a-s/b) - (a+1)\sum_{i=1}^{k}\frac{1}{c+t_i^b} + \frac{(n-k)\partial\Xi(a,b,c,s)}{\partial c}, \tag{10}$$

and,

$$\frac{\partial \ln\ell}{\partial s} = \frac{k}{b}\Psi(a-s/b) - \frac{k}{b}\Psi(1-s/b) - \frac{k}{b}\ln c + \sum_{i=1}^{k}\ln(t_i) + \frac{(n-k)\partial\Xi(a,b,c,s)}{\partial s}, \tag{11}$$

where $\Psi(e) = \Gamma'(e)/\Gamma(e)$ is the digamma function. The ML estimators of the unknown parameters of the WPLo distribution can be obtained by solving the following non-linear equations: $\partial\ln\ell/\partial a = 0, \partial\ln\ell/\partial b = 0,$ $\partial\ln\ell/\partial c = 0, \text{and}\, \partial\ln\ell/\partial s = 0.$ Unfortunately these equations cannot be solved analytically, so numerical technique is employed. Additionally, the ML estimators of parameters are obtained by solving the non-linear Equations (8)-(11) for $k = n$ in case of complete sample.

4.2 Maximum Product Spacing Estimator

The MPS technique was presented by Cheng and Amin (1979) and independently notable with Ranneby (1984) as an alternative method to ML estimation technique for continuous distributions. The product spacing's under the considered TIIC scheme can be written as

$$D(a,b,c,s)=\prod_{i=1}^{k+1}\Big(F(x_{(i)}-F(x_{(i-1)})\Big)\Big(1-F(x_{(k)})\Big)^{n-k}, \tag{12}$$

where $F(x_{(0)}) \equiv 0$ and $F(x_{(k+1)}) \equiv 1$. The MPS estimators can be obtained from maximizing $D(a,b,c,s)$ with respect to a, b, c and s subject to the constraint $ab > s$. Further, under complete sample, the MPS estimates are obtained by maximizing (12) for $k = n$.

4.3 Approximate Confidence Interval

It is known that under regularity condition that the asymptotic distribution of ML estimators of elements of unknown parameters a, b, c and s is given by

$$(\hat{a}-a),(\hat{b}-b),(\hat{c}-c),(\hat{s}-s)\rightarrow N(0,I^{-1}(a,b,c,s)),$$

where $I^{-1}(a, b, c, s)$ is the variance covariance matrix of population parameters; a, b, c and s. The elements of Fisher information matrix are obtained for complete and TIIC scheme. Therefore, the two-sided approximate α 100 percent limits for the ML estimators of population parameters for a, b, c and s are obtained, respectively, as follows:

$$L_a=\hat{a}-Z_{\alpha/2}\sqrt{\text{var}(\hat{a})},\ \ U_a=\hat{a}+Z_{\alpha/2}\sqrt{\text{var}(\hat{a})},\ \ L_b=\hat{b}-Z_{\alpha/2}\sqrt{\text{var}(\hat{b})},\ \ U_b=\hat{b}+Z_{\alpha/2}\sqrt{\text{var}(\hat{b})},$$

$$L_c=\hat{c}-Z_{\alpha/2}\sqrt{\text{var}(\hat{c})},\ U_c=\hat{c}+Z_{\alpha/2}\sqrt{\text{var}(\hat{c})},\text{ and, }\ L_s=\hat{s}-Z_{\alpha/2}\sqrt{\text{var}(\hat{s})},U_s=\hat{s}+Z_{\alpha/2}\sqrt{\text{var}(\hat{s})},$$

where Z is the $100(1-\alpha/2)\%^{th}$ standard normal percentile and var(.)'s denote the diagonal elements of variance covariance matrix corresponding to the model parameters.

5. Simulation Study

Here, we give up with a numerical study to assess the attitude of the ML and MPS estimates of the WPLo distribution and their length biased version based on complete sample and TIIC scheme. The algorithm used here is done via R package and the steps are summarized as follows:

- 1000 random sample of sizes n = 50, 100 and 200 are generated from the WPLo distribution by solving numerically Equation (7) under complete and TIIC.
- The number of failure items; k, based on TIIC are selected as 60%, 80% and 100% (complete sample).
- Exact values of parameters are chosen as; Case 1 = (a = 2.5, b = 1.5, c = 0.75, s = 1), Case 2 = (a = 2.5, b = 1.5, c = 0.75, s =2), Case 3 = (a = 3, b = 0.9, c=3, s=1) and Case 4=(a = 3, b = 2, c = 0.5, s = 3).
- The ML estimates of the model parameters are obtained by solving the non-linear Equations (8)-(11). based on complete (k=n) and TIIC scheme. Also, the MPS estimates of the population parameters are obtained by maximizing Equation (12) with respect to a, b, c and s.
- The average length (**AL**) of CIs with confidence level 0.95 for all samples sizes and the corresponding coverage probability (**CP**) are computed.
- The absolute bias (**AB**), mean square errors (**MSE**), AL and CP at α = 0.05 of all estimates are calculated.
- The numerical outcomes of the simulated data are listed in Tables 2, 3, 4 and 5.

Table 2: AB, MSE, AL and CP of ML and MPS estimates under complete and TIIC in Case 1

***a* = 2.5, *b* = 1.5, *c* = 0.75, *s* = 1**										
n	***k***		**ML**				**MPS**			
			AB	**MSE**	**AL**	**CP**	**AB**	**MSE**	**AL**	**CP**
50	1	*a*	0.0722	0.3158	0.5546	95.4	0.0952	0.2829	0.5509	97.5
		b	0.2156	0.2459	1.1565	93.7	0.0226	0.1254	1.3857	96.1
		c	0.2115	0.1042	0.4331	95.1	0.2268	0.1401	0.4320	94.7
	0.8	*a*	0.6764	0.4902	0.7083	94.9	0.1045	0.4836	0.75619	96.5

$a = 2.5, b = 1.5, c = 0.75, s = 1$										
n	**k**		**ML**				**MPS**			
			AB	**MSE**	**AL**	**CP**	**AB**	**MSE**	**AL**	**CP**
		b	0.9111	1.0028	1.6301	94.9	0.1473	0.2662	1.9392	95.6
		c	0.3260	0.1258	0.5476	95.4	0.3141	0.4834	0.5433	94.7
	0.6	*a*	0.8293	1.1887	0.7954	95.0	0.1075	0.7800	0.7438	95.9
		b	1.1595	2.8566	2.1870	95.2	0.1640	0.3350	2.1770	95.0
		c	0.6233	0.3909	0.6924	94.9	0.2613	0.4970	0.7179	95.1
100	1	*a*	0.0336	0.3637	0.1237	96.7	0.0439	0.3461	0.3657	98.0
		b	0.1561	0.7564	1.1025	95.6	0.0414	0.0744	1.0577	95.3
		c	0.2061	0.1036	0.3046	96.1	0.1742	0.0922	0.3709	94.0
	0.8	*a*	0.6294	0.4126	0.5030	95.0	0.2154	0.3559	0.5809	98.5
		b	0.8567	0.8180	1.1376	95.3	0.1467	0.1554	1.4349	95.5
		c	0.3251	0.1136	0.3492	95.2	0.2680	0.1323	0.2967	94.4
	0.6	*a*	1.0538	1.1186	0.3534	95.6	0.2603	0.6139	0.3899	98.4
		b	1.5176	2.4548	1.5281	94.9	0.2033	0.2494	1.78917	95.0
		c	0.6241	0.3904	0.4166	94.5	0.2631	0.2908	0.5846	94.3
200	1	*a*	0.1903	0.3541	0.3094	94.7	0.0204	0.2296	0.3131	94.8
		b	0.1094	0.0615	0.7150	95.0	0.0032	0.0268	0.6424	94.4
		c	0.1121	0.0913	0.2454	95.0	0.0646	0.0625	0.3947	95.7
	0.8	*a*	0.6078	0.3768	0.3368	94.9	0.0621	0.2598	0.4984	94.5
		b	0.8298	0.7257	0.7557	94.3	0.0354	0.0375	0.7466	94.6
		c	0.3172	0.1048	0.2535	95.0	0.0875	0.0659	0.2946	94.6
	0.6	*a*	0.8060	1.0127	0.4707	95.5	0.0771	0.2845	0.65714	93.5
		b	1.5134	2.3774	1.1574	95.5	0.0378	0.0484	0.8502	95.7
		c	0.6247	0.3091	0.3058	97.4	0.0782	0.0715	0.2359	95.4

Table 3: AB, MSE, AL and CP of ML and MPS estimates under complete and TIIC in Case 2

$a = 2.5, b = 1.5, c = 0.75, s = 2$										
n	**k**		**ML**				**MPS**			
			Bias	**MSE**	**AL**	**CP**	**AB**	**MSE**	**AL**	**CP**
50	1	*a*	0.4906	0.3901	3.7000	95.9	0.0555	0.3403	2.4814	95.9
		b	0.1226	0.0818	2.7563	95.0	0.0862	0.0718	1.6102	93.9
		c	0.4860	0.2578	1.5760	95.6	0.1791	0.1515	1.3723	95.6
	0.8	*a*	0.5763	0.4107	2.9457	94.2	0.2311	0.4092	2.5978	98.2
		b	0.1371	0.0850	2.2317	94.3	0.2376	0.1353	2.1352	93.8
		c	0.6107	0.3751	1.3952	94.7	0.1688	0.1532	1.3851	95.5
	0.6	*a*	0.2875	0.5018	2.1985	94.2	0.2151	0.5043	2.4171	96.6
		b	0.2943	0.1635	2.3300	95.7	0.2242	0.1359	2.1792	94.7
		c	0.6832	0.4671	1.9395	94.4	0.1479	0.1628	1.4723	95.6
100	1	*a*	0.8246	0.3648	3.8582	97.0	0.0866	0.3453	2.2794	95.5
		b	0.0986	0.0439	1.6195	94.7	0.0058	0.0561	0.9694	96.0
		c	0.5263	0.2971	1.5556	96.5	0.0679	0.0548	0.8787	96.3
	0.8	*a*	0.5385	0.3849	3.1094	94.1	0.0346	0.3917	2.1138	94.3
		b	0.1005	0.0504	1.1457	94.1	0.0566	0.0851	1.1224	96.5
		c	0.6080	0.3508	0.8943	96.2	0.0709	0.0563	0.8878	95.6
	0.6	*a*	0.2813	0.0834	1.8945	95.8	0.0406	0.4522	1.8394	94.1
		b	0.2365	0.0649	1.1536	95.5	0.0460	0.1672	1.0005	96.2
		c	0.6834	0.4167	0.9922	93.8	0.0764	0.0692	0.9876	95.4
200	1	*a*	0.5113	0.3571	2.6922	96.2	0.0564	0.2468	1.9358	96.2
		b	0.2746	0.0289	1.4950	95.3	0.0024	0.1346	0.8431	95.9
		c	0.4882	0.2479	1.3836	96.3	0.0389	0.0264	0.6187	95.5
	0.8	*a*	0.5569	0.3713	1.9995	93.0	0.0072	0.2528	1.9321	95.5
		b	0.0912	0.0308	0.8927	94.0	0.0354	0.0527	0.8899	96.3
		c	0.5141	0.2784	0.6914	94.8	0.0401	0.0258	0.6096	94.8
	0.6	*a*	0.2099	0.4698	1.6791	93.0	0.0097	0.3825	1.6750	94.1
		b	0.2075	0.0556	0.8745	93.9	0.0274	0.1483	0.8556	96.8

$a = 2.5, b = 1.5, c = 0.75, s = 2$										
n	***k***		**ML**				**MPS**			
			Bias	**MSE**	**AL**	**CP**	**AB**	**MSE**	**AL**	**CP**
		c	0.6838	0.3676	0.6797	94.7	0.0429	0.0310	0.6698	95.8

Table 4: AB, MSE, AL and CP of ML and MPS estimates under complete and TIIC in Case 3

$a = 3, b = 0.9, c = 3, s = 1$										
n	*k*		**ML**				**MPS**			
			AB	**MSE**	**AL**	**CP**	**AB**	**MSE**	**AL**	**CP**
50	1	*a*	0.2628	0.4976	2.5676	93	0.1678	0.1472	1.3533	95.1
		b	0.0240	0.0507	0.8777	94.4	0.0457	0.0190	0.5097	95.7
		c	0.3197	0.6879	3.0014	97.3	0.3038	0.3948	2.1569	93.1
	0.8	*a*	0.5731	0.5417	2.6675	95.2	0.2382	0.3485	2.3105	94.2
		b	0.4636	0.2928	2.0944	95.6	0.0790	0.0227	0.5894	96
		c	0.5052	1.3781	4.1559	97.8	0.3122	1.2196	4.1545	95
	0.6	*a*	1.2169	1.5360	2.9412	95.8	0.0185	0.3486	2.9344	94.2
		b	0.9362	1.0561	2.6623	95.7	0.0062	0.0313	0.6931	95.3
		c	1.5916	3.1306	4.4317	95.9	0.2740	1.8039	4.3483	94.5
100	1	*a*	0.2100	0.4613	2.4070	94.9	0.1221	0.0810	1.0083	94.7
		b	0.0217	0.0190	0.5336	96	0.0268	0.0086	0.3492	95.4
		c	0.3783	0.5083	2.3699	96.1	0.1956	0.2126	1.6374	93
	0.8	*a*	0.5167	0.4705	2.8180	94.4	0.2263	0.2150	1.8155	94.9
		b	0.4261	0.2148	0.7147	95.4	0.0331	0.0133	0.4517	96.7
		c	0.6349	0.7983	4.4652	95.7	0.2642	0.8387	3.4390	94.7
	0.6	*a*	1.1786	2.4171	2.9561	94.90	0.3536	0.2167	1.8707	94.30
		b	0.8709	0.8462	1.1620	95.20	0.0409	0.0205	0.5604	94.80
		c	1.7117	3.1252	4.7325	95.00	0.3199	0.8961	3.4942	94.90
200	1	*a*	0.3543	0.2405	1.3299	94.04	0.0747	0.0382	0.7079	95.72
		b	0.0312	0.0076	0.3188	94.57	0.0130	0.0041	0.2448	95.30
		c	0.4241	0.2800	1.2411	94.88	0.1264	0.1063	1.1788	94.25
	0.8	*a*	0.5073	0.2971	0.7819	94.20	0.0288	0.1875	1.6945	94.80
		b	0.4218	0.2080	0.6810	95.20	0.0225	0.0116	0.4226	96.60
		c	0.6349	0.7544	4.3247	96.30	0.2458	0.7522	3.2620	94.50
	0.6	*a*	1.1719	1.3991	2.8285	94.00	0.0426	0.1890	1.7006	94.00
		b	0.8604	0.8192	1.1018	94.70	0.0262	0.0177	0.5210	95.30
		c	1.7167	3.1209	3.6346	95.20	0.2731	0.7302	3.2756	94.80

Table 5: AB, MSE, AL and CP of ML and MPS estimates under complete and TIIC in Case 4

$a = 3, b = 2, c = 0.5, s = 3$										
n	***k***		**ML**				**MPS**			
			AB	**MSE**	**AL**	**CP**	**AB**	**MSE**	**AL**	**CP**
50	1	*a*	0.2068	0.0679	2.7121	95.5	0.1499	0.0503	1.7183	97.5
		b	0.0527	0.2058	1.7673	95.4	0.1599	0.2037	1.3003	94.0
		c	0.2001	0.1157	2.8500	94.3	0.0635	0.1086	1.2681	95.4
		s	0.1412	0.1069	5.1009	96.0	0.2146	0.0975	3.2931	95.6
	0.8	*a*	0.2503	0.0816	3.5403	96.9	0.2126	0.0760	2.9112	97.3
		b	0.4026	0.2854	1.3773	94.9	0.3161	0.2720	1.0887	95.3
		c	0.3788	0.1446	1.1298	93.7	0.0792	0.0876	1.1186	95.4
		s	0.1522	0.1112	3.1635	94.6	0.0419	0.1539	2.8750	95.8
	0.6	*a*	0.0590	0.9140	3.4013	93.3	0.2489	0.6006	2.8783	96.9
		b	0.5950	0.5000	3.4995	94.0	0.3484	0.6936	2.9667	95.3
		c	0.4460	0.1990	0.0604	94.6	0.0701	0.0878	0.0294	94.8
		s	0.2020	0.1070	1.0084	93.6	0.0973	0.4446	2.5872	95.6
100	1	*a*	0.7795	0.0697	2.3446	96.5	0.1595	0.0493	2.0764	95.7
		b	0.0661	0.0718	1.0182	96.7	0.1138	0.0685	1.5467	95.9
		c	0.1138	0.1239	1.8636	95.8	0.0285	0.0399	0.7750	95.9
		s	0.1410	0.0905	1.7579	97.0	0.1376	0.0838	2.3713	95.8

a = 3, *b* = 2, *c* = 0.5, *s* = 3										
n	**k**		**ML**				**MPS**			
			AB	**MSE**	**AL**	**CP**	**AB**	**MSE**	**AL**	**CP**
	0.8	*a*	0.2601	0.0795	0.5201	97.1	0.2096	0.0638	1.2710	96.0
		b	0.3846	0.2149	1.0154	95.1	0.2325	0.2034	1.0921	94.7
		c	0.3777	0.1433	0.2975	95.3	0.0562	0.0410	0.1763	95.5
		s	0.1862	0.0974	0.9004	95.1	0.0871	0.3363	2.2487	95.6
	0.6	*a*	0.0710	0.0920	0.3337	94.1	0.2313	0.0836	0.9165	94.3
		b	0.5560	0.3920	1.1224	94.8	0.2531	0.3866	2.2273	95.2
		c	0.4450	0.1980	0.0473	93.4	0.0528	0.1383	0.7395	95.8
		s	0.1650	0.0980	0.7905	94.6	0.1279	0.1024	1.8648	95.3
200	1	*a*	0.7669	0.0754	1.5999	96.3	0.0793	0.0815	1.4889	95.5
		b	0.1500	0.0474	0.6188	94.8	0.0416	0.0471	1.0327	94.6
		c	0.0521	0.0989	1.2164	94.3	0.0123	0.0201	0.5534	95.2
		s	0.1362	0.0362	2.7944	96.3	0.1157	0.0321	1.7551	94.5
	0.8	*a*	0.2729	0.0777	0.2209	96.1	0.1090	0.2027	1.7131	95.1
		b	0.3588	0.1567	0.6565	94.6	0.1099	0.1361	1.3812	94.9
		c	0.3763	0.1419	0.0680	94.9	0.0334	0.0194	0.5299	94.6
		s	0.1881	0.0535	0.5281	95.5	0.0366	0.1844	1.6779	95.8
	0.6	*a*	0.0720	0.0900	0.2365	94.3	0.1074	0.2140	1.7650	94.4
		b	0.5660	0.3610	0.7910	95.1	0.1156	0.1581	1.4924	95.2
		c	0.4460	0.1825	0.0349	95.5	0.0258	0.0172	0.5050	95.2
		s	0.1720	0.0555	0.5661	95.2	0.0393	0.1618	1.5698	95.3

From the numerical outcomes listed in Tables 2–5 we conclude the following

- The MSE of the MPS estimates of *a* is less than the corresponding of the ML estimates based on complete and TIIC in all cases (see for example Fig. 4 and Fig. 5).

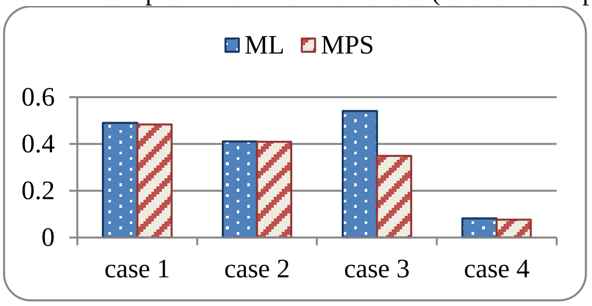


Fig. 4: MSE of the ML and MPS of *a* estimates when $k = 0.8$ at $n = 50$

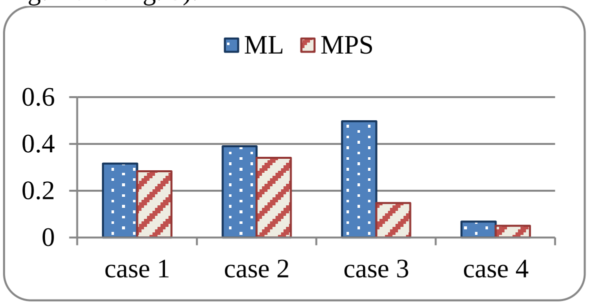


Fig. 5: MSE of the ML and MPS of *a* estimates under complete sample at $n = 50$

- The MSE for MPS and ML estimates of *c* decreases as the value of *k* increases in Case 2 (ABPLo distribution) at $n = 200$. The MSE of the ML estimate of *c* is greater than the corresponding MSE of the MPS estimate of *c* (see Fig. 6).
- Fig. 7 demonstrates that the CP for the MPS and ML estimates for *a* increases as the value of *k* increases. Also, the CP of the MPS estimates is greater than the corresponding CP of the ML estimate in Case 1 (LBPLo distribution) at $n = 50$.

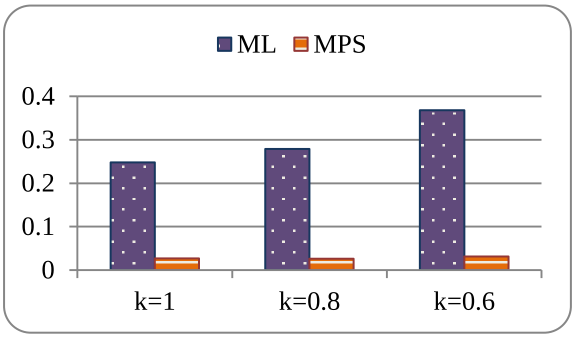


Fig. 6: MSE of the ML and MPS estimate of *c* at $n = 200$ in Case 2(ABPLo distribution)

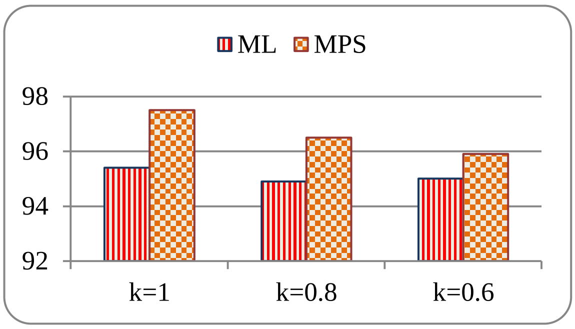


Fig. 7: The CP of *a* estimates at $n = 50$ in Case 1 (LBPLo distribution)

- Fig. 8 shows that the CP of the ML estimate of c increases as the value of k increases in Case 1 (LBPLo distribution) at $n = 100$.
- The MSE for the ML estimate of b in Case 1 (LBPLo distribution) decreases as n increases. Also it shows that as the value of k increases, the MSE for the ML estimate of b decreases for all sample sizes (see Fig. 9).

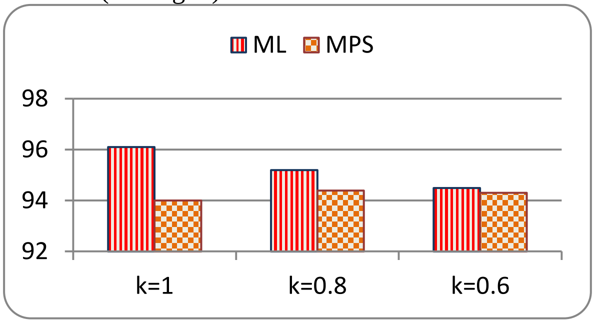


Fig. 8: The CP of c when $n = 100$ in Case 1 (LBPLo distribution)

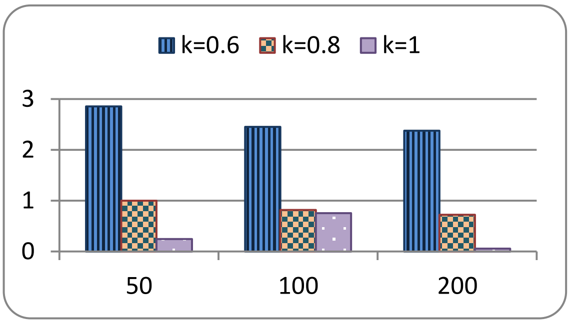


Fig. 9: MSE for the ML estimate of b in Case 1 (LBPLo distribution)

- The MSE of the MPS estimate of a in Case 1 (LBPLo distribution) decreases as n increases. Also as the value of k increases, the MSE for the MPS estimate of a decreases for all n as shown in Fig. 10.

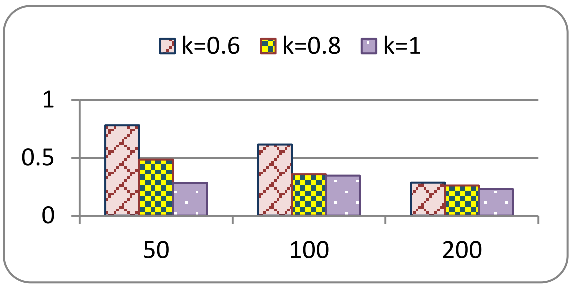


Fig. 10: MSE for the MPS estimate of parameter a in Case 1 (LBPLo distribution)

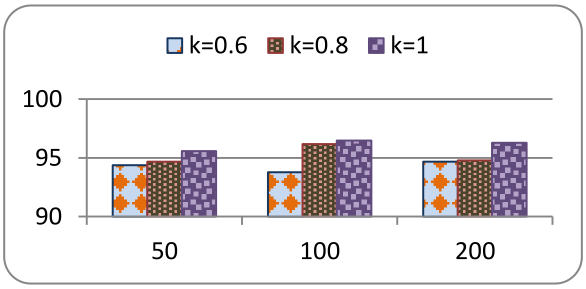


Fig. 11: CP for ML estimate of parameter c in Case 2 (ABPLo distribution)

- Fig. 11 illustrates that the CP for ML estimate of parameter c in Case 2 (ABPLo distribution) increases as n increases. Also it shows that as the value of k increases, the CP for the ML estimate of a increases for all n.
- Generally, the MPS of all parameters are preferable than the corresponding ML estimates in almost most of the situations. As the censoring level; k and sample size increase the MSE and AL of all estimates decrease.
- The CP is very close to the considered significance level for all sample size.

6. Application to Real Data

To demonstrate the adequacy of one special model of the WPLo distribution, namely; LBPLo distribution, is done using the beta Lomax (**BLo**) distribution, Weibull Lomax (**WLo**) distribution Kumaraswamy Lomax (**KLo**) distribution, exponentiated generalized Lomax (**EGLo**), gamma Lomax (**GLo**), log gamma Lomax (**LGaLo**), exponentiated Lomax (**ELo**) and generalized exponential (**GE**). The following criteria are utilized to detect the distribution with the best fit: negative log-likelihood (−LL) value, Akaike information criteria (**AIC**), Bayesian information criteria (**BIC**), consistent AIC (**AICC**), and Hannan and Quinn information criteria (**HQIC**).
The first data relating to the strengths of 1.5 cm glass fibres which was obtained by workers at the UK National Physical Laboratory are used. The data have previously been used by Smith and Naylor (1987), Merovci *et al.* (2016), Oguntunde *et al.* (2017), Khaleel *et al.* (2018) and Oguntunde *et al.* (2018a); (2018b). The first observations are as follows:

0.55	0.74	0.77	0.81	0.84	1.24	0.93	1.04	1.11	1.13	1.30	1.25
1.27	1.28	1.29	1.48	1.36	1.39	1.42	1.48	1.51	1.49	1.49	1.50
1.50	1.55	1.52	1.53	1.54	1.55	1.61	1.58	1.59	1.60	1.61	1.63
1.61	1.61	1.62	1.62	1.67	1.64	1.66	1.66	1.66	1.70	1.68	1.68
1.69	1.70	1.78	1.73	1.76	1.76	1.77	1.89	1.81	1.82	1.84	1.84
2.00	2.01	2.24.									

The second civil engineering data consisting of 85 hailing times observations Kotz and van Dorp (2004). The second observations are as follows

4.79	4.75	5.40	4.70	6.50	5.30	6.00	5.90	4.80	6.70	6.00	4.95
7.90	5.40	3.50	4.54	6.90	5.80	5.40	5.70	8.00	5.40	5.60	7.50
7.00	4.60	3.20	3.90	5.90	3.40	5.20	5.90	4.40	5.20	7.40	5.70
6.00	3.60	6.20	5.70	5.80	5.90	6.00	5.15	6.00	4.82	5.90	6.00
7.30	7.10	4.73	5.90	3.60	6.30	7.00	5.10	6.00	6.60	4.40	6.80
5.60	5.90	5.90	8.60	6.00	5.80	5.40	6.50	4.80	6.40	4.15	4.90
6.50	8.20	7.00	8.50	5.90	4.40	5.80	4.30	5.10	5.90	4.70	3.50
6.80.											

The performances of the LBPLo distribution with the other competing distributions are shown in Tables 6 and 7.

Table 6: The performances of the LBPLo model with some competing distributions

Model	-LL	AIC	AICC	BIC	HQIC
LBPLo	14.979	37.95	38.648	46.531	41.33
Blo	24.017	56.0355	56.726	64.608	59.411
KuLo	16.338	40.676	41.365	49.248	44.042
EGLo	31.502	71.005	71.698	79.578	74.377
GLo	26.99	59.98	60.387	66.409	62.509
LgaLo	30.289	68.574	69.265	77.152	71.95
Elo	31.456	68.912	69.316	75.345	71.441
GE	31.384	66.766	66.967	71.054	68.458

Table 7: The performances of the LBPLo model with some competing distributions

Model	-LL	AIC	AICC	BIC	HQIC
LBPLo	130.5025	269.005	269.505	278.776	272.935
Blo	133.3107	274.621	275.121	284.392	278.554
KuLo	132.7561	273.512	274.012	283.232	277.442
EGLo	146.6422	301.284	301.784	311.055	305.214
GLo	138.3625	282.724	283.021	290.053	285.673
LgaLo	159.2354	326.47	326.979	336.241	330.4
Elo	137.4985	280.997	281.293	288.325	283.944
GE	137.3389	278.677	278.824	283.563	280.642

Finally, in order to assess whether the proposed model is appropriate for the above mentioned data, we display the visualization of the estimated **pdfs** in Figures 12–13. As seen we suggest that the fit of the **LBPLo** model performs better than the other competitive distributions.

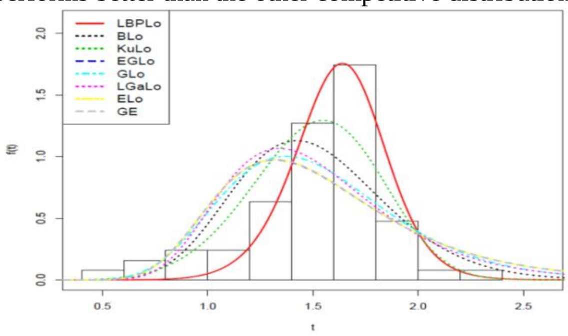


Fig. 12: Estimated pdfs for the first data

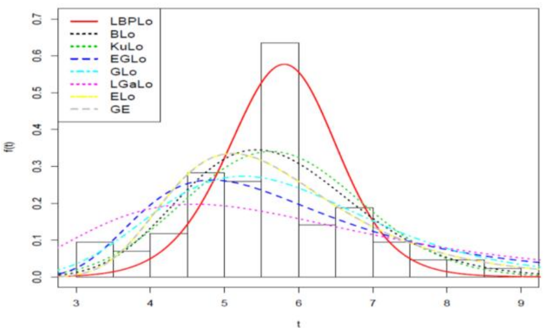


Fig. 13: Estimated pdfs for the second data

7. Concludıng Remarks

In this paper, we propose a new weighted distribution related to power Lomax model, named as weighted power Lomax distribution. The new model contains some new distributions besides it contains existing distribution. Main properties of the weighted power Lomax distribution are discussed. Based on complete and Type II censoring samples, the point and approximate confidence intervals of parameters are derived depend on maximum likelihood and maximum product spacing procedures. Due to the complicated forms of the non-linear equations, the Monte Carlo simulation study is done to assess the behaviour of the estimates for different sample sizes. Based on simulation study, we conclude that the maximum product spacing estimates are preferable than the maximum likelihood estimates in approximately most of the situations. Finally, for illustrative purpose, a real life data set is analyzed and compared with other lifetime distributions.

Conflict of Interest

The authors declare that they have no conflict of interest.

Acknowledgments

The authors are grateful to the editor and referee for their useful comments and suggestions for improving the paper which undoubtedly enhanced the presentation and quality of the article.

References

1. Abdul-Moniem, I. and Abdel-Hameed, H. (2012). A lifetime distribution with decreasing failure rate. *International Journal of Mathematical Education*, *33*(5), 1-7.
2. Ahmad, A., Ahmad, S. and Ahmed, A. (2016). Length-biased weighted Lomax distribution: statistical properties and application. *Pakistan Journal of Statistics and Operation Research, 12*(2), 245-255.
3. Atkinson, A. B. and Harrison, A. J. (1978). *Distribution of Personal Wealth in Britain*. Cambridge University Press.
4. Bantan, R., Hassan, A. S. and Elsehetry, M. (2020). Zubair Lomax distribution: properties and estimation based on ranked set sampling. *CMC-Computer, Materials and Continua, 65*(3), 2169-2187.
5. Cheng, R. and Amin, N. (1979). Maximum product of spacings estimation with application to the lognormal distribution (Mathematical Report 79-1). *Cardiff: University of Wales Institute of Science and Technology*.
6. Corbellini, A., Crosato, L., Ganugi, P. and Mazzoli, M. (2010). Fitting Pareto II distributions on firm size: Statistical methodology and economic puzzles. *In Skiadas C. (eds) Advances in Data Analysis.Statistics for Industry and Technology* (pp. 321-328): Birkhäuser Boston.
7. Cordeiro, G. M., Nadarajah, S. and Ortega, E. M. M. (2013). General results for the beta Weibull distribution. *Journal of Statistical Computation and Simulation, 83*(6), 1082-1114.
8. Gupta, R. C. and Keating, J. P. (1986). Relations for reliability measures under length biased sampling. *Scandinavian Journal of Statistics*, 49-56.
9. Harris, C. M. (1968). The Pareto distribution as a queue service discipline. *Operations Research*, *16*(2), 307-313.
10. Hassan, A. S. and Abd-Allah, M. (2018). Exponentiated Weibull-Lomax distribution: properties and estimation. *Journal of Data Science*, *16*(2), 277-298.
11. Hassan, A. S. and Abd-Allah, M. (2019). On the inverse power Lomax distribution. *Annals of Data Science*, *6*(2), 259-278.
12. Hassan, A. S. and Abdelghafar, M. A. (2017). Exponentiated Lomax geometric distribution: properties and applications. *Pakistan Journal of Statistics and Operation Research, 13*(3), 545-566.
13. Hassan, A. S. and Al-Ghamdi, A. (2009). Optimum step stress accelerated life testing for Lomax distribution. *Journal of Applied Sciences Research*, *5*(12), 2153-2164.
14. Hassan, A. S. and Mohamed, R. E. (2019a). Parameter estimation for inverted exponentiated Lomax distribution with right censored data. *Gazi University Journal of Science*, *32*(4), 1370-1386.
15. Hassan, A. S. and Mohamed, R. E. (2019b). Weibull inverse Lomax distribution. *Pakistan Journal of Statistics and Operation Research*, *15*(3), 587-603.
16. Hassan, A. S. and Nassr, S. G. (2018). Power Lomax Poisson distribution: properties and estimation. *Journal of Data Science*, *18*(1), 105-128.
17. Hassan, A. S., Assar, S. M. and Shelbaia, A. (2016). Optimum step-stress accelerated life test plan for Lomax distribution with an adaptive type-II progressive hybrid censoring. *Journal of Advances in Mathematics and Computer Science*, *13*(2), 1-19.
18. Hassan, A. S., Elgarhy, M. and Mohamed, R. E. (2020a). Statistical properties and estimation of type II half logistic Lomax distribution. *Thailand Statistician*, *18*(3), 290-305.
19. Hassan, A. S., Sabry, M. A. and Elsehetry, A. M. (2020b). Truncated power Lomax distribution with application to flood data. *Journal of Statistics Applications and Probability*, *9*(2), 347-359.

20. Holland, O., Golaup, A. and Aghvami, A. (2006). Traffic characteristics of aggregated module downloads for mobile terminal reconfiguration. *IEE Proceedings-Communications*, *153*(5), 683-690.
21. Khaleel, M. A., Ibrahim, N. A., Shitan, M. and Merovci, F. (2018). New extension of Burr type X distribution properties with application. *Journal of King Saud University-Science*, *30*(4), 450-457.
22. Kotz, S. Y. and van Dorp, J. R. (2004). *Beyond Beta: Other Continuous Families of Distributions with Bounded Support and Applications*. Singapore: World Scientific.
23. Lemonte, A. J. and Cordeiro, G. M. (2013). An extended Lomax distribution. *Statistics*, *47*(4), 800-816.
24. Merovci, F., Khaleel, M. A., Ibrahim, N. A. and Shitan, M. (2016). The beta Burr type X distribution properties with application. *SpringerPlus*, *5*(1), 697.
25. Oguntunde, P. E., Khaleel, M. A., Adejumo, A. O. and Okagbue, H. I. (2018a). A study of an extension of the exponential distribution using logistic-x family of distributions. *International Journal of Engineering & Technology*, *7*(4), 5467-5471.
26. Oguntunde, P. E., Khaleel, M. A., Ahmed, M. T., Adejumo, A. O. and Odetunmibi, O. A. (2017). A new generalization of the Lomax distribution with increasing, decreasing, and constant failure rate. *Modelling and Simulation in Engineering*, 1-7.
27. Oguntunde, P. E., Khaleel, M. A., Adejumo, A. O., Okagbue, H. I., Opanuga, A. A. and Owolabi, F. O. (2018b). The Gompertz Inverse Exponential (GoIE) distribution with applications. *Cogent Mathematics & Statistics*, *5*(1), 1507122.
28. Oluyede, B. O. (1999). On inequalities and selection of experiments for length biased distributions. *Probability in the Engineering and Informational Sciences*, *13*(2), 169-185.
29. Patil, G. P. and Ord, J. (1976). On size-biased sampling and related form-invariant weighted distributions. *Sankhyā: The Indian Journal of Statistics, Series B*, *38*, 48-61.
30. Patil, G. P. and Rao, C. R. (1978). Weighted distributions and size-biased sampling with applications to wildlife populations and human families. *Biometrics*, *34*, 179-189.
31. Rady, E.-H. A., Hassanein, W. and Elhaddad, T. (2016). The power Lomax distribution with an application to bladder cancer data. *SpringerPlus*, *5*(1), 1838.
32. Ranneby, B. (1984). The maximum spacing method. An estimation method related to the maximum likelihood method. *Scandinavian Journal of Statistics*, 93-112.
33. Shaked, M. and Shanthikumar, J. (2007). *Stochastic Orders*. New York: Springer.
34. Smith, R. L. and Naylor, J. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, *36*(3), 358-369.
35. Tahir, M. H., Cordeiro, G. M., Mansoor, M. and Zubair, M. (2015). The Weibull-Lomax distribution: properties and applications. *Hacettepe Journal of Mathematics and Statistics*, *44*(2), 461-480.
36. Tahir, M. H., Hussain, M. A., Cordeiro, G. M., Hamedani, G., Mansoor, M. and Zubair, M. (2016). The Gumbel-Lomax distribution: properties and applications. *Journal of Statistical Theory and Applications*, *15*(1), 61-79.
37. Zelen, M. and Feinleib, M. (1969). On the theory of screening for chronic diseases. *Biometrika*, *56*(3), 601-614.