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On The New Modified Burr XII Distribution: Development, Properties, Characterizations and Applications

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Abstract

A new distribution with flexible hazard rate function is introduced which is called new modified Burr XII (NMBXII) distribution. The proposed distribution is derived from the T-X family technique and compounding the generalized Nadarajah–Haghighi (GNH) and gamma distributions. We highlighted the shapes of NMBXII density and failure rate functions. The density function of NMBXII model can take shapes such as J, reverse J, positively skewed and symmetrical. The proposed model can produce almost all types of failure rates such as increasing, decreasing, increasing-decreasing, decreasing-increasing, bimodal, inverted bathtub and modified bathtub. To show the importance of the proposed distribution, we established various mathematical properties such as quantiles, moments, incomplete moments, inequality measures, residual life functions and reliability measures theoretically. We have characterized the NMBXII distribution via two techniques. We addressed the maximum likelihood estimation technique for model parameters. The precision of the MLEs is estimated via a simulation study. We have considered three real data sets for applications to demonstrate the potentiality and utility of the NMBXII model. Then, we have established empirically that the proposed model is suitable for tax revenue, time periods between successive earthquakes and flood discharges applications. Finally, various model selection criteria, the goodness of fit statistics and graphical tools were used to examine the adequacy of the NMBXII distribution.

Key Words: Moments; Reliability; Characterizations; Maximum Likelihood Estimation.

1. Introduction

In recent decades, many continuous distributions have been introduced in statistical literature. These distributions, however, are not flexible enough to be suitable for the data sets from survival analysis, life testing, reliability, finance, environmental sciences, biometry, hydrology, ecology and geology. Hence, the applications of the generalized models to these fields are clear requisite. Generalization of a distribution is the only way to increase the applicability of the parent distribution. The generalizations are derived either by inserting a shape parameter or by transforming into the parent distribution. So, the generalized distributions will be more suitable than the competing model and sub-models.

Burr (1942) suggested 12 types of distributions as Burr family to fit cumulative frequency functions on frequency data. Burr distribution type -XII (BXII) distribution has wide applications in modeling insurance data in finance and Business and failure time data in reliability, survival analysis and acceptance sampling plans, since the empirical approaches to real data are often non-monotone hazard rate function (hrf) such as three-parameter Burr XII (BXII) three-parameter BXII (Okasha and Matter; 2015).

The probability density function (pdf) of BXII has unimodal or decreasing shaped as well as monotone hrf. However, these properties are inadequate; since the empirical approaches to real data are often non-monotone hrf shapes such as inverted bathtub hazard rate, bathtub, and various shaped specifically in the lifetime applications. Thus, various modified, extended and generalized forms of BXII distribution with extra shape and scale parameters are available in the literature such as BXII (Takahasi; 1965), extended three-parameter BXII (Shao et al.; 2004), six-parameter generalized BXII (Olapade; 2008), beta BXII (Paranaíba et al.; 2011), extended BXII (Usta; 2013), Kumaraswamy BXII (Paranaíba et al.; 2013), BXII geometric (Korkmaz and Erişoğlu, 2014), BXII power series (Silva and Cordeiro; 2015), three-parameter BXII Distribution (Thupeng; 2016), BXII-Poisson (Muhammad; 2016), extensions of the BXII (Cadena; 2017), new extended BXII (Ghosh and Bourguignon; 2017), BXII (Kumar; 2017), BXII modified Weibull (Mdlongwa et al.; 2017), BXII (Kayal et al.;2017), five-parameter BXII (Mead and Afify; 2017), new BXII distribution (Yari and Tondpour; 2017), four-parameter BXII (Afify et al.; 2018), BXII system of densities (Cordeiro et al.;2018), Odd Lindley BXII (Abouelmagd et al., 2018 and Korkmaz et al., 2018), BXII (Gunasekera;2018), Modified log BXII (Bhatti et al.;2018), BXII (Chiang et al.;2018), BXII (Chen and Singh;2018) and BXII (Keighley et al.;2018).

This study focuses on the following motivations: (i) to generate distributions with symmetrical, left-skewed, right-skewed, J and reverse-J shaped as well as high kurtosis; (ii) to have monotone and non-monotone failure rate function; (iii) to study numerically descriptive measures for the NMBXII distribution based on parameter values; (iv) to derive mathematical properties such as random number generator, sub-models, ordinary moments, incomplete moments, inequality measures, residual life functions, reliability measures and characterizations; (v) to estimate the precision of the maximum likelihood estimators via a simulation study; (vi) to reveal the potentiality and utility of the NMBXII model; (vii) to work as the preeminent substitute model to other existing models to discover and model the real data in economics (tax revenue’s data), geology (time periods between successive earthquakes) and hydrology (flood discharges); and life testing analysis and new fileds of research; (viii) to deliver better fits model than the existing models; and (ix) to infer empirically from goodness of fit statistics (GOFs) and graphical tools.

The content of the article is structured as follows: In Section 2, the NMBXII model is derived on the basis of (i) T-X technique and (ii) compounding the generalized Nadarajah–Haghighi (GNH) and gamma distributions. We study basic structural properties, random number generator and sub-models for the NMBXII model. We highlight the nature of the density and failure rate functions. Section 3 presents certain mathematical properties such as the ordinary moments, incomplete moments, inequality measures, residual and reverse residual life function, stress-strength reliability and multicomponent stress-strength reliability measures. Section 4 characterizes the NMBXII distribution via (i) conditional expectation and (ii) truncated moment. Section 5, addresses the MLE (maximum likelihood estimation) for the NMBXII model. In Section 6, we evaluate the precision of the MLEs via a simulation study. In Section 7, three real data sets for applications are considered to illustrate the potentiality and utility of the NMBXII model. We test the competency of the NMBXII distribution using various model selection criteria, goodness of fit statistics (GOFs) and graphical tools. The concluding remarks are given in Section 8.

2. The NMBXII Distribution

In this section, the NMBXII distribution is derived from (i) T-X family technique and (ii) compounding the generalized Nadarajah–Haghighi (GNH) and gamma distributions. We present the basic structural properties. Then, we highlight the nature of the density and failure rate functions.

2.1 T-X Family Technique

The cumulative distribution function (cdf) of the Burr XII (BXII) distribution is

$$G(x;\beta,\kappa)=1-\left(1+x^{\beta}\right)^{-\kappa}, \qquad x\geq 0, \beta>0, \kappa>0 \tag{1}$$

The odds ratio for the BXII random variable (r.v.) X is

$$W\left(G(x,\beta,\kappa)\right)=\frac{G(x;\beta,\kappa)}{1-G(x;\beta,\kappa)}=\frac{1-\left(1+x^{\beta}\right)^{-\kappa}}{\left(1+x^{\beta}\right)^{-\kappa}}=\left(1+x^{\beta}\right)^{\kappa}-1\cdot \qquad (2)$$

To obtain a wider family of distributions, Alzaatreh et al. (2016) derived the cdf for the T-*X* family as follows:

$$F\left(x\right)=\int_{a}^{W\left[G(x;\xi)\right]}r(t)\,dt, \qquad x\in\mathbb{R}, \qquad (3)$$

where $r\left(t\right)$ is the pdf of the r.v. *T*, where $T\in\left[a,\ \mathrm{b}\right]$ for $-\infty\leq a<\mathrm{b}<\infty$, $W\left[G\left(x;\xi\right)\right]$ is a function of the baseline cdf of a r.v. *X*, subject to the vector parameter ξ and satisfying

i) $W\left[G\left(x;\xi\right)\right]\in\left[a,\ \mathrm{b}\right]$,

ii) $W\left[G\left(x;\xi\right)\right]$ is differentiable and monotonically non-decreasing and

iii) $\lim_{x\to-\infty}W\left[G\left(x;\xi\right)\right]=a$ and $\lim_{x\to\infty}\left[G\left(x;\xi\right)\right]=b$.

For the T-*X* family of distributions, the pdf of X is given by

$$f\left(x\right)=\left\{\frac{\partial}{\partial x}W\left[G\left(x;\xi\right)\right]\right\}r\left\{W\left[G\left(x;\xi\right)\right]\right\}, \qquad x\in\mathbb{R} . \qquad (4)$$

We derive the NMBXII distribution via the T-X family technique by setting $r\left(t\right)=\alpha\eta t^{\eta-1}\left(1+\gamma t^{\eta}\right)^{-\frac{\alpha}{\gamma}-1},t>0,\alpha,\gamma,\eta>0$ and $W\left[G\left(x;\xi\right)\right]=\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]$. Then, the cdf of the NMBXII distribution is

$$F\left(x\right)=1-\left\{1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right\}^{-\frac{\alpha}{\gamma}},x\geq0, \qquad (5)$$

where $\alpha,\ \beta,\gamma,\kappa,\eta>0$ are parameters. The pdf corresponding to (5) is given by

$$f\left(x\right)=\alpha\beta\kappa\eta x^{\beta-1}\left(1+x^{\beta}\right)^{\kappa-1}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta-1}\left\{1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right\}^{-\frac{\alpha}{\gamma}-1},x>0. \qquad (6)$$

In future, a r.v. with pdf (6) is denoted by X~ NMBXII $\left(\alpha,\beta,\gamma,\kappa,\eta\right)$. The NMBXII model is also well-known as modified Burr XII Burr XII (MBXII-BXII) distribution.

2.1 Compounding Generalized Nadarajah–Haghighi (GNH) and Gamma Distributions

Here, we derive the NMBXII distribution through compounding generalized Nadarajah–Haghighi (GNH) and gamma distributions.

Lemma (i). If $Y|\beta,\kappa,\eta,\theta$ ~GNH(β,κ,η,θ) i.e. $g\left(y|\beta,\kappa,\eta,\theta\right)$ and $\theta|\alpha,\gamma$ ~gamma $\left(\theta;\alpha,\gamma\right)$,i.e. $g\left(\theta|\alpha,\gamma\right)$ then integrating the effect of θ with the help of $f\left(y,\alpha,\beta,\gamma,\kappa,\eta\right)=\int_{0}^{\infty}g\left(y|\beta,\kappa,\eta,\theta\right)g\left(\theta|\alpha,\gamma\right)d\theta$, we have Y~NMBXII$\left(\alpha,\beta,\gamma,\kappa,\eta\right)$.

2.2 Basic Structural Properties

For X~NMBXII$(\alpha,\beta,\gamma,\kappa,\eta)$, the survival, hazard, reverse hazard and cumulative hazard functions, the Mills ratio and elasticity of X are given respectively by

$$S(x)=\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-\frac{\alpha}{\gamma}},\quad x\geq 0, \tag{7}$$

$$h(x)=\alpha\beta\kappa\eta x^{\beta-1}\left(1+x^{\beta}\right)^{\kappa-1}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta-1}\left\{1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right\}^{-1}, \tag{8}$$

$$r(x)=\frac{d}{dx}\ln\left\{1-\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-\frac{\alpha}{\gamma}}\right\}, \tag{9}$$

$$H(x)=\frac{\alpha}{\gamma}\ln\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right],\quad x>0, \tag{10}$$

$$m(x)=\frac{1}{\alpha\beta\kappa\eta}x^{-\beta+1}\left(1+x^{\beta}\right)^{-\kappa+1}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{-\eta+1}\left\{1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right\}, \tag{11}$$

and

$$e(x)=\frac{d}{d\ln x}\ln\left\{1-\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-\frac{\alpha}{\gamma}}\right\}. \tag{12}$$

The quantile function of the NMBXII distribution is

$$x_q=\left[\left(\left\{\frac{1}{\gamma}\left[\left(1-q\right)^{-\frac{\gamma}{\alpha}}-1\right]\right\}^{\frac{1}{\eta}}+1\right)^{\frac{1}{\kappa}}-1\right]^{\frac{1}{\beta}}. \tag{13}$$

The NMBXII r.v. generator is

$$X=\left\{\left[\left[\gamma^{-1}\left[\left(1-Z\right)^{-\frac{\gamma}{\alpha}}-1\right]\right]^{\frac{1}{\eta}}+1\right]^{\frac{1}{\kappa}}-1\right\}^{\frac{1}{\beta}}, \tag{14}$$

where the random variable Z has the uniform distribution on (0,1).

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Table 1: Sub-Models of the NMBXII Distribution

α	β	γ	κ	η	NMBXII(MBXII-BXII) distribution
α	β	1	κ	η	NBXII(BXII-BXII) distribution
α	β	γ	1	1	MBXII distribution
α	β	1	1	1	BXII distribution
α	1	1	1	1	Lomax distribution
1	β	1	1	1	Log-logistic distribution
α	β	$\gamma \to 0$	1	1	Weibull distribution
1	β	1	κ	η	Log-logistic-Power distribution
α	β	$\gamma \to 0$	κ	η	Weibull-BXII distribution
α	1	$\gamma \to 0$	κ	1	Nadarajah–Haghighi (NH) distribution
α	β	$\gamma \to 0$	κ	1	GNH distribution

2.3 Shapes of NMBXII Density and Hazard Rate Functions

The following graphs show that shapes of NMBXII density are J, reverse J, left-skewed, right-skewed and symmetrical (Fig. 1). The NMBXII distribution has increasing, decreasing, increasing-decreasing, decreasing-increasing, bimodal, inverted bathtub and modified bathtub hazard rate function (Fig. 2).

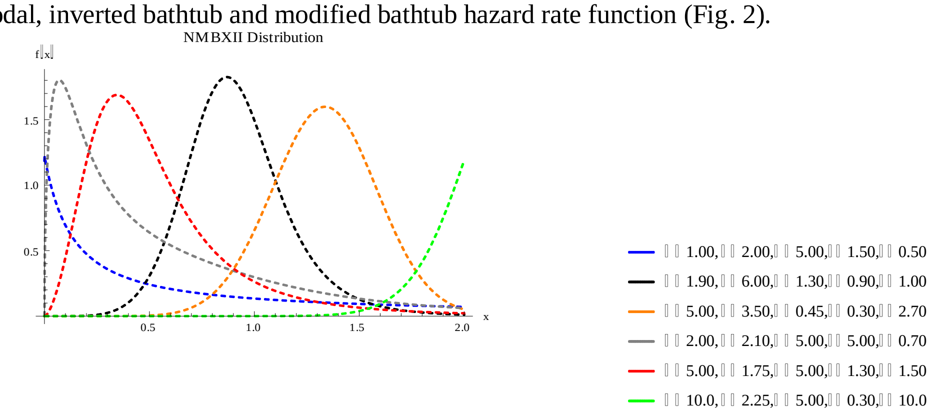


Fig. 1: Plots of pdf of NMBXII distribution

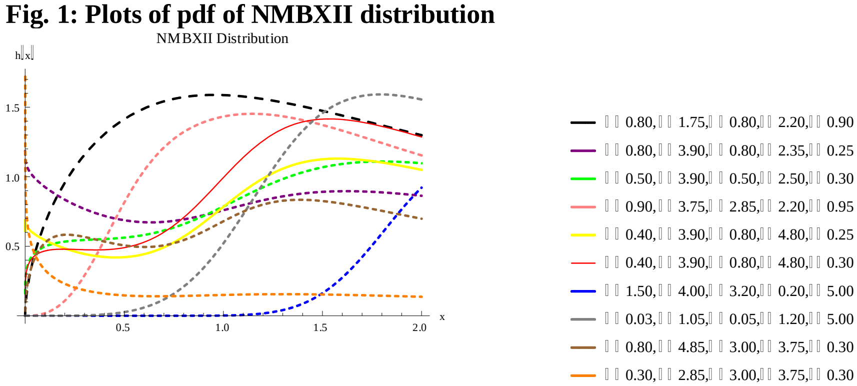


Fig. 2: Plots of hrf of NMBXII distribution

From Fig. 2, we see that new hrf has monotone hrf shapes as well as it has non monotone hrf shapes such as unimodal (upside-down bathtub) shaped, firstly bathtub then decreasing shaped and bi-modal (M) shaped. These properties are advantages of new model for data modeling.

3. Linear representation

In this subsection, we provide a useful linear representation for the density of X, which can be used to derive some mathematical properties of the NMBXII model. The cdf (5) can be expressed as

$$F_{\Theta}(x)=1-\underbrace{\left\{1+\gamma\left[\frac{1-\left(x^{\beta}+1\right)^{-\kappa}}{\left(x^{\beta}+1\right)^{-\kappa}}\right]^{\eta}\right\}^{-\frac{\alpha}{\gamma}}}_{A(x)}. \quad (15)$$

First, we shall consider the three power series

$$\left(1+c\right)^{-\beta}=\sum_{\zeta=0}^{\infty}\left(-1+c\right)^{\zeta}\left(\frac{1}{2}\right)^{\beta+\zeta}\binom{-\beta}{\zeta}, \quad (16)$$

$$\left(1-c\right)^{-\beta}=\sum_{j=0}^{\infty}\frac{\Gamma\left(\beta+j\right)}{j!\Gamma\left(\beta\right)}c^{j}\,|_{\left(|c|<1,\,\beta>0\right)}, \quad (17)$$

and the generalized binomial series given by

$$\left(1-c\right)^{\eta-1}=\sum_{r=0}^{\infty}\frac{(-1)^{r}\,\Gamma\left(\eta\right)}{r!\Gamma\left(\eta-r\right)}c^{r}\,|_{\left(|c|<1\ and\ \eta>0\ real\ non-integer\right)}. \quad (18)$$

Applying (16) for $A(x)$ in (15), we obtain

$$F_{\Theta}(x)=1-\sum_{\zeta=0}^{\infty}\left\{\gamma\left[\frac{1-\left(x^{\beta}+1\right)^{-\kappa}}{\left(x^{\beta}+1\right)^{-\kappa}}\right]^{\eta}-1\right\}^{\zeta}\left(\frac{1}{2}\right)^{\frac{\alpha}{\gamma}+\zeta}\binom{-\frac{\alpha}{\gamma}}{\zeta}.$$

Second, using the binomial expansion, the last equation can be expressed as

$$F_{\Theta}(x)=1-\sum_{\zeta=0}^{\infty}\sum_{i=0}^{\zeta}\frac{(-1)^{i}\,\gamma^{\zeta-i}\left(\frac{1}{2}\right)^{\frac{\alpha}{\gamma}+\zeta}}{\left[1-\left(x^{\beta}+1\right)^{-\kappa}\right]^{-\eta(\zeta-i)}}$$
$$\times\binom{\zeta}{i}\binom{-\frac{\alpha}{\gamma}}{\zeta}\underbrace{\left\{1-\left[1-\left(x^{\beta}+1\right)^{-\kappa}\right]\right\}^{-\eta(\zeta-i)}}_{B(x)}.$$

Third, applying (17) for $B(x)$ in the last equation, we can write

$$F_{\Theta}(x)=1-\sum_{j,\zeta=0}^{\infty}\sum_{i=0}^{\zeta}\tau_{i,j,\zeta}\,\Pi_{\eta(\zeta-i)+j}(x), \quad (19)$$

where

$$\Pi_{\eta(\zeta-i)+j}(x)=\left[1-\left(x^{\beta}+1\right)^{-\kappa}\right]^{\eta(\zeta-i)+j}$$

is the cdf of the exponentiated BXII with parameters $\beta,\kappa,\left[\left(\zeta-i\right)\eta+j\right]>0$ and

$$\tau_{i,j,\zeta}=\gamma^{\zeta-i}\left(\frac{1}{2}\right)^{\frac{\alpha}{\gamma}+\zeta}\frac{(-1)^{i}\,\Gamma\left[\left(\zeta-i\right)\eta+j\right]}{j!\Gamma\left[\left(\zeta-i\right)\eta\right]}\binom{\zeta}{i}\binom{-\frac{\alpha}{\gamma}}{\zeta}.$$

Upon differentiating (19) and applying (18), we obtain

$$f_{\Theta}(x)=\sum_{r=0}^{\infty}\zeta_{r}\,g_{\beta,\kappa(1+r)}(x), \tag{20}$$

where $g_{\beta,\kappa(1+r)}(x)$ is the BXII density with parameters $\beta,\ \kappa(1+r)$ and

$$\zeta_{r}=\frac{(-1)^{r+1}}{r!(1+r)\Gamma\left(\eta\left(\zeta-i\right)+j-r\right)}$$
$$\times\sum_{j,\zeta=0}^{\infty}\sum_{i=0}^{\zeta}\tau_{i,j,\zeta}\left(\eta\left(\zeta-i\right)+j\right)\Gamma\left(\eta\left(\zeta-i\right)+j\right)|_{(j+\zeta\geq 1)}.$$

Equation (20) reveals that the NMBXII density is a linear combination of BXII densities. So, some of its mathematical properties can be determined from those of the BXII distribution.

3. Moments

In this section, we derive theoretically ordinary moments, incomplete moments, inequality measures, residual and reverse residual life function, reliability measures and some other properties.

3.1 Moments about the Origin

The n^{th} ordinary moment of X is given by

$$\mu_{n}^{'}=E(X^{n})=\sum_{r=0}^{\infty}\zeta_{r}\int_{0}^{\infty}x^{n}\ g_{\beta,\kappa(1+r)}(x)dx. \tag{21}$$

or

$$\mu_{n}^{'}=E(X^{n})=\sum_{r=0}^{\infty}\zeta_{r}\kappa\left(1+r\right)B\left(\kappa\left(1+r\right)-\frac{n}{\beta},\frac{n}{\beta}+1\right)|_{\left(n<\beta\kappa(1+r)\right)},$$

where $B\left(.,.\right)$ is the beta function.

Setting $n=1$ in (21), we have the mean of X. The s^{th} central moment (M_s) and cumulants (κ_s) of X, are, respectively, given by

$$M_{s}=E(X-\mu_{1}^{'})^{s}=\sum_{i=0}^{s}(-1)^{i}\binom{s}{i}(\mu_{1}^{'})^{s}\mu_{s-i}^{'},$$

and

$$\kappa_{s}=\mu_{s}^{'}-\sum_{i=0}^{s-1}\binom{s-1}{i-1}\kappa_{r}\mu_{s-r}^{'},$$

where $\kappa_1=\mu_1'$.

Moment generating function

The moment generating function (MGF) of X, say $M_X(t)=E[exp(tX)]$, can be obtained from (20) as

$$M_X(t)=\sum_{r=0}^{\infty}\zeta_r M_{\beta,\kappa(1+r)}(t),$$

where $M_{\beta,\kappa(1+r)}(t)$ is the MGF of the BXII distribution with parameters β and $\kappa(1+r)$. Paranaíba et al. (2011) provided a simple representation for the MGF of the three-parameter BXII distribution. For $t<0$, we can write

$$M(t)=\beta\kappa\int_0^{\infty}\exp(yt)\,y^{\beta-1}\left(y^{\beta}+1\right)^{-\kappa-1}dy.$$

Next, we require the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x\,|\,\begin{matrix}\beta_1,\dots,\beta_p\\ \kappa_1,\dots,\kappa_q\end{matrix}\right)=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma\left(\kappa_j+t\right)\prod_{j=1}^{n}\Gamma\left(1-\beta_j-t\right)}{\prod_{j=n+1}^{p}\Gamma\left(\beta_j+t\right)\prod_{j=m+1}^{p}\Gamma\left(1-\kappa_j-t\right)}x^{-t}dt,$$

where $i=\sqrt{-1}$ is the complex unit and L denotes an integration path (Gradshteyn and Ryzhik (2000)). The Meijer **G**-function contains, as particular cases, many integrals with elementary and special functions (Prudnikov et al. (1986)). We now assume that $\beta\ =m/\kappa$, where m and κ are positive integers. This condition is not restrictive since every positive real number can be approximated by a rational number. We have the following result, which holds for m and k positive integers, $\mu\ >-1$ and $p>0$ (Prudnikov et al. (1992))

$$I\left(p,\mu,\frac{m}{\left(\frac{\alpha}{\gamma}\right)},v\right)|_0^{\infty}=\int_0^{\infty}\exp(-px)\,x^{\mu}\left(1+x^{\frac{m}{\kappa}}\right)^{v}dx$$

$$=\Upsilon_{(p,\mu,m,v)}G_{\kappa+m,\kappa}^{\kappa,\kappa+m}\left(\frac{m^m}{p^m}\,|\,\begin{matrix}\Delta(m,-\mu),\Delta(\kappa,v+1)\\ \Delta(\kappa,0)\end{matrix}\right),$$

where

$$\Upsilon_{(p,\mu,m,v)}=\frac{\kappa^{-v}m^{\mu+\frac{1}{2}}}{(2\pi)^{\frac{m-1}{2}}\Gamma(-v)\,p^{\mu+1}}$$

and

$$\Delta(\beta,\kappa)=\frac{\beta}{\kappa},\frac{\beta+1}{\kappa},\frac{\beta+2}{\kappa},\dots,\frac{\beta+\kappa}{\kappa}.$$

We can write (for $t<0$)

$$M(t)=m\times I\left(-t,\frac{m}{\kappa}-1,\frac{m}{\kappa},-\kappa-1\right),$$

where $I(.,.,.,.)$ is integral result due to Prudnikov et al. (1992).

Hence, the MGF of X can be expressed as

$$M_X(t)=m\times\sum_{r=0}^{\infty}\zeta_r I\left(-t,\frac{m}{\kappa(1+r)}-1,\frac{m}{\kappa(1+r)},-\left[1+\kappa(1+r)\right]\right).$$

3.2 Incomplete Moments

The s^{th} incomplete moment, say $\phi_s(t)$, of the NMBXII distribution is given by

$$\phi_s(t)=\int_0^t x^s f(x)dx.$$

From equation (20), we have

$$I_s(t)=\sum_{r=0}^{\infty}\zeta_r\int_0^t x^s g_{\beta,\kappa(1+r)}(x)dx,$$

and using the lower incomplete gamma function, we obtain

$$I_s(t)=\sum_{r=0}^{\infty}\zeta_r\kappa(1+r)B\left(t^{\eta};\kappa(1+r)-\frac{s}{\beta},\frac{s}{\beta}+1\right),$$

where $B(t,.,.)$ is the incomplete beta function.

The first incomplete moment of X, denoted by $\phi_1(t)$, is simply determined from the above equation by setting $s=1$. The first incomplete moment has important applications related to the Bonferroni and Lorenz curves and the mean residual life and the mean waiting time. Furthermore, the amount of scatter in a population is evidently measured, to some extent, by the totality of deviations from the mean and median. The mean deviations, about the mean and about the median of X, depend on $I_1(t)$.

3.3 Residual and reversed residual life functions

The n^{th} moment of the residual life (RL), denoted by

$$m_n(t)=E\left[(X-t)^n\right]|_{(X>t,\, n=1,2,\ldots)},$$

or

$$m_n(t)=\frac{1}{1-F_{\Theta}(t)}\int_t^{\infty}(x-t)^n dF_{\Theta}(x).$$

Then, we can write

$$m_n(t)=\frac{1}{1-F_{\Theta}(t)}\sum_{i=0}^{n}\sum_{r=0}^{\infty}\frac{(-1)^{n-i}n!t^{n-i}}{i!\Gamma(n-i+1)}\zeta_r \times\kappa(1+r)B\left(t^{\eta};\kappa(1+r)-\frac{i}{\beta},\frac{i}{\beta}+1\right).$$

The n^{th} moment reversed residual life, say

$$M_n(t)=E\left[(t-X)^n\right]|_{(X\le t,t>0\text{ and }n=1,2,\ldots,)},$$

or

$$M_n(t)=\frac{1}{F_{\Theta}(t)}\int_0^t(t-x)^n dF_{\Theta}(x).$$

The n^{th} moment of the reversed residual life of X

$$M_n(t)=\frac{1}{F_{\Theta}(t)}\sum_{i=0}^{n}\sum_{r=0}^{\infty}\frac{(-1)^i n!}{i!(n-i)!}\zeta_r\kappa(1+r)B\left(t^{\eta};\kappa(1+r)-\frac{i}{\beta},\frac{i}{\beta}+1\right).$$

The mean (μ_1'), median $(\tilde{\mu})$, standard deviation (σ), skewness (γ_1) and kurtosis (γ_2) for the NMBXII distribution for selected values of $\alpha,\beta,\gamma,\kappa$ and η are listed in Table 2. We also depict that the NMBXII model can be effective to model data sets in terms of the descriptive measures.

Table 2: μ_1', $\tilde{\mu}$, σ, γ_1 and γ_2 of the NMBXII Distribution

Parameters $\alpha,\beta,\gamma,\kappa,\eta$	μ_1'	$\tilde{\mu}$	σ	γ_1	γ_2
1.5,1.5,1.5,1.5,1.5	0.7527	0.5952	0.6708	8.0759	364.836
2.5,1.5,1.5,1.5,1.5	0.5197	0.4528	0.3338	2.2472	16.6231
5,1.5,1.5,1.5,1.5	0.3501	0.3227	0.187	1.0751	5.439
5,3.5,0.45,0.30,2.7	1.335	1.335	0.2511	0.0325	3.0441
5,3,0.3,0.3,3	1.4485	1.45	0.2864	0.0018	2.9692
5,5,0.5,0.5,3	1.0493	1.0573	0.1048	-0.4448	3.415
5,5,0.3,0.3,3	1.2428	1.2496	0.1501	-0.253	3.1207
5,3.5,0.3,0.3,3	1.3704	1.375	0.2336	-0.0859	2.9912
5,1.5,0.5,0.5,1.5	1.4794	1.4888	0.2973	-0.1383	2.9893
5,0.5,0.5,0.5,5	3.6247	3.298	2.0469	1.007	4.5925
5,1.5,0.5,1.5,1.5	0.547	0.5539	0.0834	-0.461	3.3298
5,5,0.5,0.5,5	1.1196	1.1268	0.0708	-0.6306	3.7676
5,5,5,5,5	0.6491	0.649	0.038	0.0285	4.0665

3.4 Stress-Strength Reliability of NMBXII Distribution

Let X_1 be strength, X_2 be stress, X_1~NMBXII$(\alpha_1,\beta,\gamma,\kappa,\eta)$ and X_2~NMBXII $(\alpha_2,\beta,\gamma,\kappa,\eta)$, then the reliability parameter (Kotz et al.; 2003) of a component is

$$\mathrm{R}=\Pr\left(\mathrm{X}_2<X_1\right)=\int_0^\infty f_{X_1}(x)F_{X_2}(x)\,dx\,,$$

$$=\int_0^\infty \alpha_1\beta\kappa\eta x^{\beta-1}\left(1+x^\beta\right)^{\kappa-1}\left[\left(1+x^\beta\right)^\kappa-1\right]^{\eta-1}\left\{1+\gamma\left[\left(1+x^\beta\right)^\kappa-1\right]^\eta\right\}^{-\frac{\alpha_1}{\gamma}-1}\left\{1-\left\{1+\gamma\left[\left(1+x^\beta\right)^\kappa-1\right]^\eta\right\}^{-\frac{\alpha_2}{\gamma}}\right\}dx,$$

$$=\frac{\alpha_2}{\alpha_1+\alpha_2}. \qquad (22)$$

Therefore R is independent of β,κ,γ *and* η.

3.5 Reliability Estimation of Multicomponent Stress-Strength model

Consider a system that has m identical components out of which *s* components are functioning. The strengths of *m* components are $\mathrm{X}_i, i=1,2...m$ with common cdf F while, the stress Y imposed on the components has cdf G. The strengths $\mathrm{X}_i's$ and stress Y are independently and identically distributed (i.i.d.). The probability that system operates properly is reliability of the system i.e.

$$R_{s,m}=\Pr\,[\textit{Strengths } (X_i,\ i=1,2,\dots,m) > \textit{stress } (Y)],$$

$= \Pr$ [*at least "s" of (X_i, i=1,2,…,m) exceed stress (Y)]*,

$$R_{s,m} = \sum_{l=s}^{m} \binom{m}{\ell} \int_0^{\infty} [1-F(y)]^l \, [F(y)]^{m-l} dG(y),$$ (Bhattacharyya and Johnson; 1974). (23)

Let X ~NMBXII$(\alpha_1, \beta, \gamma, \kappa, \eta)$ and Y ~NMBXII $(\alpha_2, \beta, \gamma, \kappa, \eta)$, with common parameters β, κ *and* γ and unknown shape parameters α_1 and α_2. The multicomponent stress- strength reliability for the NMBXII distribution is

$$R_{s,m} = \sum_{l=s}^{m} \binom{m}{l} \int_0^{\lambda} \left(\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa} -1 \right]^{\eta} \right]^{-\frac{\alpha_1}{\gamma}} \right)^{l} \left(1 - \left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa} -1 \right]^{\eta} \right]^{-\frac{\alpha_1}{\gamma}} \right)^{(m-l)} \times$$

$$\alpha_2 \beta\kappa\eta x^{\beta-1} \left(1+x^{\beta}\right)^{\kappa-1} \left[\left(1+x^{\beta}\right)^{\kappa} -1 \right]^{\eta-1} \left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa} -1 \right]^{\eta} \right]^{-\frac{\alpha_2}{\gamma}-1} dx.$$

Letting $u = \left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa} -1 \right]^{\eta} \right]^{-\frac{\alpha_2}{\gamma}}$, we obtain

We obtain $R_{s,m} = \sum_{\ell=s}^{m} \binom{m}{\ell} \int_0^1 \left(u^{\nu}\right)^{\ell} \left(1-u^{\nu}\right)^{(m-\ell)} du \quad where \ \ \nu = \frac{\alpha_1}{\alpha_2}.$

Let $u^{\nu} = w$, we have

$$R_{s,m} = \sum_{\ell=s}^{m} \binom{m}{i} \int_0^1 w^{\ell} \left(1-w\right)^{(m-\ell)} \frac{1}{\nu} w^{\frac{1}{\nu}-1} dw$$

$$R_{s,m} = \frac{1}{\nu} \sum_{\ell=s}^{m} \binom{m}{\ell} B\left(\ell + \frac{1}{\nu}, m-\ell+1 \right). \quad (24)$$

The probability in (24) is known as reliability of multicomponent stress-strength model.

4. Characterizations

In this section, we characterize the NMBXII distribution via: (i) conditional expectation and (ii) truncated moment.

We present our characterizations in two subsections.

4.1 Conditional Expectation

We characterize the NMBXII distribution via conditional expectation.

Proposition 4.1.1: Let $X : \Omega \to (0,\infty)$ be a continuous r.v. with cdf $F(x)$, then for $\alpha > \gamma$, X has cdf (5) if and only if

$$E\left[\left[\left(1+x^{\beta}\right)^{\kappa} -1 \right]^{\eta} \middle| X > t \right] = \frac{1}{(\alpha-\gamma)} \left\{ 1+\alpha\left[\left(1+t^{\beta}\right)^{\kappa} -1 \right]^{\eta} \right\}, \ \ t > 0. \quad (25)$$

Proof. If X has cdf (5), then

$$E\left[\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\middle|X>t\right]=\left[1-F(t)\right]^{-1}\int_t^{\infty}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}f(x)dx,$$

$$E\left[\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\middle|X>t\right]=\left[1-F(t)\right]^{-1}\int_t^{\infty}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\alpha\beta\kappa\eta x^{\beta-1}\left(1+x^{\beta}\right)^{\kappa-1}\times$$

$$\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta-1}\left\{1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right\}\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-\frac{\alpha}{\gamma}-1}dx.$$

Upon integration by parts and simplification, we arrive at

$$E\left[\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\middle|X>t\right]=\frac{1}{(\alpha-\gamma)}\left\{1+\alpha\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right\}, for\ \alpha>\gamma\ \text{and}\, t>0.$$

Conversely, if (25) holds, then

$$\frac{1}{\bar{F}(t)}\int_t^{\infty}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}f(x)dx=\frac{1}{(\alpha-\gamma)}\left\{1+\alpha\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}\right\},$$

$$\frac{1}{\bar{F}(t)}\int_t^{\infty}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}f(x)dx=\left\{\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}+\frac{1}{(\alpha-\gamma)}\left[1+\gamma\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]\right\}. \quad (26)$$

Differentiating (26) with respect to t, we obtain

$$-\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}f(t)=\begin{cases}\bar{F}(t)\left[\beta\kappa\eta x^{\beta-1}\left(1+x^{\beta}\right)^{\kappa-1}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta-1}+\frac{1}{(\alpha-\gamma)}\gamma\beta\kappa\eta x^{\beta-1}\left(1+x^{\beta}\right)^{\kappa-1}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta-1}\right]- \\ f(t)\left[\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}+\frac{1}{(\alpha-\gamma)}\left[1+\gamma\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]\right].\end{cases}$$

After simplification and integration, we arrive at

$$F(t)=1-\left[1+\gamma\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-\frac{\alpha}{\gamma}}, t\geq 0.$$

4.2 Truncated Moment of a Function of the Random Variable

Here, we characterize the NMBXII distribution via relationship between truncated moments of a function of X with another function. This characterization is stable in the sense of weak convergence (Glänzel; 1990).

Proposition 4.2.1: Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable and let $g(x)=\left[1+\gamma\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-1}, x>0.$ The pdf of X is (6) if and only if the function $h(x)$, in Theorem G (Glänzel; 1990), has the form $h(x)=\frac{\alpha}{\alpha+\gamma}\left[1+\gamma\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-1},\quad x>0$.

Proof If X has pdf (6), then

$$\left(1-F\left(x\right)\right)E\left(g\left(X\right)\middle| X\geq x\right)=\frac{\alpha}{\alpha+\gamma}\left[1+\gamma\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-(\frac{\alpha}{\gamma}+1)},x>0,$$

or

$$E\left(g\left(X\right)\middle| X\geq x\right)=\frac{\alpha}{\alpha+\gamma}\left[1+\gamma\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-1},\ x>0,$$

and

$$h\left(x\right)-g\left(x\right)=-\frac{\gamma}{\alpha+\gamma}\left[1+\gamma\left[\left(1+t^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-1},x>0.$$

Conversely, if $h\left(x\right)$ is given as above, then

$$h'\left(x\right)=-\frac{\alpha}{\alpha+\gamma}\gamma\beta\kappa\eta x^{\beta-1}\left(1+x^{\beta}\right)^{\kappa-1}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta-1}\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-2}<0,\ for\ x>0,$$

and

$$s'\left(x\right)=\frac{h'\left(x\right)}{h\left(x\right)-g\left(x\right)}=\frac{\frac{\alpha}{\gamma}\gamma\beta\kappa\eta x^{\beta-1}\left(1+x^{\beta}\right)^{\kappa-1}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta-1}}{\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]},x>0,$$

and hence

$$s\left(x\right)=\ln\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{\frac{\alpha}{\gamma}},x>0,$$

and

$$e^{-s\left(x\right)}=\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{-\frac{\alpha}{\gamma}},x>0.$$

In view of Theorem G, X has density (6).

Corollary 4.2.1: Let $X:\Omega\rightarrow\left(0,\infty\right)$ be a continuous random variable. The pdf of X is (6) if and only if there exist functions $h\left(x\right)$ and $g\left(x\right)$ (defined in Theorem G of Glänzel; (1990)) satisfying the differential equation

$$s'\left(x\right)=\frac{\frac{\alpha}{\gamma}\gamma\beta\kappa\eta x^{\beta-1}\left(1+x^{\beta}\right)^{\kappa-1}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta-1}}{\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]},\ x>0.$$

Remark 4.2.1: The general solution of the differential equation in Corollary 4.2.1 is

$$h\left(x\right)=\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{\frac{\alpha}{\gamma}}\left[-\int\frac{\frac{\alpha}{\gamma}\gamma\beta\kappa\eta x^{\beta-1}\left(1+x^{\beta}\right)^{\kappa-1}\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta-1}}{\left[1+\gamma\left[\left(1+x^{\beta}\right)^{\kappa}-1\right]^{\eta}\right]^{\frac{\alpha}{\gamma}+1}}g\left(x\right)dx+D\right],$$

where D is a constant.

5. Maximum Likelihood Estimation

Here, we adopt MLE technique for estimating the NMBXII parameters. Let $\xi=(\alpha,\beta,\gamma,\kappa,\eta)^T$ be the unknown parameter vector. The log likelihood function $\ell(\xi)$ for the NMBXII distribution is given by

$$\ell=\ell(\xi)=n\ln(\alpha)+n\ln(\beta)+n\ln(\kappa)+n\ln(\eta)+(\beta-1)\sum_{i=1}^{n}\ln x_i+(\kappa-1)\sum_{i=1}^{n}\ln\left(1+x_i^{\beta}\right)+ (\eta-1)\sum_{i=1}^{n}\ln\left[\left(1+x_i^{\beta}\right)^{\kappa}-1\right]-\left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\ln\left[1+\gamma\left(\left(1+x_i^{\beta}\right)^{\kappa}-1\right)^{\eta}\right]. \quad (27)$$

We can compute the MLEs of $\alpha,\beta,\gamma,\kappa$ and η by solving equations (28)-(32) either directly or using quasi-Newton procedure, computer packages/ softwares such as R, SAS, Ox, MATHEMATICA, MATLAB and MAPLE.

$$\frac{\delta}{\delta\alpha}\ell(\xi)=\frac{n}{\alpha}-\frac{1}{\gamma}\sum_{i=1}^{n}\ln\left[1+\gamma\left(\left(1+x_i^{\beta}\right)^{\kappa}-1\right)^{\eta}\right]=0, \quad (28)$$

$$\frac{\delta}{\delta\beta}\ell(\xi)=\frac{n}{\beta}+\sum_{i=1}^{n}\ln x_i+(\kappa-1)\sum_{i=1}^{n}\frac{\ln x_i}{\left(1+x_i^{-\beta}\right)}+(\eta-1)\kappa\sum_{i=1}^{n}\frac{x_i^{\beta}\left(1+x_i^{\beta}\right)^{\kappa-1}\ln x_i}{\left(1+x_i^{\beta}\right)^{\kappa}-1}-(\alpha+\gamma)\kappa\eta\sum_{i=1}^{n}\frac{\left(\left(1+x_i^{\beta}\right)^{\kappa}-1\right)^{\eta-1}x_i^{\beta}\left(1+x_i^{\beta}\right)^{\kappa-1}\ln x_i}{\left[1+\gamma\left(\left(1+x_i^{\beta}\right)^{\kappa}-1\right)^{\eta}\right]}, \quad (29)$$

$$\frac{\delta}{\delta\gamma}\ell(\xi)=\frac{\alpha}{\gamma^2}\sum_{i=1}^{n}\ln\left[1+\gamma\left(\left(1+x_i^{\beta}\right)^{\kappa}-1\right)^{\eta}\right]-\left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\left[\left(\left(1+x_i^{\beta}\right)^{\kappa}-1\right)^{-\eta}+\gamma\right]^{-1}, \quad (30)$$

$$\frac{\delta}{\delta\kappa}\ell(\xi)=\frac{n}{\kappa}+\sum_{i=1}^{n}\ln\left(1+x_i^{\beta}\right)+(\eta-1)\sum_{i=1}^{n}\frac{\ln\left(1+x_i^{\beta}\right)}{\left[1-\left(1+x_i^{\beta}\right)^{-\kappa}\right]}-(\alpha+\gamma)\eta\sum_{i=1}^{n}\frac{\left(\left(1+x_i^{\beta}\right)^{\kappa}-1\right)^{\eta-1}\left(1+x_i^{\beta}\right)^{\kappa}\ln\left(1+x_i^{\beta}\right)}{\left[1+\gamma\left(\left(1+x_i^{\beta}\right)^{\kappa}-1\right)^{\eta}\right]}, \quad (31)$$

$$\frac{\delta}{\delta\eta}\ell(\xi)=\frac{n}{\eta}+\sum_{i=1}^{n}\ln\left[\left(1+x_i^{\beta}\right)^{\kappa}-1\right]-(\alpha+\gamma)\sum_{i=1}^{n}\frac{\ln\gamma\left(\left(1+x_i^{\beta}\right)^{\kappa}-1\right)}{\left[1+\gamma\left(\left(1+x_i^{\beta}\right)^{\kappa}-1\right)^{-\eta}\right]}. \quad (32)$$

6. Simulation Study

In this section, we perform the simulation study by using selected the NMBXII distributions. To see the performance of MLE's of these distributions, we generate 1,000 samples of sizes 20, 60 and 100 with quantile function of the NMBXII distribution. The results of the simulations are reported in Table 3. From this Table, we observe that the estimates approach true values as the sample size increases whereas the standard deviations of the estimates decrease, as expected.

Table 3. Empirical means and standard deviations (in parenthesis) for selected parameters values of the NMBXII distribution

Sample	Parameters	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\eta}$	$\hat{\kappa}$
n=20	3,10,2,5,0.5	3.1486 (0.5411)	10.0326 (0.4321)	1.5890 (0.8544)	5.0119 (0.9063)	0.4933 (0.0393)
	1,5,3,10,1	1.1902 (0.6081)	5.2655 (1.0842)	2.8633 (0.3033)	10.1680 (0.5602)	1.0252 (0.1042)
	5,1,1,3,10	4.9530 (0.2825)	0.9960 (0.0402)	0.9246 (1.0767)	3.2318 (0.6055)	10.0276 (0.0579)
	0.5,5,5,5,0.5	0.5521 (0.4390)	5.5003 (1.1710)	4.9131 (0.2906)	5.6872 (1.1822)	0.5306 (0.2771)
	5,5,5,5,5	5.0543 (1.7875)	5.2190 (0.2925)	3.9855 (2.8571)	5.5277 (1.9942)	5.3805 (1.0088)

Sample	Parameters	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\eta}$	$\hat{\kappa}$
n=60	3,10,2,5,0.5	3.0278 (0.2786)	10.0209 (0.1852)	1.8999 (0.4733)	5.0467 (0.4127)	0.4956 (0.0227)
	1,5,3,10,1	1.0761 (0.2630)	5.1544 (0.8034)	2.9663 (0.0949)	10.0932 (0.4166)	1.0088 (0.0506)
	5,1,1,3,10	4.9964 (0.1145)	1.0031 (0.0259)	0.9556 (0.6601)	3.0439 (0.3782)	10.0088 (0.0379)
	0.5,5,5,5,0.5	0.5328 (0.2204)	5.1339 (0.6979)	4.9853 (0.1488)	5.1479 (0.7074)	0.4874 (0.0663)
	5,5,5,5,5	4.9574 (1.1620)	5.0750 (0.2037)	4.8282 (1.4416)	5.2292 (1.0902)	5.1906 (0.5288)

Sample	Parameters	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\eta}$	$\hat{\kappa}$
n=100	3,10,2,5,0.5	3.0175 (0.2615)	9.9867 (0.1843)	1.9054 (0.4758)	4.9695 (0.4066)	0.4980 (0.0221)
	1,5,3,10,1	1.0026 (0.2519)	4.9905 (0.5320)	2.9841 (0.0780)	10.0032 (0.2652)	0.9935 (0.0390)
	5,1,1,3,10	5.0035 (0.0793)	1.0001 (0.0196)	0.9664 (0.4639)	2.9998 (0.3221)	10.0027 (0.0320)
	0.5,5,5,5,0.5	0.5219 (0.1555)	5.0609 (0.6701)	4.9934 (0.1407)	5.0938 (0.6718)	0.4974 (0.0498)
	5,5,5,5,5	5.0480 (0.8472)	5.0553 (0.1677)	4.9347 (0.9685)	5.0605 (0.6785)	5.0814 (0.3714)

7. Applications

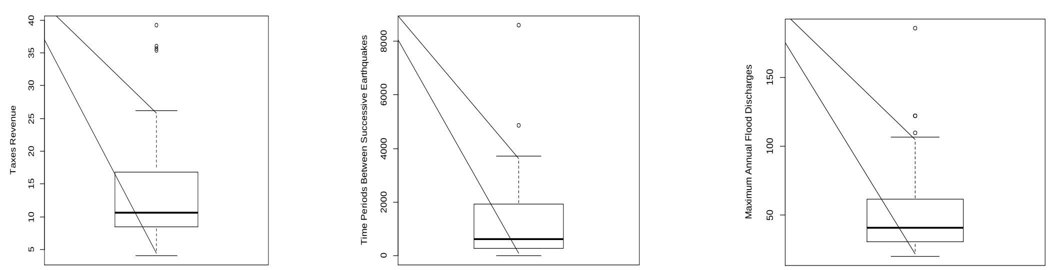
We consider five data sets (taxes revenue, time periods between successive earthquakes, rains and flood discharges) to endorse the potentiality of the NMBXII distribution. The first data set is about the about actual taxes (in 1000 million pounds) for Egyptian (Nassar and Nada; 2011). The second data set represents the time periods between successive earthquakes at North Anatolia Zone (Kus; 2007) during the last century. In third data set, Van Montfort (1970) studied the maximum annual flood discharges (1000 ft^3/sec) data for 48 years of the North Saskachevan River (Edmonton).

We compare the NMBXII distribution with models such as NBXII, New Lomax, MBXII, BXII, Lomax, Log Logistic distributions. For selection of the optimum distribution, we compute the estimate of likelihood ratio statistics ($-2\hat{\ell}$), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Cramer-von Mises (W*), Anderson Darling (A*), and Kolmogorov- Smirnov [K-S] statistics with p-values for all competing and sub distributions. We compute the MLEs and their standard errors (in parentheses). We also compute goodness of fit statistics **(GOFs)** values for the NMBXII, NBXII, new Lomax, MBXII, BXII, Lomax and Log Logistic models. Table 4 reports some descriptive measures for three data sets.

Table 4: Descriptive Statistics

Data Sets	N	Min	Max	Mean	Median	Standard deviation	Skewness	Kurtosis
Taxes Revenue	59	4.1	39.2	13.4881	10.6	8.0515	1.6083	5.2560
Time Periods between Successive Earthquakes	24	9	8592	1429.625	624.5	1980.727	2.3223	8.3507
Max. Annual Flood Discharges	48	19.885	185.56	51.4952	40.4	32.3768	2.0686	7.9507

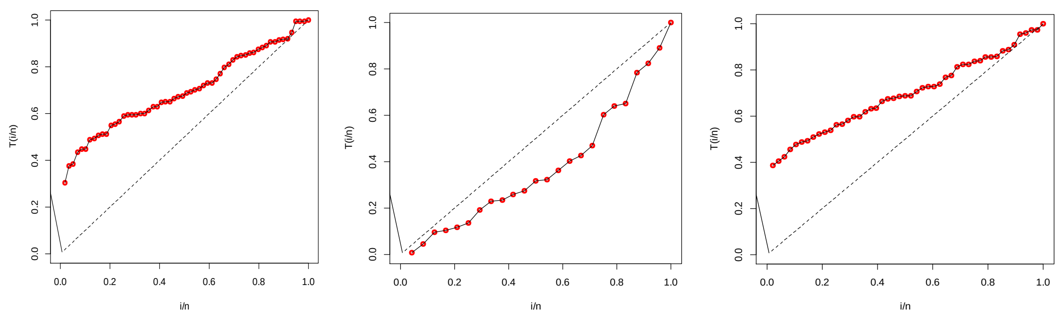
Table 4 shows that the taxes revenue data set is right-skewed, with significant positive kurtosis. About the time periods between successive earthquakes data set, it is right-skewed, with moderate positive kurtosis. Maximum annual flood discharges data set is significantly right-skewed, with significantly positive kurtosis.



(a) (b) (c)

Figure 3: Boxplots of the (a) Taxes Revenue (b) Time Periods between Successive Earthquakes (c) maximum annual flood discharges

The nature of the three data sets differs in numerous features. Some extreme points are also present in these data sets. Here, we study the statistical analysis by total time on test (TTT) for the five data sets in Figure 4.



(a) (b) (c)

Figure 4: TTT plots of the (a) Taxes Revenue (b) Time Periods between Successive Earthquakes (c) maximum annual flood discharges

Figure 4(a) presents that the TTT plot for taxes revenue data is concave, which infers increasing failure rate.. In figure 4(b), the TTT curve is convex, which infers decreasing failure rate for time periods between successive earthquakes data. For maximum annual flood discharges data, the TTT curve (Figure 4(c)) is concave, which also suggests increasing failure intensity. It should also be noted that the NMBXII distribution covers increasing and decreasing failure intensities. So, the NMBXII distribution is suitable to model these data sets

7.1 Taxes Revenue: Table 5 reports the MLEs (standard errors in parentheses) and measures W*, A*, K-S (p-values). Table 6 displays the values of measures $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC.

Table 5: MLEs (standard errors) and W*, A*, KS (p-values) for taxes revenue data

Model	α	β	κ	η	γ	W*	A*	K-S (p-value)
NMBXII	0.0991 (0.8444)	0.1644 (0.424)	0.8112 (0.4450)	35.0547 (84.8271)	0.2479 (2.1128)	**0.0456**	**0.2984**	**0.0847 (0.7913)**
NBXII	0.0438 (0.1062)	5.0188 (9.0647)	3.0400 (8.9280)	0.6079 (1.7838)	1	0.0560	0.3254	0.467 (1.333e-11)
New Lomax	0.0206 (0.0448)	4.1988 (5.4628)	4.6917 (12.3485)	1	1	0.0561	0.3257	0.4676 (1.24e-11)
MBXII	0.0035 (0.001)	2.29436 (0.1636)	---	---	0.00215 (0.0005)	0.1444	0.8347	0.1263 (0.3031)
BXII	0.0669 (0.2651)	6.0806 (24.1034)	---	---	---	0.0560	0.3254	0.4674 (1.269e-11)
Lomax	0.392 (0.0510)	1	---	---	---	0.0739	0.4245	0.4908 (9.009e-13)
LL	1	0.6221 (0.0628)	---	---	---	0.0890	0.5157	0.7168 (< 2.2e-16)

Table 6: $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC for tax revenue

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
NMBXII	**378.3576**	**388.3575**	**389.4896**	**398.7452**	**392.4124**
NBXII	514.4646	522.4647	523.2054	530.7748	525.7086
New Lomax	514.4816	520.4816	520.9180	526.7142	522.9146
MBXII	387.7828	393.7829	394.2192	400.0155	396.2158
BXII	514.4602	518.4602	518.6745	522.6153	520.0822
Lomax	529.5340	531.5340	531.6042	533.6116	532.3450
LL	574.8004	576.8005	576.8707	578.8780	577.6115

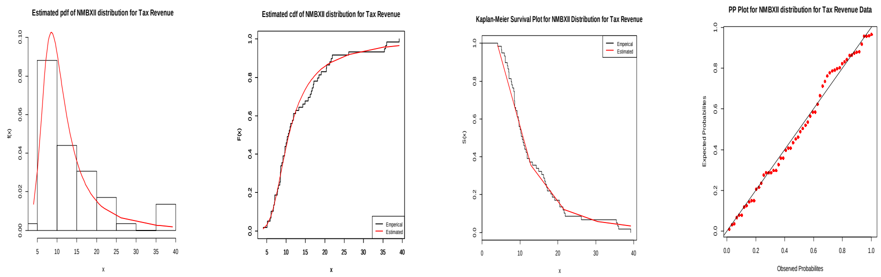


Fig. 5: Fitted pdf, cdf, survival and pp plots of the NMBXII distribution for tax revenue data

Tables 5--6, infer that the NMBXII distribution is the best distribution because it has smallest values for W*, A*, K-S, $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC statistics and maximum p-value. Figure 5 also approves that the proposed distribution is closely fitted to taxes revenue.

7.2 Time Periods between Successive Earthquakes: Table 7 reports the MLEs (standard errors in parentheses) and measures W*, A*, K-S (p-values). Table 8 displays the values of measures $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC.

Table 7: MLEs (standard errors) and W*, A*, KS (p-values) for time periods between successive earthquakes

Model	α	β	κ	η	γ	W	A	K-S (p-value)
NMBXII	5.7815 (112.3016)	0.3247 (7.9350)	0.2267 (6.2599)	5.1023 (60.2430)	0.5120 (36.3887)	**0.0276**	**0.1769**	**0.0767 (0.9969)**
NBXII	0.0503 (0.2050)	2.7600 (8.9116)	3.3700 (58.7240)	0.3333 (5.7746)	1	0.0818	0.6340	0.4553 (4.582e-05)
New Lomax	0.0225 (0.0930)	2.3857 (6.71141)	2.9098 (15.8539)	1	1	0.0817	0.6334	0.4554 (4.546e-05)
MBXII	0.0218 (0.0288)	17.0000 (14.9528)	---	---	2.4000	0.0809	0.6279	0.456 (4.415e-05)
BXII	0.0565 (0.1753)	2.7601 (8.5549)	---	---	---	0.0818	0.6342	0.4552 (4.576e-05)
Lomax	0.1556 (0.0318)	1	---	---	---	0.0743	0.5812	0.4553 (7.045e-12)
LL	1	0.2374 (0.0376)	---	---	---	0.0339	0.2889	0.689 (4.595e-05)

Table 8: $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC for time periods between successive earthquakes

Model	$-2\hat{\ell}$	**AIC**	**CAIC**	**BIC**	**HQIC**
NMBXII	**393.3944**	**403.3944**	**406.7277**	**409.2846**	**404.9571**
NBXII	445.3058	453.3058	455.411	458.018	454.5559
New Lomax	445.31	451.3099	452.5099	454.8441	452.2475
MBXII	445.6822	451.6821	452.8821	455.2163	452.6197
BXII	445.3034	449.3035	449.8749	451.6596	449.9285
Lomax	445.6814	447.6814	447.8632	448.8594	447.9939
LL	470.058	472.058	472.2398	473.236	472.3705

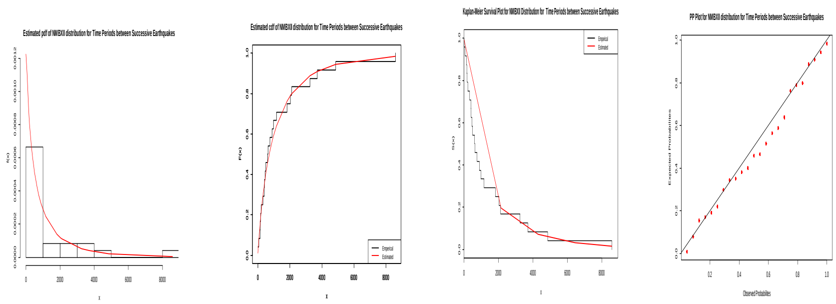


Fig. 6: Fitted pdf, cdf, survival and pp plots of the NMBXII distribution for time periods between successive earthquakes

Tables 7--8, infer that the NMBXII model is the distribution because it has smallest values for W*, A*, K-S, $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC statistics and maximum p-value. Figure 6 also approves that the proposed distribution is closely fitted to time periods between successive earthquakes data.

7.3 Maximum Annual Flood Discharges: Table 9 reports the MLEs (standard errors in parentheses) and measures W*, A*, K-S (p-values). Table 10 displays the values of measures $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC.

Table 9: MLEs (standard errors) and W*, A*, KS (p-values) for maximum annual flood discharges

Model	α	β	κ	η	γ	W	A	K-S (p-Value)
NMBXII	1.4118 (4.1455)	1.0084 (0.4477)	0.1858 (0.0800)	14.1535 (5.2020)	3.1603 (9.9300)	**0.0330**	**0.232**	**0.079 (0.9255)**
NBXII	0.0836 (0.2730)	2.7610 (5.3052)	3.3708 (35.2326)	0.3384 (3.51694)	1	0.0534	0.3687	0.5449 (8.35e-13)
New Lomax	0.0349 (0.0597)	2.4811 (2.7794)	3.0288 (5.4099)	1	1	0.0534	0.3687	0.5438 (9.365e-13)
MBXII	0.0122 (0.0103)	63.9754	--	---	2.9752	0.0535	0.3693	0.5453 (8.01e-13)
BXII	0.0954 (0.1831)	2.7607 (5.2864)	---	---	---	0.0534	0.3686	0.5449 (8.372e-13)
Lomax	0.2616 (0.0378)	1	---	---	---	0.0576	0.3957	0.5484 (5.803e-13)
LL	1	0.4049 (0.0451)	---	---	---	0.0743	0.5047	0.7704 (< 2.2e-16)

Table 10: $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC for maximum annual flood discharges

Model	$-2\hat{\ell}$	AIC	CAIC	BIC	HQIC
NMBXII	**432.814**	**442.8141**	**444.2426**	**452.1701**	**446.3497**
NBXII	588.779	596.779	597.7093	604.2638	599.6076
New Lomax	588.7874	594.7874	595.3328	600.401	596.9088
MBXII	589.2012	595.2012	595.7467	600.8148	597.3226
BXII	588.7774	592.7774	593.0441	596.5198	594.1917
Lomax	591.7706	593.7705	593.8575	595.6417	594.4776
LL	637.0264	639.0265	639.1134	640.8977	639.7336

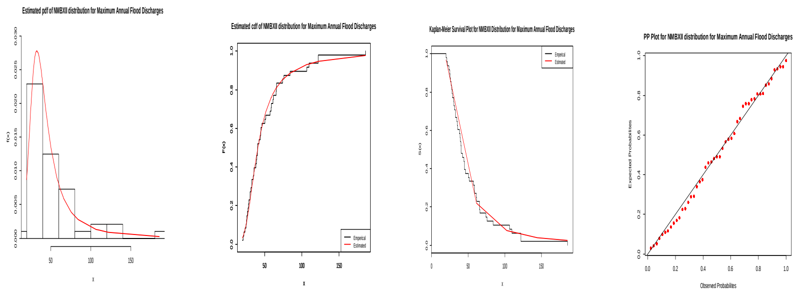


Fig. 7: Fitted pdf, cdf, survival and pp plots of the NMBXII distribution for maximum annual flood discharges

From tables 9--10, the NMBXII distribution can be considered as the best distribution for maximum annual flood discharges (1000 ft^3/sec) data because the values of all criteria (W*, A*, K-S, $-2\hat{\ell}$, AIC, CAIC, BIC and HQIC) are smaller for the proposed model. Figure 7 endorses this claim via the graphical display of fitted pdf, estimated cdf, survival and PP plots of the NMBXII distribution. From this figure, we can infer that the proposed distribution is closely fitted to maximum annual flood discharges.

8. Concluding Remarks

The NMBXII distribution is derived from the basis of the T-X family technique and compounding the generalized Nadarajah–Haghighi (GNH) and gamma distributions. The NMBXII density function is symmetrical, left-skewed, right-skewed, J and reverse-J which is indicative of its applicability. The NMBXII model can produce all types of failure rates such as bimodal, inverted bathtub, modified bathtub, increasing, decreasing, increasing-decreasing, decreasing-increasing and increasing-decreasing-increasing. To show the importance of the NMBXII distribution, we established various mathematical properties such as random number generator, sub-models, ordinary moments, incomplete moments, inequality measures, residual life functions and reliability measures. We characterized the NMBXII distribution via two techniques. We addressed the maximum likelihood estimation technique for the model parameters. We performed the simulation study to demonstrate of the performance of maximum likelihood estimators of the NMBXII parameters. We demonstrated the potentiality and utility of the NMBXII distribution by considering three real data sets for applications such as tax revenue, time periods between successive earthquakes and flood discharges. We applied various model selection criteria, GOFs and graphical tools to examine the adequacy of the proposed distribution. We inferred that the NMBXII model is empirically suitable for the lifetime applications. Therefore, the NMBXII model is quite flexible and can be applied excellently in evaluating numerous data sets. It has the potential to be superior to other competing distributions. Future projects include (i) unit NMBXII; (ii) bivariate extension of NMBXII; (iii) modeling the wind speed data with NMBXII and (iv) the study of the complexity of the NMBXII via Bayesian methods.

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