Transient Analysis of an $M/M/1$ Queue with Multiple Vacations

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Abstract

In this paper, we have obtained explicit expressions for the time dependent probabilities of the $M/M/1$ queue with server vacations under a multiple vacation scheme. The corresponding steady state probabilities have been obtained. We also obtain the time dependent performance measures of the systems.

Keywords: Markovian queue, Transient probabilities, Multiple vacations, Performance measures.

1. Introduction

In the literature, analytical results for the transient behavior of queuing models are not as widely available as the steady-state results. The steady-state measures cannot give insight into the transient behavior of the system. The steady-state results are well suited to study the performance measures of the system on a long time scale, while the transient solutions are more useful for studying the dynamical behavior of systems over a finite period. Moreover, stationary results are mainly used within the system design process. However, due to both variability and uncertainty of the traffic offered to the system, satisfactory resource utilization and robust performance are in general difficult to achieve together within the design process. Thus, some management actions have to be taken with the aim of dynamically adapting the resource assignment to the system load. These management actions can often be based on transient analytical results, for a simplified queueing model, accurately representing the system of interest. In general, transient analytical results are useful for studying the finite-time properties of queueing systems (see Kelton (1985)).

Many methods have been derived for obtaining transient solutions: the method of generating functions of Bailey (1954), the spectral method of Lederman and Reuter (1956), the combinatorial method of Champernowne (1956), the difference equation technique of Conolly (1958) etc. The transient analysis of a queueing system demands a methodologically simple and easily numerically implementable approach.

Parthasarathy and Lenin (1997), (1998) and (1991) used continued fractions to analyse the transient behavior of birth death processes. Krishna Kumar and Arivudainambi (2002), studied the transient behavior of an $M/M/1$ queue with catastrophes. Parthasarathy and Selvaraju (2001) analyzed the transient behavior of an $M/M/1$ queue in which potential customers are discouraged by the queue length. Parthasarathy et. al (2002) studied the transient behavior of a single server driven fluid queue using a continued fractions approach. Krishna Kumar and Pavai Madheswari (2005) considered the time dependent analysis of an $M/M/1$ queue subject to catastrophes and server failures. Tarabia and El-Baz have studied (2006) the exact transient solutions to non empty Markovian queues by using the power series technique. Griffiths et. al (2006) have studied the transient behaviur of the modified Bessel function of the second kind. Parathasarathy and Sudhesh (2006) considered the exact transient solution of a discrete time queue with state dependent rates. Krishna Kumar et al. (2007) studied a single server queue with catastrophes, failures and repairs. Parthasarathy and Sudhesh (2007) studied the time dependent analysis of a single server retrial queue with state dependent rates. Krishna Kumar et al. (2008) considered the transient behavior of an $M/M/1$ queue with state dependent rates and catastrophes. Parthasarathy and Sudhesh (2008) have obtained tarnsient solution of $M/M/c$ queue with N-policy with the help of modified Bessel function of the second kind. Leonenko (2009) studied a new approcah to study the $M/E_k/1$ queue. Thangaraj and Vanitha (2010), considered the transient analysis of a $M/M/1$ queue with Bernulli feedback. Recently, Sudhesh (2010) has examined the transient behavior of a single server queue with catastrophes and customer impatience.

Queueing systems with server vacations have been studied extensively. A comprehensive review of vacation models, methods, results, examples and applications can be found in the survey of Doshi (1986) and Ke et.al (2010) and the monographs of Takagi (1991) and Tian and Zhang (2006). However, results are not available in the literature on the time dependent behavior of a vacation queueing system. In this paper we have obtained explicit expressions for the transient state probabilities of the $M/M/1$ queue with multiple vacations. We have considered the multiple vacation scheme where the server takes a vacation each time the system becomes empty. We have also considered queueing systems with server vacations.

The rest of the paper is organized as follows. In section 2, we examine the transient behavior of the $M/M/1$ queue with multiple vacations. In section 3, we obtain the expressions for the mean and variance at time t. In section 4, cases, we have obtained the corresponding steady state results.

2. The $M/M/1$ queue with multiple vacations

Consider an $M/M/1$ queueing system with first come first served discipline. Customers arrive at the service station according to a Poisson process with arrival rate λ. There is infinite room for customers to wait. Customer service times are independently and identically distributed (i.i.d.) exponential random variables with parameter μ. However, the server takes a vacation after completion of all the services. As before, the server vacation time is a exponential random variable with a parameter γ. After returning from the vacation, if the server finds no customers in the system, the server is permitted to take another vacation. This process continues until the server returns to a non empty system.

Let $C(t)$ denote the state of the server at any instant of time t.

$$C(t) = \begin{cases} 0, & \text{if the server is on vacation,} \\ 1, & \text{if the server is available.} \end{cases}$$

Let $N(t)$ denote the number of customers in the system at time t. Then $\{(C(t), N(t)), t \geq 0\}$ is a continuous time Markov chain.

Let

$$p_{0,n}(t) = Prob\{C(t) = 0, N(t) = n\}, \ \ n \geq 0,$$
$$p_{1,n}(t) = Prob\{C(t) = 1, N(t) = n\}, \ \ n \geq 1.$$

2.1 Transient analysis

The forward Chapman Kolmogorov equations for the system are

$$p'_{0,0}(t) = -\lambda p_{0,0}(t) + \mu p_{1,1}(t), \tag{1}$$
$$p'_{0,n}(t) = -(\lambda+\gamma)\, p_{0,n}(t) + \lambda p_{0,n-1}(t), n \geq 1, \tag{2}$$
$$p'_{1,1}(t) = -(\lambda+\mu)\, p_{1,1}(t) + \gamma p_{0,1}(t) + \mu p_{1,2}(t), \tag{3}$$
$$p'_{1,n}(t) = -(\lambda+\mu)\, p_{1,n}(t) + \gamma p_{0,n}(t) + \mu p_{1,n+1}(t) + \lambda p_{1,n-1}(t), n \geq 2. \tag{4}$$

Without loss of generality, assume that initially the server is busy with i customers. i.e.,

$$p_{1,i}(0) = \alpha_i, i \geq 1 \ and \ \ p_{0,i}(0) = 0, i \geq 0, \tag{5}$$

where α_i be the probability that there are i customers in the system at time $t = 0$. Let

$$\alpha(z) = \sum_{i=1}^{\infty} \alpha_i z^i, |z| \leq 1.$$

Taking Laplace transforms of (1) and (2) and using (5), we get

$$p^*_{0,0}(s) = \frac{1}{s+\lambda} + \frac{\mu}{s+\lambda}\, p^*_{1,1}(s), \tag{6}$$
$$p^*_{0,n}(s) = \frac{\lambda}{s+\lambda+\gamma}\, p^*_{0,n-1}(s). \tag{7}$$

By recursively using equation (7), we obtain

$$p^*_{0,n}(s) = \left(\frac{\lambda}{s+\lambda+\gamma}\right)^n p^*_{0,0}(s). \tag{8}$$

Define partial probability generating functions as follows, for $|z|\le 1$,

$$P_1(z,t)=\sum_{n=1}^{\infty}P_{1,n}(t)z^n,$$

$$P_0(z,t)=\sum_{n=0}^{\infty}P_{0,n}(t)z^n.$$

Multiplying (3) and (4) by appropriate powers of z and summing over $n\ge 1$, we obtain

$$\frac{\partial}{\partial t}P_1(z,t)=\left\{\lambda z+\frac{\mu}{z}-(\lambda+\mu)\right\}P_1(z,t)+\gamma P_0(z,t) \\ -\mu P_{1,1}(t)-\gamma P_{0,0}) \tag{9}$$

Taking Laplace transforms on both sides of (9), we get,

$$\left((s+\lambda+\mu)-(\lambda z+\frac{\mu}{z})\right)P_1^*(z,s)=\alpha(z)+\gamma P_0^*(z,s)-\gamma p_{0,0}^*(s)-\mu p_{1,1}^*(s)$$

Hence

$$P_1^*(z,s)=\frac{z\left\{\alpha(z)+\gamma P_0^*(z,s)-\gamma p_{0,0}^*(s)-\mu p_{1,1}^*(s)\right\}}{(s+\lambda+\mu)z-\lambda z^2-\mu} \tag{10}$$

The denominator put to zero

$$\lambda z^2-(s+\lambda+\mu)z+\mu=0$$

has two roots

$$z_1=\frac{w-\sqrt{w^2-4\lambda\mu}}{2\lambda}$$

$$z_2=\frac{w+\sqrt{w^2-4\lambda\mu}}{2\lambda}$$

$$where\ w=s+\lambda+\mu$$

of which z_1 is of modulus less than 1. Considering that $P_1^*(z,s)$ exists in the unit circle, we see that the numerator of (10) must also vanish for $z=z_1$. Putting $z=z_1$ in the numerator of (10) equated to zero , we get,

$$\alpha(z_1)+\gamma P_0^*(z_1,s)-\gamma p_{0,0}^*(s)-\mu p_{1,1}^*(s)=0$$

$$\alpha(z_1)+\gamma\frac{2(s+\lambda+\gamma)}{2(s+\lambda+\gamma)-(w-\sqrt{w^2-4\lambda\mu})}p_{0,0}^*(s)-(s+\lambda+\gamma)p_{0,0}^*(s)=0$$

Hence

$$p^{*}_{0,0}(s) = \frac{\alpha(z_1)}{s+\lambda+\gamma}\frac{2(s+\lambda+\gamma)-(w-\sqrt{w^2-4\lambda\mu})}{w+\sqrt{w^2-4\lambda\mu}-2\mu}$$
$$= \alpha(z_1)F(s) \qquad (11)$$

where $F(s) = \frac{w-\sqrt{w^2-4\lambda\mu}}{-2\mu s} + \frac{(w-\sqrt{w^2-4\lambda\mu})^2}{4\mu s(s+\lambda+\gamma)} + \frac{1}{s} - \frac{w-\sqrt{w^2-4\lambda\mu}}{2s(s+\lambda+\gamma)}$

On inversion (2.11), yields

$$p_{0,0}(t) = \int_0^t \alpha(u)F(t-u)du \qquad (12)$$

where $\alpha(t) = \sum_{n=1}^{\infty}\alpha_n I_n(2\sqrt{\lambda\mu}t)\frac{n}{t}(\sqrt{\frac{\mu}{\lambda}})^n$

$$F(t) = 1-\sqrt{\frac{\lambda}{\mu}}\int_0^t I_1(2\sqrt{\lambda\mu}u)e^{-(\lambda+\mu)u}du$$
$$+\int_0^t\left(\frac{\lambda}{\lambda+\gamma}I_2(2\sqrt{\lambda\mu}u)-\frac{\sqrt{\lambda}}{\lambda+\gamma}I_2(2\sqrt{\lambda\mu}u)\right)(1-e^{-(\lambda+\gamma)(t-u)})e^{-(\lambda+\mu)u}du$$

By using the expression for $p_{0,0}(t)$, the other state probabilities $p_{0,n}(t), n\geq 0$ and $p_{1,n}(t), n\geq 1$ can be obtained as follows.

From (8) and (1), we get

$$p_{0,n}(t) = \lambda^n\int_0^t p_{0,0}(u)\frac{(t-u)^{n-1}e^{-(\lambda+\gamma)(t-u)}}{(n-1)!}du \qquad (13)$$

$$p_{1,1}(t) = \frac{1}{\mu}\left(p'_{0,0}(t)+\lambda p_{0,0}(t)\right) \qquad (14)$$

The equation (9) can be considered as a first order differential equation in $P_1(z,t)$ and t. Integrating, we get

$$P_1(z,t) = \alpha(z)e^{[\lambda z+\frac{\mu}{z}-(\lambda+\mu)]t} + \gamma\int_0^t P_0(z,u)e^{[\lambda z+\frac{\mu}{z}-(\lambda+\mu)](t-u)}du$$
$$-\gamma\int_0^t p_{0,0}(u)e^{[\lambda z+\frac{\mu}{z}-(\lambda+\mu)](t-u)}du - \mu\int_0^t p_{1,1}(u)e^{[\lambda z+\frac{\mu}{z}-(\lambda+\mu)](t-u)}du$$

It is well known that if $\theta = 2\sqrt{\lambda\mu}$ and $\beta = \sqrt{\frac{\lambda}{\mu}}$ then

$$e^{(\lambda z+\frac{\mu}{z})t} = \sum_{n=-\infty}^{\infty} I_n(\theta t)(\beta z)^n$$

where $I_n(.)$ is the modified Bessel function of the first kind. Comparing the coefficients of z^n on both sides of (15), we get

$$p_{1,n}(t)=\sum_{k=1}^{\infty}\alpha_k I_{n-k}(\theta t)\beta^{n-k}e^{-(\lambda+\mu)t}$$
$$+\gamma\int_0^t\sum_{k=0}^{\infty}p_{0,k}(u)I_{n-k}(\theta(t-u))\beta^{n-k}e^{-(\lambda+\mu)(t-u)}du$$
$$-\gamma\int_0^t p_{0,0}(u)I_n(\theta(t-u))\beta^n e^{-(\lambda+\mu)(t-u)}du$$
$$-\mu\int_0^t p_{1,1}(u)I_n(\theta(t-u))\beta^n e^{-(\lambda+\mu)(t-u)}du$$

The above equation can be written as

$$p_{1,n}(t)=\sum_{k=1}^{\infty}\alpha_k I_{n-k}(\theta t)\beta^{n-k}e^{-(\lambda+\mu)t}+\sum_{r=0}^{\infty}\alpha_{n+r}I_r(\theta t)\beta^{-r}e^{-(\lambda+\mu)t}$$
$$+\gamma\int_0^t\sum_{k=0}^{n-1}p_{0,k}(u)I_{n-k}(\theta(t-u))\beta^{n-k}e^{-(\lambda+\mu)(t-u)}du$$
$$+\gamma\int_0^t\sum_{r=0}^{\infty}p_{0,n+r}(u)I_r(\theta(t-u))\beta^{-r}e^{-(\lambda+\mu)(t-u)}du$$
$$-\gamma\int_0^t p_{0,0}(u)I_n(\theta(t-u))\beta^n e^{-(\lambda+\mu)(t-u)}du$$
$$-\mu\int_0^t p_{1,1}(u)I_n(\theta(t-u))\beta^n e^{-(\lambda+\mu)(t-u)}du \qquad (15)$$

Consider

$$\gamma\sum_{r=0}^{\infty}p^*_{0.n+r}(s)\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^r\frac{1}{\sqrt{w^2-\alpha^2}}=\gamma\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n P^*_{0,0}(s)\frac{1}{\sqrt{w^2-\alpha^2}}$$
$$\times\sum_{r=0}^{\infty}\left(\frac{\lambda}{s+\lambda+\gamma}\right)^r\left(\frac{w-\sqrt{w^2-\alpha^2}}{2\lambda}\right)^r$$
$$=-\frac{\gamma}{2\mu s}\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n\frac{[(w-\sqrt{w^2-\alpha^2})-2\mu]}{\sqrt{w^2-\alpha^2}}$$
$$\times\sum_{k=0}^{\infty}\alpha_k\left(\frac{w-\sqrt{w^2-\alpha^2}}{\sqrt{w^2-\alpha^2}}\right)^k$$
$$(using\ eqn\ (11))$$

Inverting the above equation and substituting in (16), we get

$$p_{1,n}(t)=\sum_{r=1}^{n-1}\alpha_r I_{n-r}(\theta t)\beta^{n-r}e^{-(\lambda+\mu)t}+\sum_{r=0}^{\infty}\alpha_{n+r}I_r(\theta t)\beta^{-r}e^{-(\lambda+\mu)t}$$

$$+\gamma\int_0^t\sum_{k=0}^{n-1}p_{0,k}(u)I_{n-k}(\theta(t-u))\beta^{n-k}e^{-(\lambda+\mu)(t-u)}du$$

$$-\frac{\gamma}{\beta^2}\sum_{k=1}^{\infty}\int_0^t\alpha_k\chi_n(t-u)\theta^{k+1}I_{k+1}(\theta u)e^{-(\lambda+\mu)u}du$$

$$+\frac{\gamma}{2\lambda}\sum_{k=1}^{\infty}\int_0^t\alpha_k\chi_n(t-u)\theta^{k}I_{k}(\theta u)e^{-(\lambda+\mu)u}du$$

$$-\gamma\int_0^t p_{0,0}(u)I_n(\theta(t-u))\beta^n e^{-(\lambda+\mu)(t-u)}du$$

$$-\mu\int_0^t p_{1,1}(u)I_n(\theta(t-u))\beta^n e^{-(\lambda+\mu)(t-u)}du$$

where $\chi_n(t)=\int_0^t\frac{u^{n-1}e^{-(\lambda+\gamma)u}}{(n-1)!}du$

The probabilities $p_{0,0}(t)$, $p_{0,n}(t)(n\geq 1)$ and $p_{1,1}(t)$ can be obtained form (12), (13) and (14) respectively

2.2 Performance measures

In this section we derive the expression for the expected number of customers in the system at time t. Now

$$E[N(t)]=m(t)=\sum_{n=1}^{\infty}n\left(p_{0,n}(t)+p_{1,n}(t)\right)$$

Then

$$m'(t)=\sum_{n=1}^{\infty}n\left(p'_{0,n}(t)+p'_{1,n}(t)\right)$$

From equations (2), (3) and (4), we get,

$$m'(t)=\lambda-\mu+\mu h(t)$$

$$where\ h(t)=\sum_{n=0}^{\infty}p_{0,n}(t)$$

Now

$$h^*(s)=p^*_{0,0}(s)+\lambda p^*_{0,0}(s)\frac{1}{s+\gamma}$$

Hence

$$h(t) = p_{0,0}(t) + \lambda\int_0^t p_{0,0}(u)e^{-\gamma(t-u)}du$$

where $p_{0,0}(t)$ is given in (12). Now

$$m(t) = (\lambda-\mu)t + \mu\int_0^t h(x)dx + m(0)$$

$$i.e.,\ E[N(t)] = (\lambda-\mu)t + \mu\int_0^t h(x)dx + \sum_{n=1}^{\infty} n\alpha_n$$

Now

$$Var(X(t)) = \eta(t) - (m(t))^2$$

$$where\ \eta(t) = \sum_{n=1}^{\infty} n^2\left(p_{0,n}(t) + p_{1,n}(t)\right)$$

From equations (2), (3) and (4), we get,

$$\eta^{'}(t) = 2(\lambda+\mu)m(t) + \lambda - 4\mu p_{1,1}(t) - \mu p_{1,2}(t) + \mu(1-p_{0,0}(t)) - 2\mu g(t)$$

where $g(t) = \sum_{n=1}^{\infty} np_{0,n}(t)$

Now, $g^*(s) = \sum_{n=1}^{\infty} np^*_{0,n}(s)$

$$= \sum_{n=1}^{\infty} n\left(\frac{\lambda}{s+\lambda+\gamma}\right)^n p^*_{0,0}(s)$$

$$= \lambda p^*_{0,0}(s)\left(\frac{1}{s+\gamma} + \frac{\lambda}{(s+\gamma)^2}\right)$$

Therefore $g(t) = \lambda\int_0^t p_{0,0}(u)e^{-\gamma(t-u)}du + \lambda^2\int_0^t p_{0,0}(u)e^{-\gamma(t-u)}(t-u)du$

Hence $\eta(t) = 2(\lambda+\mu)\int_0^t m(x)dx + (\lambda+\mu)t - 4\mu\int_0^t p_{1,1}(x)dx - \mu\int_0^t p_{1,2}(x)dx$

$$-\mu\int_0^t p_{0,0}(x)dx - 2\mu\int_0^t g(x)dx$$

2.3 The steady state probabilities

In this section we shall discuss the steady state results of the $M/M/1$ queue with server vacations under a multiple vacation scheme. Multiplying (11) by s on both sides and taking limits as $s \to 0$, we get,

$$\lim_{s\to 0} sp^*_{0,0}(s) = \frac{\gamma}{\lambda+\gamma}(1-\rho)$$

$$p_{0,0} = \frac{\gamma}{\lambda+\gamma}(1-\rho)\ (using\ Tauberian\ theorem)$$

Similarly we obtain

$$p_{0,n} = \left(\frac{\lambda}{\lambda+\gamma}\right)^n \frac{\gamma}{\lambda+\gamma}(1-\rho),\ \ n \geq 1,$$

$$p_{1,n} = \frac{\rho^n (1-\rho)\gamma}{\lambda+\gamma-\mu}\left(1-\left(\frac{\mu}{\lambda+\gamma}\right)^n\right),\ \ n \geq 1.$$

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