Pak.j.stat.oper.res. Vol.18 No. 2 2022 pp 417-426 DOI: http://dx.doi.org/10.18187/pjsor.v18i2.3319

Pakistan Journal of Statistics and Operation Research

# Mathematical Modeling of Age-Specific Fertility Rates of Nepali Mothers: A Polynomial Approach

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**Abstract**

In this paper, polynomial models have been formulated to describe the distribution pattern of age-specific fertility rates (ASFRs) and forward-cumulative ASFRs of Nepali mothers. The former follows the bi-quadratic polynomial and the latter follows the quadratic one. Velocity and elasticity equations of the fitted models have been formulated. The areas covered by the curves of the fitted models have been evaluated, and the area covered by the curve of ASFRs is equivalent to the total fertility rate (TFR). The elasticity curve of ASFRs helps to measure the sensitivity of ASFRs to the age of women at the birth of a child. Furthermore, the mode of the fitted ASFRs has been estimated. To test the stability and validity of fitted models, cross-validation prediction power, shrinkage of the model, F-test statistics, and the coefficient of determination have been applied. All the test statistics suggest that the proposed models are significantly fit the ASFRs.

**Key Words:** Fertility rates, polynomial model, cross-validity prediction power, shrinkage, coefficient of determination.

**Mathematical Subject Classification:** 62-07, 00A71, 97M70, 91D20

## 1. INTRODUCTION

The natural capability of women to produce offspring is termed fertility, which is one of the fundamental components of population growth. Fertility is the key demographic indicator among others, including migration and mortality. Age-specific fertility rate (ASFR) is the total number of children birth to women in a specific age group, expressed as births per thousand years of exposure in that age group in some defined area. ASFR patterns are specified using various mathematical and stochastic models [discussed in the following paragraphs**]**. However, the modeling of ASFRs in the case of Asian countries has been used on a limited scale (Islam, 2011). In Nepal, few researchers try to model the fertility data most of the researchers are mainly confined to finding the differential and determinant of demographical variables. So in this research, a polynomial model is proposed to fit the ASFRs of Nepal. The velocity curve measures the rate of change of ASFRs with respect to the age of the mother at the birth of a child and the elasticity curve of ASFRs which measures the sensitivity or the percentage change in the curve of ASFRs are formulated. Also, the peak of the fertility curve and the area covered by the curve are computed. The validity tools are used to measure the suitability of the proposed model.

ASFRs provide a clear picture of the fertility behavior of women in different age groups. The prime concern of many researchers and policymakers is on different age groups of women, especially adolescents. According to the Nepal Demographic and Health (NDH) Survey report, 17% of women of Nepal begin childbearing during the ages between 15 and 19 (NDHS, 2016). Adolescence is considered an inadequate period for pregnancy. Early childbearing among adolescents is likely to cause adverse health consequences for mothers and babies. Some of the consequences of adolescent pregnancy are lacking in pursuing educational opportunities, increased risk of sickness, death of both mothers and children. Jeha et al. (2015) include increased risk of anemia, postpartum depression and infections for the teen mother, higher risk of cesarean delivery, and lacking proper breastfeeding initiation. Santos et al. (2014) found a significant association between the underweight of children and early pregnancy of adolescents. According to the Global Health Estimates (2016) report of the World Health Organization, adolescent birth children face a higher risk of low birth weight, preterm delivery, and severe neonatal conditions. Recent estimates (2011 and 2016 DHS estimates) of adolescent fertility rates for municipalities show higher and stagnant levels in many less developed areas of Nepal (Greenbaum and KC, 2001). Therefore, the adolescent fertility issue is important for health and social reasons.

In literature, a variety of fertility models have been found. Some demographers have proposed deterministic models and others proposed stochastic models. Hoem et al. (1981) presented a fertility curve in stochastic form; they expressed several specifications of probability density function by using Hadwiger, Gamma, Beta, Coal-Trussel, Brass, and the Gompertz function. Furthermore, they defined two models based on regression spline and polynomial functions. Gilje (1969) used a generalized Hadwiger function to fit the fertility data of Hungary and Norway, which has three parameters and is useful for bi-models ASFRs data. Luther (1984) used the Makeham curve to fit the ASFRs of Pakistan. Chandola et al. (1999) used the two-component mixture model of Hadwiger to capture the modern fertility pattern in the population of the UK, Ireland, and the USA. Schmertmann (1999) forwarded a thirteen parameters piecewise quadratic splines function to be estimated, and it showed a good fit to the wide variety of fertility schedules. But Beer (2011) claimed that the parameters were not interpretable demographically. Gayawan et al. (2010) presented the adjusted error model to fit the ASFRs of some African countries. Kostaki et al. (2009) proposed the support vector mechanics, a modern nonparametric graduation technique to fit a single year ASFRs for the population of ten countries: Sweden, Norway, Denmark, Belgium, Greece, Italy, UK, Ireland, Spain, and the white and the black population of the USA.

Similarly, skew-normal distribution and its generalization have been investigated and applied to ASFRs by Azzalini (1985, 2005). Mazzuco and Scarpa (2011) studied a skew-symmetric model to fit the fertility pattern of countries that experienced a bimodal-fertility schedule viz the USA, the UK, Ireland, and countries that keep a classic uni-modal fertility pattern, viz Italy and the Czech Republic. Asili et al. (2014) used a skew-logistic probability model to fit ASFRs of Italy, and the same model has been applied to fit ASFRs of India (Mishra et al., 2017). Gaire et al. (2019) proposed and used the skew-log-logistic probability distribution and Gaire and Aryal (2015) applied the Inverse Gaussian distribution model to fit the ASFRs of Nepali mothers. Peristera and Kostaki (2007) proposed a new model based on the normal mixture model to capture both traditional and modern distorted ASFRs. Islam (2011) used a polynomial model to fit the age-specific marital fertility rate of Bangladesh, and ASFRs of Indonesia. Islam and Ali (2004) used a bi-quadratic polynomial model to fit ASFRs of a rural area of Bangladesh. Singh et al. (2015) applied the third-degree polynomial of age and reciprocal of age for the ASFRs of India. The observed ASFRs of Nepal is still classical uni-modal so, at this juncture, a polynomial model is proposed and studied for the ASFRs of Nepali mothers with the following objectives:

**i.** To build the polynomial models for ASFRs of Nepali mothers.
**ii.** To apply validity tools to test the stability and validity of the proposed models.

Different researchers and demographers chose different validity tools to validate their model to fit ASFRs. Here for the fitting of the polynomial model to ASFRs of Nepal, we choose cross-validity prediction power (CVPP), F-test, shrinkage of the fitted model, and coefficient of determination have been used as the validity tools to test the significance of the polynomial model to fit the ASFRs of Nepal. Furthermore, Velocity curve to measure the rate of change of ASFRs with respect to the age of mother at the birth of a child and the elasticity curve of ASFRs to measures the sensitivity as well as the percentage change in ASFRs to the age of women at the birth of a child have been used.

The rest of the paper is organized as follows: Section 2 includes methods, the theoretical concept of polynomial models, the velocity curve, elasticity curve, the area covered by the curve, the peak of the fertility curve, and the validity tools. The bi-quadratic polynomial model for ASFRs and quadratic polynomial model for forward-cumulative ASFRs of Nepali mothers have been presented and the mathematical results are discussed in Section 3 and finally, Section 4 concludes the paper.

## 2. METHODS

To study and evaluate the proposed models and to fulfill the above objectives, two real data sets of ASFRs of Nepali women aged 15 – 49 have been taken from the Nepal Demographic and Health Surveys (NDHS-2011, NDHS-2016). Both surveys are the national representative surveys, the 2011 NDH survey covers the completed interviews with 12,674 women aged 15 – 49 in a sample of 10,826 households whereas the 2016 NDH survey consists of completed interviews with 12,674 women aged 15 – 49 in the 11,040 selected households. The polynomial model, velocity and elasticity equation (curve) of these models, the area and maxima of the ASFRs curve along with the estimation of parameters are discussed in brief. Estimation of the parameters of the polynomial model with its related graphs and calculation of the area under different fitted curves has been carried out by using poly-function in R-Programming software (R-Studio, 2015) and using polyfit function in MATLAB software.

## 2.1. POLYNOMIAL MODEL

According to Spiegel (1992), the polynomial relationship between independent variable x and function of this as dependent variable y is defined in the form

$$y = f(x) = a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + \cdots + a_r\, x^r \qquad (1)$$

Here, we consider x as the age of the mother at the birth of the child. ASFRs are considered as a function of the age of the mother at the birth of the child. This function is described by the following polynomial model.

$$f(x) = a_0 + \sum_{i=1}^{r} a_i x^i + e \qquad (2)$$

where '$e$' is the error term, follows Normal distribution with mean zero and variance ($\square^2$) i.e. E(e) = 0 and V(e)= ($\square^2$), $a_i$ are constants ($a_0 \neq 0$) and $'r'$ is a positive integer.

Equation (2) is called the polynomial of degree $'r'$. We obtain constant, linear, quadratic, cubic, and higher-order polynomial functions if the value of 'r' is chosen as 0, 1, 2, 3, and so on respectively.

## 2.2. VELOCITY AND ELASTICITY CURVE

The velocity curve is just the first derivative of the fitted polynomial regression concerning the age of the mother at the birth of the child (Gasser et al., 1984). To draw the velocity and elasticity curves, we fit the regression models of ASFRs of Nepali mothers. The velocity curve is the first derivative of the fitted polynomial curve of ASFRs, which is the rate of change of ASFRs with respect to the age of the mother at the birth of a child as follows:

$$\frac{dy}{dx} = f'(x) = a_1 + 2a_2\, x + 3a_3\, x^2 + \cdots + na_n\, x^{n-1} \qquad (3)$$

The elasticity curve is estimated by using the relation mentioned by Dewett (2015) as

$$\text{Elasticity} = \frac{dlogy}{dlogx} = \frac{x\, dy}{y\, dx} = \frac{x}{y} f'(x) \qquad (4)$$

## 2.3. THE AREA COVERED BY ASFRS CURVE

If f(x) is a single-valued bounded continuous function defined in the interval $(a, b)$, both a and b being finite quantities with $b > a$, then the area is defined as the definite integral of function f(x) with respect to x within the limits of a and b expressed as

$$Area\ = \int_a^b f(x)dx \qquad (5)$$

In this study, the function f(x) is a fitted model of ASFRs and forward-cumulative ASFRs which is the function of the age of the mother at the birth of the child. In this paper, we set lower bound $a = 15$ and upper bound $b = 49$ years of mothers since the reproductive age of women is considered as $(15 - 49)$ years (WHO, 2006). The total area covered by the curve of ASFRs is the equivalent total fertility rate (TFR).

## 2.4. PEAK OF CHILD-BEARING OF THE ASFRS CURVE

If f(x) is a continuous and differentiable plane curve, then the peak of the function is at the point where the first derivative has vanished and the second derivative is negative at that point. That is,

$$\frac{dy}{dx} = 0 \ \ and \ \ \frac{d^2y}{dx^2} < 0 \qquad (6)$$

## 2.5. CROSS-VALIDATION PREDICTION POWER AND F-TEST STATISTICS

To test the stability and validity of the proposed model, we use cross-validation prediction power (CVPP), F-Test Statistics, and shrinkage of the model. From the relationship provided by Stevens (1996), the CVPP is defined as follows:

$$\rho_{CV}^2 = 1 - \frac{(n^2-1)(n-2)}{n(n-k-1)(n-k-2)}(1-R^2) \quad (7)$$

where n is the number of classes, k is the number of variables in the model. The cross-validated R is the correlation between observed ASFRs and expected ASFRs obtained from the fitting polynomial model for Nepali women. The higher the value of R better the model fits the data. For the stability of the coefficient of determination of the model, we use the relationship as (1- shrinkage) of the model where the expression $|\rho_{CV}^2 - R^2|$ is the shrinkage of the model. Furthermore, to verify the overall measure of the significance of the model and the significance of $R^2$, we chose the formula of $F-$test statistics taken from Gujarati (2009) as follows:

$$F = \frac{ESS/_{(m-1)}}{RSS/_{n-m}} = \frac{R^2/_{(m-1)}}{(1-R^2)/_{n-m}} \quad (8)$$

where $n$ is the number of cases, m is the number of parameters of the fitted model and ESS is the error sum of a square and RSS is the regression sum of a square. The model will be significant with a p-value less than 5% for the computed value of F-test statistics.

## 3. NUMERICAL RESULT AND DISCUSSION

The fitted bi-quadratic polynomial models assumed for ASFRs data of Nepali mothers for the years 2011 and 2016 are as follows:

$$y_{(2011)} = -4.072 + 0.5442x - 0.02495x^2 + 0.0004836x^3 - 0.0000034x^4 \quad (9)$$

$$y_{(2016)} = -3.581 + 0.4795x - 0.02187x^2 + 0.0004195x^3 - 0.000002909x^4 \quad (10)$$

The quadratic polynomial is considered for the forward-cumulative ASFRs of Nepali mothers for the years 2011 and 2016, which are as follows:

$$z_{(2011)} = -0.7545 + 0.06137x - 0.000733x^2 \quad (11)$$

$$z_{(2016)} = -0.7049 + 0.05878x - 0.0007248x^2 \quad (12)$$

where y and z are ASFRs and forward-cumulative ASFRs of Nepali mothers respectively.

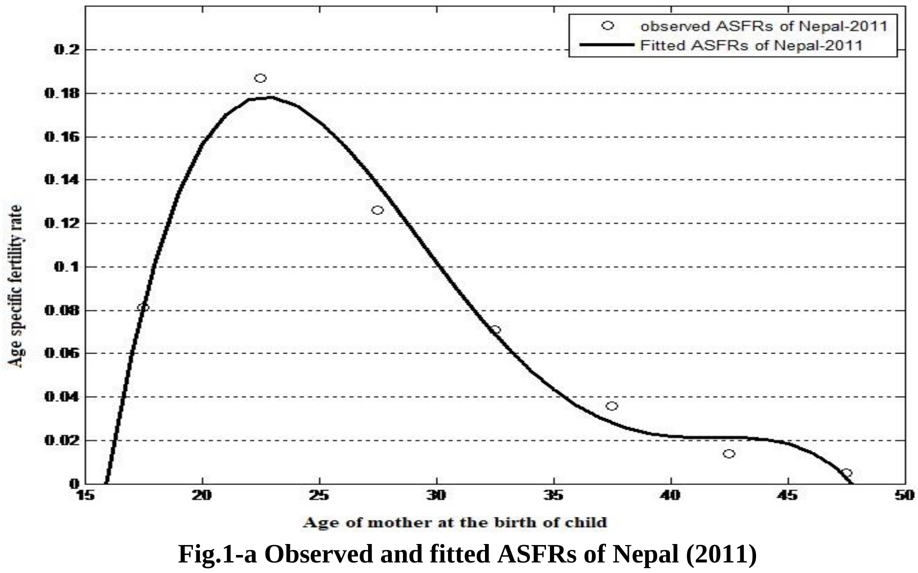


**Fig.1-a Observed and fitted ASFRs of Nepal (2011)**

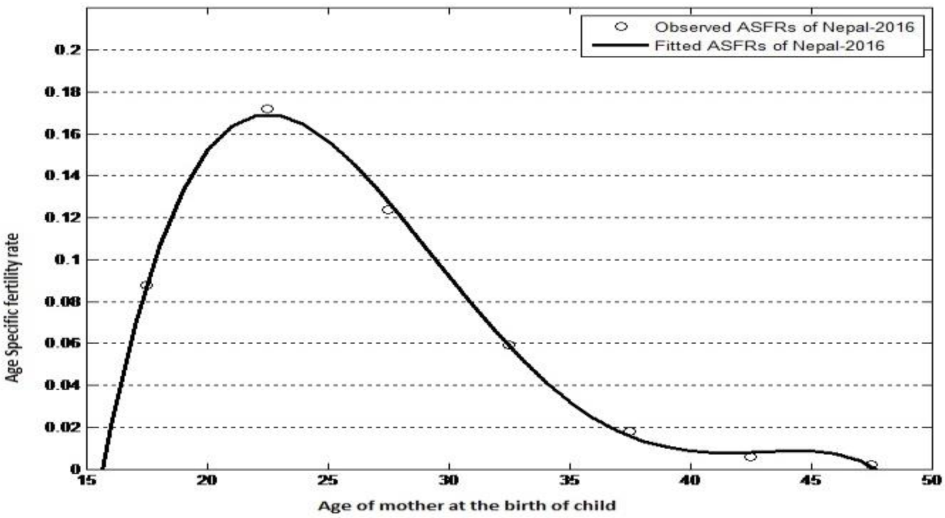


**Fig.1-b Observed and fitted ASFRs of Nepal (2016)**

It has been observed from equations (9) and (10) that the fitted bi-quadratic polynomial has a maximum value at the point $x = 22.48$ years for ASFRs of 2011 and $x = 22.70$ years for ASFRs of 2016. Thus, the result shows that the fertility of Nepali women is the highest at the year 22 which is consistent with the observed ASFRs which is the highest for the year group $20 - 24$ for both 2011 and 2016. Figure-1a and Figure-1b show the observed and fitted ASFRs of Nepali mothers. Table-1 and Table-2 give the observed, predicted value of ASFRs as well as forward cumulate ASFRs and the residual for the years 2011 and 2016. The error terms (residual) in both cases have a mean zero.

**Table-1: Observed and expected value of ASFRs by using the bi-quadratic Polynomial**

| Age group | Age-Specific Fertility Rates | | | | | |
|---|---|---|---|---|---|---|
| | Observed (2011) | Predicted (2011) | Residual (2011) | Observed (2016) | Predicted (2016) | Residual (2016) |
| 15 – 19 | 0.081 | 0.0831 | -0.0021 | 0.088 | 0.0886 | -0.0006 |
| 20 – 24 | 0.187 | 0.1784 | 0.0086 | 0.172 | 0.1695 | 0.0025 |
| 25 – 29 | 0.126 | 0.1377 | -0.0117 | 0.124 | 0.1273 | -0.0033 |
| 30 – 34 | 0.071 | 0.0686 | 0.0024 | 0.059 | 0.0584 | 0.0006 |
| 35 – 39 | 0.036 | 0.0279 | 0.0081 | 0.018 | 0.0156 | 0.0024 |
| 40 – 44 | 0.014 | 0.0212 | -0.0072 | 0.006 | 0.0081 | -0.0021 |
| 45 – 49 | 0.005 | 0.0031 | 0.0019 | 0.002 | 0.0015 | 0.0005 |
| Total | 0.520 | 0.520 | | 0.469 | 0.469 | |

**Table-2: Observed and expected value of forward-cumulative ASFRs by using quadratic Polynomial**

| Age group | Forward-Cumulative Age-Specific Fertility Rates | | | |
|---|---|---|---|---|
| | Observed (2011) | Predicted (2011) | Observed (2016) | Predicted (2016) |
| 15 – 19 | 0.081 | 0.0948 | 0.088 | 0.1018 |
| 20 – 24 | 0.268 | 0.2550 | 0.260 | 0.2507 |
| 25 – 29 | 0.394 | 0.3785 | 0.384 | 0.3634 |
| 30 – 34 | 0.465 | 0.4653 | 0.443 | 0.4399 |
| 35 – 39 | 0.501 | 0.5155 | 0.461 | 0.4801 |
| 40 – 44 | 0.515 | 0.529 | 0.467 | 0.4841 |
| 45 – 49 | 0.520 | 0.5058 | 0.469 | 0.4519 |

By applying the definite integral to the fitted models, the area between the fitted curves for the age $15-49$ years are found as $2.54$, $2.26$, $13.12$and $12.38$ respectively. Here, the area covered by the models of ASFRs represents the total fertility rate (TFR). The observed TFR of Nepali women for the year 2011 is 2.6 and from the fitted model, it has been found as 2.54. Similarly, the observed TFR of Nepali women for the year 2016 is 2.3 and from the fitted model, it has been found as 2.26. Figure-2a and Figure-2b show the observed and fitted ASFRs and cumulative ASFRs of Nepali mothers.

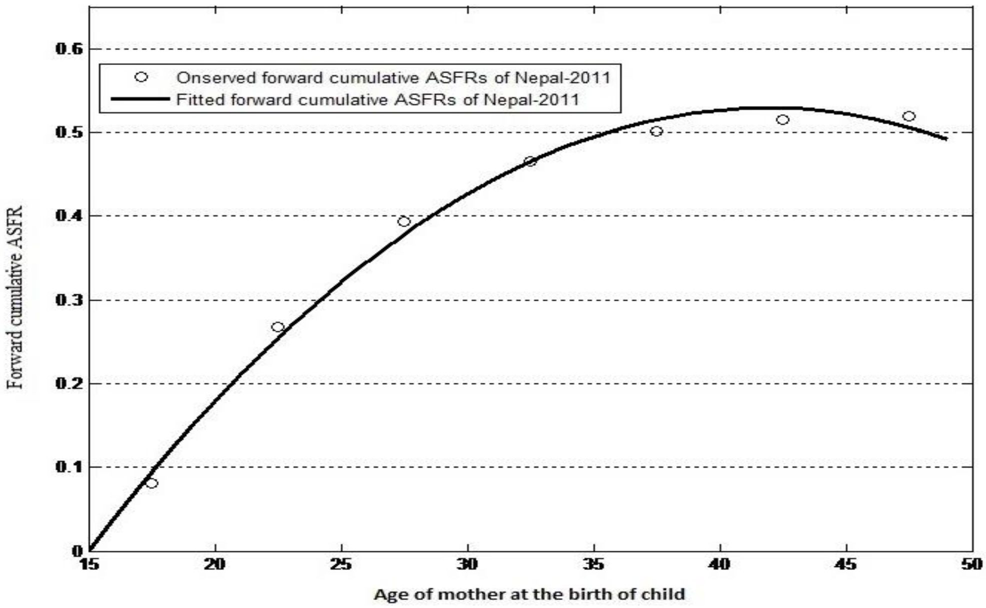


**Figure-2-a: Observed and Fitted Forward-Cumulative ASFRs of Nepal (2011)**

The values of the goodness of fit of the models (9) to (12) as the residual standard error, the value of CVPP ($\rho^2_{CV}$), the coefficient of determination ($R^2$), the calculated value of F-test statistics and their p-values are presented in Table-3. The F-test statistics and p- values indicate that the models are statistically significant at 5% level of significance.

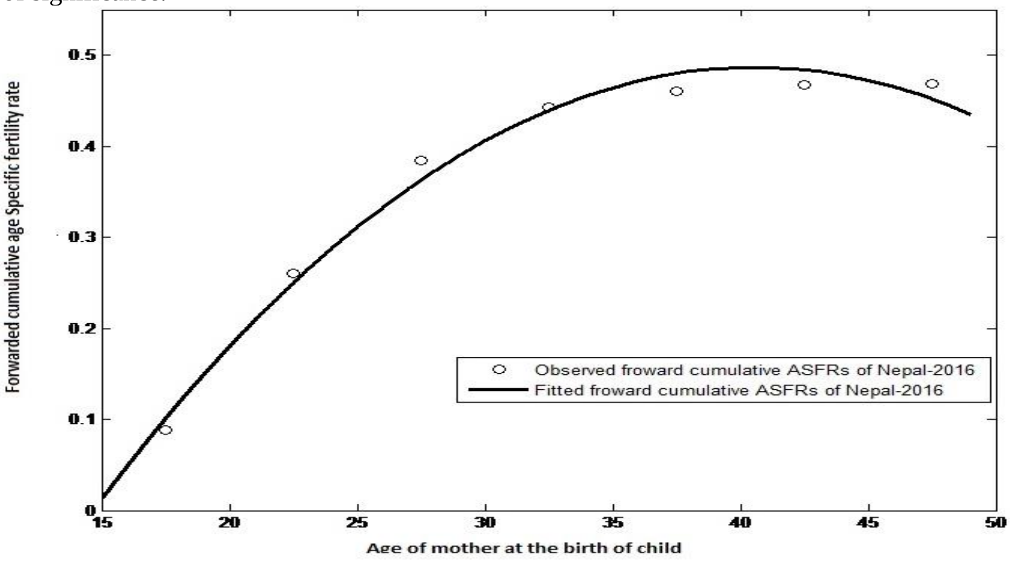


**Figure-2-b: Observed and Fitted Forward-Cumulative ASFRs of Nepal (2016)**

From the observed value, it is found that the fitted models (9) to (12) are highly cross-validated and their shrinkages are found as 0.2179, 0.0178, 0.0139, and 0.0247 respectively. The values of CVPP obtained by using equation (7) indicate that the fitted models are stable more than 76%, 98%, 97%, and 96% respectively. The Stability of value of $R^2$ of proposed models have been found more than 78%, 98%, 98%, and 97% respectively and the residual standard errors of the models as 0.01306, 0.00376, 0.01738, and 0.0208 and the value of $R^2$ as 0.9865, 0.9989, 0.9925 and 0.9987 respectively. From the observed value of $R^2$, CVPP, and p-value of F-test statistics it is to be noted that the proposed models are significantly fit the ASFRs of Nepal.

**Table 3: Estimated value of CVPP, for ASFRs and Forward-cumulative of Nepal (2011, 2016)**

| Model | $n$ | $K$ | $R^2$ | $CVPP$ | $Shrinkage$ | Calculated $F$ | Residual S.E. | $P$-value |
|---|---|---|---|---|---|---|---|---|
| Equation (9) | 7 | 4 | 0.9865 | 0.7686 | 0.2179 | 36.63 | 0.01306 | 0.026 |
| Equation (10) | 7 | 4 | 0.9989 | 0.9811 | 0.0178 | 443.8 | 0.00376 | 0.002 |
| Equation (11) | 7 | 2 | 0.9925 | 0.9786 | 0.0139 | 264.3 | 0.01738 | 0.000 |
| Equation (12) | 7 | 2 | 0.9867 | 0.9620 | 0.0247 | 148.0 | 0.02038 | 0.000 |

Figure-3a and Figure-3b depict the velocity curves of ASFRs of Nepali women for the years 2011 and 2016 which are estimated in Equations (13) and (14). These graphs indicate that the velocity of these curves is decreasing rapidly up to the mother's age 30 and slightly increased to age of mother and again drop down when the fertility age is reached 45 – 49 for both data sets.

$$\frac{dy}{dx}_{(2011)} = 0.5442 - 0.0499x + 0.0014508x^2 - 0.0000136x^3 \quad (13)$$

$$\frac{dy}{dx}_{(2016)} = 0.4795 - 0.04374x + 0.0012585x^2 - 0.000011636x^3 \quad (14)$$

Equations (15) and (16) give the velocity curves of forward-Cumulative ASFRs of Nepali women for the years 2011 and 2016 are respectively, which show that the velocity of these curves is constantly decreasing with respect to the increase in age of the mother.

$$\frac{dz}{dx}_{(2011)} = 0.06137 - 0.001466x \quad (15)$$

$$\frac{dz}{dx}_{(2016)} = 0.05878 - 0.0014496x \quad (16)$$

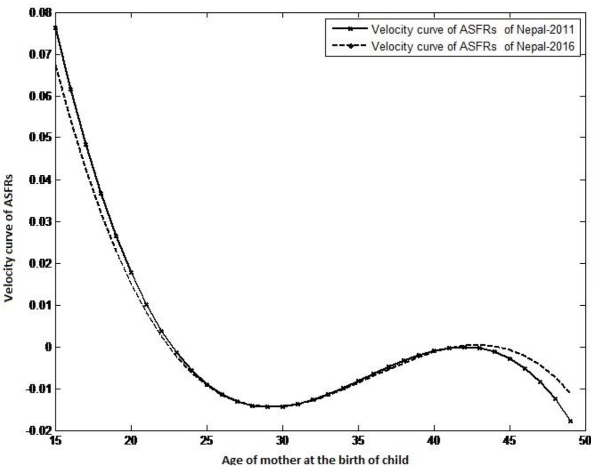


**Figure 3a: Velocity curve of ASFRs of**

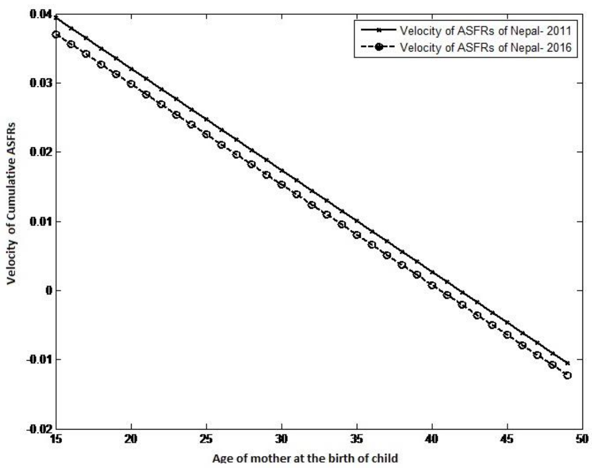


**Figure. 3b: Velocity curve of forward-cumulative ASFR**

Elasticity curve of ASFRs measures the sensitivity of ASFRs to the age of women at the birth of a child. Thus, it measures the percentage change in ASFRs in response to a change in the age of women at the birth of a child. This helps to interpret the change in ASFRs concerning the change in the age of the mother at the birth of a child. Equations (17) and (18) present the elasticity curves of fitted ASFRs of Nepali women-2011 and 2016 respectively, which show that the elasticity of the curves (Percentage change in ASFRs of Nepali women with respect of age of women at the birth of a child) remains constant except for some fluctuation at the initial and final age of mothers at the birth of a child. Similarly, the values for the velocity and elasticity curves of ASFRs and forward-cumulative ASFRs of Nepali mothers have been presented in Figure-4a and Figure-4b.

$$\text{Elasticity of } y_{(2011)} = \frac{x}{y}(0.5442 - 0.0499x + 0.0014508x^2 - 0.0000136x^3) \quad (17)$$

$$\text{Elasticity of } y_{(2016)} = \frac{x}{y}(0.4795 - 0.04374x + 0.0012585x^2 - 0.000011636x^3) \quad (18)$$

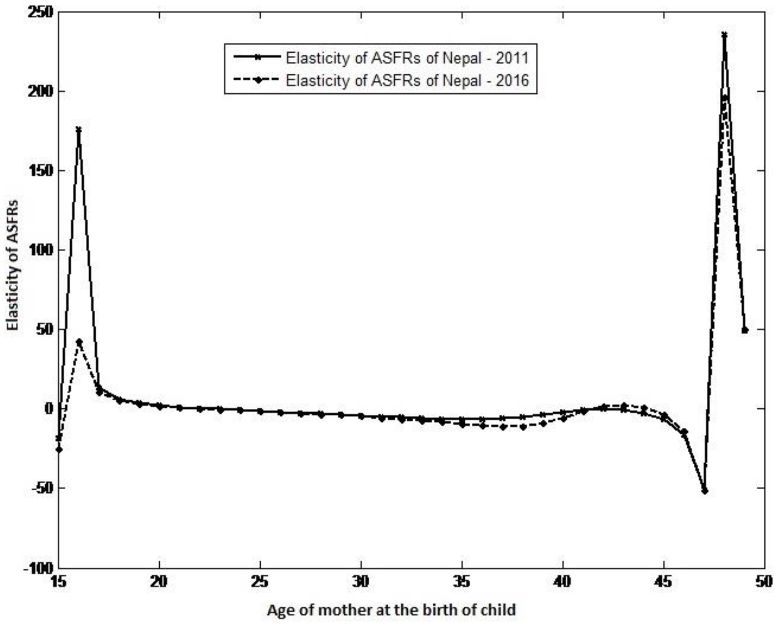


**Figure-4a: Elasticity curve of ASFRs of Nepal**

Equations (19) and (20) represent the elasticity curves of fitted forward-cumulative ASFRs of Nepali women for the years 2011 and 2016 respectively. This indicates that the elasticity curves ASFRs (that is the percentage change ASFRs) are decreasing with respect to the increase in the age of mothers at the birth of a child.

$$\text{Elasticity of } z_{(2011)} = \frac{x}{z}(0.06137 - 0.001466x) \quad (19)$$

$$\text{Elasticity of } z_{(2016)} = \frac{x}{z}(0.05878 - 0.0014496x) \quad (20)$$

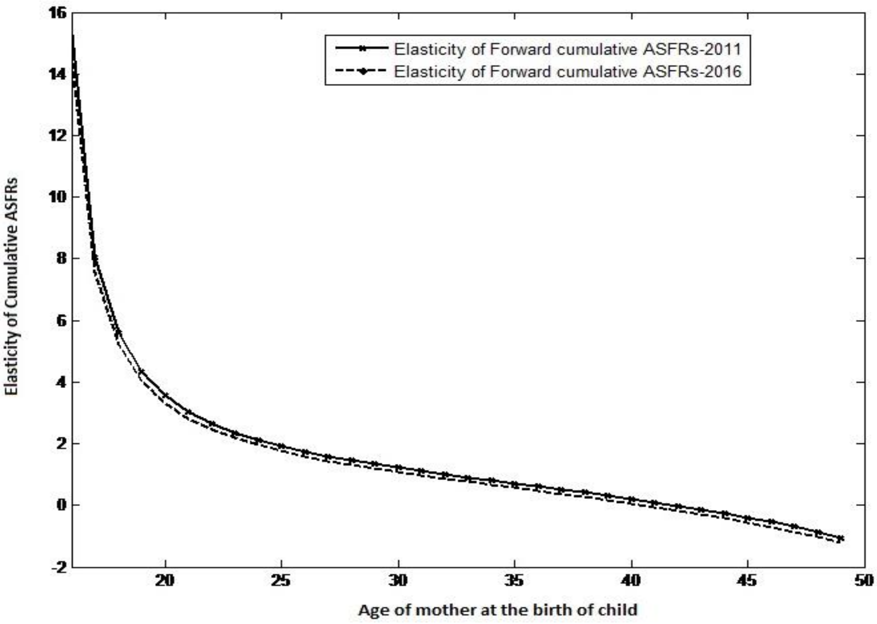

**Figure-4b: Elasticity curve of forward-cumulative ASFRs**

## 4. CONCLUSION

The ASFRs and forward-cumulative ASFRs of Nepali mothers for the years 2011 and 2016 are modeled mathematically as polynomial functions and fitted. The p-value of test statistics, the value of $R^{2,}$ and the CVPP value indicate that ASFRs followed the bi-quadratic model and the forward-cumulative ASFRs followed the quadratic model significantly. In addition to that, the velocity and elasticity curves of both ASFRs and cumulative ASFRs have been formulated and sketched. Also, the areas covered by each of the models have been estimated. It has been observed that the area covered by the ASFRs curve represents the TFR which is consistent with the observed TFR. The parameters of the model schedule may reflect the level and timing of fertility of Nepali mothers. The area covered by the curve of ASFRs may be used for the estimation of TFR, elasticity curve of ASFRs helps to measure the sensitivity of ASFRs to the age of women at the birth of a child.

## REFERENCES


1. Asili, S., Rezaei, S. & Najjar, L. (2014). Using skew-logistic probability density function as a model for age-specific fertility rate pattern. *BioMed research international*. 10, 1-5.
2. Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian journal of statistics*. 171-178.
3. Azzalini, A. (2005). The skew-normal distribution and related multivariate families. *Scandinavian Journal of Statistics*. *32*(2), 159-188.
4. Beer, J.D. (2011). A new relational method for smoothing and projecting age-specific fertility rates: TOPALS. *Demographic Research*. 24, 409-454.
5. Chandola, T., Coleman, D.A. & Hiorns, R.W. (1999). Recent European fertility patterns: Fitting curves to distorted distributions. Population Studies. 53(3), 317-329.
6. Dewett, K.K. (2015). *Modern Economic Theory*. S. Chand and Company Ltd. (22nd Revised edition) New Delhi, India.
7. DHS, M. (2011). *Population Division Ministry of Health and Population*, Ramshah Path, Kathmandu Nepal.
8. Gaire, A.K. & Aryal, R. (2015). Inverse Gaussian model to describe the distribution of age-specific fertility rates of Nepal. *Journal of Institute of Science and Technology*. 20(2), 80-83.
9. Gaire, A.K., Thapa G.B. & KC, S. (2019). Preliminary results of Skew Log-logistic distribution, properties, and application. *Proceeding of the 2nd International Conference on Earthquake Engineering and Post Disaster Reconstruction Planning* (ICEE-PDRP-2019), 25–27 April 2019, Bhaktapur, Nepal, 37-43.
10. Gasser, T., Köhler, W., Müller, H.G., Kneip, A., Largo, R., Molinari, L. & Prader, A. (1984). Velocity and acceleration of height growth using kernel estimation. Annals of Human Biology. 11(5), 397-411.
11. Gayawan, E., Adebayo, S.B., Ipinyomi, R.A. & Oyejola, B.A. (2010). Modeling fertility curves in Africa. *Demographic Research*, 22, 211-236.
12. Gilje, E. (1969). Fitting curves to age-specific fertility rates: Some examples. *Statistical Review of the Swedish National Central Bureau of Statistics*. 3(7), 118-134.
13. Global Health Estimates. (2016). Deaths by cause, age, sex, by country and by region, 2000-2016. Geneva: WHO.
14. Greenbaum, C, Samir KC (2001). Population Reference Bureau (PRB). Municipality-level estimates of adolescent fertility in Nepal. New York: Retrieved from: https://www.prb.org/resources/municipality-level-estimates-of-adolescent-fertility-in-nepal.
15. Gujarati, D.N. (2009). *Basic Econometrics*. Tata McGraw-Hill Education, India.
16. Hoem, J.M., Madien, D., Nielsen, J.L., Ohlsen, E.M., Hansen, H.O. & Rennermalm, B. (1981). Experiments in modeling recent Danish fertility curves. *Demography*. 18(2), 231-244.
17. Islam, M.R. & Ali, M.K. (2004). Mathematical modeling of age-specific fertility rates and study of the productivity in the rural area of Bangladesh during 1980-1998. *Pakistan Journal of Statistics*. 20(3), 379-392.

18. Islam, R. (2011). Modeling of age-specific fertility rates of Jakarta in Indonesia: A polynomial model approach. *International Journal of Scientific & Engineering Research*. 2(11), 1-5.
19. Jeha, D., Usta, I., Ghulmiyyah, L., & Nassar, A. (2015). A review of the risks and consequences of adolescent pregnancy. *Journal of neonatal-perinatal medicine*, *8*(1), 1-8.
20. Kostaki, A., Moguerza, J.M., Olivares, A. & Psarakis, S. (2009). Graduating the age-specific fertility pattern using Support Vector Machines. *Demographic Research*. 20, 599-622.
21. Luther, N. Y. (1984). Fitting age-specific fertility with the Makeham curve. In *Asian and Pacific census* forum 10(3), 5.
22. Mazzuco, S. & Scarpa, B. (2011). Fitting age-specific fertility rates by a skew-symmetric probability density function. *The University of Padova, Working paper Series Italy.* 10.
23. Ministry of Health, Nepal; New ERA & ICF (2017). *Nepal Demographic and Health Survey 2016: Key Indicators*. Kathmandu, Nepal: Ministry of Health, Nepal. https://www.newera.com.np/NDHS-2016%20Key%20Indicators.pdf.
24. Mishra, R., Singh, K.K. & Singh, A. (2017). A model for age-specific fertility rate pattern of India using skew-logistic distribution function. *American Journal of Theoretical and Applied Statistics*. 6(1), 32-37.
25. Peristera, P. & Kostaki, A. (2007). Modeling fertility in modern populations. *Demographic Research*. 16, 141-194.
26. RStudio, R.T. (2015). Integrated Development for R. RStudio, Inc., Boston, MA.
27. Santos, NLDAC, Costa, MCO, Amaral, MTR, Vieira, GO, Bacelar, EB, & Almeida, AHDVD (2014). Adolescent pregnancy: analysis of risk factors for low birth weight, prematurity, and cesarean section. *Science & Public Health*, *19*, 719-726.
28. Schmertmann, C.P. (2003). A system of model fertility schedules with graphically intuitive parameters. *Demographic Research*. 9, 81-110.
29. Singh, B.P., Gupta, K. and Singh, K. K. (2015). Analysis of fertility pattern through mathematical curves. *American Journal of Theoretical and Applied Statistics*, *4*(2), 64-70.
30. Spiegel, M.R. (1992). *Theory and Problems of Statistics* (Second Edition in SI unit). McGraw-Hill Book Company, Schaum's Outline Series, London, UK.
31. Stevens, J. (1996). *Applied Multivariate Statistics for the Social Sciences* (3rd edition), New Jersey: Lawrence Erlbaum Associates Inc. Publishers.
32. World Health Organization. (2006). Reproductive Health Indicators Reproductive Health and Research Guidelines for their generation, interpretation, and analysis for global monitoring. *Geneva: WHO*.