Pakistan Journal of Statistics and Operation Research

The four-parameter exponentiated Weibull model with Copula, properties and real data modeling

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Abstract

A new four-parameter lifetime model is introduced and studied. The new model derives its flexibility and wide applicability from the well-known exponentiated Weibull model. Many bivariate and the multivariate type versions are derived using the Morgenstern family and Clayton copula. The new density can exhibit many important shapes with different skewness and kurtosis which can be unimodal and bimodal. The new hazard rate can be decreasing, **J**-shape, **U**-shape, constant, increasing, upside down and increasing-constant hazard rates. Various of its structural mathematical properties are derived. Graphical simulations are used in assessing the performance of the estimation method. We proved empirically the importance and flexibility of the new model in modeling various types of data such as failure times, remission times, survival times and strengths data.

Key Words:
Marshall-Olkin Family**;** Lehmann Weibull Distribution; Order Statistics, Maximum Likelihood Estimation; Simulation; Generating Function; Quantile function; Moments.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction and genesis

The first generalization allowing for nonmonotone hazard rates is the exponentiated Weibull (EW) distribution due to Mudholkar and Srivastava (1993) and Mudholkar et al. (1995). A random variable Z is said to have the EW distribution if its probability density function (PDF) and cumulative distribution function (CDF) are given by

$$h_{c_2,c_1}(z) = c_1 c_2 z^{c_1-1} e^{-z^{c_1}} \left(1 - e^{-z^{c_1}}\right)^{c_2-1} |_{(z>0 \text{ and } c_2,c_2>0)},$$

and

$$H_{c_2,c_1}(z) = \left(1 - e^{-z^{c_1}}\right)^{c_2} |_{(z>0, c_2>0 \text{ and } c_2>0)},$$

respectively. We write $Z \sim \text{EW}\,(c_2, c_1)$. The case for $c_1 = 1$ is the exponentiated exponential (EE) distribution due to Gupta and Kundu (1999). The case for $c_2 = 1$ is the one parameter Weibull distribution (Weibull (1951)). The case for $c_1 = 2$ is the Burr type X (BrX) distribution or the exponentiated Rayleigh (ER) distribution. The case for $c_1 = 2$ and $c_2 = 1$ is the Rayleigh (R) distribution. Based on Weibull (1951) and Lehmann (1953), consider a baseline CDF of the Lehmann exponentiated Weibull (LEW) distribution

$$G_{\theta,c_2,c_1}(z) = 1 - \left[1 - \left(1 - e^{-z^{c_1}}\right)^{c_2}\right]^{\theta} |_{(z\geq 0 \text{ and } \theta,c_2,c_1>0)}, \quad (1)$$

The corresponding PDF of (1) can be derived as

$$g_{\theta,c_2,c_1}(z) = \theta c_2^{c_1} c_1 z^{c_1-1} \frac{e^{-z^{c_1}}\left(1-e^{-z^{c_1}}\right)^{c_2}}{\left\{1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right\}^{1-\theta}} |_{(z\geq 0 \text{ and } \theta,c_2,c_1>0)}, \quad (2)$$

For $\theta = 1$, the LEW model reduces to the EW model. For $\theta = 1 = c_1 = 1$, the LEW model reduces to the EE model. For $\theta = 1, c_1 = 2$, the LEW model reduces to the ER model. For $\theta = c_2 = 1, c_1 = 2$, the LEW model reduces to the R model. For $\theta = c_2 = c_1 = 1$, the LEW model reduces to the E model. For $c_1 = 1$ we have the Lehmann E (LE) model. For $c_1 = 1$, the LEW model reduces to the two-parameter Lehmann W model. For $c_1 = 2$, the LEW model reduces to the Lehmann R (LR) model. Marshall and Olkin (1996) pioneered a simple method of adding a positive shape parameter into a family of distributions and many authors used this method to extend many well-known models in the last few years. The CDF of the Marshall-Olkin-G (MO-G) family of distributions is defined by

$$F_{\nu,\underline{\psi}}(z) = 1 - \frac{\nu \overline{H}_{\underline{\psi}}(z)}{1-\overline{\nu} H_{\underline{\psi}}(z)} |_{(z\in\Re,\ \nu>0)}, \tag{3}$$

where $\overline{H}_{\underline{\psi}}(z) = 1 - H_{\underline{\psi}}(z)$ is the RF of the baseline model, ν is a positive shape parameter, $\overline{\nu} = 1 - \nu$ and $\underline{\psi}$ refers to the parameters vector of the baseline model. For $\nu \in (0,1)$, MOL-G family reduces to the complementary geometric -G (CGc-G) family. For $\nu = 1$, MO-G family reduces to the standard G family. The corresponding PDF of (3) is given by

$$f_{\nu,\underline{\psi}}(z) = \nu h_{\underline{\psi}}(z) \left[1 - \overline{\nu} H_{\underline{\psi}}(z)\right]^{-2} |_{(z\in\Re,\ \nu>0)}, \tag{4}$$

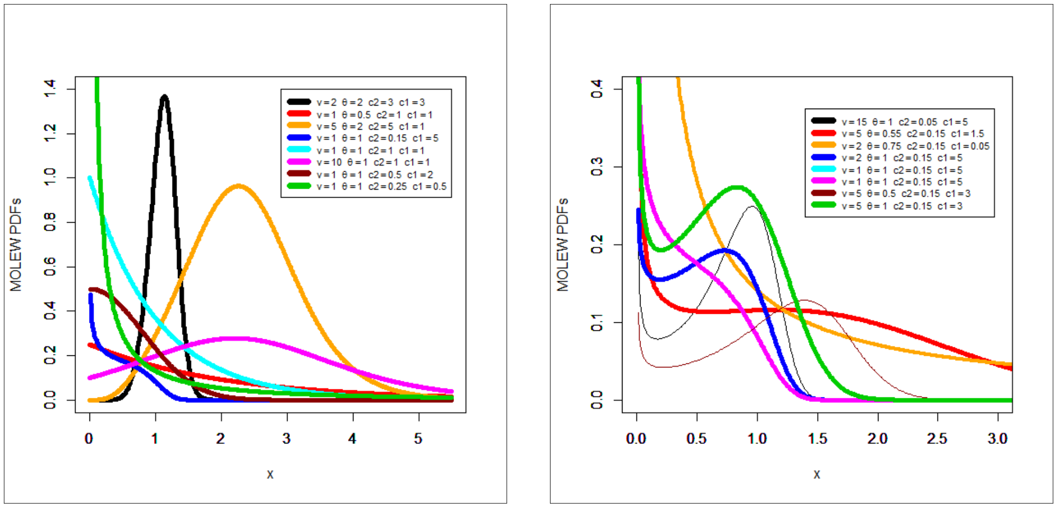
where $h_{\underline{\psi}}(z) = \frac{d}{dx} H_{\underline{\psi}}(z)$ is the PDF of the baseline model. By inserting (1) in (3), we obtain the CDF of the Marshall-Olkin Lehmann exponentiated Weibull (MOLEW) class

$$F_{\underline{\xi}}(z) = \frac{1-\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}}{1-\overline{\nu}\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}} |_{(z\geq 0 \text{ and } \nu,\theta,c_2,c_1>0)}, \tag{5}$$

where $\underline{\xi} = (\nu, \theta, c_2, c_1)$. The corresponding PDF of (5) is given by

$$f_{\underline{\xi}}(z) = \nu\theta c_2^{c_1} c_1 z^{c_1-1} \frac{e^{-z^{c_1}}\left(1-e^{-z^{c_1}}\right)^{c_2-1}}{\left\{1-\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}\right\}^{-1}} \left\{1 - \overline{\nu}\left[1 - \left(1 - e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}\right\}^{-2} |_{(z\geq 0 \text{ and } \nu,\theta,c_2,c_1>0)}. \tag{6}$$

Table 1 below gives some sub-models from the MOLEW model. Equation (5) and (6) can be also derived based on Yousof et al. (2018a). Figure 1 gives some plots of the MOLEW PDF and HRF. From Figure 1 (a and b) we conclude that the PDF MOLEW distribution can exhibit many important shapes with different skewness and kurtosis which can be unimodal and bimodal. From Figure 2, we conclude that the HRF MOLEW distribution can be decreasing hazard rate $(\nu = 1, \theta = 0.5, c_2 = 0.5, c_1 = 1)$, J-hazard rate $(\nu = 5, \theta = 2, c_2 = 5, c_1 = 20)$, bathtub shape $(\nu = 1, \theta = 0.5, c_2 = 0.5, c_1 = 1)$, constant hazard rate $(\nu = 1, \theta = 1, c_2 = 1, c_1 = 1)$, increasing hazard rate $(\nu = 20, \theta = 2.85, c_2 = 1, c_1 = 1)$, upside down (reversed bathtub shape) $(\nu = 1, \theta = 1, c_2 = 2.5, c_1 = 0.5)$, and increasing-constant hazard rate $(\nu = 1, \theta = 1, c_2 = 3, c_1 = 0.5)$.



(a) (b)

Figure 1: Plots of the MOLEW PDF.

Table 1: Some sub-models from the MOLEW model.

	ν	θ	c_2	c_1	Reduced model
1	$\nu^*\vert\nu^* \in (0,1)$				CGcLEW
2	$\nu^*\vert\nu^* \in (0,1)$			2	CGcLER
3	$\nu^*\vert\nu^* \in (0,1)$			1	CGcLEExp
4	$\nu^*\vert\nu^* \in (0,1)$		1		CGcLW
5	$\nu^*\vert\nu^* \in (0,1)$		1	2	CGcLR
6	$\nu^*\vert\nu^* \in (0,1)$		1	1	CGcLE
7	$\nu^*\vert\nu^* \in (0,1)$	1			CGcEW
8	$\nu^*\vert\nu^* \in (0,1)$	1		2	CGcER
9	$\nu^*\vert\nu^* \in (0,1)$	1		1	CGcER
10	$\nu^*\vert\nu^* \in (0,1)$	1	1		CGcW
11	$\nu^*\vert\nu^* \in (0,1)$	1	1	2	CGcR
12	$\nu^*\vert\nu^* \in (0,1)$	1	1	1	CGcExp
13				2	MOLER
14			1	2	MOLR
15				1	MOLEE
16			1	1	MOLE
17		1			MOEW
18		1		2	MOER
19		1		1	MOEE
20		1	1		MOW
21		1	1	2	MOR
23		1	1	1	MOE
24	1				LEW
25	1		1		LW
26	1			2	LER
27	1		1	2	LR
28	1			1	LEE
29	1		1	1	LEE

30	1	1			EW
31	1	1		2	ER
32	1	1		1	EE
33	1	1	1		W
34	1	1	1	2	R
35	1	1	1	1	E

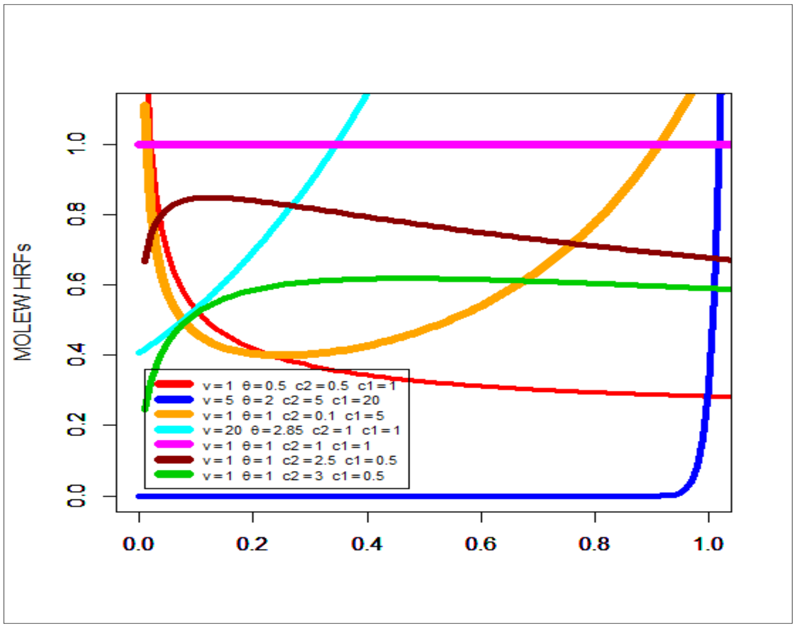


Figure 2: Plots of the MOLEW HRF.

2. Properties

Moments

First the quantity $\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}$ can expanded as

$$\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}=1+\sum_{j_1=0}^{\infty}(-1)^{1+j_1}\binom{\theta}{j_1}\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta j_1}=\sum_{j_1=0}^{\infty}\zeta_{j_1}\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta j_1},\ (7)$$

where

$$\zeta_0=2$$

and

$$\zeta_{j_1}=(-1)^{1+j_1}\binom{\theta}{j_1}|_{(j_1\geq 1)},$$

similarly, the quantity $1-\bar{v}\left(1-\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}\right)$ can expanded as

$$1-\bar{v}\left(1-\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}\right)=1-\bar{v}-\sum_{j_1=0}^{\infty}(-1)^{j_1}\binom{\theta}{j_1}\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta j_1},$$

and then

$$1-\bar{v}\left(1-\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}\right)=\sum_{j_1=0}^{\infty}\eta_{j_1}\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta j_1}, \qquad (8)$$

where $\eta_0=\nu$ and

$$\eta_{j_1} = (1-\nu)(-1)^{1+j_1}\binom{\theta}{j_1},$$

using (7) and (8) the CDF of the MOL-G family in (5) can be expressed as

$$F_{\underline{\xi}}(z) = \sum_{j_1=0}^{\infty} \frac{\zeta_{j_1}\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta j_1}}{\eta_{j_1}[1-(1-e^{-z^{c_1}})^{c_2}]^{\theta j_1}} = \sum_{j_1=0}^{\infty} \tau_{j_1}\left[1-\left(1-e^{-z^{c_1}}\right)^{c_2}\right]^{\theta j_1},$$

where $\Upsilon_0 = \frac{\zeta_0}{\eta_0}$ and

$$\Upsilon_{(j_1)} = \frac{1}{\eta_0}\left(\zeta_{j_1} - \frac{1}{\eta_0}\sum_{r=1}^{j_1} \eta_r \Upsilon_{(j_1-r)}\right)|_{(j_1\geq 1)},$$

the PDF of the MOLEW model can also be expressed as a mixture of exponentiated W (EW) PDF. By differentiating $F_{\underline{\xi}}(z)$, we obtain the same mixture representation

$$f_{\underline{\xi}}(z) = \textstyle\sum_{j_1=0}^{\infty} \Upsilon_{(j_1)} g_{\theta^*,c_2,c_1}(z), \tag{9}$$

where $\theta^* = 1+\theta j_1$ and $g_{\theta^*,c_2,c_1}(z)$ is the EW PDF with power parameter (θ^*) . Equation (9) reveals that the MOLEW density function is a linear combination of EW densities. Thus, some structural properties of the new family such as the ordinary and incomplete moments and generating function can be immediately obtained from well-established properties of the EW distribution. The rth ordinary moment of z is given by

$$\mu'_r = E(Z^r) = \int_{-\infty}^{\infty} z^r f(z)dz,$$

then we obtain

$$\mu'_r = c_2^{-r}\Gamma\left(\frac{r}{c_1}+1\right)\sum_{j_1,j_2=0}^{\infty} \boldsymbol{\mathcal{Q}}_{j_1,j_2}^{(\theta^*,r)}|_{(r>-c_1)}, \tag{10}$$

where

$$\boldsymbol{\mathcal{Q}}_{j_1,j_2}^{(\theta^*,r)} = \Upsilon_{(j_1)}\,\boldsymbol{\mathcal{Q}}_{j_2}^{(\theta^*,r)},$$

and

$$\boldsymbol{\mathcal{Q}}_m^{(\nabla,r)} = \nabla\frac{(-1)^m}{(m+1)^{-\left(\frac{r}{c_1}+1\right)}}\binom{\nabla-1}{m},$$

setting $r=1,2,3,4$ in (11) we get

$$E(z) = \mu'_1 = c_2^{-1}\Gamma\left(\frac{1}{c_1}+1\right)\sum_{j_1,j_2=0}^{\infty} \boldsymbol{\mathcal{Q}}_{j_1,j_2}^{(\theta^*,1)}|_{(1>-c_1)},$$

$$E(z^2) = \mu'_2 = c_2^{-2}\Gamma\left(\frac{2}{c_1}+1\right)\sum_{j_1,j_2=0}^{\infty} \boldsymbol{\mathcal{Q}}_{j_1,j_2}^{(\theta^*,2)}|_{(2>-c_1)},$$

$$E(z^3) = \mu'_3 = c_2^{-3}\Gamma\left(\frac{3}{c_1}+1\right)\sum_{j_1,j_2=0}^{\infty} \boldsymbol{\mathcal{Q}}_{j_1,j_2}^{(\theta^*,3)}|_{(3>-c_1)},$$

and

$$E(z^4) = \mu'_4 = c_2^{-4}\Gamma\left(\frac{4}{c_1}+1\right)\sum_{j_1,j_2=0}^{\infty} \boldsymbol{\mathcal{Q}}_{j_1,j_2}^{(\theta^*,4)}|_{(4>-c_1)}.$$

The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The moment generating function (MGF) $M_z(\tau) = E(e^{\tau z})$ of z . Clearly, the first one can be derived from equation (9) as

$$M_z(\tau) = \Gamma\left(\frac{r}{c_1}+1\right)\sum_{j_1,j_2,r=0}^{\infty} c_2^{-r}\boldsymbol{\mathcal{Q}}_{j_1,j_2,r}^{(\theta^*,r)}|_{(r>-c_1)},$$

where

$$\boldsymbol{\mathcal{Q}}_{j_1,j_2,r}^{(\theta^*,r)} = \frac{\tau^r}{r!}\,\boldsymbol{\mathcal{Q}}_{j_1,j_2}^{(\theta^*,r)}.$$

The sth incomplete moment, say $I_s(\tau)$, of z can be expressed from (9) as

$$I_s(\tau) = \int_{-\infty}^{\tau} z^s f(z)\, dz = c_2^{-s}\gamma\left(\frac{r}{c_1}+1,\left(\frac{1}{\tau}\right)^{-c_1}\right)\sum_{j_1,j_2=0}^{\infty} \mathcal{Q}_{j_1,j_2}^{(\theta^*,r)}|_{(s>-c_1)}, \tag{11}$$

setting $s = 1,2,3,4$ in (11) we get

$$I_1(\tau) = \int_{-\infty}^{\tau} z f(z)\, dz = c_2^{-1}\gamma\left(\frac{1}{c_1}+1,\left(\frac{1}{\tau}\right)^{-c_1}\right)\sum_{j_1,j_2=0}^{\infty} \mathcal{Q}_{j_1,j_2}^{(\theta^*,1)}|_{(1>-c_1)},$$

$$I_2(\tau) = \int_{-\infty}^{\tau} z^2 f(z)\, dz = c_2^{-2}\gamma\left(\frac{2}{c_1}+1,\left(\frac{1}{\tau}\right)^{-c_1}\right)\sum_{j_1,j_2=0}^{\infty} \mathcal{Q}_{j_1,j_2}^{(\theta^*,2)}|_{(2>-c_1)},$$

$$I_3(\tau) = \int_{-\infty}^{\tau} z^3 f(z)\, dz = c_2^{-3}\gamma\left(\frac{3}{c_1}+1,\left(\frac{1}{\tau}\right)^{-c_1}\right)\sum_{j_1,j_2=0}^{\infty} \mathcal{Q}_{j_1,j_2}^{(\theta^*,3)}|_{(3>-c_1)},$$

and

$$I_4(\tau) = \int_{-\infty}^{\tau} z^4 f(z)\, dz = c_2^{-4}\gamma\left(\frac{4}{c_1}+1,\left(\frac{1}{\tau}\right)^{-c_1}\right)\sum_{j_1,j_2=0}^{\infty} \mathcal{Q}_{j_1,j_2}^{(\theta^*,4)}|_{(4>-c_1)}.$$

Order statistics

Suppose $z_1,\dots,z_m$ is any random sample from any MOLEW distribution. Let $z_{\hbar:\ m}$ denote the ith order statistic. The PDF of $z_{\hbar:\ m}$ can be expressed as

$$f_{\hbar:\ m}(z) = \frac{f(z)}{B(\hbar, m-\hbar+1)}\sum_{s=0}^{m-\hbar} (-1)^s \binom{m-\hbar}{s} F(z)^{s+\hbar-1}.$$

Following Nadarajah et al. (2015), we have

$$f_{\hbar:\ m}(z) = \textstyle\sum_{r,j_1=0}^{\infty} \vartheta_{r,j_1}\, g_{1+r+\theta j_1}(z), \tag{12}$$

where

$$\vartheta_{r,j_1} = \frac{m!\,(r+1)(\hbar-1)!\,\Upsilon_{(r+1)}}{(1+r+\theta j_1)}\sum_{s=0}^{m-\hbar} \frac{(-1)^s}{(m-\hbar-s)!\,s!}\xi_{s+\hbar-1,j_1},$$

$\Upsilon_{(j_1)}$ is given in Section 3 and the quantities $\xi_{s+\hbar-1,j_1}$ can be determined with

$$\xi_{s+\hbar-1,0} = w_0^{s+\hbar-1}$$

and recursively for $j_1 \geq 1$

$$\xi_{s+\hbar-1,j_1} = (j_1\tau_0)^{-1}\sum_{m=1}^{j_1} \tau_m\ [m(s+\hbar)-j_1]\,\xi_{s+\hbar-1,j_1-m}.$$

Based on (12) we have

$$E\left(z_{\hbar:\ m}^{\zeta}\right) = c_2^{-\zeta}\Gamma\left(\frac{\zeta}{c_1}+1\right)\sum_{r,j_1,j_2=0}^{\infty} \mathcal{Q}_{r,j_1,j_2}^{(1+r+\theta j_1,\zeta)}|_{(\zeta>-c_1)},$$

where

$$\mathcal{Q}_{r,j_1,j_2}^{(1+r+\theta j_1,\zeta)} = \vartheta_{r,j_1}\mathcal{Q}_{j_2}^{(1+r+\theta j_1,\zeta)}.$$

Residual life and reversed residual life functions

The mth moment of the residual life, say

$$\nu_{m,\tau} = E\left[(z-\tau)^m\ |_{(z>\tau \text{ and } m=1,2,\dots)}\right],$$

the mth moment of the residual life of z is given by

$$\nu_{m,\tau} = \frac{\int_{\tau}^{\infty}(z-\tau)^m\, dF(z)}{1-F(\tau)},$$

therefore

$$\nu_{m,\tau} = \frac{\gamma\left(\frac{m}{c_1}+1,\left(\frac{1}{\tau}\right)^{-c_1}\right)}{c_2^m[1-F(\tau)]}\sum_{j_1,j_2=0}^{\infty}\sum_{r=0}^{m} c_{j_1,j_2,r}^{(\theta^*,m)}|_{(m>-c_1)},$$

where

$$c_{j_1,j_2,r}^{(\theta^*,m)} = (1-\tau)^m \boldsymbol{Q}_{j_1,j_2}^{(\theta^*,m)}.$$

The mean residual life (MRL) at age τ can be defined as

$$\nu_{m=1,\tau} = E\left[(z-\tau)|_{(z>\tau \text{ and } m=1)}\right],$$

which represents the expected additional life length for a unit which is alive at age τ. The m[th] moment of the reversed residual life, say $V_{m,\tau}$

$$V_{m,\tau} = E\left[(\tau-z)^m \ |_{(z\le\tau,\tau>0 \text{ and } m=1,2,\ldots)}\right],$$

we obtain

$$V_{m,\tau} = \frac{\int_0^\tau \ (\tau-z)^m dF(z)}{F(\tau)}.$$

Then, the m[th] moment of the reversed residual life of z becomes

$$V_{m,\tau} = \frac{\gamma\left(\frac{m}{c_1}+1, \left(\frac{1}{\tau}\right)^{-c_1}\right)}{c_2^m F(\tau)} \sum_{j_1,j_2=0}^{\infty} \sum_{r=0}^{m} C_{j_1,j_2,r}^{(\theta^*,m)}|_{(m>-c_1)},$$

where

$$C_{j_1,j_2,r}^{(\theta^*,m)} = (-1)^r \binom{m}{r} \tau^{m-r} \boldsymbol{Q}_{j_1,j_2}^{(\theta^*,m)}.$$

The mean inactivity time (MIT) or mean waiting time (MWT) also called the mean reversed residual life function is given by

$$V_{m=1,\tau} = E[(\tau-z)|_{(z\le\tau \text{ and } m=1)}],$$

and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,\tau)$.

3. Simple type Copula based construction

In this Section, we consider several approaches to construct the bivariate and the multivariate MOLEW type distributions via copula (or with straightforward bivariate CDFs form, in which we just need to consider two different MOLEW CDFs). Many other useful details can be found in Alizadeh et al. (2017 and 2018), Al-Babtain et al. (2020a,b), Yadav et al. (2020)

Via Morgenstern family

First, we start with CDF for Morgenstern family of two random variables $(\boldsymbol{Z_1}, \boldsymbol{Z_2})$ which has the following form

$$F_\lambda(z_1,z_2)|_{(|\lambda|\le1)} = F_1(z_1)F_2(z_2)\{1+\lambda[1-F_1(z_1)][1-F_2(z_2)]\},$$

let

$$F_1(z_1) = \frac{1-\left[1-\left(1-e^{-z_1^{c_1}}\right)^{c_2}\right]^{\theta_1}}{1-\bar{\nu}_1\left[1-\left(1-e^{-z_1^{c_1}}\right)^{c_2}\right]^{\theta_1}},$$

and

$$F_2(z_2) = \frac{1-\left[1-\left(1-e^{-z_2^{a_1}}\right)^{a_2}\right]^{\theta_2}}{1-\bar{\nu}_2\left[1-\left(1-e^{-z_2^{a_1}}\right)^{a_2}\right]^{\theta_2}},$$

then we have a 9-dimension parameter model.

Via Clayton copula

The bivariate extension

The bivariate extension via Clayton copula can be considered as a weighted version of the Clayton copula, which is of the form

$$C(u,v)|_{\delta_1+\delta_2\ge0} = \left[u^{-(\delta_1+\delta_2)} + v^{-(\delta_1+\delta_2)} - 1\right]^{-\frac{1}{\delta_1+\delta_2}}.$$

This is indeed a valid copula. Next, let us assume that $X \sim$ MOLEW $(\nu_1, \theta_1, c_2, c_1)$ where

$$u = \frac{1-\left[1-\left(1-e^{-z_1^{c_1}}\right)^{c_2}\right]^{\theta_1}}{1-\overline{\nu}_1\left[1-\left(1-e^{-z_1^{c_1}}\right)^{c_2}\right]^{\theta_1}}$$

and $Y \sim$ MOLEW $(\nu_1, \theta_1, a_2, a_1)$ where

$$v = \frac{1-\left[1-\left(1-e^{-z_2^{a_1}}\right)^{a_2}\right]^{\theta_2}}{1-\overline{\nu}_2\left[1-\left(1-e^{-z_2^{a_1}}\right)^{a_2}\right]^{\theta_2}},$$

the associated CDF bivariate MOLEW type distribution will be

$$H(x,y) = \left(\begin{array}{c} \left\{\frac{1-\left[1-\left(1-e^{-z_1^{c_1}}\right)^{c_2}\right]^{\theta_1}}{1-\overline{\nu}_1\left[1-\left(1-e^{-z_1^{c_1}}\right)^{c_2}\right]^{\theta_1}}\right\}^{-(\delta_1+\delta_2)} \\ +\left\{\frac{1-\left[1-\left(1-e^{-z_2^{a_1}}\right)^{a_2}\right]^{\theta_2}}{1-\overline{\nu}_2\left[1-\left(1-e^{-z_2^{a_1}}\right)^{a_2}\right]^{\theta_2}}\right\}^{-(\delta_1+\delta_2)} \\ -1 \end{array} \right)^{-\frac{1}{\delta_1+\delta_2}}.$$

The Multivariate extension

A straightforward m -dimensional extension IRom the above will be

$$H(x_1, x_2, \cdots, x_m) = \left(\sum_{i=1}^{m} \left\{\frac{1-\left[1-\left(1-e^{-z_i^{c_i}}\right)^{a_i}\right]^{\theta_i}}{1-\overline{\nu}_i\left[1-\left(1-e^{-z_i^{c_i}}\right)^{a_i}\right]^{\theta_1}}\right\}^{-(\delta_1+\delta_2)} + 1 - m \right)^{-\frac{1}{\delta_1+\delta_2}}.$$

Further future works could be be allocated for studying the bivariate and the multivariate extensions of the MOLEW model.

4. Estimation

Let $z_1, \dots, z_m$ be a random sample from the MOLEW distribution with parameters ν, θ, c_2 and c_1 . Let $\underline{\xi} = ($ ν, θ, c_2, c_1) $\boldsymbol{Q}$ be the 4×1 parameter vector. For determining the MLE of $\underline{\xi}$, we have the log-likelihood function

$$\begin{aligned} \ell = \ell(\underline{\xi}) &= m \log \nu + m \log \theta + m \log c_1 + m c_1 \log c_2 \\ &+ (c_1 - 1) \sum_{\hbar=1}^{m} \log(z_\hbar) + (c_2 - 1) \sum_{\hbar=1}^{m} \log\left(1 - e^{-z^{c_1}}\right) \\ &+ \sum_{\hbar=1}^{m} \log\left\{1 - \left[1 - \left(1 - e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}\right\} - 2 \sum_{\hbar=1}^{m} \log\left\{1 - \overline{\nu}\left[1 - \left(1 - e^{-z^{c_1}}\right)^{c_2}\right]^{\theta}\right\}. \end{aligned}$$

The components of the score vector are easily to be derived.

5. Graphical assessment

Graphically, we can perform the simulation experiments to assess of the finite sample behavior of the MLEs. The assessment was based on the following algorithm:

I. use

II. $z = \left\{-\frac{1}{\theta c_2} ln\left[\frac{1-U}{1-(1-\nu)U}\right]\right\}^{\frac{1}{c_1}}$

III. we generate 1000 samples of size m from the MOLEW distribution.

IV. compute the MLEs for the 1000 samples,

V. compute the SEs of the MLEs for the 1000 samples,

VI. The standard errors (SEs) were computed by inverting the observed information matrix.

VII. compute the biases and mean squared errors given for $\underline{\xi} = \nu, \theta, c_2, c_1$. We repeated these steps for $m = 50,100, \dots, 1000$ with $\nu = \theta = c_1 = c_2 = 1$, so computing biases $\left(B_{\underline{\xi}}(m)\right)$, mean squared errors ($MSEs$) $\left(MSE_{j_2}(m)\right)$ for ν, θ, c_2, c_1 and $m = 50,100, \dots, 1000$.

Figure 3 (left panel) shows how the four biases vary with respect to m . Figure 3 (right panel) shows how the four MSEs vary with respect to m . The broken lines in Figure 3 corresponds to the biases being 0 . From Figure 3, the biases for each parameter are generally negative and decrease to zero as $m \to \infty$, the MSEs for each parameter decrease to zero as $m \to \infty$.

6. Applications

In this section, we provide four applications of the OLEW distribution to show empirically its potentiality. In order to compare the fits of the MOLEW distribution with other competing distributions, we consider the Cramér-von Mises CVM and the Anderson-Darling (AD). These two statistics are widely used to determine how closely a specific CDF fits the empirical distribution of a given data set. These statistics are given by

$$\text{CVM} = \left[(1/12m) + \sum_{s=1}^{m} \quad [z_{\hbar} - (2s-1)/2m]^2\right](1 + 1/2m),$$

and

$$\text{AD} = \left(1 + \frac{9}{4m^2} + \frac{3}{4m}\right)\left\{m + \frac{1}{m}\sum_{s=1}^{m} \quad (2s-1)\, log[z_{\hbar}(1 - z_{m-s+1})]\right\},$$

respectively, where $z_{\hbar} = F(z_s)$ and the z_s 's values are the ordered observations. The smaller these statistics are, the better the fit. The required computations are carried out using the R software. The MLEs and the corresponding standard errors (in parentheses) of the model parameters are given in Tables 2, 4, 6 and 8. The numerical values of the statistics CVM and AD are listed in Tables 3, 5, 7 and 9. The Estimated PDF (EPDF), ECDF, P-P plots for data sets **I**, **II, III** and **IV** of the proposed model are displayed in Figures 5, 7, 9 and 11. The TTT plot and the EHRF for data sets **I**, **II, III** and **IV** of the proposed model are displayed in Figures 4, 6, 8 and 10.

Modeling failure times

The data consist of 84 observations. The data are given in Appendix (a). Here, we shall compare the fits of the MOLEW distribution with those of other competitive models, namely: the Odd Lindley Exponentiated W (OLE-W), Burr X Exp W (BXE-W), Poisson Topp Leone-W (PTL-W), MO extended-W (MOE-W) (Ghitany et al., (2005)), Gamma-W (Ga-W) (Provost et al., (2011)), Kumaraswamy-W (Kw-W) (Cordeiro et al., (2010)), Beta-W (B-W) (Lee et al., (2007)), Transmuted modified-W (TM-W) (Khan and King, (2013)), W-Fréchet (W-Fr) (Afify et al., (2016b)), Kumaraswamy transmuted-W (KumT-W) (Afify et al., (2016a)), Modified beta-W (MB-W) (Khan, (2015)) Mcdonald-W (Mac-W) (Cordeiro et al., (2014)), transmuted exponentiated generalized W (TExG-W) (Yousof et al., (2015)) distributions. Some other extensions of the W distribution can also be used in this comparison, but are not limited to Yousof et al. (2015), Alizadeh et al. (2016), Yousof et al. (2016a,b), Cordeiro et al. (2017a,b), Yousof et al. (2017a,b,c,d), Brito et al. (2017), Aryal et al. (2017a,b), Nofal et al. (2017), Korkmaz et al. (2017), Yousof et al. (2018a,b) and Hamedani et al. (2018). Based on the figures in Table 3 we conclude that the new lifetime model provides adequate fits as compared to other W models with small values for CVM and AD. The proposed MOLEW lifetime model is much better than the BXE-W, PTL-W, MOE-W, Ga-W, Kw-W, W-Fr, B-W, TM-W, KumT-W, MB-W, Mac-W, TExG-W models, and a good alternative to these models.

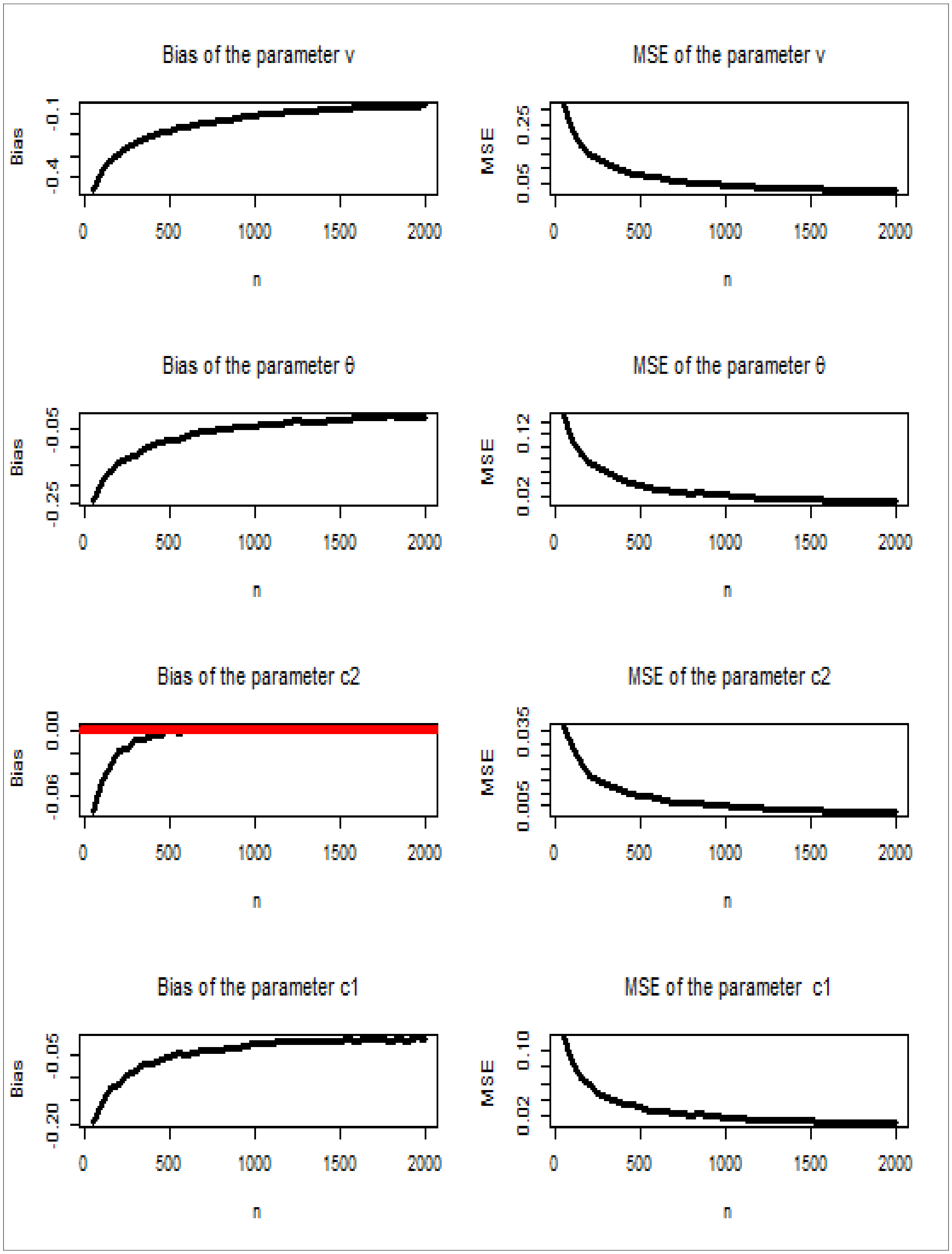


Figure 3: biases and mean squared errors for all parameters

Modeling cancer data

This data set represents the remission times (in months) of a random sample of 128 bladder cancer patients as reported in Lee and Wang (2003). This data is given in Appendix (b). We compare the fits of the MOLEW distribution with other competitive models, namely: The TMW, MBW, transmuted additive W distribution (TA-W) (Elbatal and Aryal, (2013)), exponentiated transmuted generalized Rayleigh (ETGR) (Afify et al., (2015)), and the W (W, (1951)) distributions with corresponding densities (for $z > 0$). Based on the figures in Table 5 we conclude that the proposed MOLEW lifetime model is much better than the W, TM-W, MB-W, TA-W, ETG-R models with small values for CVM and A.D in modeling cancer patient's data.

Modeling survival times

The second real data set corresponds to the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by Bjerkedal (1960). This data is given in Appendix (c). We shall compare the fits of the MOLEW distribution with those of other competitive models, namely: Odd Lindley exponentiated W (OLEW), the Odd W-W (OW-W) (Bourguignon et al., (2014)), the gamma exponentiated-exponential (GaE-E) (Ristic and Balakrishnan (2012)) distributions, whose PDFs (for $z > 0$). Based on the figures in Table 7 we conclude that the proposed MOLEW model is much better than the OLEW, OW-W, WLog-W, GaE-E models, and a good alternative to these models in modeling survival times of Guinea pigs.

Modeling glass fibers data

This data consists of 63 observations of the strengths of 1.5 cm glass fibres, originally obtained by workers at the UK National Physical Laboratory. This data is given in Appendix (d). These data have also been analyzed by Smith and Naylor (1987). For this data set, we shall compare the fits of the new distribution with some competitive models like OLEW, E-W, T-W and OLL-W. Based on the figures in Table 9 we conclude that the proposed MOLEW model is much better than the OLEW, E-W, T-W, OLL-W models, and a good alternative to these models in modeling glass fibers data.

Table 2: MLEs (standard errors (SEs) in parentheses) for data set I.

Distribution	Estimates (SEs)				
OLE-W(c,ν,θ)	0.15935	0.7322	0.765		
	(0.3712)	(1.778)	(0.041)		
BXE-W(c_1,ν,θ)	0.63684	4.2622	0.5364		
	(0.356)	(1.757)	(0.0997)		
PTL-W(λ,ν,c_2)	-5.78175	4.22865	0.65801		
	(1.395)	(1.167)	(0.039)		
MOE-W(γ,θ,ν)	488.899	0.2832	1261.97		
	(189.358)	(0.013)	(351.07)		
Ga-W(ν,θ,γ)	2.37697	0.84809	3.5344		
	(0.378)	(0.00053)	(0.665)		
MOLEW(ν,θ,c_2,c_1)	1.2726	0.0992	0.936	2.3183	
	(0.437)	(0.0188)	(0.0207)	(0.0036)	
Kw-W(ν,θ,c,c_2)	14.4331	0.2041	34.6599	81.8459	
	(27.095)	(0.042)	(17.527)	(52.014)	
W-Fr(ν,θ,c,c_2)	630.9384	0.3024	416.097	1.1664	
	(697.94)	(0.032)	(232.359)	(0.357)	
B-W(ν,θ,c,c_2)	1.36	0.2981	34.1802	11.4956	
	(1.002)	(0.06)	(14.838)	(6.73)	
TM-W($\nu,\theta,\gamma,\lambda$)	0.2722	1	4.6×10^{-6}	0.4685	
	(0:014)	(5.2×10^{-5})	(1.9×10^{-4})	(0.165)	
KumT-W(ν,θ,λ,c,c_2)	27.7912	0.178	0.4449	29.5253	168.06
	(33.401)	(0.017)	(0.609)	(9.792)	(129.17)
MB-W(ν,θ,c,c_2,c)	10.1502	0.1632	57.4167	19.3859	2.0043
	(18.697)	(0.019)	(14.063)	(10.019)	(0.662)
Mac-W(ν,θ,c,c_2,c)	1.9401	0.306	17.686	33.6388	16.7211
	(1.011)	(0.045)	(6.222)	(19.994)	(9.722)
TExG-W(ν,θ,λ,c,c_2)	4.2567	0.1532	0.0978	5.2313	1173.3277
	(33.401)	(0.017)	(0.609)	(9.792)	(6.999)

Table 3: CVM and AD for data set I.

Distribution	CVM	AD
MOLEW(ν,θ,c_2,c_1)	0.05788	0.5498
OLE-W(c,ν,θ)	0.0723	0.6086
BXE-W(c_1,ν,θ)	0.0744	0.6420
PTL-W(λ,ν,c_2)	0.1397	1.1939
MOE-W(γ,θ,ν)	0.3995	4.4477
Ga-W(ν,θ,γ)	0.2553	1.9489
Kw-W(ν,θ,c_2,c_1)	0.1852	1.5059
W-Fr(ν,θ,c_2,c_1)	0.25372	1.95739
B-W(ν,θ,c_2,c_1)	0.4652	3.2197
TM-W($\nu,\theta,\gamma,\lambda$)	0.8065	11.2047
KumT-W(ν,θ,λ,c,c_2)	0.1640	1.3632
MB-W(ν,θ,c_2,c_1,c)	0.4717	3.2656
Mac-W(ν,θ,c_2,c_1,c)	0.1986	1.5906
TExG-W($\nu,\theta,\lambda,c_2,c_1$)	1.0079	6.2332

Table 4: MLEs (SEs) for data set II.

Distribution	Estimates (SEs)				
W(ν,θ)	9.5593	1.0477			
	(0.853)	(0.068)			
MOLEW(ν,θ,c_2,c_1)	2.6679	2.2816	5.7043	0.3717	
	(2.3345)	(3.938)	(4.6088)	(0.2386)	
TM-W($\nu,\theta,\gamma,\lambda$)	0.1208	0.8955	0.0002	0.2513	
	(0.024)	(0.626)	(0.011)	(0.407)	
ETG-R(ν,θ,ν,λ)	7.3762	0.0473	0.0494	0.118	
	(5.389)	(3.97×10^{-3})	(0.036)	(0.26)	
MB-W(ν,θ,c_2,c_1,c)	0.1502	0.1632	57.4167	19.3859	2.0043
	(22.437)	(0.044)	(37.317)	(13.49)	(0.789)
TA-W($\nu,\theta,\gamma,c,\lambda$)	0.1139	0.9722	3.0936×10^{-5}	1.0065	-0.163
	(0.032)	(0.125)	(6.106×10^{-3})	(0.035)	(0.28)

Table 5: CVM and AD for data set II.

Distribution	CVM	AD
MOLEW(ν,θ,c_2,c_1)	0.04328	0.27189
W(ν,θ)	0.1055	0.6628
TM-W($\nu,\theta,\gamma,\lambda$)	0.1251	0.7603
MB-W(ν,θ,c_2,c_1,c)	0.1068	0.7207
TA-W($\nu,\theta,\gamma,c,\lambda$)	0.1129	0.7033
ETG-R(ν,θ,ν,λ)	0.3979	2.3608

Table 6: MLEs (SEs) for data set III.

Distribution	Estimates (SEs)			
OLE-W(c,ν,θ)	0.0018	0.0716	0.2813	
	(0.0004)	(0.025)	(0.009)	
OW-W(θ,γ,λ)	11.1576	0.0881	0.457	
	(4.5449)	(0.036)	(0.08)	
GaE-E(λ,ν,c_1)	2.1138	2.6006	0.0083	
	(1.3288)	(0.5597)	(0.005)	
MOLEW(ν,c,c_2,c_1)	9.2743	0.1500	0.7497	0.566
	(2.285)	(0.0147)	(0.0044)	(0.004)

DOI: http://dx.doi.org/10.18187/pjsor.v17i3.3311

Table 7: CVM and AD for data set III.

Distribution	CVM	AD
MOLEW(ν,θ,c_2,c_1)	0.10081	0.59126
OLE-W(c,ν,θ)	0.25172	1.47502
OW-W(θ,γ,λ)	0.4494	2.4764
GaE-E(λ,ν,c)	0.3150	1.7208

Table 8: MLEs (SEs) for data set IV.

Distribution	Estimates (SEs)			
OLE-W(c,ν,θ)	0.50878	2.534	1.7122	
	(0.397)	(1.8298)	(0.0959)	
E-W(c,ν,θ)	0.671	7.285	1.718	
	(0.249)	(1.707)	(0.086)	
T-W(c,ν,θ)	-0.5010	5.1498	0.6458	
	(0.2741)	(0.6657)	(0.0235)	
OLL-W(c_1,ν,θ)	0.9439	6.0256	0.6159	
	(0.2689)	(1.3478)	(0.0164)	
MOLEW(ν,θ,c_2,c_1)	8.8328	0.3605	0.6288	3.9958
	(6.3173)	(0.1697)	(0.3913)	(0.562)

Table 9: CVM and AD for data set IV.

Distribution	CVM	AD
MOLEW(ν,θ,c_2,c_1)	0.09161	0.52802
OLEW(c,ν,θ)	0.27113	1.49645
E-W(c,ν,θ)	0.6361	3.4842
T-W(c,ν,θ)	1.0358	0.1691
OLL-W(c_1,ν,θ)	1.2364	0.2194

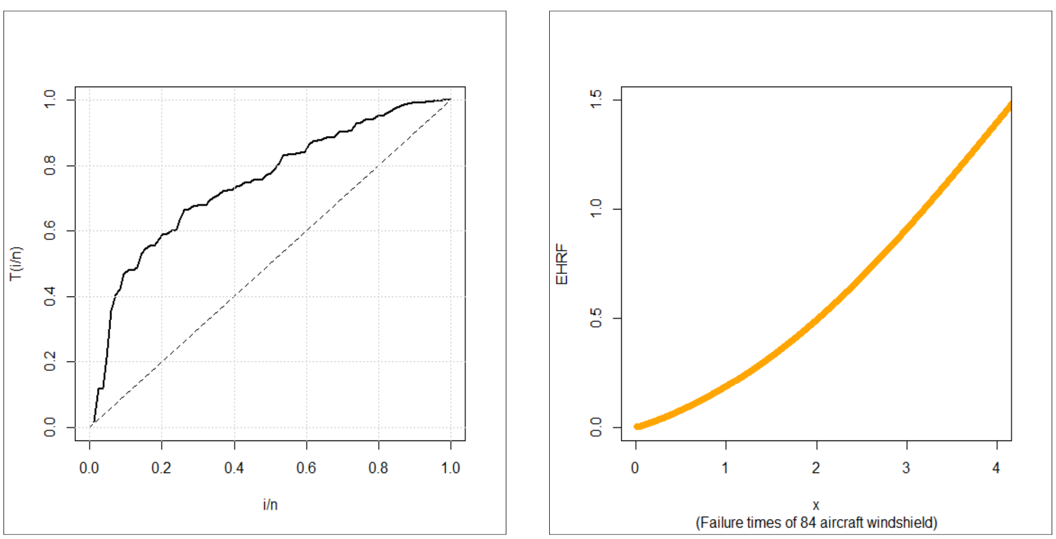


Figure 4: TTT plot and the EHRF plot for data sets **I**.

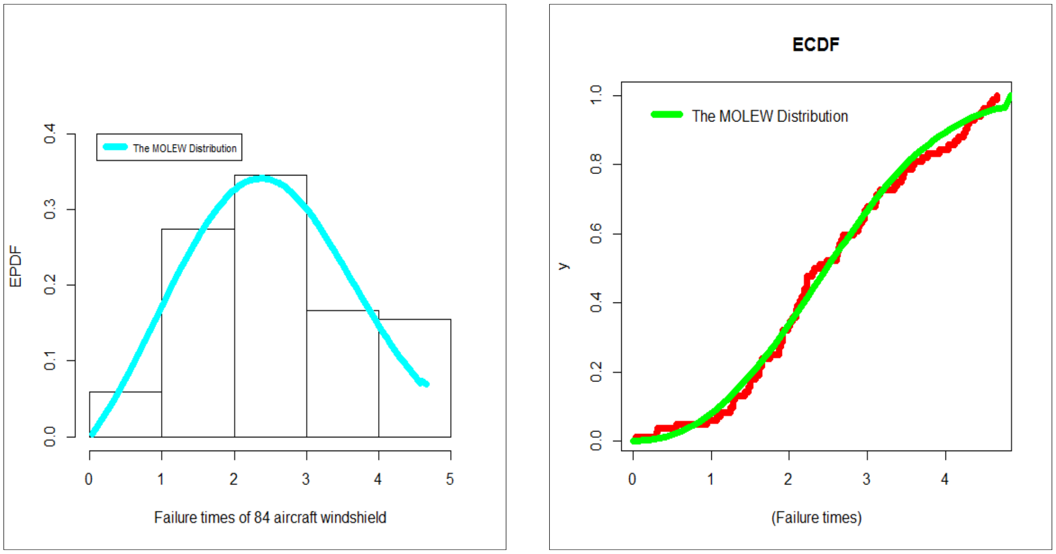


Figure 5: EPDF and ECDF plots for data set **I**.

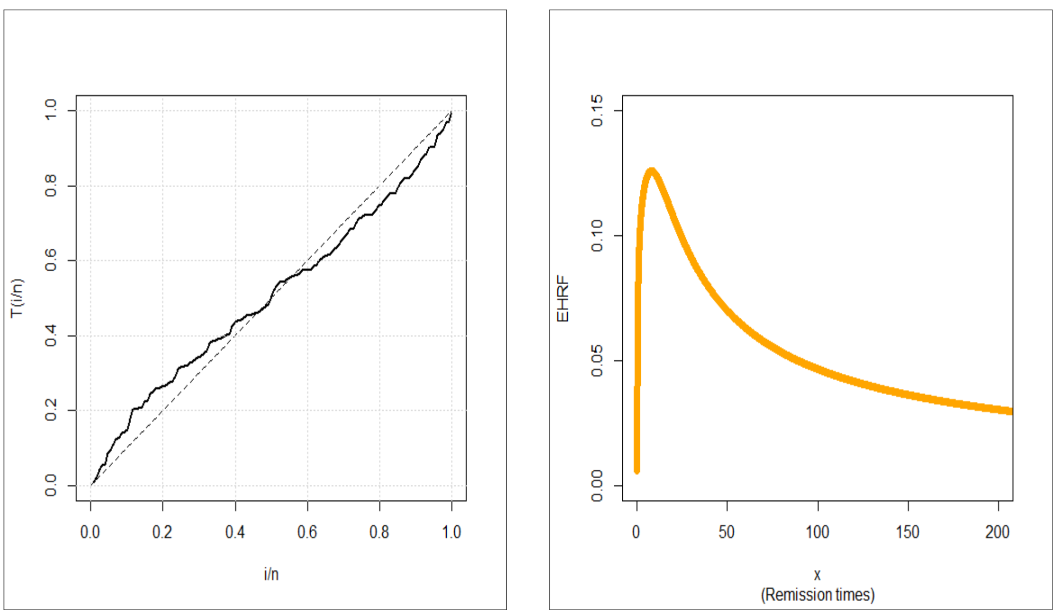


Figure 6: TTT plot and the EHRF plot for data sets **II**.

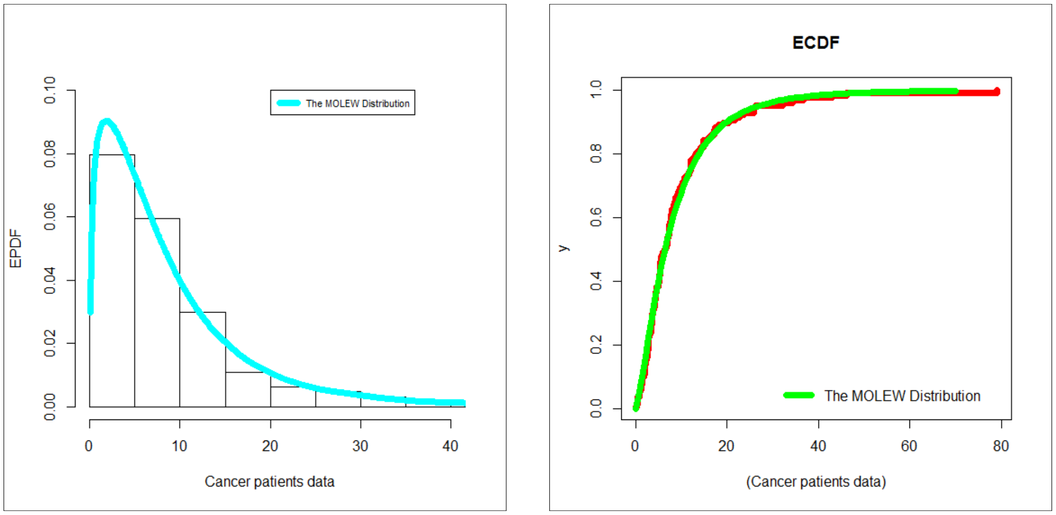


Figure 7: EPDF and ECDF plots for data set **II**.

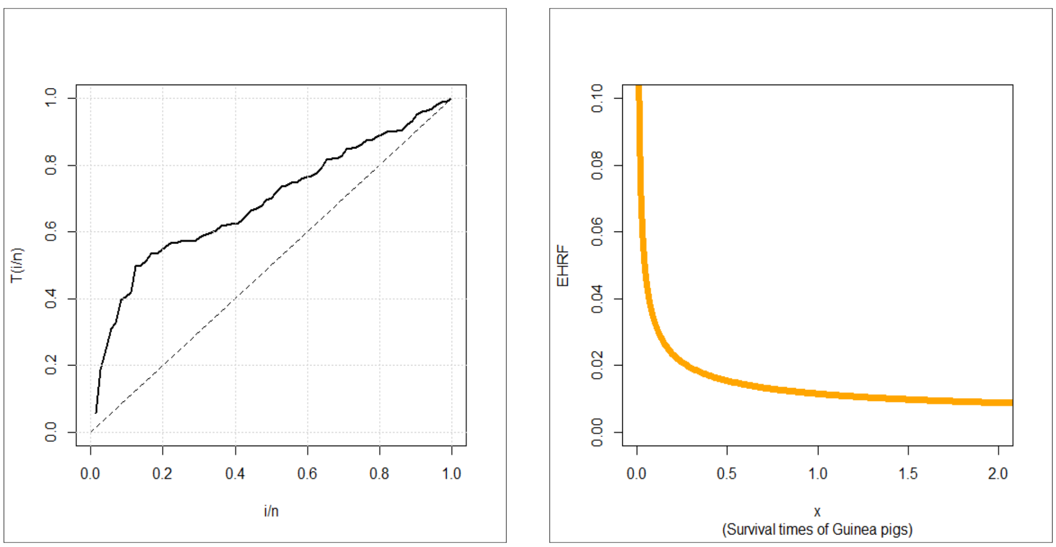


Figure 8: TTT plot and the EHRF plot for data sets **III**.

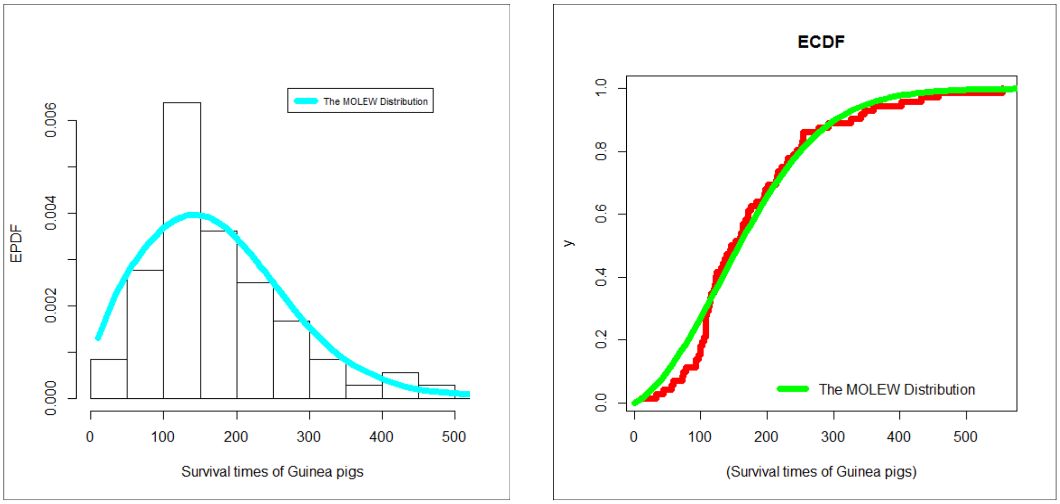


Figure 9: EPDF and ECDF plots for data set **III**.

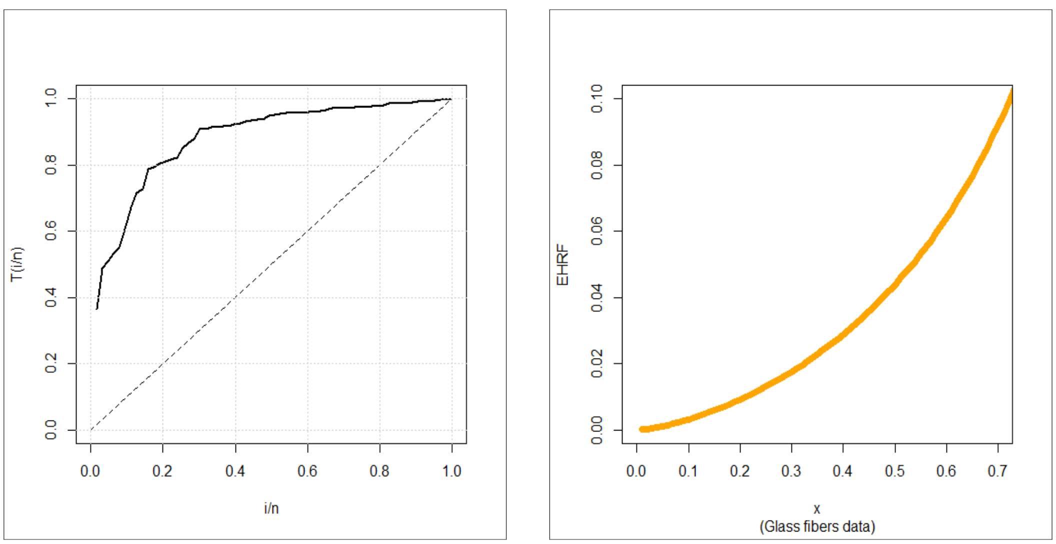


Figure 10: TTT plot and the EHRF plot for data sets **IV**.

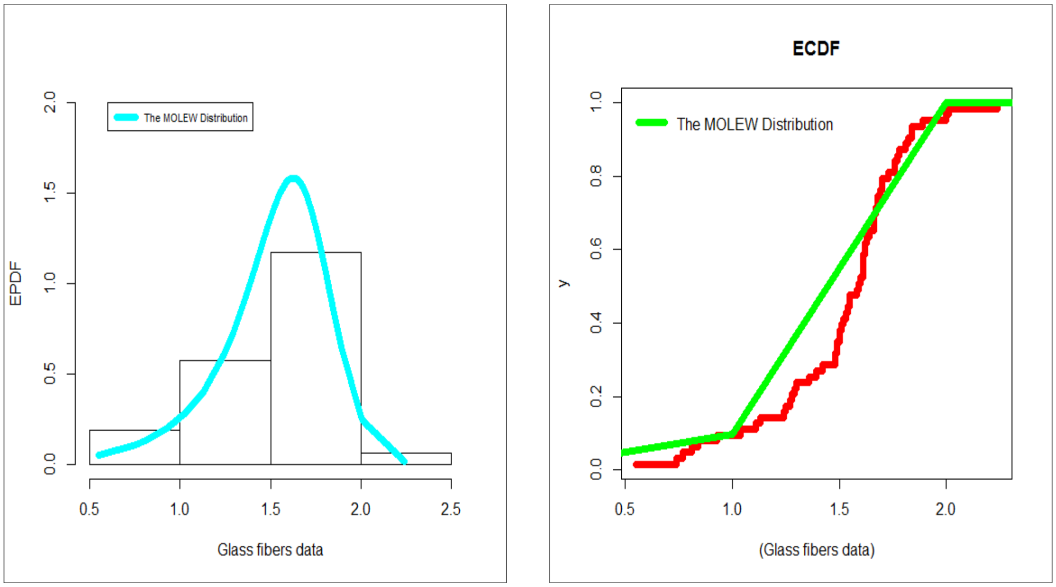


Figure 11: EPDF and ECDF plots for data set **IV**.

7. Concluding remarks

This paper introduces a new four-parameter lifetime model. The new model derives its flexibility and wide applicability from the exponentiated Weibull model. The new density can exhibit many important shapes with different skewness and kurtosis which can be unimodal and bimodal. The new hazard rate can be decreasing, **J**-hazard rate, bathtub shape (**U**-hazard rate), constant hazard rate, increasing hazard rate, upside down (reversed bathtub shape) and increasing-constant hazard rate. Various of its structural mathematical properties are derived with details. The new density is expressed as a linear mixture of well-known exponentiated Weibull density. The method of the maximum likelihood is used to estimate the model parameters. Graphical simulation is used in assessing the performance of the estimation method. We proved empirically the importance and flexibility of the new model in modeling various types of data such as failure times, remission times, survival times and strengths data. Many bivariate and the multivariate type versions are derived using the Morgenstern family and Clayton copula. Future works could be allocated to study these new bivariate and the multivariate types.

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Appendix

(a): 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 2.688, 3.924, 1.281, 2.038, 2.823, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 2.324, 3.376, 4.663.

(b): 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39,10.34, 14.83, 34.26, 0.90 , 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

(c): 10, 33, 44, 56, 59, 72,74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230,231, 240, 245, 251, 253, 254, 255, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555.

(d): 0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.