A Generalization of Reciprocal Exponential Model: Clayton Copula, Statistical Properties and Modeling Skewed and Symmetric Real Data Sets

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Abstract

We introduce a new extension of the reciprocal Exponential distribution for modeling the extreme values. We used the Morgenstern family and the clayton copula for deriving many bivariate and multivariate extensions of the new model. Some of its properties are derived. We assessed the performance of the maximum likelihood estimators (MLEs) via a graphical simulation study. The assessment was based on the sample size. The new reciprocal model is employed for modeling the skewed and the symmetric real data sets. The new reciprocal model is better than some other important competitive models in statistical modeling.

Keywords: Reciprocal Exponential Distribution; Morgenstern Family Moments; Clayton Copula; Estimation; Simulations; Odd Log-Logistic Family.

1. Introduction

A random variable (RV) \( Y \) is said to have the reciprocal Exponential (RE) distribution if its probability density function (P-D-F) and cumulative distribution function (C-D-F) are given by

\[
g_{c_3}(y) = c_3 y^{-2} e^{-\frac{c_3}{y}} \quad y > 0 \tag{1}
\]

and

\[
G_{c_3}(y) = e^{-\frac{c_3}{y}} \quad y > 0. \tag{2}
\]

where \( c_3 > 0 \) is a scale parameter. The P-D-F in (1) is a special case from the well-known reciprocal Weibull (RW) model. The RW model is also known as the Fréchet distribution (see Fréchet (1927)) and also contains the reciprocal Rayleigh (RR) model as a special case. Cordeiro (2017) proposed a new class of continuous distributions called the generalized odd log-logistic-G (GOLL-G) family with two extra shape parameters \( c_1 > 0 \) and \( c_2 > 0 \). For an arbitrary baseline C-D-F \( G_\delta(y) \), the C-D-F of the the GOLL-G family is given by

\[
F_{c_1,c_2,\delta}(y) = \frac{G_\delta(y)^{c_1} + c_2}{[1-G_\delta(y)]^{c_1}}. \tag{3}
\]

The P-D-F corresponding to (3) is given by
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\[ f_{c_1,c_2,c_3}(y) = \frac{c_1c_2e^{c_3}(y)G_2(y)\gamma^{c_1}e^{-c_1y}}{[c_1G_2(y)\gamma^{c_1}e^{-c_1y}]^{c_1-1}}. \]  

(4)

For \( c_2 = 1 \) we get the OLL-G family Gleaton and Lynch (2006). For \( c_1 = 1 \) we get the Proportional reversed hazard rate G family Gupta and Gupta (2007). Here, we define a new RE model based on Cordeiro (2017) called generalized odd log-logistic RE (GOLL-RE) family and provide some plots of its P-D-F and hazard rate function (H-R-F) \([h_{c_1,c_2,c_3}(y)]\). The GOLL-RE C-D-F is given by

\[ F_{c_1,c_2,c_3}(y) = \frac{e^{-c_1c_2y}}{e^{-c_1c_2y} + (1 - e^{-c_1c_3y})}. \]  

(5)

The P-D-F corresponding to (5) is given by

\[ f_{c_1,c_2,c_3}(y) = \frac{c_1c_2e^{c_3}(y) - e^{-c_1c_2y}}{[e^{-c_1c_2y} + (1 - e^{-c_1c_3y})]^{c_1-1}}. \]  

(6)

The H-R-F for the new model can be get from \( f_{c_1,c_2,c_3}(y)/[1 - F_{c_1,c_2,c_3}(y)] \). Let \( c = \inf \{y \mid G(y) > 0\} \), the asymptotics of the C-D-F, P-D-F and H-R-F as \( y \to c \) are given by

\[ F_{c_1,c_2,c_3}(y) \sim e^{-c_1c_2y} |_{y \to c}, \]

\[ f_{c_1,c_2,c_3}(y) \sim c_1c_2y^{c_3} e^{-c_1c_3y} |_{y \to c}, \]

and

\[ h_{c_1,c_2,c_3}(y) \sim c_1c_2y^{c_3} e^{-c_1c_3y} |_{y \to c}. \]

The asymptotics of C-D-F, P-D-F and H-R-F as \( y \to \infty \) are given by

\[ 1 - F_{c_1,c_2,c_3}(y) \sim \left(c_2 \left(1 - e^{-c_2y}\right)\right)^{c_1} |_{y \to \infty}, \]

\[ f_{c_1,c_2,c_3}(y) \sim c_1c_2y^{c_3} e^{-c_1c_3y} \left(1 - e^{-c_1c_3y}\right)^{c_1-1} |_{y \to \infty}, \]

and

\[ h_{c_1,c_2,c_3}(y) \sim c_1c_2y^{c_3} e^{-c_1c_3y} |_{y \to \infty}. \]

Some plots of the GOLL-RE P-D-F and H-R-F are given in Figure 1 (see Appendix A for all Tables and Appendix B for all Figures) to illustrate some of its characteristics. For simulation of this new model, we obtain the quantile function (QF) of \( y \) (by inverting (5)), say \( y_u = Q(u) = F^{-1}(u) \), as

\[ y_u = c_3 \left[-\ln \left(\frac{1 - \frac{u}{c_2}}{1 + \frac{u}{c_2}}\right)\right]^{\frac{1}{c_2}}. \]  

(7)


2. Copula

2.1 Via Morgenstern family

Consider the C-D-F of the Morgenstern \((\lambda|\{\alpha\leq1\})\) family which has the following form

\[ F_2(y_1, y_2|\{\alpha\leq1\}) = F_1(y_1)F_2(y_2)[1 + \alpha[1 - F_1(y_1)][1 - F_2(y_2)]]. \]

setting

\[ F_{c_1,c_2,c_3}(y_1) = \frac{e^{-c_1c_2y_1}}{e^{-c_1c_2y_1} + (1 - e^{-c_1c_3y_1})^{c_1}}. \]
and

\[ F_{a_1,a_2,a_3}(y_2) = \frac{e^{-a_1a_2y_2}}{e^{-a_1a_2y_2} + (1 - e^{-a_2y_2})} \]

then we have a 7-dimensional bivariate GOLL-RE model model as

\[ F_\Xi(y_1, y_2) = \frac{e^{-c_1y_1}e^{-c_2y_2}}{e^{-c_1y_1} + (1 - e^{-c_2y_2})e^{-a_1a_2y_2} + (1 - e^{-a_2y_2})} \]

\[ \times \left( 1 + \lambda \left( 1 - \frac{e^{-c_1y_1}e^{-c_2y_2}}{e^{-c_1y_1} + (1 - e^{-c_2y_2})e^{-a_1a_2y_2} + (1 - e^{-a_2y_2})} \right) \right) \]

Where

\[ \Xi = |\lambda| \leq 1, c_1 > 0, c_2 > 0, c_3 > 0, a_1 > 0, a_2 > 0, a_3 > 0. \]

### 2.2 Via clayton copula

#### 2.2.1 The bivariate extension

The bivariate extension via clayton copula can be considered as a weighted version of the clayton copula, which is of the form

\[ C(u, v) = \left[ u^{-(\delta_1+\delta_2)} + v^{-(\delta_1+\delta_2)} - 1 \right]^{-\frac{1}{\delta_1+\delta_2}} \]

This is indeed a valid copula. Next, let us assume that \( Y \sim \text{GOLL-RE}(c_1, c_2, c_3) \) and \( Z \sim \text{GOLL-RE}(a_1, a_2, a_3) \). Then, setting

\[ u = \frac{e^{-c_1y}e^{-c_2y}}{e^{-c_1y} + (1 - e^{-c_2y})e^{-a_1a_2y} + (1 - e^{-a_2y})} \]

and

\[ v = \frac{e^{-a_1a_2y}}{e^{-a_1a_2y} + (1 - e^{-a_2y})} \]

the associated bivariate C-D-F of the GOLL-RE type distribution will be

\[ H(y, z) = \left\{ \begin{array}{c}
\left( \frac{e^{-c_1y}e^{-c_2y}}{e^{-c_1y} + (1 - e^{-c_2y})e^{-a_1a_2y} + (1 - e^{-a_2y})} \right)^{-(\delta_1+\delta_2)} + \left( \frac{e^{-a_1a_2y}}{e^{-a_1a_2y} + (1 - e^{-a_2y})} \right)^{-(\delta_1+\delta_2)} - 1 \end{array} \right\}^{-\frac{1}{\delta_1+\delta_2}} \]

Where

\[ (\delta_1 + \delta_2) \geq 0, c_1 > 0, c_2 > 0, c_3 > 0, a_1 > 0, a_2 > 0, a_3 > 0. \]

#### 2.2.2 The Multivariate extension

A straightforward \( d \)-dimensional extension from the above will be

\[ H(y_1, y_2, \cdots, y_d) = \left\{ \sum_{i=1}^{d} \left( \frac{e^{-c_1c_2/2y_i}}{e^{-c_1c_2/2y_i} + (1 - e^{-c_2y_i})e^{-a_1a_2y_i} + (1 - e^{-a_2y_i})} \right)^{-(\delta_1+\delta_2)} + 1 - \delta \right\}^{-\frac{1}{\delta_1+\delta_2}} \]

where \((\delta_1 + \delta_2) \geq 0, c_{1i} > 0, c_{2i} > 0, c_{3i} > 0.\) Future works may be allocated for studying the new bivariate and multivariate extensions of the GOLL-RE model.

### 3. Properties

#### 3.1 Representations

Based on Cordeiro et al. (2016), the P-D-F in (6) can be expressed as

\[ f(y) = \sum_{k=0}^{\infty} c_k \pi(Y; c_3), \]

where \((\delta_1 + \delta_2) \geq 0, c_{1i} > 0, c_{2i} > 0, c_{3i} > 0.\) Future works may be allocated for studying the new bivariate and multivariate extensions of the GOLL-RE model.
where
\[
C_k = \frac{c_1 c_2}{1 + k} \sum_{k=0}^{\infty} \sum_{k_1=0}^{k} (-1)^{k_3+k_1+k_4} \binom{k}{k_2} \binom{k}{k_3} \binom{k}{k_4} \times \binom{k}{k_2+1} c_1 c_2 (k_2 + 1) + \binom{k}{k_3} c_3 - 1 \binom{k}{k_1}.
\]
and \(\pi_{(1+k_1)}(y; c_3)\) is the P-D-F of the Fr model with scale parameter \(c_3(1 + k_1)\) and shape parameter \(b\). So, the new P-DF in (6) can be expressed as a double linear mixture of the RE P-D-F. Then, several structural properties of the new model can be obtained from Equation (8) and those properties of the RE model. By integrating Equation (8), the C-D-F of \(y\) becomes
\[
F(y) = \sum_{k=0}^{\infty} C_k \Pi_{(1+k_1)}(y; c_3),
\]
where \(\Pi_{(1+k_1)}(y; c_3)\) is the C-D-F of the R-E distribution with scale parameter \(c_3 k_1\).

### 3.2 Moments and incomplete moments

The \(s^{th}\) ordinary moment of \(y\) is given by
\[
\mu'_s = E(Y^s) = \int_{-\infty}^{\infty} y^s f(y) dy,
\]
then we obtain
\[
\mu'_s = \sum_{k=0}^{\infty} C_k c_3 (1 + k_1)^s \Gamma(1 - s), \forall 1 > s,
\]
where
\[
\Psi_{1}^{-1} \Gamma(1 + \Psi_{1})|_{(\Psi_1 \in \mathbb{R}^+)} = \Psi_1! = \prod_{h=0}^{\Psi_1} (\Psi_1 - h).
\]
The \(s^{th}\) incomplete moment, say \(\varphi_s(\tau)\), of \(Y\) can be expressed, from (9), as
\[
\varphi_s(\tau) = \int_{-\infty}^{\tau} y^s f(y) dy = \sum_{k=0}^{\infty} \kappa_{k} \int_{-\infty}^{\tau} y^s \pi_{(1+k_1)}(y; c_3) dy
\]
\[
= \sum_{k=0}^{\infty} \kappa_{k} c_3 (1 + k_1)^s \gamma \left(1 - s, (1 + k_1) \left(\frac{c_3}{\tau}\right)^b\right), \forall 1 > s,
\]
where \(\gamma(\Psi_1, \Psi_2)\) is the incomplete gamma function.

\[
\gamma(\Psi_1, \Psi_2)|_{(\Psi_1 \neq 0, -1, -2, \ldots)} = \int_{0}^{\Psi_2} \tau^{\Psi_1-1} \exp(-\tau) d\tau
\]
\[
= \frac{\Psi_2^{\Psi_1}}{\Psi_1} \left[ 1_F([\Psi_1; \Psi_1 + 1; -\Psi_2]) \right]
\]
\[
= \sum_{k_1=0}^{\infty} (-1)^{k_1} k_1! (\Psi_1 + k_1) \Psi_2^{\Psi_1+k_1},
\]
and \(1_F(\tau; c)\) is a confluent hypergeometric function. The first incomplete moment given by (11) with \(s = 1\) as
\[
\varphi_1(\tau) = \sum_{k_1=0}^{\infty} \kappa_{k_1} c_3 (1 + k_1) \Gamma \left(1 - \frac{1}{b}, (1 + k_1) \left(\frac{c_3}{\tau}\right)\right).
\]

### 3.3 Moment generating function (MGF)

The MGF \(M_Y(\tau) = E(e^{\tau Y})\) of \(Y\) can be derived from equation (8) as
\[
M_Y(\tau) = \sum_{k=0}^{\infty} C_k M_{(1+k_1)}(\tau; c_3),
\]
where \(M_{(1+k_1)}(\tau; c_3)\) is the MGF of the Fr model with scale parameter \(c_3(1 + k_1)\), then
\[
M_Y(\tau) = \sum_{k_1=0}^{\infty} \sum_{s=0}^{\infty} C_k \frac{\tau^s}{s!} c_3 ((1 + k_1)^s \Gamma(1 - s), \forall 1 > s.
\]
3.4 Residual life and reversed residual life functions

The $m^{th}$ moment of the residual life

$$A_m(\tau) = E[(Y - \tau)^k]_{Y^{\tau}x_{1,2,\ldots}}$$

the $m^{th}$ moment of the residual life of $y$ is given by

$$A_m(\tau) = \frac{\int_0^\infty (y - \tau)^m dF(y)}{1 - F(\tau)}.$$ 

Therefore,

$$A_m(\tau) = \frac{c_3}{1 - F(\tau)} \sum_{k_1=0}^\infty \zeta_{k_1} (1 + k_1)^s \Gamma\left(1 - s, (1 + k_1)\left(\frac{c_3}{\tau}\right)\right), \forall 1 > m,$$

where

$$\zeta_{k_1} = c_{k_1} \sum_{s=0}^m \binom{m}{s} (-\tau),$$

and

$$\Gamma(\psi_1, \psi_2)|_{\psi > 0} = \int_{\psi_2}^{\infty} \psi_1^{-1} \exp(-\psi) d\psi,$$

The $m^{th}$ moment of the reversed residual life, say

$$B_m(\tau) = E[(\tau - Y)^m|_{Y^{\tau}\tau > 0} \text{ and } m = 1, 2, \ldots]$$

uniquely determines $F(y)$. We obtain

$$B_m(\tau) = \frac{\int_0^\infty (\tau - y)^m dF(y)}{F(\tau)}.$$ 

Then, the $m^{th}$ moment of the reversed residual life of $y$ becomes

$$B_m(\tau) = \frac{c_3}{F(\tau)} \sum_{k_1=0}^\infty \eta_{k_1} (1 + k_1)^s \gamma\left(1 - s, (1 + k_1)\left(\frac{c_3}{\tau}\right)\right), \forall 1 > m,$$

where

$$\eta_{k_1} = c_{k_1} \sum_{s=0}^m (-1)^s \binom{m}{s} \tau^{m-s}.$$ 

4. The MLE

Let $y_1, \ldots, y_n$ be a random sample from the GOLL-RE distribution with parameters $c_1, c_2, c_3$. Let $\Theta = (c_1, c_2, c_3)^T$ be the $3 \times 1$ parameter vector. For determining the MLE of $\Theta$, we have the log-likelihood function

$$\ell = \ell(\Theta) = n \log(c_1 c_2 c_3) - 2 \sum_{i=1}^n \log(y_i) - c_1 c_2 \sum_{i=1}^n \frac{c_3}{y_i}$$

$$+ 2 \sum_{i=1}^n \log\left(e^{-c_1 c_2 c_3 y_i} + (1 - e^{-c_2 c_3 y_i})\right)$$

$$+ (c_1 - 1) \sum_{i=1}^n \log\left(1 - e^{-c_2 c_3 y_i}\right).$$

The components of the score vector

$$L(\Theta) = \frac{\partial \ell}{\partial \Theta} = \left(\frac{\partial c_1}{\partial c_1}, \frac{\partial c_2}{\partial c_2}, \frac{\partial c_3}{\partial c_3}\right)^T,$$

are available if needed. Setting $L_{c_1} = L_{c_2} = L_{c_3} = 0$ and solving them simultaneously yields the MLE $\hat{\epsilon} = (\hat{c}_1, \hat{c}_2, \hat{c}_3)^T$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize $\ell$. For interval estimation of the parameters, we obtain the $3 \times 3$ observed information matrix

$$J(\Theta) = \left(\frac{\partial^2 \ell}{\partial s_p \partial s_p}\right) (\forall s, p = c_1, c_2, c_3).$$
whose elements can be computed numerically.

5. Graphical simulations
In this Section, we can performed the simulation experiments graphically to assess of the finite sample behavior of the MLEs.

The graphical assessment was based on the following algorithm:

1. use (7) for generating 10000 samples of size $n$ from the GOLL-RE model;
2. compute the MLEs for the 1000 samples, say $\left(\hat{c}_1 ,\hat{c}_2, \hat{c}_3 \right)$ where $i=1,2,...,10000$;
3. compute the SEs of the MLEs for the 1000 samples, say $\left(\hat{S}_c(\hat{c}_1),\hat{S}_c(\hat{c}_2),\hat{S}_c(\hat{c}_3) \right)$ where $i=1,2,...,10000$.

The standard errors (SEs) were computed by inverting the observed information matrix.

4. compute the biases and mean squared errors given for $p = c_1, c_2, c_3$. We repeated these steps for $n = 50,100,..., 500$ with $c_1 = 1$, $c_2 = 1$, $c_3 = 1$, so computing biases (Bias$_p(n)$), mean squared errors (MSEs) (MSE$_p(n)$) for $c_1, c_2, c_3$ and $n = 50,100,..., 500$ where

$$\text{Bias}_p(n)\big|_{p=c_1,c_2,c_3} = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{p}_i - p),$$
and

$$\text{MSE}_p(n)\big|_{p=c_1,c_2,c_3} = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{p}_i - p)^2.$$

Figures 2-4 give the biases and MSEs for $c_1, c_2$ and $c_3 \forall n = 50,100,..., 500$ for the GOLL-RE model. Figures 2-4 shows how the biases and MSEs vary with respect to $n$. The broken line in Figure 4 corresponds to the biases being 0. From Figures 2-4, the biases for each parameter are generally negative and decrease to zero as $n \rightarrow \infty$, the MSEs for each parameter decrease to zero as $n \rightarrow \infty$.

6. Real data modeling
We consider the Cramér-Von Mises (C-V-M) and the Anderson-Darling (A-D) and the Kolmogorov-Smirnov (KS) statistic. We compare the fits of the GOLL-RE distribution with other models such as RW, Kumaraswamy-RW (KFr), exponentiated RW (E-RW), beta-RW (B-RW), transmuted-RW (T-RW), Marshall-Olkin-RW (MO-RW) and McDonald-RW (Mc-RW) distributions given by:

E-RW : 
$$f_{ERW}(z) = c_1 b c_3 y^{-(b+1)} \exp[-(c_3/y)^b][1 - \exp[-(c_3/y)^b]]^{c_1-1};$$
B-RW : 
$$f_{BWR}(z) = b c_3 B^{-1}(c_1, c) y^{-(b+1)} \exp[-c_1 (c_3/y)^b][1 - \exp[-(c_3/y)^b]]^{c_1-1};$$
K-RW : 
$$f_{KRW}(z) = c_1 b c_3 y^{-(b+1)} \exp[-c_1 (c_3/y)^b][1 - \exp[-c_1 (c_3/y)^b]]^{c_1-1};$$
T-RW : 
$$f_{T-RW}(z) = b c_3 y^{-(b+1)} \exp[-c_3/y]^b [1 + c_1 - 2 c_1 \exp[-c_3/y]^b];$$
MO-RW : 
$$f_{MO-RW}(z) = c_1 b c_3 y^{-(b+1)} \exp[-c_3/y]^b [c_1 + (1 - c_1) \exp[-c_3/y]^b]^{-2};$$
Mc-RW : 
$$f_{Mc-RW}(z) = \lambda b c_3 y^{-(b+1)} B^{-1}(c_1, c) \exp[-c_3/y]^b [\exp[-c_3/y]^b]^{c_1d-1} \times (1 - \exp[-c_3/y]^b)^{c_1-1};$$
OLLE-RW : 
$$f_{OLLE-RW}(z) = c_1 c_2 b a^b z^{-(b+1)} \exp[-c_3 (a/y)^b] \times \exp[-c_3 (a/y)^b][1 - \exp[-c_3 (a/y)^b]]^{-1+c_2} \times (\exp[-c_3 c_2 (a/y)^b] + (1 - \exp[-c_3 (a/y)^b])^{c_2})^{-2};$$

The parameters of the above densities are all positive real numbers except for the T-RW distribution for which $|c_1 | \leq 1$. The 1st data set is an uncensored data set consisting of 100 observations on breaking stress of carbon fibers (in Gba) given by Nichols and Padgett (2006). Figure 5 gives the total time test (TTT) plot (See Aarset (1987)) for data set I. It indicates that the empirical H-R-Fs of data sets I is increasing.
The 2\textsuperscript{nd} data set is generated data to simulate the strengths of glass fibers which was given by Smith and Naylor (1987). Figure 6 gives the TTT plot for data set \textbf{II}. It indicates that the empirical H-R-Fs of data sets \textbf{II} is increasing.

The 3\textsuperscript{rd} data set (wingo data) represents a complete sample from a clinical trial describe a relief time (in hours) for 50 arthritic patients. Many other useful full data sets can be found and analyzed (see Basikhasteh et al. (2018), Elbiely and Yousof (2018 and 2019), Khalil et al. (2019), Gad et al. (2019), Ibrahim et al. (2019), Ibrahim (2019, 2020a, b), Ibrahim et al. (2020), Ibrahim and Yousof (2020), Mansour et al. (2020) and Al-Babtain et al. (2020a, b)). Figure 7 gives the TTT plot for data set \textbf{III}. It indicates that the empirical H-R-Fs of data sets \textbf{III} is increasing.

The statistics (C-V-M, A-D, K-S and P-value) of all fitted models are presented in Table 1, Table 3 and Table 5 for data set \textbf{I}, \textbf{II} and \textbf{III} respectively. The MLEs and corresponding standard errors (SEs) are given in Table 2, Table 4 and Table 6 for data set \textbf{I}, \textbf{II} and \textbf{III} respectively. Figure 8 gives the estimated density (E-P-D-F), Figure 9 gives the estimated C-D-F (E-C-D-F), Figure 10 gives the estimated H-R-F (E-H-R-F) and Figure 8 gives the P-P (P-P) plots. The GOLL-RE distribution in Tables 1, 3 and 5 give the lowest values the C-V-M, A-D, K-S and the biggest value of the P-value statistics as compared to other extensions of the RW models, and therefore the GOLL-RE can be chosen as the best model.

7. Concluding remarks
We introduce a new extension of the reciprocal Exponential distribution for modeling the extreme values. We used the Morgenstern family and the clayton copula for deriving many bivariate and multivariate extensions of the new model. Some of its mathematical properties are derived. We assessed the performance of the maximum likelihood estimators via a graphical simulation study. Based on the graphical simulations, the biases for each parameter are generally negative and decrease to zero as \( n \to \infty \), the MSEs for each parameter decrease to zero as \( n \to \infty \). The assessment was based on the sample size. The new reciprocal model is better than some other important competitive models in modeling the breaking stress data, the glass fibers data and the relief time data.

References
Appendix: A

Table 1: C-V-M, A-D, K-S and P-value for data set I.

<table>
<thead>
<tr>
<th>Model</th>
<th>Goodness of fit criteria</th>
<th>C-V-M</th>
<th>A-D</th>
<th>K-S</th>
<th>P-value</th>
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<tr>
<td>GOLL-RE</td>
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Table 2: MLEs and their SEs for data set I.

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Table 3: C-V-M, A-D, K-S and P-value for data set II.

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### Table 4: MLEs and their SEs for data set II.

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### Table 5: C-V-M, A-D, K-S and P-value for data set III.

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### Table 6: MLEs and their SEs for data set III.

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Appendix B

Figure 1: Plots of the GOLL-RE P-D-F and H-R-F.

Figure 2: Biases and MSEs for $c_1$ and $n = 50, 100, \ldots, 500$ for the GOLL-RE model.
A Generalization of Reciprocal Exponential Model: Clayton Copula, Statistical Properties and Modeling skewed and symmetric Real Data Sets

Figure 3: Biases and MSEs for $c_2$ and $n = 50, 100, ..., 500$ for the GOLL-RE model.

Figure 4: Biases and MSEs for $c_3$ and $n = 50, 100, ..., 500$ for the GOLL-RE model.
A Generalization of Reciprocal Exponential Model: Clayton Copula, Statistical Properties and Modeling skewed and symmetric Real Data Sets
A Generalization of Reciprocal Exponential Model: Clayton Copula, Statistical Properties and Modeling skewed and symmetric Real Data Sets

Figure 10: E-H-R-Fs.

Figure 11: P-P plots.