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Characterizations of Three (2020) Introduced Discrete Distributions

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Abstract

The problem of characterizing a probability distribution is an important problem which has attracted the attention of many researchers in the recent years. To understand the behavior of the data obtained through a given process, we need to be able to describe this behavior via its approximate probability law. This, however, requires to establish conditions which govern the required probability law. In other words we need to have certain conditions under which we may be able to recover the probability law of the data. So, characterization of a distribution plays an important role in applied sciences, where an investigator is vitally interested to find out if their model follows the selected distribution. In this short note, certain characterizations of three recently introduced discrete distributions are presented to complete, in some way, the works of Hussain(2020), Eliwa et al.(2020) and Hassan et al.(2020).

Key Words: Zero truncated discrete distribution; Discrete Lindley distribution; Discrete Gompertz distribution; Poisson Aliamujia distribution; Characterizations.

1. Introdution

To understand the behavior of the data obtained through a given process, we need to be able to describe this behavior via its approximate probability law. This, however, requires to establish conditions which govern the required probability law. In other words we need to have certain conditions under which we may be able to recover the probability law of the data. So, characterization of a distribution is important in applied sciences, where an investigator is vitally interested to find out if their model follows the selected distribution. Therefore, the investigator relies on conditions under which their model would follow a specified distribution. A probability distribution can be characterized in different directions one of which is based on the truncated (or conditional) moments. This type of characterization initiated by Galambos and Kotz(1978) and followed by other authors such as Kotz and Shanbhag(1980), Glänzel et al.(1984), Glänzel(1987), Glänzel and Hamedani(2001) and Kim and Jeon(2013), to name a few. For example, Kim and Jeon(2013) proposed a credibility theory based on the truncation of the loss data to estimate conditional mean loss for a given risk function. It should also be mentioned that characterization results are mathematically challenging and elegant. Hussain(2020) introduced a new discrete probability model called Zero Truncated Discrete Lindley (ZTDL) distribution, Eliwa et al.(2020) proposed a discrete version of the Gompertz-G distribution called Discrete Gompertz-G (DG$_z$-G) distribution and Hassan et al.(2020) introduced a new flexible discrete

distribution called Poisson Aliamujia (PA) distribution. We intend to present certain characterizations of these distributions to complete, in some way, the works mentioned above. These characterizations presented below are based on (i) the conditional expectation of certain function of the random variable and (ii) the hazard function.

The cumulative distribution function (cdf), $F(x)$, the corresponding probability mass function (pmf), $f(x)$, and the hazard function, $h_F(x)$, of each of the distributions ZTDL, DG$_z$-G and PA are given, respectively, by

$$F(x) = 1 - \frac{((1-p)(1+\beta x)+\beta)p^x}{1+\beta-p}, \quad x = 1, 2, \ldots, \tag{1}$$

$$f(x) = \frac{(1-p)^2}{1+\beta-p}(1+\beta x)p^{x-1}, \quad x = 1, 2, \ldots, \tag{2}$$

$$h_F(x) = \frac{(1-p)}{p} - \frac{(1-p)\beta}{p((1-p)(1+\beta x)+\beta)}, \quad x = 1, 2, \ldots. \tag{3}$$

where $\beta > 0$ and $p \in (0, 1)$ are parameters;

$$F(x) = 1 - p^{\frac{1}{c}\left[\left(\overline{G}(x+1;\psi)\right)^{-c}-1\right]}, \quad x = 0, 1, \ldots, \tag{4}$$

$$f(x) = p^{-\frac{1}{c}}\left[p^{\frac{1}{c}\left(\overline{G}(x;\psi)\right)^{-c}} - p^{\frac{1}{c}\left(\overline{G}(x+1;\psi)\right)^{-c}}\right], \quad x = 0, 1, \ldots, \tag{5}$$

$$h_F(x) = \frac{p^{\frac{1}{c}\left(\overline{G}(x;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(x+1;\psi)\right)^{-c}}} - 1, \quad x = 0, 1, \ldots, \tag{6}$$

where $c > 0$ and $p \in (0, 1)$ are parameters; and

$$F(x) = 1 - \frac{4\alpha + 2\alpha x + 1}{(1+2\alpha)^{x+2}}, \quad x = 0, 1, \ldots, \tag{7}$$

$$f(x) = \frac{4\alpha^2(1+x)}{(1+2\alpha)^{x+2}}, \quad x = 0, 1, \ldots, \tag{8}$$

$$h_F(x) = 2\alpha - \frac{2\alpha(2\alpha+1)}{4\alpha+2\alpha x+1}, \quad x = 0, 1, \ldots, \tag{9}$$

where $\alpha > 0$ is a parameter.

Remarks 1. (a) The cdf (and consequently the hazard function) given by Hussain(2020) was incorrect. The corrected versions are given in (1) (and (3)) above. (b) The hazard functions (3), (6) and (9) have been rewritten for the sake of the simplicity of the related computations.

2. Characterization results

We present our characterizations (i) and (ii) in the following two sub-sections.

2.1. Characterization based on the conditional expectation of a function of the random variable

In this sub-section, we use the fact that the hazard function uniquely determines the distribution of a random variable (see Nair et al.(2018)).

Proposition 2.1.1. Let $X : \Omega \to \mathbb{N}$ ($\mathbb{N}$ is the set of all positive integers) be a random variable and let $\varphi(X) = (1+\beta X)^{-1}$. The pmf of X is (2) if and only if the conditional expectation of $\varphi(X)$ given $X > k$, is of the form

$$E\left\{\left[(1+\beta X)^{-1}\right]\mid X>k\right\}=\frac{(1-p)}{(1-p)(1+\beta k)+\beta}\ ,\quad k\in\mathbb{N}. \tag{10}$$

Proof. If X has pmf (2), then the left-hand side of (10) will be

$$(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\left[(1+\beta x)^{-1}f(x)\right]\right\}$$

$$=(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\frac{(1-p)^2}{1+\beta-p}\left\{p^{x-1}\right\}$$

$$=\frac{(1-p)}{(1-p)(1+\beta k)+\beta},\quad k\in\mathbb{N}.$$

Conversely, if (10) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[(1+\beta x)^{-1}\right]f(x)\right\}$$

$$=(1-F(k))\frac{(1-p)}{(1-p)(1+\beta k)+\beta}$$

$$=[1-F(k+1)+f(k+1)]\frac{(1-p)}{(1-p)(1+\beta k)+\beta}, \tag{11}$$

using the fact that $F(k)=P(X\le k)=P(X\le k+1)-P(X=k+1)=F(k+1)-f(k+1)$.

From (11), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[(1+\beta x)^{-1}\right]f(x)\right\}=(1-F(k+1))\frac{(1-p)}{(1-p)(1+\beta(k+1))+\beta}. \tag{12}$$

Now, subtracting (12) from (11), we arrive at

$$\left[\frac{1}{(1+\beta(k+1))}-\left(\frac{(1-p)}{(1-p)(1+\beta k)+\beta}\right)\right]f(k+1)$$

$$=(1-F(k+1))\left[\left(\frac{(1-p)}{(1-p)(1+\beta k)+\beta}\right)-\left(\frac{(1-p)}{(1-p)(1+\beta(k+1))+\beta}\right)\right],$$

or

$$\left[\frac{(1-p)(1+\beta k)+\beta-(1-p)(1+\beta(k+1))}{(1+\beta(k+1))[(1-p)(1+\beta k)+\beta]}\right]f(k+1)$$

$$=(1-F(k+1))\left[\frac{\beta(1-p)^2}{[(1-p)(1+\beta k)+\beta][(1-p)(1+\beta(k+1))+\beta]}\right],$$

or

$$\left[\frac{\beta p}{(1+\beta(k+1))[(1-p)(1+\beta k)+\beta]}\right]f(k+1)$$

$$=(1-F(k+1))\left[\frac{\beta(1-p)^2}{[(1-p)(1+\beta k)+\beta][(1-p)(1+\beta(k+1))+\beta]}\right],$$

or

$$\left[\frac{\beta p}{(1+\beta(k+1))}\right] f(k+1)$$

$$= (1 - F(k+1))\left[\frac{\beta(1-p)^2}{[(1-p)(1+\beta(k+1))+\beta]}\right],$$

or

$$\frac{f(k+1)}{1-F(k+1)} = \frac{\frac{\beta(1-p)^2}{[(1-p)(1+\beta(k+1))+\beta]}}{\frac{\beta p}{(1+\beta(k+1))}},$$

From the last equality, after some algebric calculations, we obtain

$$h_F(k+1) = \frac{f(k+1)}{1-F(k+1)} = \frac{(1-p)}{p} - \frac{(1-p)\beta}{p((1-p)(1+\beta(k+1))+\beta)},$$

which, in view of (3), implies that X has pmf (2).

Proposition 2.1.2. Let $X : \Omega \to \mathbb{N}^*$ ($\mathbb{N}^* = \mathbb{N} \cup \{0\}$) be a random variable and let $\varphi(X) = p^{\frac{1}{c}\left(\overline{G}(X;\psi)\right)^{-c}} + p^{\frac{1}{c}\left(\overline{G}(X+1;\psi)\right)^{-c}}$. The pmf of X is (5) if and only if the conditional expectation of $\varphi(X)$ given $X > k$, is of the form

$$E\left\{\left[p^{\frac{1}{c}\left(\overline{G}(X;\psi)\right)^{-c}} + p^{\frac{1}{c}\left(\overline{G}(X+1;\psi)\right)^{-c}}\right] \mid X > k\right\} = p^{\frac{1}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}}, \quad k \in \mathbb{N}^*. \tag{13}$$

Proof. If X has pmf (5), then the left-hand side of (13) will be

$$(1-F(k))^{-1}\sum_{x=k+1}^{\infty}\left\{\left[\left(p^{\frac{1}{c}\left(\overline{G}(X;\psi)\right)^{-c}} + p^{\frac{1}{c}\left(\overline{G}(X+1;\psi)\right)^{-c}}\right)f(x)\right]\right\}$$

$$= (1-F(k))^{-1}p^{-\frac{1}{c}}\sum_{x=k+1}^{\infty}\left[p^{\frac{2}{c}\left(\overline{G}(X;\psi)\right)^{-c}} - p^{\frac{2}{c}\left(\overline{G}(X+1;\psi)\right)^{-c}}\right]$$

$$= p^{-\frac{1}{c}\left\{\left(\overline{G}(k+1;\psi)\right)^{-c}-1\right\}}p^{-\frac{1}{c}+\frac{2}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}} = p^{\frac{1}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}}, \quad k \in \mathbb{N}^*.$$

Conversely, if (13) holds, then

$$\sum_{x=k+1}^{\infty}\left\{\left[p^{\frac{1}{c}\left(\overline{G}(x;\psi)\right)^{-c}} + p^{\frac{1}{c}\left(\overline{G}(x+1;\psi)\right)^{-c}}\right]f(x)\right\}$$

$$= (1-F(k))p^{\frac{1}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}}$$

$$= [1 - F(k+1) + f(k+1)]p^{\frac{1}{c}\left(\overline{G}(k;\psi)\right)^{-c}}. \tag{14}$$

From (14), we also have

$$\sum_{x=k+2}^{\infty}\left\{\left[p^{\frac{1}{c}\left(\overline{G}(x;\psi)\right)^{-c}} + p^{\frac{1}{c}\left(\overline{G}(x+1;\psi)\right)^{-c}}\right]f(x)\right\}$$

$$= (1-F(k+1))p^{\frac{1}{c}\left(\overline{G}(k+2;\psi)\right)^{-c}}. \tag{15}$$

Now, subtracting (15) from (14), we arrive at

$$
\left[p^{\frac{1}{c}\left(\overline{G}(k+2;\psi)\right)^{-c}} \right] f(k+1)
$$
$$
= (1 - F(k+1)) \left[p^{\frac{1}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}} - p^{\frac{1}{c}\left(\overline{G}(k+2;\psi)\right)^{-c}} \right],
$$

or

$$
\frac{f(k+1)}{1 - F(k+1)} = \frac{p^{\frac{1}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}} + p^{\frac{1}{c}\left(\overline{G}(k+2;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(k+2;\psi)\right)^{-c}}}
$$
$$
= \frac{p^{\frac{1}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(k+2;\psi)\right)^{-c}}} - 1,
$$

which, in view of (6), implies that X has pmf (5).

Proposition 2.1.3. Let $X : \Omega \to \mathbb{N}^*$ ($\mathbb{N}^* = \mathbb{N} \cup \{0\}$) be a random variable and let $\varphi(X) = \frac{(1+2\alpha)^2}{1+X}$. The pmf of X is (8) if and only if the conditional expectation of $\varphi(X)$ given $X > k$, is of the form

$$
E\left\{ \left[\frac{(1+2\alpha)^2}{1+X} \right] \mid X > k \right\} = \frac{2\alpha(1+2\alpha)^2}{4\alpha + 2\alpha k + 1}, \quad k \in \mathbb{N}^*. \tag{16}
$$

Proof. If X has pmf (8), then the left-hand side of (16) will be

$$
(1 - F(k))^{-1} \sum_{x=k+1}^{\infty} \left\{ \left[\frac{(1+2\alpha)^2}{1+x} \right] f(x) \right\}
$$
$$
= (1 - F(k))^{-1} \sum_{x=k+1}^{\infty} 4\alpha^2 \left(\frac{1}{1+2\alpha} \right)^x
$$
$$
= \frac{(1+2\alpha)^{k+2}}{4\alpha + 2\alpha k + 1} \left[4\alpha^2 \left(\frac{1}{2\alpha(1+2\alpha)^k} \right) \right] = \frac{2\alpha(1+2\alpha)^2}{4\alpha + 2\alpha k + 1}, \quad k \in \mathbb{N}^*.
$$

Conversely, if (16) holds, then

$$
\sum_{x=k+1}^{\infty} \left\{ \left[\frac{(1+2\alpha)^2}{1+x} \right] f(x) \right\}
$$
$$
= (1 - F(k)) \frac{2\alpha(1+2\alpha)^2}{4\alpha + 2\alpha k + 1}
$$
$$
= [1 - F(k+1) + f(k+1)] \frac{2\alpha(1+2\alpha)^2}{4\alpha + 2\alpha k + 1}. \tag{17}
$$

From (17), we also have

$$
\sum_{x=k+2}^{\infty} \left\{ \left[\frac{(1+2\alpha)^2}{1+x} \right] f(x) \right\}
$$
$$
= (1 - F(k+1)) \frac{2\alpha(1+2\alpha)^2}{4\alpha + 2\alpha(k+1) + 1}. \tag{18}
$$

Now, subtracting (18) from (17), we arrive at

$$\left[\frac{(1+2\alpha)^2}{1+(k+1)} - \frac{2\alpha(1+2\alpha)^2}{4\alpha+2\alpha k+1} \right] f(k+1)$$

$$= (1-F(k+1)) \left[\frac{2\alpha(1+2\alpha)^2}{4\alpha+2\alpha k+1} - \frac{2\alpha(1+2\alpha)^2}{4\alpha+2\alpha(k+1)+1} \right],$$

or

$$\frac{f(k+1)}{1-F(k+1)} = \frac{\frac{4\alpha^2(1+2\alpha)^2}{(4\alpha+2\alpha k+1)(4\alpha+2\alpha(k+1)+1)}}{\frac{(1+2\alpha)^2}{(k+2)(4\alpha+2\alpha k+1)}}$$

$$= \frac{4\alpha^2(k+2)}{4\alpha+2\alpha(k+1)+1}$$

$$= \frac{2\alpha[2\alpha(k+1)+4\alpha+1-(1+2\alpha)]}{4\alpha+2\alpha(k+1)+1}$$

$$= 2\alpha - \frac{2\alpha(1+2\alpha)}{4\alpha+2\alpha(k+1)+1},$$

which, in view of (9), implies that X has pmf (8).

2.2.   Characterizations based on hazard function

Proposition 2.2.1. Let $X : \Omega \to \mathbb{N}$ be a random variable. The pmf of X is (2) if and only if its hazard rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = \frac{(1-p)\beta}{p} \left(\frac{1}{(1-p)(1+\beta k)+\beta} - \frac{1}{(1-p)(1+\beta(k+1))+\beta} \right), \quad k \in \mathbb{N}, \qquad (19)$$

with the initial condition $h_F(1) = \frac{(1-p)^2(1+\beta)}{p((1-p)(1+\beta)+\beta)}$.

Proof. If X has pmf (2), then clearly (19) holds. Now, if (19) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=1}^{x-1} \{h_F(k+1) - h_F(k)\}$$

$$= \frac{(1-p)\beta}{p} \sum_{k=1}^{x-1} \left\{ \left[\frac{1}{(1-p)(1+\beta k)+\beta} - \frac{1}{(1-p)(1+\beta(k+1))+\beta} \right] \right\},$$

or

$$h_F(x) - h_F(1) = \frac{(1-p)\beta}{p} \left(\frac{1}{(1-p)(1+\beta)+\beta} - \frac{1}{(1-p)(1+\beta x)+\beta} \right).$$

In view of the fact that $h_F(1) = \frac{(1-p)^2(1+\beta)}{p((1-p)(1+\beta)+\beta)}$, from the last equation we have

$$h_F(x) = \frac{(1-p)}{p} - \frac{(1-p)\beta}{p((1-p)(1+\beta x)+\beta)}, \quad x \in \mathbb{N},$$

which, in view of (3), implies that X has pmf (2).

Proposition 2.2.2. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (5) if and only if its hazard

rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = \frac{p^{\frac{1}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(k+2;\psi)\right)^{-c}}} - \frac{p^{\frac{1}{c}\left(\overline{G}(k;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}}}, \quad k \in \mathbb{N}^* \tag{20}$$

with the initial condition $h_F(0) = \frac{p^{\frac{1}{c}\left(\overline{G}(0;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(1;\psi)\right)^{-c}}} - 1$.

Proof. If X has pmf (5), then clearly (20) holds. Now, if (20) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\}$$

$$= \sum_{k=0}^{x-1} \left\{ \left[\frac{p^{\frac{1}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(k+2;\psi)\right)^{-c}}} - \frac{p^{\frac{1}{c}\left(\overline{G}(k;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(k+1;\psi)\right)^{-c}}} \right] \right\},$$

or

$$h_F(x) - h_F(0) = \left\{ \frac{p^{\frac{1}{c}\left(\overline{G}(x;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(x+1;\psi)\right)^{-c}}} - \frac{p^{\frac{1}{c}\left(\overline{G}(0;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(1;\psi)\right)^{-c}}} \right\}.$$

In view of the fact that $h_F(0) = \frac{p^{\frac{1}{c}\left(\overline{G}(0;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(1;\psi)\right)^{-c}}} - 1$, from the last equation we have

$$h_F(x) = \frac{p^{\frac{1}{c}\left(\overline{G}(x;\psi)\right)^{-c}}}{p^{\frac{1}{c}\left(\overline{G}(x+1;\psi)\right)^{-c}}} - 1, \quad x \in \mathbb{N}^*,$$

which, in view of (6), implies that X has pmf (5).

Proposition 2.2.3. Let $X : \Omega \to \mathbb{N}^*$ be a random variable. The pmf of X is (8) if and only if its hazard rate function satisfies the difference equation

$$h_F(k+1) - h_F(k) = 2\alpha(2\alpha+1)\left(\frac{1}{4\alpha + 2\alpha k + 1} - \frac{1}{4\alpha + 2\alpha(k+1) + 1} \right), \quad k \in \mathbb{N}^* \tag{21}$$

with the initial condition $h_F(0) = \frac{4\alpha^2}{4\alpha+1}$.

Proof. If X has pmf (8), then clearly (21) holds. Now, if (21) holds, then for every $x \in \mathbb{N}$, we have

$$\sum_{k=0}^{x-1} \{h_F(k+1) - h_F(k)\}$$

$$= 2\alpha(2\alpha+1) \sum_{k=0}^{x-1} \left[\frac{1}{4\alpha + 2\alpha k + 1} - \frac{1}{4\alpha + 2\alpha(k+1) + 1} \right],$$

or

$$h_F(x) - h_F(0) = 2\alpha(2\alpha+1)\left[\frac{1}{4\alpha+1} - \frac{1}{4\alpha + 2\alpha x + 1} \right].$$

In view of the fact that $h_F(0) = \frac{4\alpha^2}{4\alpha+1}$, from the last equation we have

$$h_F(x) = 2\alpha - \frac{2\alpha(2\alpha+1)}{4\alpha + 2\alpha x + 1}, \quad x \in \mathbb{N}^*,$$

which, in view of (9), implies that X has pmf (8).

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