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A novel four-parameter log-logistic model: mathematical properties and applications to breaking stress, survival times and leukemia data

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Abstract
In this paper, we introduce a new continuous log-logistic extension. Several of its properties are established. A numerical analysis for skewness and kurtosis is presented. The new failure rate can be "bathtub or **U** shaped", "increasing", "decreasing-constant", "**J** shaped", "constant" and "decreasing". Many bivariate and Multivariate type distributions are derived using the Clayton Copula and the Morgenstern family. To assess of the finite sample behavior of the estimators, we performed a graphical simulation. Some useful applications are considered for supporting the new model.

Key Words: Burr XII Distribution; Log-Logistic Model; Maximum Likelihood; Leukemia Data; Modeling; Simulation.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1.Introduction and motivation

A continues random variable (RV) Y is said to has the log-logistic (LL) model if its survival/reliability function (SF) be written as

$$\overline{S}_{b}(y) = 1 - S_{b}(y) = \left(1 + y^{b}\right)^{-1}, \tag{1}$$

where $b > 0$ is a shape parameter and $S_{b}(y)$ refer to the cumulative distribution functions (CDF) of the LL model. The density function (PDF) due to (1) is given as

$$s_{b}(y) = b\, y^{b-1}\left(1 + y^{b}\right)^{-2}. \tag{2}$$

The CDF and PDF in (1) and (2) is a sub-model from the Burr model from the type-XII (BUXII) model (Burr (1942, 1968 and 1973), Tadikamalla (1980) and Rodriguez (1977)). Due to Yousof et al. (2018), the CDF of the new Weibull Generalized log-logistic (WG-LL) is defined as

$$F(y) = 1 - e^{-a\left[\left(1+y^{b}\right)^{\beta}-1\right]^{\alpha}}, \tag{3}$$

where $y > 0$, $\alpha, \beta, a > 0$ and its corresponding PDF is given by

$$f(y) = \alpha\beta a b\, y^{b-1}\left(1 + y^{b}\right)^{\beta-1}\left[\left(1 + y^{b}\right)^{\beta} - 1\right]^{\alpha-1} e^{-a\left[\left(1+y^{b}\right)^{\beta}-1\right]^{\alpha}}. \tag{4}$$

Figure 1 below gives some graphical results for the novel PDF and its corresponding HRF for the WG-LL model. Due to Fig. 1(left chart) it is seen that the novel PDF of the WG-LL model can be monotonical and unimodal PDF, symmetric PDF or negative skewed PDF. From Figure 1(right chart) the HRF can be "bathtub" ($\alpha = 0.85, \beta =$

$0.5, a = 0.001, b = 0.55)$ or "increasing (monotone-HRF)" $(\alpha = 0.95, \beta = 7, a = 0.0001, b = 0.55)$ or "decreasing-constant" ($\alpha = 0.85, \beta = 3, a = 0.05, b = 0.5$) or "**J**-shaped" ($\alpha = 2, \beta = 5, a = 0.01, b = 1.5$) or "constant" ($\alpha = 1, \beta = 1, a = 1, b = 1$) or "decreasing (monotone-HRF)" ($\alpha = 1, \beta = 0.5, b = a = 1$).

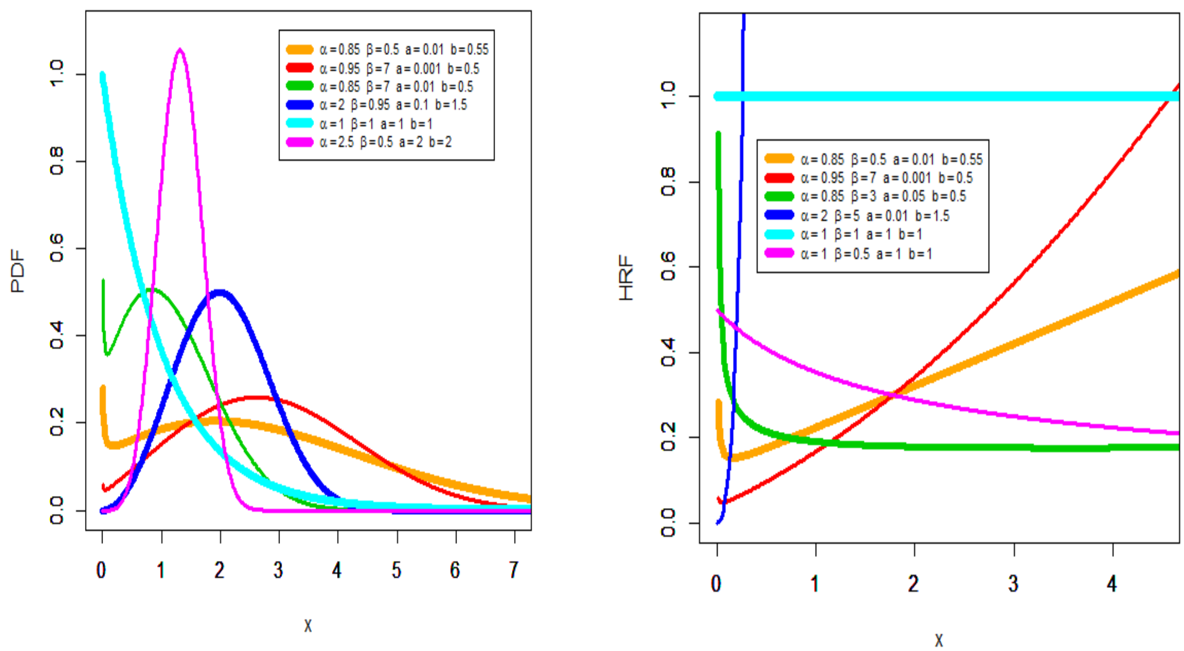

Figure 1: graphical charts of the novel PDF and it corresponding HRF.

Due to Yousof et al. (2018), the new PDF in (4) can be re-expressed as

$$f(y) = \sum_{j_4=0}^{\infty} C_{[j_4]} s_{[b,(1+j_4)]}(y),$$

where $s_{[b,(1+j_4)]}(y) = (1 + j_4) b y^{b-1} (1 + y^{b})^{-2-j_4}$ is the LL PDF with b and $(1 + j_4)$ where

$$C_{[j_4]} = \sum_{j_1,j_2,j_3=0}^{\infty} (-1)^{1+j_1+j_2+j_3+j_4} \frac{1}{j_1!\, j_4!} \binom{j_1\alpha}{j_2} \binom{\beta(j_2 - j_1\alpha)}{j_3} \binom{1 + j_3}{j_4},$$

Analogously, the new CDF (3) can be re-expressed as

$$F(y) = \sum_{j_4=0}^{\infty} C_{[j_4]} S_{[b,(1+j_4)]}(y),$$

where

$$S_{[b,(1+j_4)]}(y) = -(1 + y^{b})^{-1-j_4} + 1,$$

represents the CDF of the well-known LL model with parameters b and $(1 + j_4)$. This work can be motivated via the following applied justifications:

i. The new model in its novel pattern will be useful in mathematical modeling of the engineering real-life datasets such as the “monotonically-increasing HRF” engineering breaking stress real-life dataset.

ii. The novel model in its novel version can be used for statistical modeling of the reliability real-life datasets such as the “monotonically-increasing HRF” reliability real-life dataset.

iii. The new current version of the log-logistic version can be used in modeling the medical real-life datasets such as the “**U**- HRF” medical real-life dataset.

2. Properties

Moments and generating function

The m^th^ ordinary moment of Y is given by

$$\mu'_{m,Y}=E(Y^m)=\int_{-\infty}^{\infty} f(y)y^m dy.$$

Then, we obtain

$$\mu'_{m,Y}=\sum_{j_4=0}^{\infty} C_{[j_4]}(1+j_4)B\left(1+\frac{m}{b},(1+j_4)-\frac{m}{b}\right)|_{(m<(1+j_4)b)},$$

where

$$B(d_1,d_2)=\int_0^{\infty} t^{d_1-1}(1+t)^{-(d_1+d_2)}dt,$$

is the second type of beta function. By fixing $m=1$ in $\mu'_{m,Y}$, we obtain the mean of the model. The m^th^ incomplete moment ($I_n(t)$) of Y can be expressed from (3) as

$$I_{m,Y}(t)=\int_{-\infty}^{t} y^m f(y)dy=\sum_{j_4=0}^{\infty} C_{[j_4]}(1+j_4)B\left(t^{b};1+\frac{m}{b},(1+j_4)-\frac{m}{b}\right)|_{(m<(1+j_4)b)},$$

where

$$B(q;a_1,a_2)=\int_0^{q} t^{a_1-1}(1+t)^{-(a_1+a_2)}dt$$

Is the second type of the incomplete beta function. The moment generating function (mgf) $M_Y(t)=E(exp(tY))$ of Y can be derived from (3) as

$$M_Y(t)=\sum_{j_4,n=0}^{\infty} \frac{t^n}{n!}C_{[j_4]}(1+j_4)B\left(1+\frac{n}{b},(1+j_4)-\frac{n}{b}\right)|_{(n<(1+j_4)b)},$$

Probability weighted moments (PWMs)

The (m,r)^th^ PWM of Y can be expressed as

$$p_{m,r,Y}=\sum_{r=0}^{\infty} C_{[r]}(1+r)B\left(\frac{m}{b}+1,(1+r)-\frac{m}{b}\right)|_{(m<(1+r)b)},$$

where

$$C_{[r]}=\alpha\beta a\sum_{i,j_1,j_2,j_3=0}^{\infty}(-1)^{i+j_1+j_2+j_3}\frac{(1+i)^{j_1}}{\Gamma(1+r)\Gamma(1+j_1)!}(r)_{j_3}$$
$$\times\binom{1+j_3}{r}\binom{(1+j_1)\alpha-1}{j_2}\binom{\beta[-(1+j_1)\alpha+j_2]-1}{j_3},$$

and $(c_1)_{c_2}=c_1(c_1-1)\ldots(1+c_1-c_2)$ is the common factorial for the descending king and c_2 should be integer and also positive.

Reversed residual life Moment (MRRL) function

The m ^th^ MRRL, say

$$A_{m,Y}(t)=E\left[(t-y)^m\ |_{(y\le t,\ t>0,m=1,2,\ldots)}\right]$$

Then, we have

$$A_m(t)=F^{-1}(t)\int_0^{t}(t-y)^m dF(y).$$

Then, the m ^th^ MRRL of Y becomes

$$A_{m,Y}(t)=F^{-1}(t)\sum_{r=0}^{\infty} C_{[j_4]}^{(\omega)}(1+j_4)B\left(t^{b};(1+j_4)-\frac{m}{b},1+\frac{m}{b}\right),$$

Where

$$C_{[j_4]}^{(\omega)}=C_{[j_4]}\sum_{r=0}^{m}(-1)^r\binom{m}{r}t^{m-r}.$$

3. Checking flexibility numerically

In this section, the impacts of the parameters on the model mean (μ_1'), variance of the model (V (X)), skewness of the model (S (X)) and kurtosis of the model (K (X)) are given below in Table 1. The impacts of $\mathscr{b}$ for the standard LL model on the μ_1' , V (X) , S (X) and K (X) are provided in Table 2.

For the new WG-LL model, S $(x) \in (-1.08107, 16.7425)$. However, for the LL model, S $(X) \in (0.0872, 2.4853)$. For the novel WG-LL model, K $(x) \in (3.2451, 702.5)$. However, for the LL model, K $(X) \in (3.7409, 29.5562)$.

Table 1: μ_1', V(X), S(x), K(X) for the WG-LL model.

α	β	a	b	μ_1'	V(X)	S(X)	K(X)
0.35	1	1	1	5.029144	399.0459	16.74246	702.4982
0.5				2	20	6.618761	87.72
1				1	1	2	9
2				0.8862269	0.2146018	0.6311107	3.245089
10				0.9513508	0.01310046	-0.6376371	3.570166
35				0.984295	0.001249773	-0.978317	4.684532
50				0.9888442	0.0006253426	-1.024853	4.877788
75				0.9924775	0.0002825923	-1.062093	5.040069
100				0.9943259	0.0001603049	-1.081074	5.125489
5	0.1	3	2	17.25458	63.62621	0.749758	3.620892
	0.5			1.414735	0.04437565	-0.4251616	3.159187
	1			0.8523708	0.01051628	-0.6376371	3.570166
	2.5			0.4929942	0.00267863	-0.7801194	3.933523
	5			0.3387453	0.001151769	-0.8305135	4.079192
	7.5			0.2739872	0.000730451	-0.8549788	4.244319
	10			0.2361687	0.0005340107	-0.8562963	4.157227
2.5	1.5	0.00001	3	2.618206	0.0872134	-0.7195030	3.804983
		0.0001		2.108410	0.0601449	-0.7321432	3.837628
		0.001		1.684376	0.0419942	-0.7383334	3.839446
		0.01		1.329646	0.0294111	-0.7251535	3.773603
		0.1		1.033039	0.0201868	-0.6837104	3.636145
		0.5		0.856814	0.0150825	-0.6422962	3.525361
		1		0.788435	0.0131753	-0.6236186	3.481136
		10		0.592367	0.0080480	-0.5684401	3.367608
		50		0.481838	0.0055071	-0.5412732	3.320269
		200		0.402187	0.0039072	-0.5256268	3.295386
		500		0.356572	0.0030960	-0.5184591	3.284576
		5000		0.262971	0.0017034	-0.5079508	3.269382
		10000		0.239855	0.0014198	-0.5061509	3.266805
10	0.5	100	0.1	180.4745	51898.910	2.8789170	16.50961
			0.5	2.504478	0.4792982	-0.05866376	2.752959
			1	1.566053	0.0519556	-0.49187000	3.259029
			2	1.247898	0.0088024	-0.7519055	3.903483
			3	1.158345	0.0034531	-0.8483738	4.206997
			4	1.116276	0.0018266	-0.8989813	4.393305
			5	1.091856	0.0011271	-0.9301086	4.509344

Table 2: μ_1', V(X), S(X), K(X) for the LL model.

b	μ_1'	V(X)	S(X)	K(X)
5	1.068959	0.1786323	2.48528	29.5562
7.5	1.029853	0.0667170	1.33004	9.18867
10	1.016641	0.0354009	0.93667	6.51021
12.5	1.010606	0.0220617	0.72919	5.563852
15	1.007348	0.01510236	0.598998	5.10838
17.5	1.005391	0.01100040	0.509077	4.85121
20	1.004124	0.00837532	0.443015	4.69083
25	1.002637	0.00532522	0.352162	4.50848
30	1.001830	0.00368497	0.292463	4.41214
35	1.001344	0.00270154	0.250166	4.35495
40	1.001029	0.00206550	0.218603	4.31819
45	1.000813	0.00163045	0.194136	4.29314
50	1.000658	0.00131982	0.172998	4.33383
55	1.000544	0.00109017	0.158659	4.26215
60	1.000457	0.00091569	0.14538480	4.25213
65	1.000389	0.00078000	0.13416210	4.24442
70	1.000336	0.00067239	0.12455090	4.23827
75	1.000292	0.00058562	0.11622630	4.23332
80	1.000257	0.000514623	0.10894580	4.22928
85	1.000228	0.00045580	0.10252450	4.22593
90	1.000203	0.00040652	0.09681852	4.22313
95	1.000182	0.00036482	0.09171449	4.22077
100	1.000164	0.00032925	0.08716923	3.74092

4. Copula

Bivariate WG-LL via Morgenstern family

First, consider the CDF for Morgenstern model

$$F_\lambda(y_1, y_2)|_{(|\lambda|\leq 1)} = F_1(y_1)F_2(y_2)\{1 + \lambda[1 - F_1(y_1)][1 - F_2(y_2)]\}.$$

Let

$$F_1(y_1) = 1 - e^{-a_1\left[\left(1+y_1^{b_1}\right)^{\beta_1}-1\right]^{\alpha_1}}$$

and

$$F_2(y_2) = 1 - e^{-a_2\left[\left(1+y_2^{b_2}\right)^{\beta_2}-1\right]^{\alpha_2}}$$

then we have new bivariate model as

$$F_\lambda(y_1, y_2)|_{(|\lambda|\leq 1)} = \left(1 - e^{-a_1\left[\left(1+y_1^{b_1}\right)^{\beta_1}-1\right]^{\alpha_1}}\right)\left(1 - e^{-a_2\left[\left(1+y_2^{b_2}\right)^{\beta_2}-1\right]^{\alpha_2}}\right)$$
$$\times\left\{1 + \lambda\left[e^{-a_1\left[\left(1+y_1^{b_1}\right)^{\beta_1}-1\right]^{\alpha_1}}\right]\left[e^{-a_2\left[\left(1+y_2^{b_2}\right)^{\beta_2}-1\right]^{\alpha_2}}\right]\right\}.$$

Bivariate WG-LL via Clayton Copula

Consider the following Clayton Copula

$$C(u, v) = \left[u^{-(\delta_1+\delta_2)} + v^{-(\delta_1+\delta_2)} - 1\right]^{-\frac{1}{\delta_1+\delta_2}}.$$

Then, setting

$$u = 1 - e^{-a_1\left[\left(1+x^{b_1}\right)^{\beta_1}-1\right]^{\alpha_1}}$$

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and

$$v = 1 - e^{-a_2\left[\left(1+y^{b_2}\right)^{\beta_2}-1\right]^{\alpha_2}}$$

The bivariate CDF can be written as

$$C(u,v) = \left[\left(1 - e^{-a_1\left[\left(1+x^{b_1}\right)^{\beta_1}-1\right]^{\alpha_1}}\right)^{-(\delta_1+\delta_2)} + \left(1 - e^{-a_2\left[\left(1+y^{b_2}\right)^{\beta_2}-1\right]^{\alpha_2}}\right)^{-(\delta_1+\delta_2)} - 1\right]^{-\frac{1}{\delta_1+\delta_2}}.$$

The Multivariate extension via Copula of Clayton

The d-dimensional model can be expressed as

$$H(x_i) = \left(\sum_{i=1}^{d} \left\{1 - e^{-a_i\left[\left(1+x_i^{b_i}\right)^{\beta_i}-1\right]^{\alpha_i}}\right\}^{-(\delta_1+\delta_2)} + 1 - d\right)^{-\frac{1}{\delta_1+\delta_2}},$$

where $x_i = x_1, x_2, \cdots, x_d$. For other copulas see Ali et al. (2020a), Ali et al. (2020b), Elgohari et al. (2021), Elgohari and Yousof (2020a,b and 2021).

5. Simulations

Assessing the behavior of the maximum likelihood estimations (MLEs) is discussed in this section. For this purpose, consider the following active algorithm:

1) Use

$$y_U = \left(\left\{\left[-\frac{1}{a} ln(1-U)\right]^{\frac{1}{\alpha}} + 1\right\}^{\frac{1}{\beta}} - 1\right)^{\frac{1}{b}}$$

for generating 5000 group of size n from the WG-LL model.

2) Get the MLEs for the 5000 groups.

3) Using the inverting the observed information matrix, compute the standard errors (SEs) of the MLEs for the 1000 samples, where the SEs).

4) Compute the biases $(B_h(n)|h = \alpha, \beta, a, b)$ and mean squared errors ($MSE_h(n)|h = \alpha, \beta, a, b$) given for $h = \alpha, \beta, a, b$. These steps must be repeated for $n = 50,100, \dots, 5000$ with $1 = \alpha = b = \beta = a$ for getting the values of biases and the values of the MSE_h for a, b, α and $n = 50,100, \dots, 5000$.

Figure 2, Figure 3, Figure 4 and Figure 5 (left charts) show how the four biases of α, β, a and b vary with respect to $n|n = 50,100, \dots, 5000$. Figures 2, 3, 4 and 5 (right charts) show how the four MSEs of α, β, a and b vary with respect to $n|n = 50,100, \dots, 5000$. From Figure 2, 3 Figure, Figure 4 and Figure 5, the biases f decrease to zero as $n \rightarrow \infty$, the valued of the obtained MSEs of α, β, a and b decrease to zero as $n \rightarrow \infty$. Based on this assessment, the ML method performs well and can be used in estimating the model parameter. The following Section provide some useful real data applications using the ML method for comparing the competitive models.

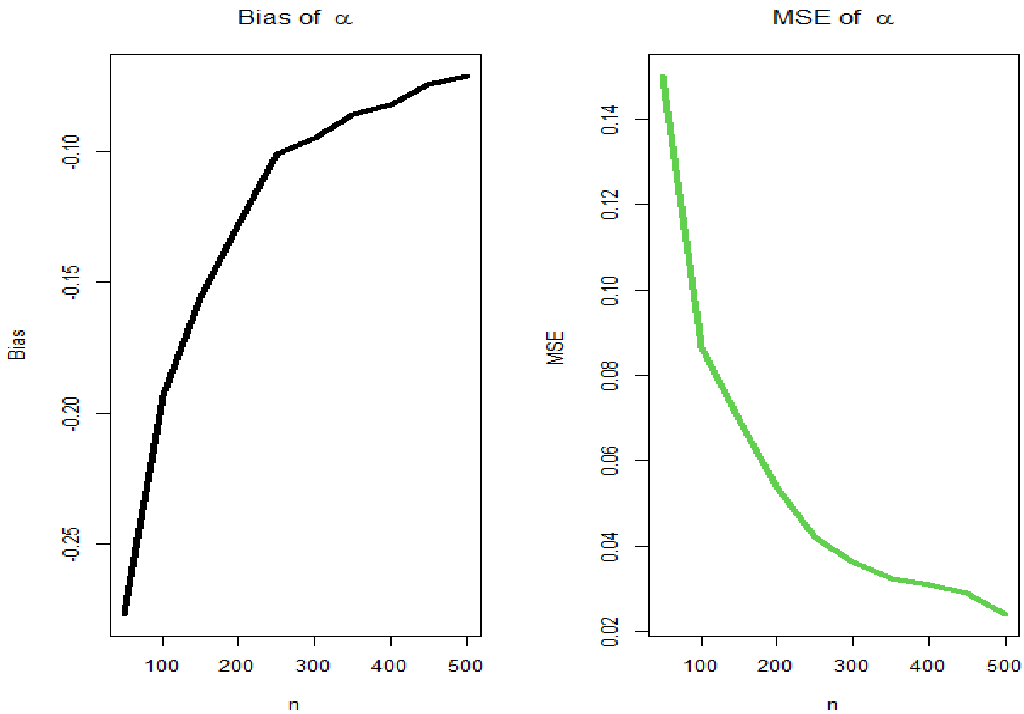

Figure 2: Bias and MSE for the parameter α.

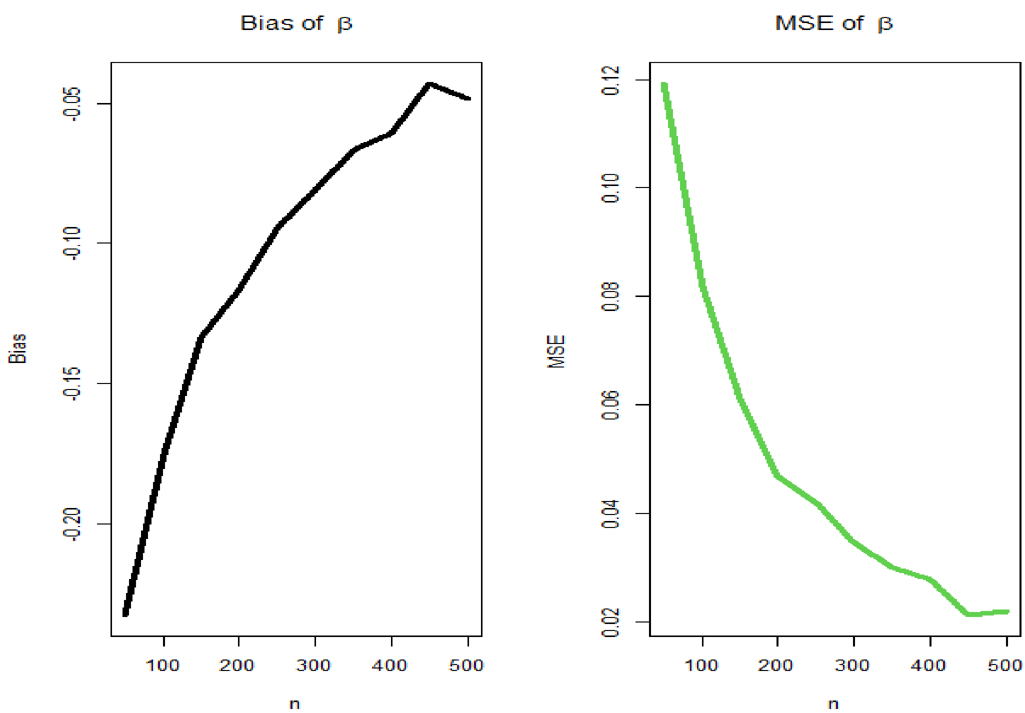

Figure 3: Bias and MSE for the parameter β.

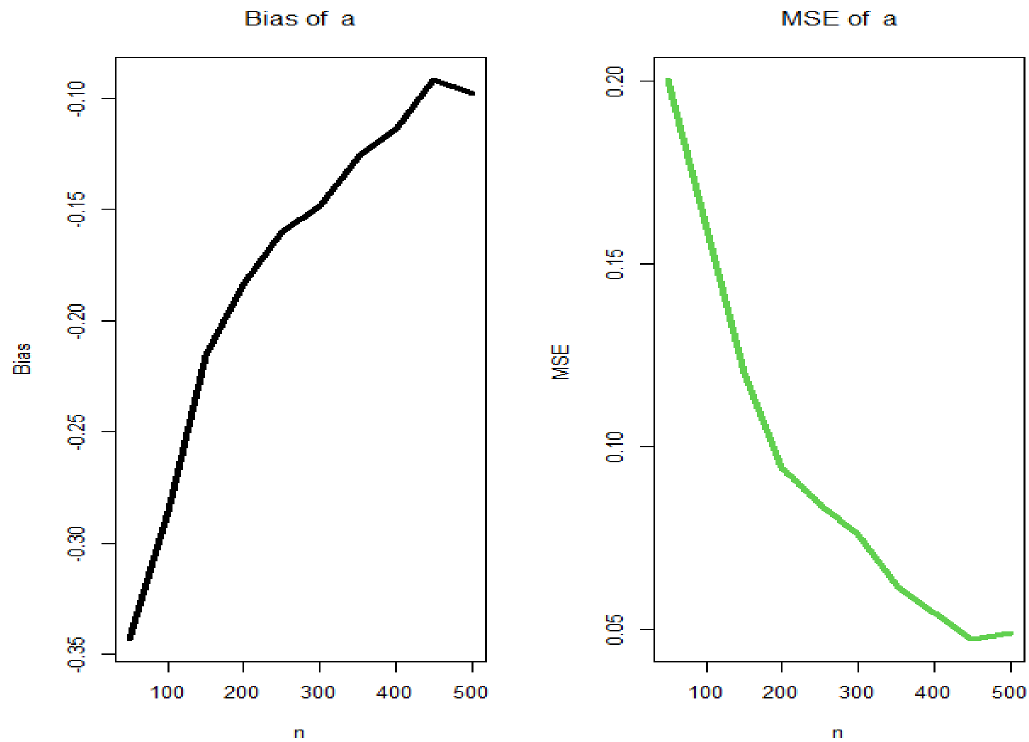

Figure 4: Bias and MSE for the parameter a.

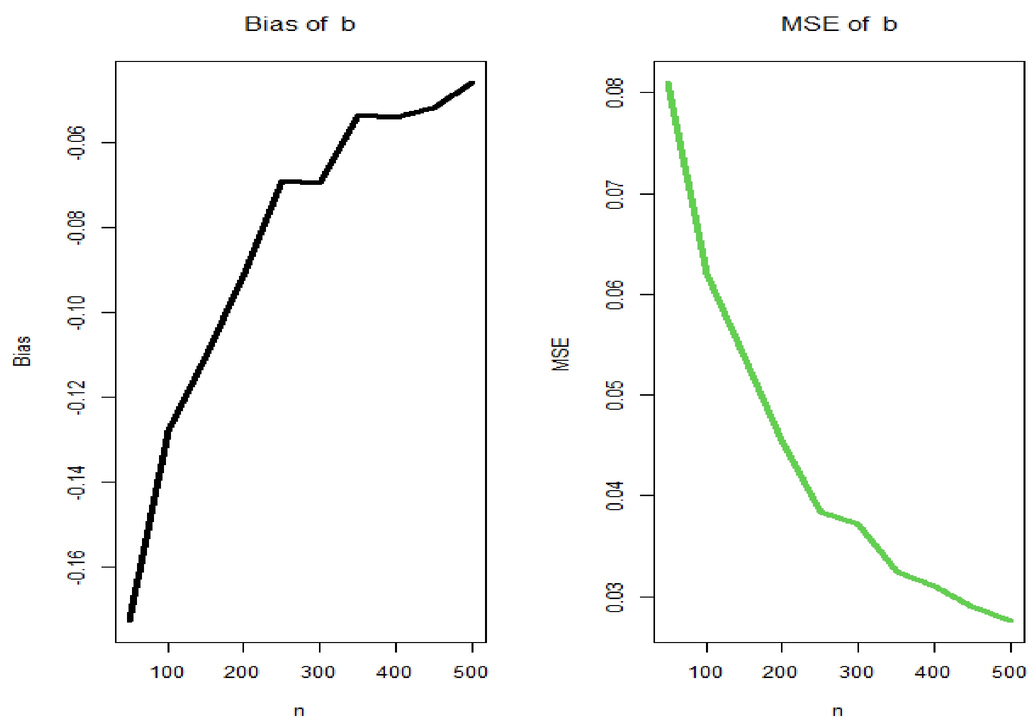

Figure 5: Bias and MSE for the parameter b.

6. Real data modeling and analysis

In this part, some different real-life data sets are modeled and analyzed for demonstrating applied importance, applicable potentiality, and wide flexibility of the WG-LL model. For these data, we compare the WG-LL distribution, with other models such as the Burr model of the kind XII (BUXII), the WLL, Marchall-Olkan-BUXII (MOBUXII), Topp-Leone-BUXII (TLBUXII), Zografos-Balakrishnan-BUXII (ZOBBUXII), the beta-BUXII which has five parameters (FBBUXII), Beta BUXII b(BBUXII), Beta-exponentiated-BUXII (BEBUXII), five-parameters Kumaraswamy-BUXII (FKwBUXII), Kumaraswamy-BUXII (KwBUXII) and Weibull-log-logistic (WLL). All versions are recently modeled by Altun et al.(2018a) Yousof et al. (2018 and 2019) and Altun et al. (2018b). Other real-life datasets are in Aryal and Yousof (2017), Yousof et al. (2017), Hamedani et al. (2017,2018 and 2019), Merovci et al. (2017 and 2020), Korkmaz et al. (2018a), Korkmaz et al. (2018b), Nascimento et al. (2019), Alizadeh et al. (2020a,b), Korkmaz et al. (2020) and Karamikabir et al. (2020).

Real-life data set **I:** the breaking stress data. It consists of 100 observations of breaking stress of carbon fibers (in Gba) (see Nichols and Padgett (2006)). Real-life data set **II:** the survival times in days of 72 pigs from guinea which was infected with the virulent tubercle bacilli(Bjerkedal (1960)). Real-life data set **III:** the leukemia data. It represents the times of survival, in weeks, of 33 patients suffering from the wee-known acute myelogenous leukemia.

The total time test (TTT) plots (Aarset(1987)) for the three real data sets are presented in Figure 2. It is seen that the HRFs of data sets **I**, **II** are monotonical increasing and U-HRF for data set **III**. We consider the following goodness-of-fit statistics: the Akaike-criterion (T_1), Bayesian-criterion (T_2), consistent-criterion (T_3) and Hannan-Quinn criterion (T_4). Generally, the smaller these statistics are, the better the fit. Tables 3, 4 and 5 give the MLEs, standard errors (SEs), confidence interval (CIs95%) with for the data set **I**, **II** and **III**. Tables 6, 7 and 8 give the statistics T_1, T_2, T_3, and T_4 values for the data set **I**, **II** and **III**. Due to Table 6, Table 7 and Table 8 and Figure 3-6 the WG-LL model has the best results with small values of the T_1, T_2, T_3, and T_4.

Table 3: Estimation results for the data set **I**.

Model		**Estimates, SEs and CI(95%)**
BUXII(β,α)	Estimates	5.941, 0.187
	SEs	(1.2792), (0.0443)
	CI(95%)	(3.431,8.455), (0.100,0.277)
MOBUXII (β,α,γ)	Estimates	1.1921, 4.8394, 838.713
	SEs	(0.9522), (4.8961), (229.341)
	CI(95%)	(0, 3.064), (0, 14.438), (389.224,1288.246)
TLBUXII(β,α,γ)	Estimates	1.3501, 1.0614, 13.7284
	SEs	(0.3788), (0.3844), (8.405)
	CI(95%)	(0.614, 2.09), (0.313,1.811), (0, 30.199)
KwBUXII (λ,θ,α,)	Estimates	48.1032 ,79.5162 ,0.3519 ,2.7305
	SEs	(19.340), (58.185), (0.098), (1.080)
	CI(95%)	(10.10,86.038), (0,193.568), (0.16,0.547), (0.62,4.845)
BBUXII(λ,θ,α,)	Estimates	359.7, 260.1, 0.1752, 1.1233
	SEs	(57.9), (132,4), (0.013), (0.24)
	CI(95%)	(246,473), (0.96,519.24), (0.14,0.204), (0.65,1.66)
BEBUXII(λ,θ,α,,γ)	Estimates	0.3813, 11.9492, 0.9372, 33.4026, 1.7057
	SEs	(0.0783), (4.6353), (0.2674), (6.2871),(0.4789)
	CI(95%)	(0.2,0.535), (2.86,215), (0.41,1.55), (21,457), (0.8,2.67)
FBBUXII(λ,θ,β,α,γ)	Estimates	0.4212, 0.834, 6.2, 1.67, 3.452
	SEs	(0.01), (0.94), (2.31), (0.23), (1.96)
	CI(95%)	(0.4,0.44), (0. 2.7), (1.57, 10.7), (1.23, 2.1), (0, 7)
FKwBUXII(λ,θ,β,α,γ)	Estimates	0.542, 4.223, 5.313, 0.411, 4.152
	SEs	(0.1375), (1.88), (2.32), (0.497), (1.9954)
	CI(95%)	(0.3, 0.88), (0.53,7.96), (0.9,9.5), (0, 1.74), (0.2,8.5)
ZOBBUXII(λ,β,α)	Estimates	123, 0.368, 139.23
	SEs	(243), (0.34), (319)
	CI(95%)	(0, 599.401), (0, 1.043), (0, 763.599)
LL(a)	Estimates	1.6292359
	SEs	(0.128801)

	CI(95%)	(1.3611, 1.847)
ExpLL(β,a)	Estimates	5.2214, 2.31971
	SEs	(0.5981), (0.14966)
	CI(95%)	(4.2, 6.19), (2, 2.59)
WLL(β,a)	Estimates	1.1159, 0.709
	SEs	(6.511), (4.1259)
	CI(95%)	(0, 16.09), (0, 8.938)
WG-LL(β,α,b,a)	Estimates	2.206, 0.667, 0.137, 1.581
	SEs	(2.509), (0.424), (0.351), (1.822)
	CI(95%)	(0, 7.5), (0, 1.5), (0, 0.84), (0, 5.1)

Table 4: Estimation results for the data set **II**.

Model		**Estimates, SEs and CI(95%)**
BUXII(β,α)	Estimates	3.1024, 0.4649,
	SEs	(0.5379), (0.0774)
	CI(95%)	(2.051,4.160), (0.310,0.624)
MOBUXII(β,α,γ)	Estimates	2.25911,1.5335, 6.7603
	SEs	(0.8641), (0.9075), (4.5871)
	CI(95%)	(0.571,3.955), (0,3.311), (0, 15.757)
TLBUXII(β,α,γ)	Estimates	2.396,0.4584,1.7964
	SEs	(0.91), (0.24), (0.92)
	CI(95%)	(0.6,4.18), (0, 0.95), (0.002,3.6)
KwBUXII (λ,θ,β,α)	Estimates	14.1049,7.42, 0.53, 2.273
	SEs	(10.81), (11.85), (0.28), (0.991)
	CI(95%)	(0, 35.3), (0.30.7), (0, 1.1), (0.33, 4.26)
BBUXII(λ,θ,β,α)	Estimates	2.6, 6.0580,1.8,0.29434,
	SEs	(1.86), (10.39), (0.96), (0.47)
	CI(95%)	(0, 6.3), (0, 26.5), (0, 3.7), (0, 1.2)
BEBUXII(λ,θ,β,α,γ)	Estimates	1.88,2.99, 1.781, 1.342, 0.573
	SEs	(0.09), (1.73), (0.7), (0.82), (0.33)
	CI(95%)	(1.7,2.1), (0, 6.4), (0.40, 3.19), (0, 3), (0, 1.22)
FBBUXII(λ,θ,β,α,γ)	Estimates	0.62, 0.55,3.838, 1.38, 1.67
	SEs	(0.54), (1.01), (2.79), (2.31), (0.44)
	CI(95%)	(0, 1.73), (0, 2.53), (0, 9.31), (0, 5.99), (0.8, 4.55)
FKwBUXII(λ,θ,β,α,γ)	Estimates	0.5583,0.308, 3.9991, 2.1312, 1.4754
	SEs	(0.44), (0.3143), (2.08), (1.83), (0.364)
	CI(95%)	(0, 1.29), (0, 0.89), (0, 3.12), (0, 5.69), (0.76, 2.3)
LL(a)	Estimates	2.27532
	SEs	(0.22327)
	CI(95%)	(1.92, 2.78)
ExpLL(β,a)	Estimates	1.9513, 2.254
	SEs	(0.2288), (0.20679)
	CI(95%)	(1.51, 2.33), (2.14, 2.90)
WLL(β,a)	Estimates	0.78545, 1.25401
	SEs	(0.00), (0.00)
	CI(95%)	--, --
WG-LL(β,α,b,a)	Estimates	0.869, 0.328, 0.879, 3.486
	SEs	(0.729), (0.262), (1.477), (2.886)
	CI(95%)	(0, 2.33), (0, 0.852), (0, 3.879), (0, 9.1)

Table 5: Estimation results for the data set **III**.

Model		**Estimates, SEs and CI(95%)**
BUXII(β,α)	Estimates	58.7110,0.0062
	SEs	(42.3821), (0.0046)
	CI(95%)	(0, 141.782), (0, 0.014)
MOBUXII(β,α,γ)	Estimates	11.8381, 0.0783, 12.2510

	SEs	(4.3681), (0.0129), (7.771)
	CI(95%)	(0, 141.8), (0, 0.011), (0, 27.50)
TLBUXII(β,α,γ)	Estimates	0.2814, 1.8823 ,50.2147
	SEs	(0.29), (2.40), (176.5)
	CI(95%)	(0, 0.855), (0, 6.599), (0, 396)
KwBUXII(λ,θ,β,α)	Estimates	9.201, 36.428, 0.242,0.941
	SEs	(10.06), (35.651), (0.168), (1.0455)
	CI(95%)	(0, 28.9123), (0, 106.3), (0, 0.57), (0, 3)
BBUXII(λ,θ,β,α)	Estimates	96.1, 52.12, 0.104, 1.2278
	SEs	(41.2), (33.49), (0.02), (0.33)
	CI(95%)	(15.4,177), (0, 118), (0.6, 0.15), (0.59,2)
BEBUXII(λ,θ,β,α,γ)	Estimates	0.087, 5.01, 1.56, 31.27, 0.323
	SEs	(0.09), (3.85), (0.012), (12.94), (0.0344)
	CI(95%)	(0, 0.33), (0, 12.66), (1.5, 1.66), (5.9, 56.58), (0.3,0.44)
FBBUXII(λ,θ,β,α,γ)	Estimates	15.1943, 32.0482, 0.233, 0.5801, 21.8555
	SEs	(11.59), (9.868), (0.092), (0.07), (35.55)
	CI(95%)	(0, 37.77), (12.7,51), (0.05,0.4), (0.45,0.7), (0, 91.5)
FKwBUXII(λ,θ,β,α,γ)	Estimates	14.73, 15.28, 0.29, 0.84, 0.0342
	SEs	(12.39), (18.87), (0.22), (0.85), (0.08)
	CI(95%)	(0, 39), (0, 52.3), (0, 0.7), (0, 2.5), (0, 0.2)
ZOBBUXII(λ,β,α,γ)	Estimates	41.9733,0.1572, 44.2632
	SEs	(38.7875), (0.0824), (47.65)
	CI(95%)	(0, 118), (0, 0.32,) (0, 137.65)
LL(a)	Estimates	0.507323
	SEs	(0.07090)
	CI(95%)	(0.366, 0.644)
ExpLL(β,a)	Estimates	5.5941, 0.76478
	SEs	(1.18343), (0.0933)
	CI(95%)	(3.1, 7.99), (0.5, 0.888)
WLL(β,a)	Estimates	1.0699, 0.2245
	SEs	(0.000), (0.000)
	CI(95%)	--,--
WG-LL(β,α,b,a)	Estimates	0.643, 0.160, 0.091, 6.701
	SEs	(0.253), (0.076), (0.048), (0.009)
	CI(95%)	(0.1, 1.1), (0.02, 0.3), (0, 0.186), (6.68, 6.718)

Table 4: T_1, T_2, T_3, and T_4 values for the data set **I**.

Model	T_1, T_2, T_3, and T_4
BUXII	382.942, 388.155, 383.063, 385.053
WLL	510.693, 515.911, 510.820, 512.844
TLBUXII	323.524, 331.354, 323.771, 326.701
MOBUXII	305.781, 313.614, 306.030, 308.960
FBBUXII	304.260, 317.311, 304.892, 309.564
ExpLL	325.931, 331.144, 326.066, 328.040
BBUXII	305.644, 316.060, 306.063, 309.853
KwBUXII	303.761, 314.201, 304.185, 308.001
BEBUXII	305.822, 318.845, 306.462, 311.093
LL	469.633, 472.233, 469.670, 470.680
FKwBUXII	305.503, 318.551, 306.145, 310.802
ZOBBUXII	302.960, 310.781, 303.214, 306.130
WG-LL	**290.750, 301.173, 291.174, 294.970**

Table 6: T_1, T_2, T_3, and T_4 values for the data set **II**

Model	T_1, T_2, T_3, and T_4
BUXII	209.600, 214.154, 209.771, 211.401
LL	231.850, 234.134, 231.914, 232.765
FBBUXII	206.803, 218.204, 207.716, 211.303
ExpLL	207.833, 212.385, 208.003, 209.644
MOBUXII	209.743, 216.564, 210.091, 212.443
TLBUXII	211.803, 218.633, 212.154, 214.520
WLL	270.441, 274.981, 270.602, 272.230
KwBUXII	208.760, 217.865, 209.366, 212.380
BBUXII	210.443, 219.545, 211.036, 214.060
FKwBUXII	206.501, 217.905, 207.413, 211.000
BEBUXII	212.103, 223.503, 213.001, 216.600
WG-LL	**205.862, 214.970, 206.460, 209.490**

Table 8: T_1, T_2, T_3, and T_4 values for the data set **III.**

Model	T_1, T_2, T_3, and T_4
BUXII	328.203, 331.193, 328.601, 329.194
LL	362.711, 364.200, 362.833, 363.215
MOBUXII	315.541, 320.010, 316.370, 317.043
TLBUXII	316.264, 320.733, 317.092, 317.761
BBUXII	316.461, 322.454, 317.893, 318.471
KwBUXII	317.363, 323.301, 318.793, 319.343
ExpLL	315.4652, 318.450, 315.850, 316.47
FBBUXII	317.863, 325.342, 320.081, 320.364
BEBUXII	317.580, 325.062, 319.801, 320.090
FKwBUXII	317.760, 325.213, 319.982, 320.260
ZOBBUXII	313.862, 318.354, 314.39, 315.3603
WLL	378.8212, 381.7902, 379.27, 379.84
WG-LL	**310.900, 316.891, 312.332, 312.923**

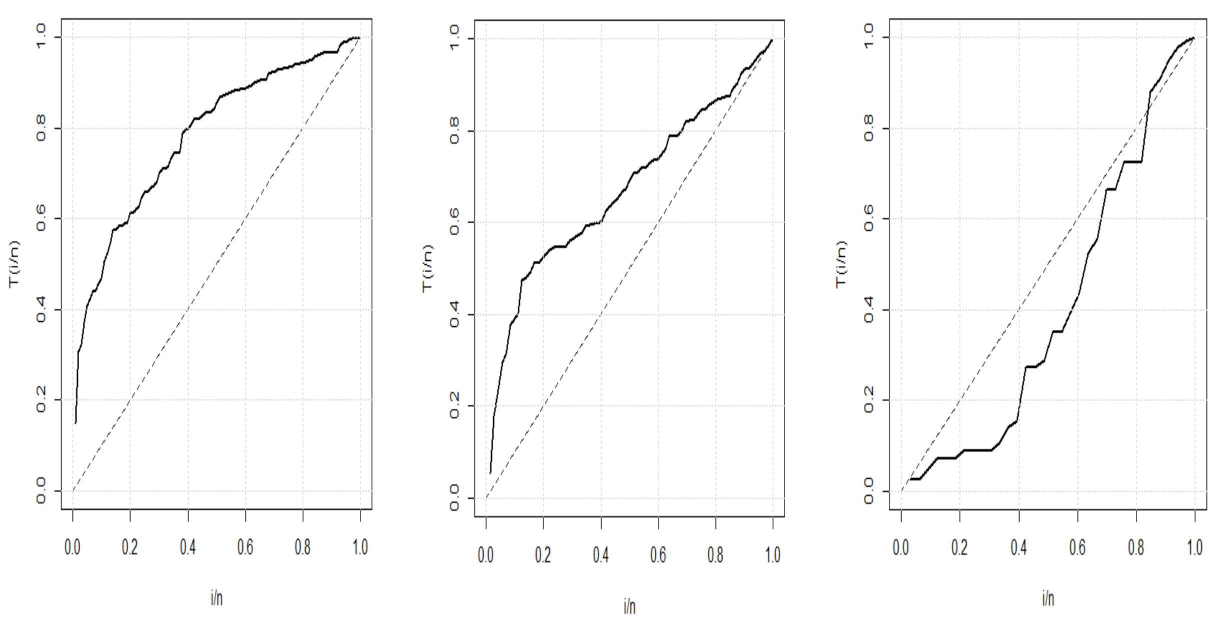

Figure 6: TTT plots.

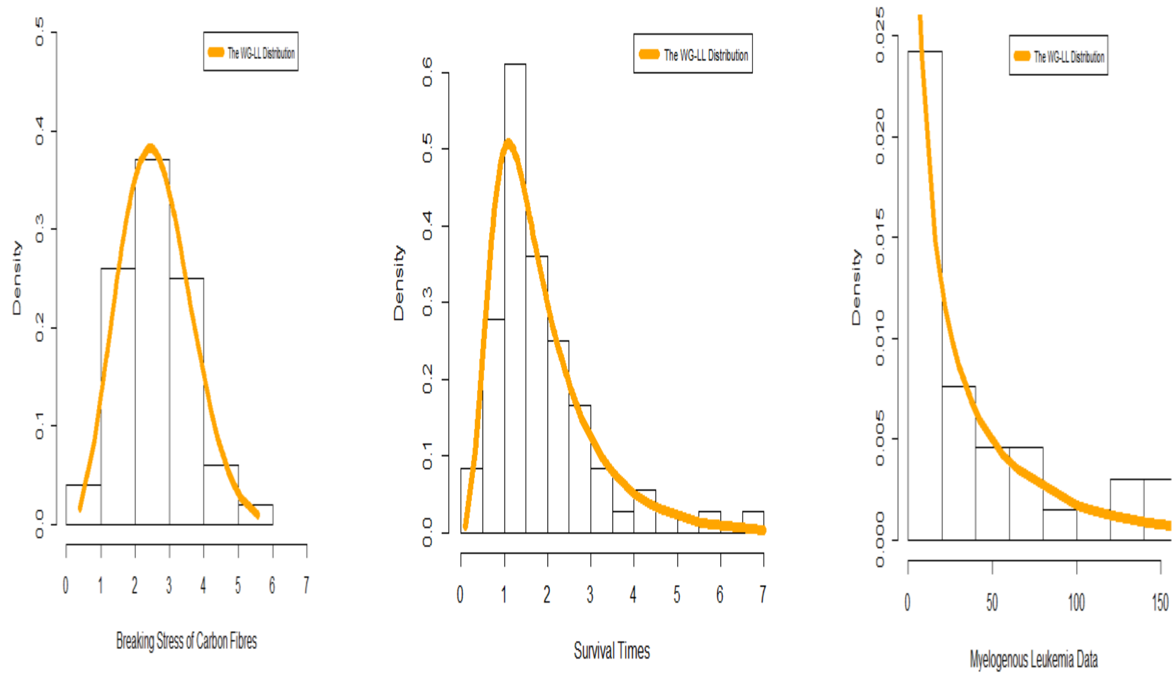

Figure 7: Estimated PDFs.

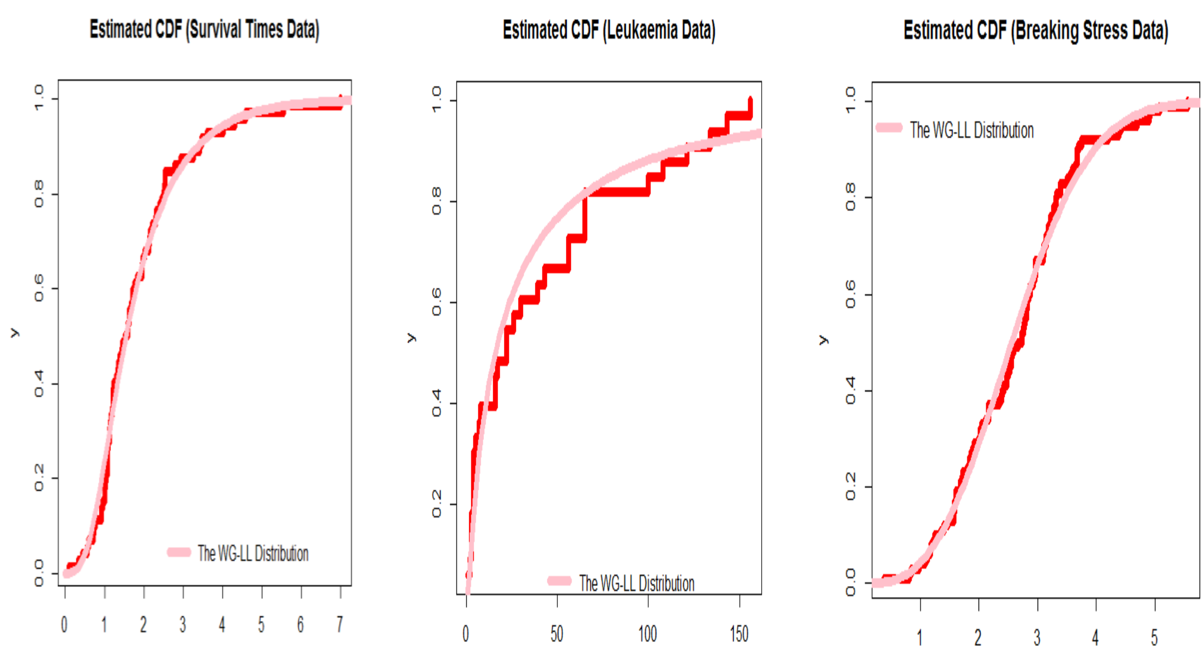

Figure 8: Estimated CDFs.

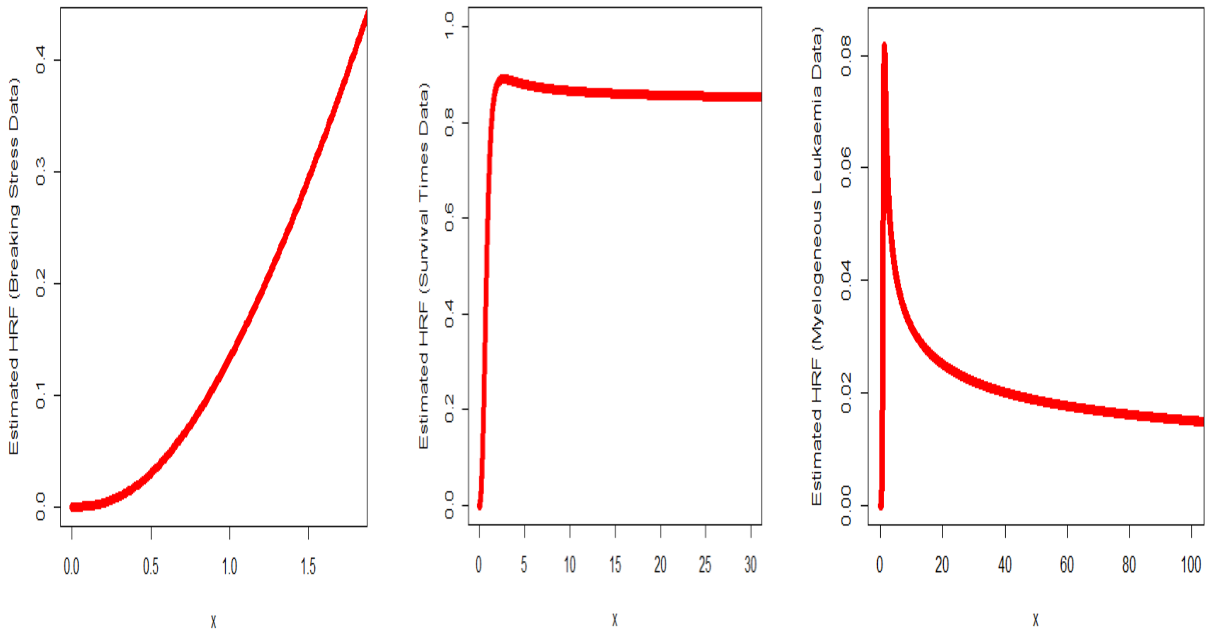

Figure 9: Estimated HRFs.

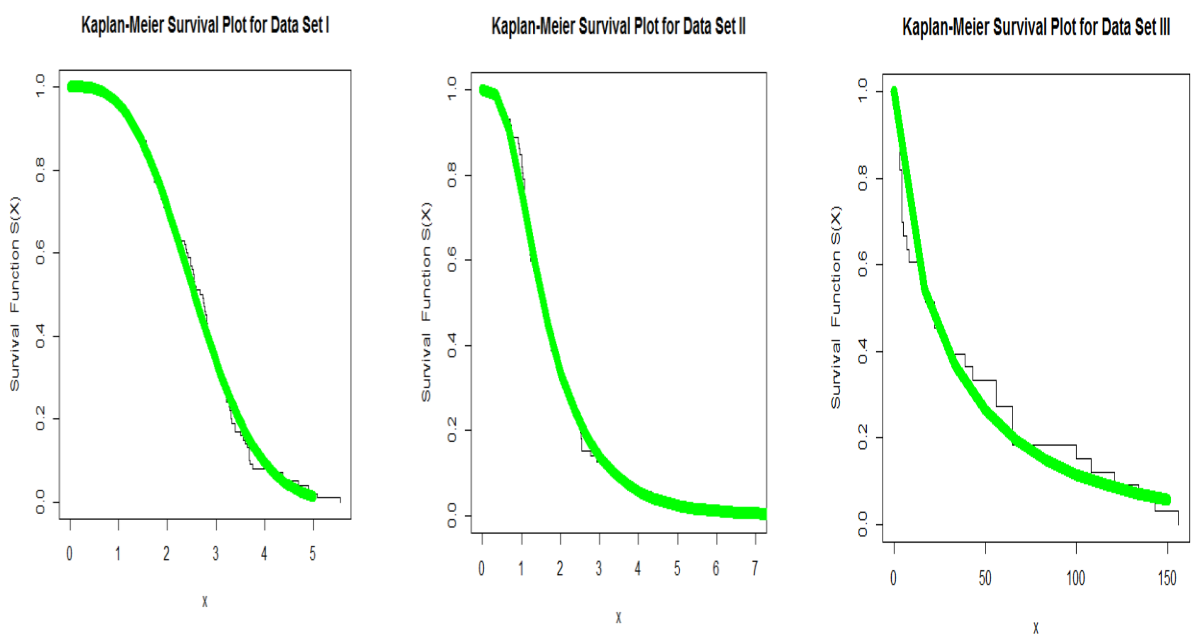

Figure 10: Kaplan-Meier survival plots.

7. Concluding remarks

In the present paper, we introduced a novel continuous log-logistic model. Several of its main characteristic properties such as the moments, the generating function, the weighted moments, the reversed residual life are mathematically derived. Numerical analysis for the skewness and the kurtosis is presented and useful comments are added. For the new log-logistic model, the skewness $\in(-1.081, 16.74)$. However, for the standard log-logistic model, the skewness

∈(0.087, 2.4853). hence, the novel model can be negative skewed and positive skewed while the standard model can only be negative skewness. For the new log-logistic model, kurtosis ∈(3.245089, 702.498). However, for the standard log-logistic model, kurtosis ∈(3.741, 29.56). The new PDF can be unimodal, symmetric, or left skewed. The new failure rate can be "bathtub or **U**-failure rate", "increasing failure rate", "decreasing-constant failure rate", "**J**-failure rate", "constant failure rate " and "decreasing failure rate".

Many bivariate and extensions are derived. To assess the estimators, we performed a graphical simulation. Three different real-life data are modeled under some statistical tests. For all these real-life datasets, we compare the novel function with many relevant extensions. The new model is better than all other competitive models in modeling breaking stress data, survival times data and leukemia data.
Future points:

1- Presenting a novel discrete model for modeling count real-life data (see Aboraya et al. (2020), Chesneau et al. (2021), Ibrahim et al. (2021) and Yousof et al. (2021) for more details).
2- Appling the Nikulin-Rao-Robson and Bagdonavičius-Nikulin tests (see Ibrahim et al. (2019), Goual et al. (2019, 2020), Mansour et al. (2020a,b,c,d,e,f), Yadav et al. (2020), Goual and Yousof (2020) and Aidi et al. (2021), among others.).

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