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A Generalization of Lomax Distribution with Properties, Copula and Real Data Applications

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Abstract
A new generalization of Lomax distribution is derived and studied. Some of its useful properties are derived. A simple clayton copula is used to generate many bivariate and multivariate type models. We performed graphical simulations to assess the finite sample behavior of the estimations. The new model is employed in modelling three real data sets.

Key Words: Lomax Distribution; Copula; Kaplan-Meier; Maximum Likelihood; Simulation; Modeling; Applications; Real Data.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1.Introduction

The Lomax (Lx) distribution or Pareto type II (PaII) distribution was presented for modeling business failure data by Lomax (1954). The Lx distribution has found wide attention and applications in a variety of fields, for instance, biological sciences, actuarial science, income, wealth inequality, engineering, medical, engineering, lifetime and reliability modeling. It has been applied for modeling data obtained from income and wealth (see Harris (1968) and Asgharzadeh and Valiollahi (2011)), firm size (see Corbellini et al. (2007)), reliability and life testing (see Hassan and Al-Ghamdi (2009)). A random variable (RV) is said to have the Lomax distribution if its survival function (SF) is given by

$$S_\alpha(y) = 1 - \mathbf{\Pi}_\alpha(y) = (1+y)^{-\alpha}|_{y\geq 0}, \tag{1}$$

where $\mathbf{\Pi}_\alpha(y) = 1-(1+y)^{-\alpha}|_{y\geq 0}$ is the cumulative distribution function (CDF) of the standard Lx model and

$$h_\alpha(y) = \alpha(1+y)^{-\alpha-1}|_{y\geq 0}, \tag{2}$$

is the probability density function (PDF) of the standard Lx model, where α is the shape parameters. The PDF in (2) is a special case from the well-known Burr XII (BXII) model. Many useful details about the Lx model and its relationship with other models can be found in Burr (1942, 1968 and 1973), Lomax (1954), Burr and Cislak (1968), Harris (1968), Rodriguez (1977), Tadikamalla (1980). Cordeiro et al. (2016) investigated a new flexible class of continuous distributions called the generalized odd log-logistic-G (GOLL-G) family with only two extra shape parameters. In the research, we introduce a new version of the BXII model using the GOLL-G family called the generalized odd log-logistic Lx (GOLLLx). For an arbitrary baseline CDF $\Pi_{\underline{\Psi}}(y)$, the CDF of the GOLL-G family is given by

$$F_{a,b,\underline{\Psi}}(y) = \frac{\mathbf{\Pi}_{\underline{\Psi}}(y)^{ab}}{\mathbf{\Pi}_{\underline{\Psi}}(y)^{ab} + \left[1-\mathbf{\Pi}_{\underline{\Psi}}(y)^{b}\right]^{a}}, \tag{3}$$

where $\underline{\mathbf{\Psi}}$ is the parameter vector of the base line model. For $b=1$ we get the OLL-G family (Gleaton and Lynch (2006)). For $a=1$, the OLL-G family reduces to the proportional reversed hazard rate G (PRHR-G) family (Gupta and Gupta (2007)). The CDF of the GOLLLx is given by

$$F_{\underline{\Phi}}(y)=\frac{[1-(1+y)^{-\alpha}]^{ab}}{[1-(1+y)^{-\alpha}]^{ab}+\{1-[1-(1+y)^{-\alpha}]^{b}\}^{a}}, \quad (5)$$

where $\underline{\Phi}=a,b,\alpha$. The PDF corresponding to (4) can be given as

$$f_{\underline{\Phi}}(y)=\alpha ab(1+y)^{-\alpha-1}\frac{[1-(1+y)^{-\alpha}]^{ab-1}\{1-[1-(1+y)^{-\alpha}]^{b}\}^{a-1}}{\left([1-(1+y)^{-\alpha}]^{ab}+\{1-[1-(1+y)^{-\alpha}]^{b}\}^{a}\right)^{2}}. \quad (6)$$

The hazard rate function (HRF) for the GOLLLx model can be obtained from $h_{\underline{\Phi}}(y)=f_{\underline{\Phi}}(y)/\left[1-F_{\underline{\Phi}}(y)\right]$. For $b=1$ we get the OLLLx model. For $a=1$ we get the PRHRLx model. Recently, many useful extensions are presented by Bhatti et al. (2018), Goual Yousof (2019), Ibrahim and Yousof (2020), Karamikabir et al. (2020), Mansour et al. (2020f), Ansari et al. (2020), Goual et al. (2020) and Yadav et al. (2020). Based on generalized binomial expansions and after some algebraic processes, the PDF in (6) can be rewritten as

$$f_{\underline{\Phi}}(y)=\sum_{\ell_4=0}^{\infty}c_{[\ell_4]}h_{\alpha^*}(y), \quad (6)$$

where $\alpha^*=\alpha(1+\ell_4)$ and

$$c_{[\ell_4]}=\frac{ab}{1+\ell_4}\sum_{\ell_1,\ell_2=0}^{\infty}\sum_{\ell_3=\ell_4}^{\infty}(-1)^{\ell_2+\ell_3+\ell_4}\binom{-2}{\ell_1}\binom{\ell_3}{\ell_4}\binom{1+\ell_3}{y}\binom{-a(\ell_1+1)}{\ell_2}\binom{ab(\ell_1+1)+b\ell_2-1}{\ell_3},$$

and $h_{\alpha^*}(y)$ is the PDF of the Lx model with parameters α^*. Accordingly, the PDF of the new model can be expressed as a linear mixture of the Lx PDF. So, many properties of the new Lx model can be derived from (6) and those of the standard Lx model. Let Y be a RV having the Lx distribution (2) with parameter a_2. For $m<a_2$, the m^{th} ordinary and incomplete moments (Ic-Ms) of Y are, respectively, given by

$$\mu'_m=\alpha B(\alpha-m,1+m) \text{ and } \mathrm{I}_m(t)=\alpha B(t;\alpha-m,1+m),$$

where

$$B(a_1,a_2)=\int_0^{\infty}1+y)^{-(a_1+a_2)}\,(y^{a_1-1}dy \text{ and } B(t;a_1,a_2)=\int_0^{y}(1+y)^{-(a_1+a_2)}\,y^{a_1-1}dy$$

are the beta and the incomplete beta functions of the second type, respectively.

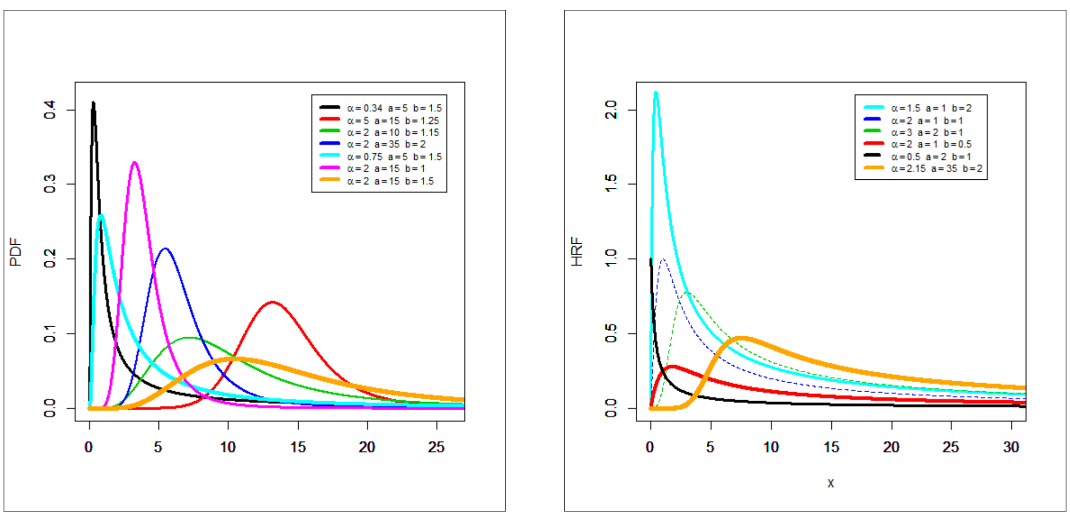

Figure 1: Plots for the new PDF and its corresponding HRF for some selected parameter values.

2.Properties

2.1 Asymptotics

Let $\varepsilon=inf\{Y|_{[\Pi_\alpha(y)>0]}\}$, the asymptotics of the CDF, PDF and HRF as $Y\to\varepsilon$ are given by

$$F_{\underline{\Phi}}(y)|_{y\to\varepsilon}\sim[1-(1+y)^{-\alpha}]^{ab}\,,f_{\underline{\Phi}}(y)|_{y\to\varepsilon}\sim\frac{\alpha ab(1+y)^{-\alpha-1}}{[1-(1+y)^{-\alpha}]^{-ab+1}}$$

and

$$h_{\underline{\Phi}}(y)|_{y\to\varepsilon}\sim\frac{\alpha ab(1+y)^{-\alpha-1}}{\{1-(1+y)^{-\alpha}\}^{-ab+1}}.$$

The asymptotics of CDF, PDF and HRF as $Y \to \infty$ are given by

$$1-F_{\underline{\Phi}}(y)|_{y\to\infty} \sim \frac{b^{a}}{(1+y)^{a\alpha}}, f_{\underline{\Phi}}(y)|_{y\to\infty} \sim \frac{ab^{a}\alpha}{(1+y)^{a\alpha+1}} \text{ and } h_{\underline{\Phi}}(y)|_{y\to\infty} \sim \frac{a\alpha}{1+y}.$$

2.2 Ordinary moment

The m^{th} ordinary moment of Y is given by

$$\mu_m' = E(y^m) = \sum_{\ell_4=0}^{\infty} c_{[\ell_4]} \int_0^{\infty} y^m \; h_{\alpha^*}(y)dy.$$

Then,

$$\mu_m' = E(y^m) = \textstyle\sum_{\ell_4=0}^{\infty} c_{[\ell_4]}\, \alpha^* B(\alpha^* - m, 1+m)|_{(m<a\alpha^*)}. \quad (7)$$

Setting $m = 1$ in (7), we have the mean of Y. The effects of the parameters a, b, α on the mean (μ_1'), variance (V (Y)), skewness (S (Y)) and kurtosis (K (Y)) for given values are listed in Table 1. Form Tables 1 and 2 we note that the new additional shape parameters and has an effect on μ_1' , V (Y) , S(W) and K (Y) . For the GOLLLx model, S (Y) can range in the interval $(-$ 183.1, 7514.7). However, for the Lx model, S (Y) can range in the interval ($-$ 0.4104, 4.6476). For the GOLLLx model, K (Y) can range in the interval ($-$ 1531.11, 56479275). However, for the Lx model, K (Y) can range in the interval (0.93244, 73.8).

Table 1: Numerical results for μ_1', V(Y), S(Y), K(Y) for the GOLLLx.

a	b	α	μ_1'	V(Y)	S(Y)	K(Y)
1	10	5	0.8598479	0.326637100	3.6522880	50.47191
2			0.7540910	0.055488670	1.6301880	9.990193
5			0.7232649	0.007758169	0.6661556	5.150165
10			0.7187439	0.001897504	0.3352351	4.442170
25			0.7174704	0.000301686	0.1343592	4.239094
40			0.7173224	0.000117759	0.0839946	4.215275
50			0.7172882	7.535265×10^{-5}	0.0672000	4.209720
60			4.871488×10^{-5}	3.725648×10^{-5}	125.43350	15750.29
75			2.314250×10^{-6}	1.777331×10^{-5}	576.40470	332410.3
80			8.330471×10^{-7}	6.403259×10^{-7}	961.01880	923925.1
90			1.071033×10^{-7}	8.242558×10^{-8}	2681.3500	7191418
100			1.364333×10^{-8}	1.050770×10^{-8}	7514.6940	5647926
3	0.01	5	1.358×10^{-6}	5.3288×10^{-8}	565.0617	617625.9
	0.1		0.001966	6.1993×10^{-5}	15.21565	471.3322
	0.5		0.070440	0.0025435	2.414007	15.80801
	1		0.160977	0.0056581	1.655923	9.734984
	5		0.519958	0.0163016	1.153708	7.010816
	10		0.733841	0.0226245	1.097702	6.769463
	20		0.984661	0.0306119	1.070302	6.655797
	50		1.378768	0.0448314	1.054058	6.589766
	100		1.730548	0.0594514	1.048677	6.568111
	150		1.960502	0.0700359	1.046887	6.560931
	500		2.765278	0.1136273	1.044384	6.550913
	1000		3.324861	0.1500068	1.043848	6.548771
	3000		4.387358	0.2328648	1.043491	6.547344
	5000		4.966800	0.2856750	1.043419	6.547059
	7000		5.382125	0.3268405	1.043389	6.546937
	9000		5.711093	0.3614103	1.043372	6.546869
	10000		5.854005	0.3769692	1.043366	6.546845
4	4	1.5	2.5216120	0.5909997000	2.0270130	17.48084
		3	0.8666202	0.0373406600	1.1463170	7.326794
		5	0.4524331	0.0078818440	0.8951852	5.88929
		10	0.2046185	0.0013274300	0.7275170	5.165168
		15	0.1320247	0.0005176865	0.6747110	4.972186
		20	0.0974268	0.0002728258	0.6488175	5.133086

25	0.0771908	0.0001679220	0.6334305	−214.1473
30	0.0639131	0.0001136188	0.6232612	792.5223
35	0.0545320	8.193953×10^{-5}	0.615986	−244.6907
40	0.0475519	6.186782×10^{-5}	−152.5364	2769.331
45	0.0421557	4.835713×10^{-5}	−98.06595	1620.058
50	0.0378593	$3.883173\times1\text{-}^{-5}$	−34.64589	1206.973
55	0.0343576	3.186594×10^{-5}	89.99558	−585.3193
60	0.0314487	2.661881×10^{-5}	129.3149	−1531.107
65	0.0289940	2.256758×10^{-5}	80.29819	−1170.643
75	0.0250788	1.681641×10^{-5}	−134.3165	2622.245
80	0.0234926	1.473220×10^{-5}	−183.0836	3437.396

Table 2: Numerical results for μ_1', V(Y), S(Y), K(Y) for the Lx.

α	μ_1'	V(Y)	S(Y)	K(Y)
5	0.2500000	0.1041667	4.647580	73.8000
10	0.1111111	0.0154321	2.811057	17.82857
20	0.0526316	0.0030779	2.343806	12.13015
50	0.0204082	0.0004339	2.126365	10.06002
75	0.0135135	0.0001876	2.08269	9.684103
100	0.0101010	0.0001041	2.059443	9.522699
200	0.0050251	2.608252×10^{-5}	1.945802	8.349871
300	0.0033445	1.038924×10^{-5}	2.956477	10.32769
400	0.0025063	4.541270×10^{-6}	4.316849	22.29550
500	0.0020040	2.703376×10^{-6}	3.090030	26.71821
750	0.0013351	2.323909×10^{-6}	−0.410385	3.747853
1000	0.0010010	2.108303×10^{-6}	−0.195204	0.932436
1500	0.0006671	1.152765×10^{-6}	0.659860	1.406983
2000	0.0005003	4.810577×10^{-7}	2.153126	4.878395

2.3 Moment generating function

The moment generating function (MGF) of Y, say $M_Y(t) = E[exp(tY)]$, can be obtained from (6) as

$$M_Y(t) = \sum_{\ell_4=0}^{\infty} c_{[\ell_4]}\, M_{\alpha^*}(t),$$

where $M_{\alpha^*}(t)$ is the MGF of the Lx distribution with parameters α^*. Then, we have

$$M_Y(t) = \sum_{\ell_4,r=0}^{\infty} \frac{t^r}{r!} c_{[\ell_4]} \alpha^* B(\alpha^* - m, 1 + m)|_{(m<a\alpha^*)}.$$

2.4 Ic-Ms

The s^{th} Ic-M, say $I_s(q)$, of the GOLLLx distribution is given by $I_s(q) = \int_0^q y^s f(y)dy$. Then, from equation (7), we have $I_s(t) = \sum_{\ell_4=0}^{\infty} c_{[\ell_4]} \int_0^t y^s h_{\alpha^*}(y)dy$ and using the lower incomplete gamma function, we obtain

$$I_s(t) = \sum_{\ell_4=0}^{\infty} c_{[\ell_4]} \alpha^* B(t^a; \alpha^* - s, 1 + s).$$

The first Ic-M of Y, referred to $I_1(t)$, is just determined from the above equation by setting $s = 1$. The first Ic-M has main applications related to the Bonferroni and Lorenz curves and the mean residual life and the mean waiting time. Moreover, the amount of scattering in a population is clearly measured, to some extent, by the totality of deviations from the mean and median. The mean deviations, about the mean and about the median of Y depend on $I_1(t)$. The Bonferroni $[B_{Y,F(y),t}]$ and Lorenz $[L_{Y,t}]$ curves have many applications especially in deconomics, demography, insurance, reliability, medicine where $L_{Y,t}=\frac{I_{(0,y)}(t)}{E(Y)}|_{\left(I_{(0,y)}(z)=\int_0^y zf(z)dz\right)}$ and $B_{Y,F(y),t}=\frac{I_{(0,y)}(t)}{E(Y)F(y)}=\frac{L_{Y,t}}{F(y)}$ where $F(y)=F_{\underline{\Phi}}(y)$. Then, we have

$$L_{Y,t}=\frac{\sum_{\ell_4=0}^{\infty}c_{[\ell_4]}\,\alpha^*B(t^a;\alpha^*-s,1+s)}{\sum_{\ell_4=0}^{\infty}c_{[\ell_4]}\,\alpha^*B(\alpha^*-s,1+s)}|_{(s<a\alpha^*)}$$

and

$$B_{Y,F(y),t}=\left(1+\frac{\{1-[1-(1+y)^{-\alpha}]^b\}^a}{[1-(1+y)^{-\alpha}]^{ab}}\right)\frac{\sum_{\ell_4=0}^{\infty}c_{[\ell_4]}\,\alpha^*B(t^a;\alpha^*-s,1+s)}{\sum_{\ell_4=0}^{\infty}c_{[\ell_4]}\,\alpha^*B(\alpha^*-s,1+s)}|_{(s<a\alpha^*)}.$$

2.5 Residual and reversed residual life functions

The m^{th} moment of the residual life (RL), denoted by $u_m(t)=E[(Y-t)^m]|_{(Y>t,\ m=1,2,\dots)}$. The m^{th} moment of the residual life of Y is given by $u_m(t)=\frac{\int_t^{\infty}(Y-t)^m f_{\underline{\Phi}}(y)dy}{1-F_{\underline{\Phi}}(t)}$. Then, we can write

$$u_m(t)=\frac{1}{1-F_{\underline{\Phi}}(t)}\sum_{i=0}^{m}\sum_{\ell_4=0}^{\infty}\frac{(-1)^{m-i}m!\,t^{m-i}}{i!\,\Gamma(m-i+1)}c_{[\ell_4]}\alpha^*B(t^a;\alpha^*-m,1+m).$$

The m^{th} moment reversed residual life, say $U_m(t)=E[(t-Y)^m]|_{(t>0,Y\leq t\text{ and }m=1,2,\dots)}$. Then, $U_m(t)$ is defined by $U_m(t)=\frac{\int_0^t(t-Y)^m f_{\underline{\Phi}}(y)dy}{F_{\underline{\Phi}}(t)}$. The m^{th} moment of the reversed residual life of Y

$$U_m(t)=\frac{1}{F_{\underline{\Phi}}(t)}\sum_{i=0}^{m}\sum_{\ell_4=0}^{\infty}\frac{(-1)^{i}m!}{i!\,(m-i)!}c_{[\ell_4]}\alpha^*B(t^a;\alpha^*-m,1+m).$$

3. Maximum likelihood method

Suppose that $(y_1,\cdots,y_m)$ is a random sample (rs) from the GOLLLx model with parameter vector $\underline{\mathbf{\Psi}}$. The log-likelihood function ($\ell_m(\underline{\mathbf{\Psi}})$) for $\underline{\mathbf{\Psi}}$ is given by

$$\ell_m(\underline{\Psi})=m\,log(\alpha ab)-(\alpha+1)\sum_{i=1}^{m}log(y_i+1)+(ab-1)\sum_{i=1}^{m}log(1-s_i^{-\alpha})$$
$$+(a-1)\sum_{i=1}^{m}log\left(1-z_i^b\right)-2\sum_{i=1}^{m}log\left[z_i^{ab}+\left(1-z_i^b\right)^a\right].$$

where $s_i=(y_i+1)$ and $z_i=[1-s_i^{-\alpha}]$. The above $\ell_m(\underline{\mathbf{\Psi}})$ can be maximized numerically via **SAS** (**PROC NLMIXED**) or **R** (**optim**) or **Ox** program (via sub-routine **MaxBFGS**), among others.

4. Copula

We derive some new bivariate type GOLLLx (BGOLLLx) model using “Farlie Gumbel Morgenstern” (FGM) Copula (see Morgenstern (1956), Gumbel (1958) and Gumbel (1960)),” Clayton Copula”, “modified FGM” and “Renyi's entropy” (Pougaza and Djafari (2011)). However, future works may be allocated to study these new models. First, we consider the joint CDF of the FGM family, where $C_{\mathbf{\Omega}}(u,d)=u\psi(1+\mathbf{\Omega}u^*d^*)|_{u^*=1-u}$, where the marginal function $u=F_1, d=F_2$, $\mathbf{\Omega}\in(-1,1)$ is a dependence parameter and for every $u,\psi\in(0,1)$, $C(u,0)=C(0,\psi)=0$ which is "grounded minimum" and $C(u,1)=u$ and $C(1,d)=d$ which is "grounded maximum", $C(u_1,d_1)+C(u_2,d_2)-C(u_1,d_2)-C(u_2,d_1)\geq 0$.

4.1 BGOLLLx type via FGM Copula

A Copula is continuous in u and d; actually, it satisfies the “stronger Lipschitz condition”, where

$$|C(u_2,d_2)-C(u_1,d_1)|\leq|u_2-u_1|+|d_2-d_1|.$$

For $0\leq u_1\leq u_2\leq 1$ and $0\leq\psi_1\leq\psi_2\leq 1$, we have

$$C(u_1,d_1)+C(u_2,d_2)-C(u_1,d_2)-C(u_2,d_1)=Pr(u_1\leq U\leq u_2,d_1\leq D\leq d_2)\geq 0.$$

Then, setting $u^*=1-F_{\underline{\mathbf{\Psi}}_1}(x_1)|_{[u^*=(1-u)\in(0,1)]}$ and $d^*=1-F_{\underline{\mathbf{\Psi}}_2}(x_2)|_{[d^*=(1-d)\in(0,1)]}$. We can esaily get the get the joint CDF of the FGM family.The joint PDF can then derived from $c_{\mathbf{\Omega}}(u,\psi)=1+\mathbf{\Omega}u^{\cdot}d^{\cdot}|_{(u^{\cdot}=1-2u\text{ and }d^{\cdot}=1-2d)}$ or from $f(x_1,x_2)=C\left(F_{\underline{\mathbf{\Psi}}_1}(x_1)_1,F_{\underline{\mathbf{\Psi}}_2}(x_2)\right)f_{\underline{\mathbf{\Psi}}_1}(x_1)_1,f_{\underline{\mathbf{\Psi}}_2}(x_2)$.

4.2 BGOLLLx and MvGOLLLx type via Clayton Copula

The "Clayton Copula" can be considered as $C_{\mathbf{\Omega}}(d_1,d_2)=[(1/d_1)^{\mathbf{\Omega}}+(1/d_2)^{\mathbf{\Omega}}-1]^{-\mathbf{\Omega}^{-1}}|_{\nabla\in(0,\infty)}$. Setting $d_1=F_{\underline{\mathbf{\Psi}}_1}(t)$ and $d_2=F_{\underline{\mathbf{\Psi}}_2}(x)$. Then, the BGOLLLx type can be derived from $C(d_1,d_2)=C(F_{\underline{\mathbf{\Psi}}_1}(t),F_{\underline{\mathbf{\Psi}}_2}(x))$. Similarly, the MGOLLLx (m-dimensional extension) from the above can be derived from $C(d_i)=\left[\sum_{i=1}^{m}d_i^{-\mathbf{\Omega}}+1-m\right]^{-\mathbf{\Omega}^{-1}}$.

4.3 BGOLLLx type via Renyi's entropy

Using the theorem of Pougaza and Djafari (2011) where $R(u,d)=x_2u+x_1d-x_1x_2$. Then, the associated BGOLLLx will be $R(u,d)=R(F_{\underline{\mathbf{\Psi}}_1}(x_1),F_{\underline{\mathbf{\Psi}}_2}(x_2))$.

4.4 BGOLLLx type via modified FGM Copula

The modified version of the bivariate FGM copula defined as (Rodriguez-Lallena and Ubeda-Flores (2004)) $C_{\mathbf{\Omega}}(u,d)=ud+\mathbf{\Omega}\widetilde{\mathbf{O}(u)}\widetilde{\mathbf{\psi}(d)}$, where $\widetilde{\mathbf{O}(u)}=u\overline{\mathbf{O}(u)}$, and $\widetilde{\mathbf{\psi}(d)}=d\overline{\mathbf{O}(d)}$. Where $\mathbf{O}(u)$ and $\mathbf{\psi}(v)$ are two continuous functions on (0,1) where $\mathbf{O}(0)=\mathbf{O}(1)=\mathbf{\psi}(0)=\mathbf{\psi}(1)=0$. Let

$$b=inf\left\{\frac{\partial}{\partial u}\widetilde{\mathbf{O}(u)}:\mathcal{H}_1(u)\right\}<0, a=sup\left\{\frac{\partial}{\partial u}\widetilde{\mathbf{O}(u)}:\mathcal{H}_1(u)\right\}<0,$$
$$c=inf\left\{\frac{\partial}{\partial \psi}\widetilde{\mathbf{\psi}(d)}:\mathcal{H}_2(d)\right\}>0, s=sup\left\{\frac{\partial}{\partial \psi}\widetilde{\mathbf{\psi}(d)}:\mathcal{H}_2(d)\right\}>0.$$

Then, $min(ba,cs)\geq 1$, where $\frac{\partial}{\partial u}\widetilde{\mathbf{O}(u)}=\mathbf{O}(u)+u\frac{\partial}{\partial u}\mathbf{O}(u)$, $\mathcal{H}_1(u)=\left\{u\in(0,1):\frac{\partial}{\partial u}\widetilde{\mathbf{O}(u)}\text{ exists}\right\}$ and $\mathcal{H}_2(d)=\left\{d\in(0,1):\frac{\partial}{\partial \psi}\widetilde{\mathbf{\psi}(d)}\text{ exists}\right\}$.

4.4.1 BGOLLLx-FGM (Type I)

The BivGOLLLx-FGM (Type-I) copula can be obtained directly from $C_{\mathbf{\Omega}}(u,v)=u\psi+\mathbf{\Omega}\widetilde{\mathbf{\phi}(u)}\widetilde{\mathbf{\psi}(\psi)}$.

4.4.2 BGOLLLx-FGM (Type II)

Consider the following functional form for both $\mathbf{O}(u)$ and $\mathbf{\psi}(\psi)$ which satisfy all the conditions stated earlier where $\mathbf{O}(u)|_{(\mathbf{\Omega}_1>0)}=u^{\mathbf{\Omega}_1}(1-u)^{1-\mathbf{\Omega}_1}$ and $\mathbf{\psi}(d)|_{(\mathbf{\Omega}_2>0)}=d^{\mathbf{\Omega}_2}(1-d)^{1-\mathbf{\Omega}_2}$. The corresponding bivariate copula (henceforth, BGOLLLx-FGM (Type-II) copula) can be derived from

$$C_{\mathbf{\Omega},\mathbf{\Omega}_1,\mathbf{\Omega}_2}(u,d)=ud[1+\mathbf{\Omega}u^{\mathbf{\Omega}_1}d^{\mathbf{\Omega}_2}(1-u)^{1-\mathbf{\Omega}_1}(1-d)^{1-\mathbf{\Omega}_2}].$$

4.4.3 BGOLLLx-FGM (Type III)

Consider the following functional form for both $\mathbf{O}(u)$ and $\mathbf{\psi}(\psi)$ which satisfy all the conditions stated earlier where $\mathbf{O}^{\bullet}(u)=u[log(1+\overline{u})]$ and $\mathbf{\psi}^{\bullet}(d)=d\left[log\left(1+\overline{d}\right)\right]$. Then, the associated CDF of the BivGOLLLx-FGM (Type-III)

$$C_{\mathbf{\Omega}}(u,d)=ud+[1+\mathbf{\Omega}\mathbf{O}^{\bullet}(u)\mathbf{\psi}^{\bullet}(d)]$$

4.4.4 BGOLLLx-FGM (Type IV)

The BGOLLLx-FGM (Type-IV) model can be derived from $C(u,d)=uF^{-1}(d)+dF^{-1}(u)-F^{-1}(u)F^{-1}(d)$where $F^{-1}(u)$ and $F^{-1}(d)$ can be easily derived.

5.Graphical simulations

To assess of the finite sample behavior of the MLEs, consider the following algorithm:

1) use $y_u=\left(1-\left[\frac{\left(\frac{u}{1-u}\right)^{\frac{1}{a}}}{1+\left(\frac{u}{1-u}\right)^{\frac{1}{a}}}\right]^{\frac{1}{b}}\right)^{-\frac{1}{\alpha}}-1$ to generate 1000 samples of size m from the GOLLLx distribution;
2) compute the MLEs for the 1000 samples
3) compute the SEs of the MLEs for the 1000 samples. The standard errors (SEs) were computed by inverting the observed information matrix;
4) compute the biases $(B_h(m))$ and mean squared errors ($MSEs$) given for $h=\underline{\mathbf{\Phi}}$. We repeated these steps for $m=50,100,\ldots,1000$ with $\alpha=1$, $a=1$, $b=1$, $=1$ so computing biases, mean squared errors $(MSE_h(m))$ for $\underline{\mathbf{\Phi}}$ and $m=50,100,\ldots,1000$.

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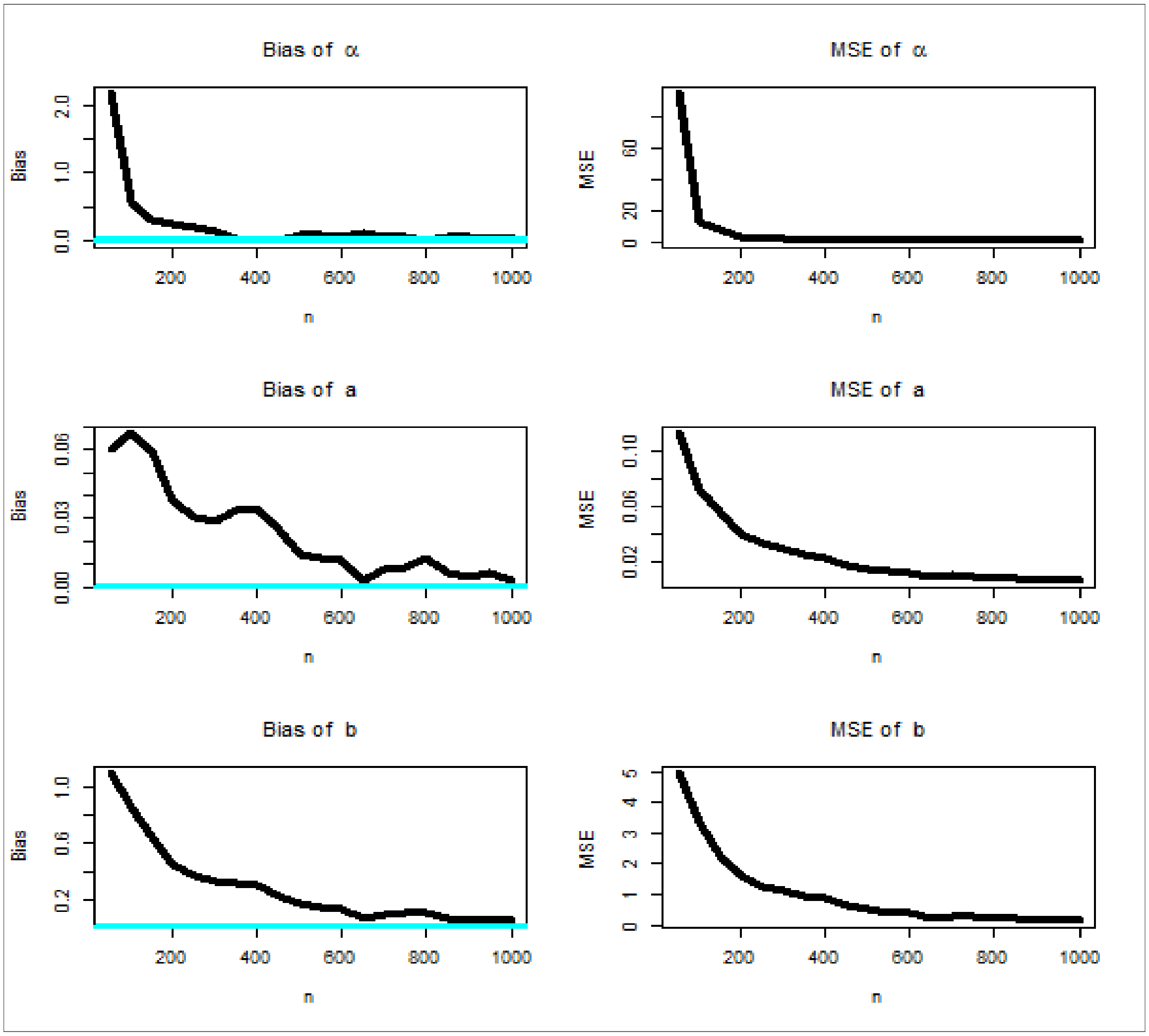

Figure 2: Biases and MSEs for a, b, α and $m = 50,100, \dots ,1000$ for the GOLLLx model.

Figure 2 (left panel) shows how the three biases vary with respect to m . Figure 2 (right panel) shows how the three MSEs vary with respect to m . The broken lines in Figure 2 corresponds to the biases being 0 . From Figure 2, the biases for each parameter decrease to zero as $m \to \infty$, the MSEs for each parameter decrease to zero as $m \to \infty$.

6. Comparing models

To illustrate the flexibility of the GOLLLx model, we provide three applications. The 1st data set called breaking stress data. This data set consists of 100 observations of breaking stress of carbon fibrrs (in Gba) given by Nichols and Padgett (2006). The 2nd data set called survival times. In this application, we work with the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, originally observed and reported by Bjerkedal, T. (1960). The 3rd data set called taxes revenue data. The actual taxes revenue data (in 1000 million Egyptian pounds) given in Altun et al. (2018a, b). For all data sets, we compare the GOLLLx distribution with the standard Lx, the exponentiated Lx (ExpLx), the Burr XII (BXII), the Marshall-Olkin Burr XII (MOBXII), the Topp-Leone Burr XII (TLBXII), the Zografos-Balakrishnan Burr XII (ZBBXII), the five-parameters beta Burr XII (FBBXII), beta Burr XII (BBXII), beta exponentiated Burr XII (BEBXII), the five-parameters Kumaraswamy Burr XII (FKumBXII) and Kumaraswamy Burr XII (KumBXII) distributions. All competitive models are given in Yousof et al. (2019) and Altun et al. (2018 a, b).

We consider the well-known G-O-F statistics: the Akaike Information Criterion $(C-AI)$, Bayesian Information Criterion $(C-Bayes)$, Hannan-Quinn Information Criterion $(C-HQ)$, Consistent Akaike Information Criterion $(C-CA)$. Tables 3, 4 and 5 gives the MLEs, standard errors (SEs), confidence interval (CL) for the data set **I, II, III** respectively. Tables 6, 7 and 8 gives $C-AI$, C-Bayes, $C-HQ$ and $C-CA$ values for the data set **I, II, III** respectively. It is noted that, for breaking stress data: $C-AI=301.44$, $C-Bayes=309.26$, $C-AI=301.69$, $C-HQ=304.61$. For survival times guinea pigs: $C-AI=204.05$, $C-Bayes=210.88$, $C-CA=206.77$, $?_{HQ}=204.405$. For the Egyptian tax's revenue data: $C-AI=384.12$, $C-Bayes=390.35$, $C-CA=386.55$, $C-HQ=384.56$. From 1, 2 and 3 we conclude that the new model has the lowest $C-AI$, $C-Bayes$, $C-CA$, $C-HQ$ for all data sets. Figure 3 gives the TTT plots. Based on Figure 3, the HRF of the three real data sets are increasing. Figure 4 gives the estimated PDFs. Figure 5 gives the estimated CDFs. Figure 6 gives the estimated HRFs. Figure 7 gives the P-P plots. Figure 8 gives Kaplan-Meier survival Plot. Based on Figures 4-8 the GOLLLx model has adequate fits to the empirical functions. Based on the values in Tables 6, 7, 8 and Figures 4-8 the GOLLLx model has the best fits as compared to BXII other models in the three applications with small values for C-AI, C-Bayes, C-CA and C-HQ. Many useful real data can be found in Mansour et al. (2020a-f), Elsayed and Yousof (2019a,b,c and 2020), Elbiely and Yousof (2018 and 2019), Yousof et al. (2018a,b,c), Cordeiro et al. (2018), Gad et al. (2018), Hamedani et al. (2019), Ibrahim et al. (2019 and 2020), Korkmaz et al. (2017 and 2018 and 2019) and Merovci (2017).

Table 3: MLEs, SEs and CIs for the data set **I**.

Model		a,b,α,θ,γ
Lx	MLEs	---,---, ---, 0.8025,---
	SEs	---,---, ---, (0.08025),---
	CIs	---,---, ---, (0.64,0.96),---
ExpLx	MLEs	---,---, 0.6437, 0.767,---
	SEs	---,---, (0.0603), (0.094),---
	CIs	---,---, (0.48, 0.72), (0.59),---
BXII	MLEs	---,---, 5.941, 0.187,---
	SEs	---,---, (1.279) ,(0.044),---
	CIs	---,---, (3.43,8.45),(0.10,0.27),---
MOBXII	MLEs	---,---, 1.192,4.834,838.73
	SEs	---,---, (0.952),(4.896),(229.34)
	CIs	---,---, 0, 3.06),(0, 14.43),(389.22,1288.24)
TLBXII	MLEs	---,---, 1.350,1.061,13.728
	SEs	---,---, 0.378) ,(0.384) ,(8.400)
	CIs	---,---, (0.61, 2.09), (0.31,1.81) ,(0, 30.19)
KumBXII	MLEs	48.103 ,79.516 ,0.351 ,2.730,---
	SEs	(19.348) ,(58.186) ,(0.098) ,(1.077) ,---
	CIs	(10.18,86.03) ,(0,193.56) ,(0.16,0.54), (0.62,4.84),---
BBXII	MLEs	359.683 ,260.097 ,0.175 ,1.123 ,---
	SEs	(57.941) ,(132.213),(0.013),(0.243),---
	CIs	(246.1,473.2), (0.96,519.2), (0.14,0.20), (0.65,1.6),---
BEBXII	MLEs	0.381, 11.949, 0.937, 33.402, 1.705
	SEs	(0.078), (4.635), (0.267), (6.287),(0.478)
	CIs	(0.23,0.53) ,(2.86,21), (0.41,1.5), (21,45), (0.8,2.6)
FBBXII	MLEs	0.421, 0.834, 6.111, 1.674, 3.450
	SEs	(0.011), (0.943), (2.314), (0.226), (1.957)
	CIs	(0.4,0.44), (0. 2.7), (1.57, 10.7), (1.23, 2.1), (0, 7)
FKumBXII	MLEs	0.542,4.223, 5.313, 0.411, 4.152
	SEs	(0.137), (1.882), (2.318), (0.497), (1.995)
	CIs	(0.3, 0.8), (0.53,7.9), (0.9,9), (0, 1.7), (0.2,8)
ZBBXII	MLEs	123.101,---,0.368, 139.247,---
	SEs	(243.011), ---, (0.343), (318.546),---
	CIs	(0, 599.40), ---,(0, 1.04), (0, 763.59),---
GOLLLx	**MLEs**	**5.013, 1.036, 0.576, ---,---**
	SEs	**(3.137), (1.046), (0.569), ---,---**
	CIs	**(3.137), (1.046), (0.569), ---,---**

Table 4: MLEs, SEs and CIs for the data set **II**.

Model		a,b,α,θ,γ
Lx	MLEs	---, ---, ---, 1.0244, ---
	SEs	---, ---, ---, (0.1207), ---
	CIs	---, ---, ---, (0.78, 1.26), ---
ExpLx	MLEs	---, ---, 0.8257, 0.9739, ---
	SEs	---, ---, (0.093), (0.1292), ---
	CIs	---, ---, (0.62, 0.98), (0.64, 1.16), ---
BXII	MLEs	---, ---, 3.102, 0.465, ---
	SEs	---, ---, (0.538), (0.077), ---
	CIs	---, ---, (2.05,4.16), (0.31,0.62), ---
MOBXII	MLEs	---, ---, 2.259,1.533, 6.760
	SEs	---, ---, (0.864), (0.907), (4.587)
	CIs	---, ---, (0.57,3.95), (0,3.31), (0, 15.75)
TLBXII	MLEs	---, ---, 2.393,0.458,1.796
	SEs	---, ---, (0.907), (0.244), (0.915)
	CIs	---, ---, (0.62,4.17), (0, 0.94), (0.002,3.59)
KumBXII	MLEs	14.105,7.424, 0.525, 2.274, ---
	SEs	(10.805), (11.850), (0.279), (0.990), ---
	CIs	(0, 35.28), (0.30.65), (0, 1.07), (0.33, 4.21), ---
BBXII	MLEs	2.555, 6.058,1.800,0.294, ---
	SEs	(1.859), (10.391), (0.955), (0.466),---
	CIs	(0, 6.28), (0, 26.42), (0, 3.67),(0, 1.21), ---
BEBXII	MLEs	1.876,2.991, 1.780, 1.341, 0.572
	SEs	(0.094), (1.731), (0.702), (0.816), (0.325)
	CIs	(1.7,2.06), (0, 6.4), (0.40, 3.2), (0, 2.9), (0, 1.21)
FBBXII	MLEs	0.621, 0.549,3.838, 1.381, 1.665
	SEs	(0.541), (1.011), (2.785), (2.312), (0.436)
	CIs	(0, 1.7), (0, 2.5), (0, 9.3), (0, 5.9), (0.8, 4.5)
FKumBXII	MLEs	0.558,0.308, 3.999, 2.131, 1.475
	SEs	(0.442), (0.314), (2.082), (1.833), (0.361)
	CIs	(0, 1.4), (0, 0.9), (0, 3.1), (0, 5.7), (0.76, 2.2)
GOLLLx	**MLEs**	**4.226, 0.663, 0.464, ---, ---**
	SEs	**(3.177), (0.701), (0.641), ---, ---**
	CIs	**(0, 10.4), (0, 2.06), (0, 1.66), ---, ---**

Table 5: MLEs, SEs and CIs for the data set **III**.

Model		a,b,α,θ,γ
Lx	MLEs	---,---, ---, 0.392,---
	SEs	---,---, ---, (0.051),---
	CIs	---,---, ---, (0.29, 0.49),---
ExpLx	MLEs	---,---, 0.332, 0.604,---
	SEs	---,---, (0.0405), (0.122),---
	CIs	---,---, (0.25, 0.41), (0.36, 0.84),---
BXII	MLEs	---,---, 5.615, 0.072,---
	SEs	---,---, (15.048), (0.194),---
	CIs	---,---, (0, 35.11), (0, 0.45),---
MOBXII	MLEs	---,---, 8.017, 0.419, 70.359
	SEs	---,---, (22.083), (0.312), (63.831)
	CIs	---,---, (0, 51.29), (0, 1.03), (0, 195.47)
TLBXII	MLEs	---,---, 91.320, 0.012, 141.073
	SEs	---,---, (15.071), (0.002), (70.028)
	CIs	---,---, (61.78,120.86) (0.008, 0.02) (3.82,278.33)
KumBXII	MLEs	18.130, 6.857, 10.694, 0.081,---
	SEs	(3.689), (1.035), (1.166), (0.012),---
	CIs	(10.89,25.36), (4.83,8.89), (8.41,12.98), (0.06,0.10),---

BBXII	MLEs	26.725, 9.756, 27.364, 0.020,---
	SEs	(9.465), (2.781), (12.351), (0.007),---
	CIs	(8.17,45.27), (4.31,15.21), (3.16,51.57), (0.006,0.03),---
BEBXII	MLEs	2.924, 2.911, 3.270, 12.486, 0.371
	SEs	(0.564), (0.549), (1.251), (6.938), (0.788)
	CIs	(1.82,4.03), (1.83,3.99), (0.82,5.72), (0, 26.08), (0, 1.92)
FBBXII	MLEs	30.441, 0.584, 1.089, 5.166, 7.862
	SEs	(91.745), (1.064), (1.021), (8.268), (15.036)
	CIs	(0, 210.26), (0, 2.67), (0, 3.09), (0, 21.37), (0, 37.33)
FKumBXII	MLEs	12.878, 1.225, 1.665, 1.411, 3.732
	SEs	(3.442), (0.131), (0.034), (0.088), (1.172)
	CIs	(6.13,19.6), (0.97,1.48), (1.56,1.73), (1.24,1.6), (1.43,6.03)
GOLLLx	**MLEs**	**1.827, 23.129, 1.42, ---,---**
	SEs	**(0.365), (14.22), (0.249), ---,---**
	CIs	**(1.1, 2.5), (0, 51), (0.9, 1.9), ---,---**

Table 6: C-AI, C-Bayes, C-CA, C-HQ values for the data **I**.

Model	C-AI, C-Bayes, C-CA, C-HQ
Lx	495.25, 497.86, 495.295, 496.31
ExpLx	470.39, 475.59, 470.51, 472.49
BXII	382.94, 388.15, 383.06, 385.05
MOBXII	305.78, 313.60, 306.03, 308.96
TLBXII	323.52, 331.35, 323.77, 326.70
KumBXII	303.76, 314.21, 304.18, 308.00
BBXII	305.64, 316.06, 306.06, 309.85
BEBXII	305.82, 318.84, 306.46, 311.09
FBBXII	304.26, 317.31, 304.89, 309.56
FKumBXII	305.50, 318.55, 306.14, 310.80
GOLLLx	**301.44, 309.26, 301.69, 304.61**

Table 7: C-AI, C-Bayes, C-CA, C-HQ values for the data **II**.

Model	C-AI, C-Bayes, C-CA, C-HQ
Lx	283.18, 285.45, 283.23, 284.08
ExpLx	282.06, 286.61, 282.23, 283.87
BXII	209.60, 214.15, 209.77, 211.40
MOBXII	209.74, 216.56, 210.09, 212.44
TLBXII	211.80, 218.63, 212.15, 214.52
KumBXII	208.76, 217.86, 209.36, 212.38
BBXII	210.44, 219.54, 211.03, 214.06
BEBXII	212.10, 223.50, 213.00, 216.60
FBBXII	206.80, 218.20, 207.71, 211.30
FKumBXII	206.50, 217.90, 207.41, 211.00
GOLLLx	**204.05, 210.88, 206.77, 204.405**

Table 7: C-AI, C-Bayes, C-CA, C-HQ values for the data **III**.

Model	C-AI, C-Bayes, C-CA, C-HQ
Lx	531.53, 533.61, 531.60, 532.34
ExpLx	397.92, 402.08, 398.13, 399.01
BXII	518.46, 522.62, 518.67, 520.08
MOBXII	387.22, 389.38, 387.66, 389.68
TLBXII	385.94, 392.18, 386.38, 388.40
KumBXII	385.58, 393.90, 386.32, 388.86
BBXII	385.56, 394.10, 386.30, 389.10

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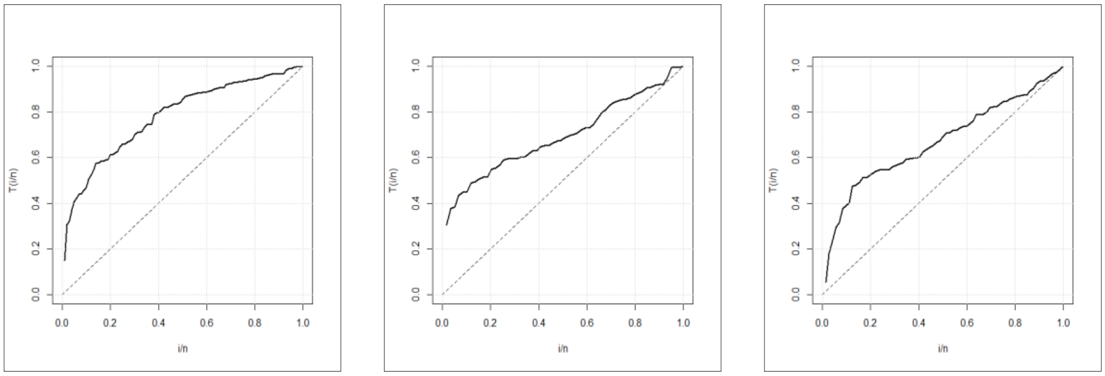
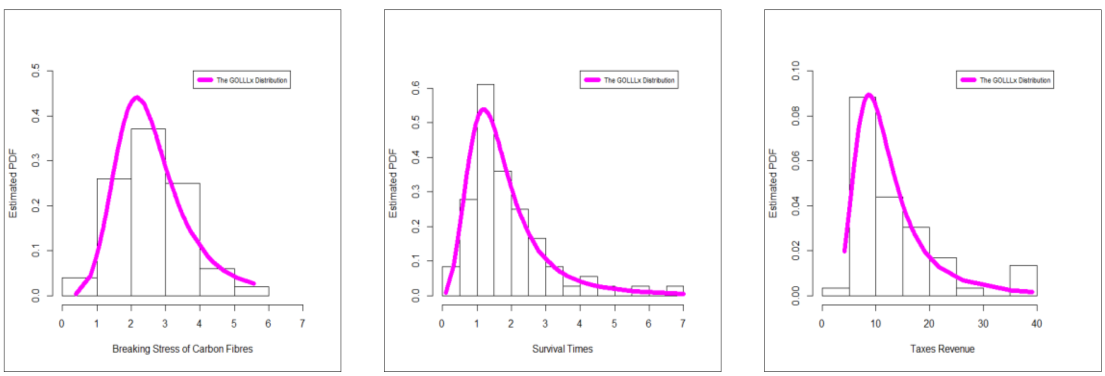
BEBXII	387.04, 397.42, 388.17, 391.09
FBBXII	386.74, 397.14, 387.87, 390.84
FKumBXII	386.96, 397.36, 388.09, 391.06
GOLLLx	**384.12, 390.35, 386.55, 384.56**



Data **I**. Data **II**. Data **III**.

Figure 3: TTT plots.



Data **I**. Data **II**. Data **III**.

Figure 4: Estimated PDFs.

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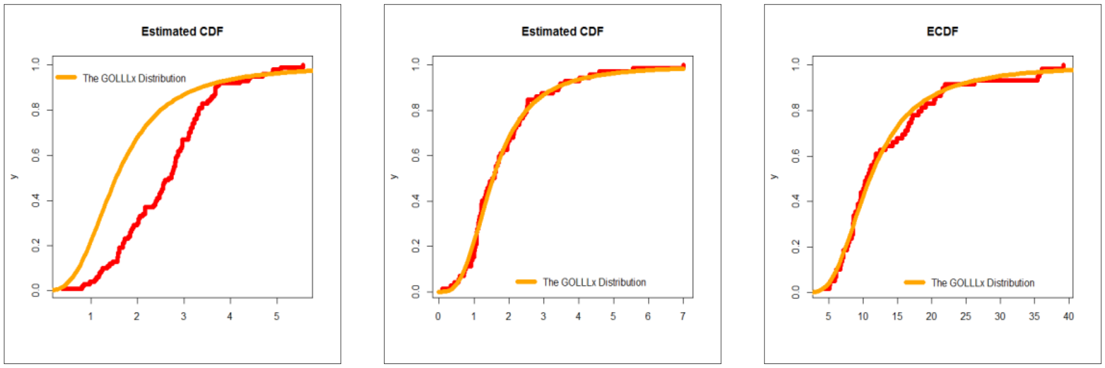
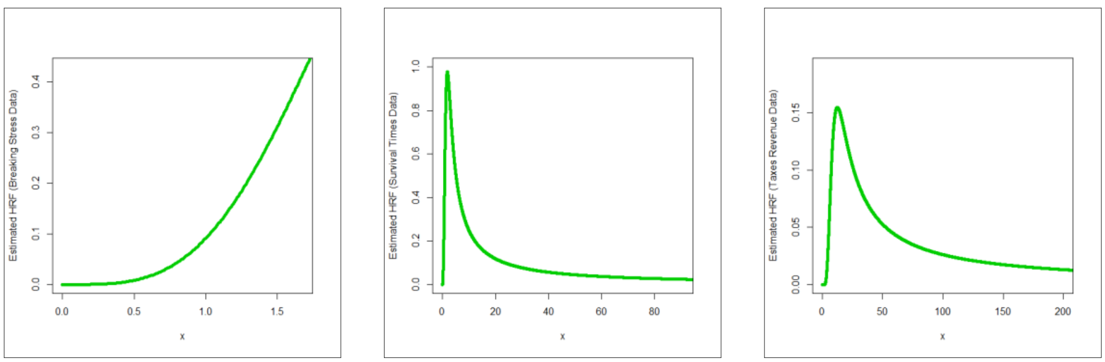


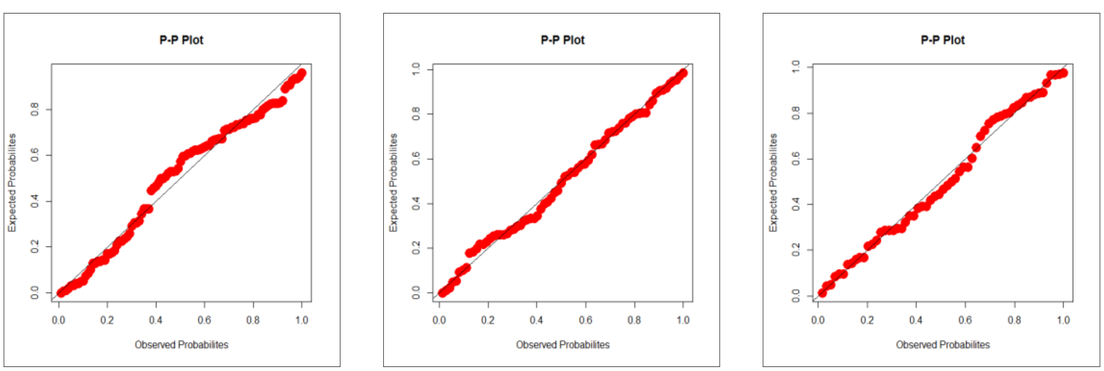
Data **I**. Data **II**. Data **III**.

Figure 5: Estimated CDFs.



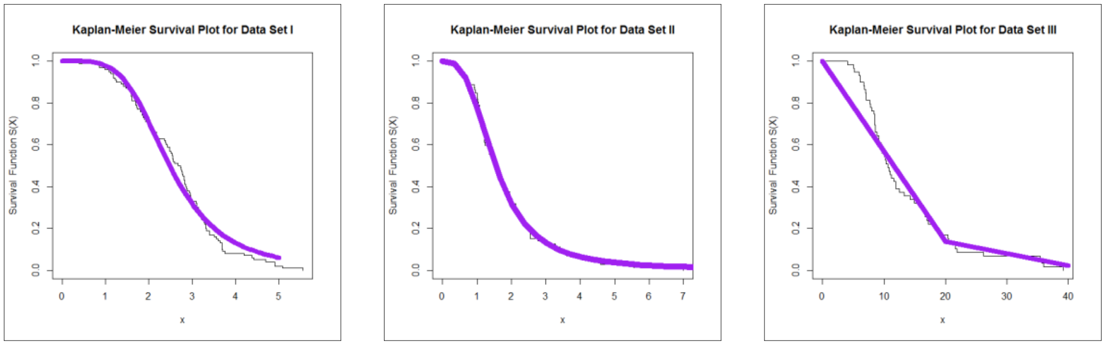
Data **I**. Data **II**. Data **III**.

Figure 6: Estimated HRFs.



Data **I**. Data **II**. Data **III**.

Figure 7: P-P plots.



Data **I**. Data **II**. Data **III**.

Figure 8: Kaplan-Meier survival Plot.

7. Conclusions

A new generalization of Lomax distribution is derived and studied. The new extension has only three parameters. Some of its useful mathematical properties are derived. We performed graphical simulations to assess the finite sample behavior of the estimations. The effects of all parameters on the mean, variance, skewness and kurtosis for given values are studied. we note that the new additional shape parameters and has an effect on the mean, variance, skewness and kurtosis. For the new Lx model, skewness can range in the interval $(-183.1, 7514.7)$. However, for the standard Lx model, skewness can range in the interval $(-0.4104, 4.6476)$. For the GOLLLx model, kurtosis can range in the interval $(-1531.11, 56479275)$. However, for the standard Lx model, kurtosis can range in the interval (0.93244, 73.8). The new model is employed in modelling three real data sets. For all data sets, we compared the new Lx distribution with the standard Lx, the exponentiated Lx, the Burr XII, beta Burr XII, the Marshall-Olkin Burr XII, the Topp-Leone Burr XII, the Zografos-Balakrishnan Burr XII, beta exponentiated Burr XII, the five-parameters beta Burr XII, the five-parameters Kumaraswamy Burr XII and Kumaraswamy Burr XII distributions. The new Lx distribution is a useful alternative for the above-mentioned models in modeling breaking stress data, survival times of guinea pig's data and the Egyptian taxes revenue data.

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