

## The Poisson Topp Leone Generator of Distributions for Lifetime Data: Theory, Characterizations and Applications



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### Abstract:

We study a new family of distributions defined by the minimum of the Poisson random number of independent identically distributed random variables having a Topp Leone-G distribution (see Rezaei et al., (2016)). Some mathematical properties of the new family including ordinary and incomplete moments, quantile and generating functions, mean deviations, order statistics, reliability and entropies are derived. Maximum likelihood estimation of the model parameters is investigated. Some special models of the new family are discussed. An application is carried out on real data set applications sets to show the potentiality of the proposed family.

**Keywords:** Topp Leone-G Family, Order Statistics, Maximum Likelihood Estimation, Quantile function, Generating Function, Moments.

**Mathematical Subject Classification:** 60E05, 97K50, 62N02.

### 1. Introduction

The statistical literature contains many new classes of distributions which have been constructed by extending common families of continuous distributions and give more flexibility by adding one or more parameters to the baseline model. These new families have been used for modeling data in many applied areas such as engineering, economics, biological studies and environmental sciences. So, several classes of distributions have been constructed by extending common families of continuous distributions. These generalized distributions give more flexibility by adding one "or more" parameters to the baseline model. They were pioneered by Gupta et al. (1998) who proposed the exponentiated-G class, which consists of raising the cumulative distribution function (CDF) to a positive power parameter. Many other classes can be cited such as the Marshall-Olkin-G family by Marshall and Olkin (1997), beta generalized-G family by Eugene et al. (2002), transmuted exponentiated generalized-G family by Yousof et al. (2015), generalized transmuted-G by Nofal et al. (2017), the Topp Leone generated family by Rezaei et al. (2017), exponentiated transmuted-G family by Merovci et al. (2017), Topp-Leone Odd Log-Logistic Family of Distributions by Brito et al. (2017), Burr X-G by Yousof et al. (2017), exponentiated generalized-G Poisson family by Aryal and Yousof (2017), type I general exponential class of distributions by Hamedani et al. (2017), Burr XII system of densities by Cordeiro et al. (2018), exponential Lindley odd log-logistic G family by Korkmaz et al. (2018a), Marshall-Olkin generalized G Poisson family by Korkmaz et al. (2018b), Burr-Hatke family of distributions by Yousof et al. (2018), a new extended G family by Hamedani et al. (2018), Type II general exponential class of distributions by Hamedani et al. (2019) and odd Nadarajah-Haghighi family of distributions by Nascimento et al. (2019), the Weibull G Poisson family by Yousof et al.(2020), among other.

Due to Rezaei et al. (2017), the cumulative distribution function (CDF) and the probability density function (PDF) of the Topp Leone (TL) generated family of distributions distribution specified by

$$H(x; \alpha, \psi) = \{G(x; \psi)[2 - G(x; \psi)]\}^\alpha = G(x; \psi)^\alpha [2 - G(x; \psi)]^\alpha, \quad (1)$$

and

$$h(x; \alpha, \psi) = 2\alpha g(x; \psi)G(x; \psi)^{\alpha-1}[1 - G(x; \psi)][2 - G(x; \psi)]^{\alpha-1}, \quad (2)$$

respectively. Suppose  $Z_1, \dots, Z_N$  be independent identically random variable (iid) with common CDF Topp Leone-G and  $N$  be random variable with

$$P(N = n) = \frac{1}{e^\lambda - 1} \times \frac{\lambda^n}{n!}, \quad n = 1, 2, \dots, \lambda > 0,$$

and define  $M_N = \max\{Z_1, \dots, Z_N\}$  then

$$F(x) = \sum_{n=0}^{\infty} p(M_N \leq x | N = n) p(N = n). \quad (3)$$

Equation (3) can be expressed as

$$F(x) = \sum_{n=0}^{\infty} \left\{ \frac{1 - \exp[-\lambda H(x; \alpha, \psi)]}{e^\lambda - 1} \right\}^n \frac{1}{e^\lambda - 1} \frac{\lambda^n}{n!}.$$

Using equations (2) and the last equation, we can write

$$F(x) = \frac{1 - \exp\{-\lambda G(x; \psi)^\alpha [2 - G(x; \psi)]^\alpha\}}{(1 - e^{-\lambda})}. \quad (4)$$

Equation (4) is called Poisson Topp Leone-G (PTL-G) family of distributions. Several new models can be generated by considering special distributions for  $G(x)$ . The corresponding PDF of reduces to

$$f(x) = \frac{2\lambda\alpha g(x; \psi)G(x; \psi)^{\alpha-1}[1 - G(x; \psi)][2 - G(x; \psi)]^{\alpha-1}}{(1 - e^{-\lambda}) \exp\{\lambda G(x; \psi)^\alpha [2 - G(x; \psi)]^\alpha\}}. \quad (5)$$

Henceforward  $G(x; \psi) = G(x)$  and  $g(x; \psi) = g(x)$ . Now we can provide a useful linear representation for the PTL-G density function in. Expanding the quantity  $\exp\{-\lambda G(x)^\alpha [2 - G(x)]^\alpha\}$  in power series, we can write

$$f(x) = \sum_{i=0}^{\infty} \frac{\lambda^{i+1} \alpha 2^{\alpha(i+1)} (-1)^i g(x) [1 - G(x)] \left[1 - \frac{G(x)}{2}\right]^{\alpha(i+1)-1}}{i! (1 - e^{-\lambda}) G(x)^{-[\alpha(i+1)-1]}}. \quad (6)$$

Consider the power series

$$(1 - n)^c = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(1 + c)}{j! \Gamma(1 + c - j)} n^j, \quad (7)$$

which holds for  $|z| < 1$  and  $b > 0$  real non-integer. Using the power series in and after some algebra the PDF of the PTL-G class in can be expressed as

$$f(x) = \sum_{i,j=0}^{\infty} \{Y_{i,j} \pi_{\alpha(i+1)+j}(x) - Y_{i,j}^* \pi_{\alpha(i+1)+j+1}(x)\}, \quad (8)$$

where  $\pi_\gamma(x) = \gamma g(x)G(x)^{\gamma-1}$ .

$$Y_{i,j} = \frac{\alpha \lambda^{i+1} (-1)^{i+j} 2^{\alpha(i+1)-j}}{i! (1 - e^{-\lambda}) [\alpha(i+1) + j]} \binom{\alpha(i+1) - 1}{j} \text{ and}$$

$$Y_{i,j}^* = \frac{\alpha \lambda^{i+1} (-1)^{i+j} 2^{\alpha(i+1)-j}}{i! (1 - e^{-\lambda}) [\alpha(i+1) + j + 1]} \binom{\alpha(i+1) - 1}{j}.$$

Equation (8) reveals that the density of  $X$  can be expressed as a linear representation of exp-G densities. So, several mathematical properties of the new family can be obtained by knowing those of the exp-G distribution. The CDF of the PTL-G family can also be expressed as a mixture of exp-G densities. By integrating, we obtain the same mixture representation

$$f(x) = \sum_{i,j=0}^{\infty} \{Y_{i,j} \pi_{\alpha(i+1)+j}(x) - Y_{i,j}^* \pi_{\alpha(i+1)+j+1}(x)\}, \quad (9)$$

where  $\pi_{\gamma}(x)$  is the CDF of the exp-G family with power parameter ( $\gamma$ ). The rest of the paper is outlined as follows. In Section 2, we define three special models and provide the plots of their PDF's and hazard rate functions (HRF's). In Section 3, we derive some of its mathematical properties including probability weighted moments (PWMs), residual life and reversed residual life functions, ordinary, incomplete moments and generating functions, finally order statistics and their moments are introduced at the end of the section. Some characterizations results are provided in Section 4. Maximum likelihood estimation of the model parameters is addressed in Section 5. In section 6, simulation results to assess the performance of the proposed maximum likelihood estimation procedure are discussed. In Section 7, we provide the applications to real data to illustrate the importance of the new family. Finally, some concluding remarks are presented in Section 8.

## 2. Special models

### 2.2 PTL-Exponential (PTL-E) distribution

The parent exponential distribution has PDF and CDF given, respectively, by

$$G(x, b) = 1 - \exp(-bx).$$

The CDF of PTE-Exponential distribution if given by ( $x > 0$ )

$$F(x) = \frac{1 - \exp\{-\lambda(1 - \exp(-bx))^{\alpha}[1 + \exp(-bx)]^{\alpha}\}}{(1 - e^{-\lambda})}.$$

Figures 1 and 2 illustrate some of the possible shapes of the PDF and HRF of the PTL-E distribution.

### 2.2 The PTL-Weibull (PTL-W) distribution

The parent Weibull distribution (see Weibull (1951)) has CDF given by

$$G(x; a, b) = 1 - \exp(-bx^a).$$

The CDF of PTL-W distribution is given by ( $x > 0$ )

$$F(x) = \frac{1 - \exp\{-\lambda(1 - \exp(-bx^a))^{\alpha}[1 + \exp(-bx^a)]^{\alpha}\}}{(1 - e^{-\lambda})}.$$

Figures 3 and 4 illustrates possible shapes of the PDF and HRF for some PTL-W distributions.

### 3.2 The PTL-generalized half normal (PTL-GHN) distribution

The parent generalized half normal distribution has CDF given by

$$G(x, a, b) = 2\Phi\left[\left(\frac{x}{a}\right)^b\right] - 1.$$

The CDF and PDF of PTL-GHN distribution are given by

$$F(x) = \frac{1 - \exp\left\{-\lambda G\left\{2\Phi\left[\left(\frac{x}{a}\right)^b\right] - 1\right\}^{\alpha}\left(2 - \left\{2\Phi\left[\left(\frac{x}{a}\right)^b\right] - 1\right\}\right)^{\alpha}\right\}}{(1 - e^{-\lambda})}.$$

## 3. Mathematical properties

### 3.1 Probability weighted moments

The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The ( $s, r$ )th PWM of  $X$  following the PTL-G family, say  $\rho_{s,r}$ , is formally defined by

$$\rho_{s,r} = E\{X^s F(X)^r\} = \int_{-\infty}^{\infty} x^s F(x)^r f(x) dx.$$

Using equations (5) and (6), we can write

$$f(x)F(x)^r = \sum_{i,j=0}^{\infty} \{p_{i,j} \pi_{\alpha(i+1)+j}(x) - p_{i,j}^* \pi_{\alpha(i+1)+j+1}(x)\},$$

where

$$p_{i,j} = \sum_{k=0}^{\infty} \frac{2^{\alpha(i+1)-j} \alpha \lambda^{i+1} (-1)^{i+j+k} (k+1)^i \binom{r}{k} \binom{\alpha(i+1)-1}{j}}{i! (1 - e^{-\lambda})^{r+1} [\alpha(i+1) + j]},$$

and

$$p_{i,j}^* = \sum_{k=0}^{\infty} \frac{2^{\alpha(i+1)-j} \alpha \lambda^{i+1} (-1)^{i+j+k} (k+1)^i \binom{r}{k} \binom{\alpha(i+1)-1}{j}}{i! (1 - e^{-\lambda})^{r+1} [\alpha(i+1) + j + 1]}.$$

Then, the  $(s, r)$  th PWM of  $X$  can be expressed as

$$\rho_{s,r} = \sum_{i,j=0}^{\infty} \{p_{i,j} E(Y_{\alpha(i+1)+j}^s) - p_{i,j}^* E(Y_{\alpha(i+1)+j+1}^s)\} dx.$$

Henceforth,  $Y_{\gamma}$  denotes the exp-G distribution with power parameter  $(\gamma)$ .

### 3.2 Residual life and reversed residual life functions

The  $n$  th moment of the residual life, say  $m_n(t) = E[(X - t)^n | X > t]$ ,  $n = 1, 2, \dots$ , uniquely determined  $F(x)$ . The  $n$  th moment of the residual life of  $X$  is given by  $m_n(t) = \frac{1}{R(t)} \int_t^{\infty} (x - t)^n dF(x)$ . Therefore,

$$m_n(t) = \frac{1}{R(t)} \sum_{r=0}^n \binom{n}{r} (-t)^{n-r} \sum_{i,j=0}^{\infty} \left\{ Y_{i,j} \int_t^{\infty} x^r \pi_{\alpha(i+1)+j}(x) dx - Y_{i,j}^* \int_t^{\infty} x^r \pi_{\alpha(i+1)+j+1}(x) dx \right\}.$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age  $t$  defined by  $m_1(t) = E[(X - t) | X > t]$ , which represents the expected additional life length for a unit which is alive at age  $t$ . The MRL of  $X$  can be obtained by setting  $n = 1$  in the last equation. The  $n$  th moment of the reversed residual life, say  $M_n(t) = E[(t - X)^n | X \leq t]$  for  $t > 0$  and  $n = 1, 2, \dots$  uniquely determines  $F(x)$ . We obtain  $M_n(t) = \frac{1}{F(t)} \int_0^t (t - x)^n dF(x)$ . Then, the  $n$  th moment of the reversed residual life of  $X$  becomes

$$M_n(t) = \frac{1}{F(t)} \sum_{r=0}^n (-1)^r \binom{n}{r} t^{n-r} \sum_{i,j=0}^{\infty} \left\{ Y_{i,j} \int_0^t x^r \pi_{\alpha(i+1)+j}(x) dx - Y_{i,j}^* \int_0^t x^r \pi_{\alpha(i+1)+j+1}(x) dx \right\}.$$

The mean inactivity time (MIT) or mean waiting time (MWT) also called the mean reversed residual life function is given by  $M_1(t) = E[(t - X) | X \leq t]$ , and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in  $(0, t)$ . The MIT of the PTL-G family of distributions can be obtained easily by setting  $n = 1$  in the above equation.

### 3.3 Moments, incomplete moments and generating function

The  $r$  th ordinary moment of  $X$  is given by  $\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx$ . Then we obtain

$$\mu_r' = \sum_{i,j=0}^{\infty} \{Y_{i,j} E(Y_{\alpha(i+1)+j}^r) - Y_{i,j}^* E(Y_{\alpha(i+1)+j+1}^r)\}. \quad (10)$$

Setting  $r = 1$  in (11), we have the mean of  $X$ . The last integration can be computed numerically for most parent distributions. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The  $n$  th central moment of  $X$ , say  $M_n$ , follows as  $M_n = E(X - \mu)^n = \sum_{h=0}^n (-1)^h \binom{n}{h} (\mu_1')^n \mu_{n-h}'$ .

The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The  $s$  th incomplete moment, say  $\phi_s(t)$ , of  $X$  can be expressed from (9) as  $\phi_s(t) = \int_{-\infty}^t x^s f(x) dx$ . Then

$$\phi_s(t) = \sum_{i,j=0}^{\infty} \left\{ \begin{array}{l} Y_{i,j} \int_{-\infty}^t x^s \pi_{\alpha(i+1)+j}(x) dx \\ - Y_{i,j}^* \int_{-\infty}^t x^s \pi_{\alpha(i+1)+j+1}(x) dx \end{array} \right\}. \quad (11)$$

The mean deviations about the mean  $[\delta_1 = E(|X - \mu'_1|)]$  and about the median  $[\delta_2 = E(|X - M|)]$  of  $X$  are given by  $\delta_1 = 2\mu'_1 F(\mu'_1) - 2\phi_1(\mu'_1)$  and  $\delta_2 = \mu'_1 - 2\phi_1(M)$ , respectively, where  $\mu'_1 = E(X)$ ,  $M = \text{Median}(X) = Q(0.5)$  is the median,  $F(\mu'_1)$  is easily calculated from (5) and  $\phi_1(t)$  is the first incomplete moment given by (12) with  $s = 1$ . A general equation for  $\phi_1(t)$  can be derived from (12) as

$$\phi_1(t) = \sum_{i,j=0}^{\infty} \{Y_{i,j} \mathbf{J}_{\alpha(i+1)+j}(x) dx - Y_{i,j}^* \mathbf{J}_{\alpha(i+1)+j+1}(x) dx\},$$

where  $\mathbf{J}_Y(x) = \int_{-\infty}^t x \pi_Y(x) dx$  is the first incomplete moment of the exp-G distribution. The moment generating function (mgf)  $M_X(t) = E(e^{tX})$  of  $X$  can be derived from equation (9) as

$$M_X(t) = \sum_{i,j=0}^{\infty} \{Y_{i,j} M_{\alpha(i+1)+j}(t) - Y_{i,j}^* M_{\alpha(i+1)+j+1}(t)\},$$

where  $M_Y(t)$  is the mgf of  $Y_Y$ . Hence,  $M_X(t)$  can be determined from the exp-G generating function.

### 3.4 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let  $X_1, \dots, X_n$  be a random sample from the PTL-G family of distributions and let  $X_{(1)}, \dots, X_{(n)}$  be the corresponding order statistics. The PDF of  $i$ th order statistic, say  $X_{i:n}$ , can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x), \quad (12)$$

where  $B(\cdot, \cdot)$  is the beta function. Substituting (5) and (6) in equation (12) and using a power series expansion, we get

$$f(x)F(x)^{j+i-1} = \sum_{w,m=0}^{\infty} \{t_{w,m} \pi_{\alpha(w+1)+m}(x) - t_{w,m}^* \pi_{\alpha(w+1)+m+1}(x)\},$$

where

$$t_{w,m} = \sum_{k=0}^{\infty} \frac{2^{\alpha(w+1)-m} \alpha \lambda^{w+1} (-1)^{w+m+k} (k+1)^w \binom{j+i-1}{k} \binom{\alpha(w+1)-1}{m}}{w! (1 - e^{-\lambda})^{j+i} [\alpha(w+1) + m]},$$

and

$$t_{w,m}^* = \sum_{k=0}^{\infty} \frac{2^{\alpha(w+1)-m} \alpha \lambda^{w+1} (-1)^{w+m+k} (k+1)^w \binom{j+i-1}{k} \binom{\alpha(w+1)-1}{m}}{w! (1 - e^{-\lambda})^{j+i} [\alpha(w+1) + m + 1]}.$$

The PDF of  $X_{i:n}$  can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{B(i, n-i+1)} \sum_{w,m=0}^{\infty} \left\{ \begin{array}{l} t_{w,m} \pi_{\alpha(w+1)+m}(x) \\ - t_{w,m}^* \pi_{\alpha(w+1)+m+1}(x) \end{array} \right\}.$$

Then, the density function of the PTL order statistics is a mixture of exp-G densities. Based on the last equation, we note that the properties of  $X_{i:n}$  follow from those properties of  $Y_{\alpha(w+1)+m}$  and  $Y_{\alpha(w+1)+m+1}$ . For example, the moments of  $X_{i:n}$  can be expressed as

$$E(X_{i:n}^q) = \sum_{j=0}^{n-i} \frac{(-1)^j \binom{n-i}{j}}{B(i, n-i+1)} \sum_{w,m=0}^{\infty} \left\{ \begin{array}{l} t_{w,m} E(Y_{\alpha(w+1)+m}^q) \\ - t_{w,m}^* E(Y_{\alpha(w+1)+m+1}^q) \end{array} \right\}. \quad (13)$$

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. They exist whenever the mean of the distribution exists, even though some higher moments may not exist, and are

relatively robust to the effects of outliers. Based upon the moments in equation (14), we can derive explicit expressions for the L-moments of  $X$  as infinite weighted linear combinations of the means of suitable PTL order statistics. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r} \sum_{d=0}^{r-1} (-1)^d \binom{r-1}{d} E(X_{r-d:r}), \quad r \geq 1.$$

#### 4. Characterizations

In this section we present certain characterizations of PTL-G distribution. The first characterization is based on hazard function and the second one is in terms of the ratio of two truncated moments.

##### 4.1 Characterization based on hazard function

It is known that the hazard function,  $h_F$ , of a twice differentiable distribution function,  $F$ , satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x). \quad (14)$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterizations establish a non-trivial characterization for (5) in terms of the hazard function which is not of the trivial form given in (14).

**Proposition 4.1.1.** Let  $X : \Omega \rightarrow R$  be a continuous random variable. Then  $X$  has pdf of (5) if and only if its hazard function  $h_F(x)$  satisfies the differential equation

$$h'_F(x) - \frac{g'(x)h_F(x)}{g(x)} = \frac{2g(x)}{(\alpha\lambda)^{-1}} \frac{d}{dx} \left\{ \frac{(G(x))^{\alpha-1} [2 - G(x)]^{\alpha-1} \bar{G}(x)}{1 - \exp\{\lambda((G(x))^\alpha [2 - G(x)]^\alpha - 1)\}} \right\}. \quad (15)$$

Proof. If  $X$  has pdf of (5), then clearly (4.1.2) holds. Now, if (15) holds, then

$$\frac{d}{dx} \left\{ (g(x))^{-1} h_F(x) \right\} = 2\alpha\lambda \frac{d}{dx} \left\{ \frac{(G(x))^{\alpha-1} [2 - G(x)]^{\alpha-1} \bar{G}(x)}{1 - \exp\{\lambda((G(x))^\alpha [2 - G(x)]^\alpha - 1)\}} \right\},$$

or

$$h'_F(x) = \frac{2\alpha\lambda g(x)(G(x))^{\alpha-1} [2 - G(x)]^{\alpha-1} \bar{G}(x)}{1 - \exp\{\lambda((G(x))^\alpha [2 - G(x)]^\alpha - 1)\}}, \quad x \in R,$$

which is the hazard function of (5).

**Remark 4.1.1.** For  $\alpha = 1$ , the differential equation (15) will have the following form

$$h'_F(x) - \frac{g'(x)h_F(x)}{g(x)} = \frac{2\lambda(g(x))^2}{(1 - \exp\{\lambda((G(x)) [1 + \bar{G}(x)] - 1)\})^2} \times \{(2G(x)\bar{G}(x) + 1) \exp\{\lambda((G(x)) [1 + \bar{G}(x)] - 1)\} - 1\}.$$

##### 4.2 Characterizations in terms of two truncated moments

In this subsection we present characterizations of (5) in terms of a simple relationship between two truncated moments. Our first characterization result employs a theorem due to Glänzel (1987), see Theorem 1 below. Note that the result holds also when the interval  $H$  is not closed. Moreover, it could be also applied when the cdf  $F$  does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

**Theorem 1.** Let  $(\Omega, F, P)$  be a given probability space and let  $H = [d, e]$  be an interval for some  $d < e$  ( $d = -\infty, e = \infty$  might as well be allowed). Let  $X : \Omega \rightarrow H$  be a continuous random variable with the distribution function  $F$  and let  $q_1$  and  $q_2$  be two real functions defined on  $H$  such that

$$E[q_2(X) | X \geq x] = E[q_1(X) | X \geq x]\eta(x), \quad x \in H,$$

is defined with some real function  $\eta$ . Assume that  $q_1, q_2 \in C^1(H)$ ,  $\eta \in C^2(H)$  and  $F$  is twice continuously differentiable and strictly monotone function on the set  $H$ . Finally, assume that the equation  $\eta q_1 = q_2$  has no real solution in the interior of  $H$ . Then  $F$  is uniquely determined by the functions  $q_1, q_2$  and  $\eta$ , particularly

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)q_1(u) - q_2(u)} \right| \exp(-s(u)) du,$$

where the function  $s$  is a solution of the differential equation  $s' = \frac{\eta' q_1}{\eta q_1 - q_2}$  and  $C$  is the normalization constant, such that  $\int_H dF = 1$ .

Here is our characterization.

**Proposition 4.2.1.** Let  $X : \Omega \rightarrow R$  be a continuous random variable and let

$$q_1(x) = \exp\{\lambda(G(x))^\alpha [2 - G(x)]^\alpha\}$$

and

$$q_2(x) = q_1(x)(G(x)[2 - G(x)])^\alpha \text{ for } x \in R.$$

The random variable  $X$  belongs to the family (5) if and only if the function  $\eta$  defined in Theorem 1 has the form

$$\eta(x) = \frac{1}{2} \{1 + (G(x)[2 - G(x)])^\alpha\}, \quad x \in R. \quad (16)$$

Proof. Let  $X$  be a random variable with pdf (5), then

$$(1 - F(x))E[q_1(x) | X \geq x] = \frac{\lambda}{1 - e^{-\lambda}} \{1 - (G(x)[2 - G(x)])^\alpha\}, \quad x \in R,$$

and

$$(1 - F(x))E[q_2(x) | X \geq x] = \frac{\lambda}{2(1 - e^{-\lambda})} \{1 - (G(x)[2 - G(x)])^{2\alpha}\} \quad x \in R,$$

and finally

$$\eta(x)q_1(x) - q_2(x) = \frac{q_1(x)}{2} \{1 - (G(x)[2 - G(x)])^\alpha\} > 0 \text{ for } x \in R.$$

Conversely, if  $\eta$  is given as above, then

$$s'(x) = \frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{2\alpha g(x)(G(x)[2 - G(x)])^{\alpha-1} \bar{G}(x)}{1 - (G(x)[2 - G(x)])^\alpha}, \quad x \in R,$$

and hence

$$s(x) = -\log\{(G(x)[2 - G(x)])^\alpha\}, \quad x \in R.$$

Now, in view of Theorem 1,  $X$  has density (5).

**Corollary 4.2.1.** Let  $X : \Omega \rightarrow R$  be a continuous random variable and let  $q_1(x)$  be as in Proposition (17). Then  $X$  has pdf (5) if and only if there exist functions  $q_2$  and  $\eta$  defined in Theorem 1 satisfying the differential equation

$$\frac{\eta'(x)q_1(x)}{\eta(x)q_1(x) - q_2(x)} = \frac{2\alpha g(x)(G(x)[2 - G(x)])^{\alpha-1} \bar{G}(x)}{1 - (G(x)[2 - G(x)])^\alpha}, \quad x \in R. \quad (17)$$

The general solution of the differential equation in Corollary 4.2.1 is

$$\eta(x) = \{1 - (G(x)[2 - G(x)])^\alpha\}^{-1} \\ \times \left[ - \int 2\alpha g(x)(G(x)[2 - G(x)])^{\alpha-1} (q_1(x))^{-1} q_2(x) dx + D \right],$$

where  $D$  is a constant.

## 5. Estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used for constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters of this family from complete samples only by maximum likelihood. Let  $x_1, \dots, x_n$  be a random sample from the PTL-G distribution with parameters  $\delta, \alpha$  and  $\psi$ . Let  $\theta$  be the  $p \times 1$  parameter vector. For determining the MLE of  $\theta$ , we have the log-likelihood function

$$\ell = \ell(\theta) = n \log(2) + n \log \lambda + n \log \alpha - n \log(1 - e^{-\lambda}) \\ + \sum_{i=1}^n \log g(x_i; \psi) + (\alpha - 1) \sum_{i=1}^n \log G(x_i; \psi) \\ + \sum_{i=1}^n \log[1 - G(x_i; \psi)] + (\alpha - 1) \sum_{i=1}^n \log[2 - G(x_i; \psi)] \\ - \lambda \sum_{i=1}^n G(x_i; \psi)^\alpha [2 - G(x_i; \psi)]^\alpha.$$

The components of the score vector are

$$U_\lambda = \frac{n}{\lambda} - \frac{ne^{-\lambda}}{(1 - e^{-\lambda})} - \sum_{i=1}^n G(x_i; \psi)^\alpha [2 - G(x_i; \psi)]^\alpha, \\ U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^n \log G(x_i; \psi) + \sum_{i=1}^n \log[2 - G(x_i; \psi)] \\ - \lambda \sum_{i=1}^n \frac{\{\log[2 - G(x_i; \psi)] + \log G(x_i; \psi)\}}{[2 - G(x_i; \psi)]^{-\alpha} G(x_i; \psi)^{-\alpha}},$$

and

$$U_\psi = \sum_{i=1}^n \frac{g'(x_i; \psi)}{g(x_i; \psi)} + (\alpha - 1) \sum_{i=1}^n \frac{G'(x_i; \psi)}{G(x_i; \psi)} \\ - \sum_{i=1}^n \frac{G'(x_i; \psi)}{1 - G(x_i; \psi)} - (\alpha - 1) \sum_{i=1}^n \frac{G'(x_i; \psi)}{2 - G(x_i; \psi)} \\ - \lambda \alpha \sum_{i=1}^n G'(x_i; \psi) \left\{ \frac{[2 - G(x_i; \psi)]^\alpha G(x_i; \psi)^{\alpha-1}}{G(x_i; \psi)^\alpha [2 - G(x_i; \psi)]^{\alpha-1}} \right\},$$

where  $g'(x_i; \psi) = \partial g(x_i; \psi) / \partial \psi$ ,  $G'(x_i; \psi) = \partial G(x_i; \psi) / \partial \psi$ . Setting the nonlinear system of equations  $U_\lambda = U_\alpha = 0$  and  $U_\psi = 0$  and solving them simultaneously yields the MLE. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize  $\ell$ .

## 6. Simulation of PTL-E distribution

In this section, we study the performance of the PTL-E distribution by conducting various simulations for different sizes ( $n=50, 150, 300$ ) by using R-Language. We simulate 500 samples for the true parameters values I:  $\lambda = 4, \alpha = 5.5, \gamma = 0.5$  in order to obtain average estimates (AEs), biases and mean square errors (MSEs) of the parameters. They are listed in Table 1. The small values of the biases and MSEs indicate that the maximum likelihood method performs quite well in estimating the model parameters of the proposed distribution.



## 7. Application

In this section, we use a real data set to show that the PTL-Weibull distribution can be a better model than one based on Beta-exponential, Kumaraswamy-exponential, PTL-Weibull, Modified Weibull and Weibull distribution. The data set given below and represents the failure times of 50 components (per 1000h)

0.036	0.058	0.061	0.074	0.078	0.086	0.102	0.103	0.114	0.116
0.148	0.183	0.192	0.254	0.262	0.379	0.381	0.538	0.570	0.574
0.590	0.618	0.645	0.961	1.228	1.600	2.006	2.054	2.804	3.058
3.076	3.147	3.625	3.704	3.931	4.073	4.393	4.534	4.893	6.274
6.816	7.896	7.904	8.022	9.337	10.940	11.020	13.880	14.730	15.080

The values in table Tab1 indicate that the PTL-W distribution leads to a better fit than the Weibull distribution, modified Weibull distribution, exponential distribution, beta exponential distribution, Kumaraswamy exponential distribution and PTL-E distribution. Also, from Table 2 PTL-G family of distribution is better than Beta-G and Kumaraswamy-G distribution.

Many other useful real data sets can be found and analyzed see Ibrahim (2019), Ibrahim et al. (2019), Ibrahim (2020a, b), Mansour et al. (2020) and Ibrahim and Yousof (2020).

## 8. Concluding remarks

We have proposed and presented results on a new family of distributions called Poisson Topp Leone-G family of distributions. This family of distributions have applications in Reliability, Economics and Survival data analysis. Properties of this family are studied including reliability properties, quantile function, series expansion of CDF and PDF, moments, moment generating function, mean deviation. Expression for  $i^{th}$  order statistics is given, and estimation of parameters are carried out by Maximum likelihood method. A special sub-model is discussed in detail for illustration propose. Finally, an application is carried out on real data set to check the performance of the proposed family which provides consistently better fit than other models.

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## Appendix

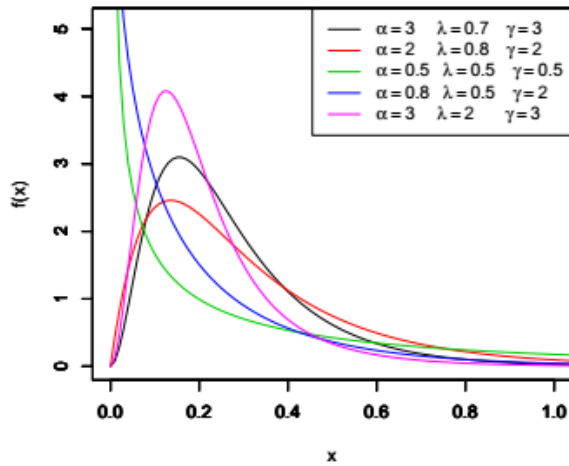


Figure 1: The PDF's of various PTL-E distributions.

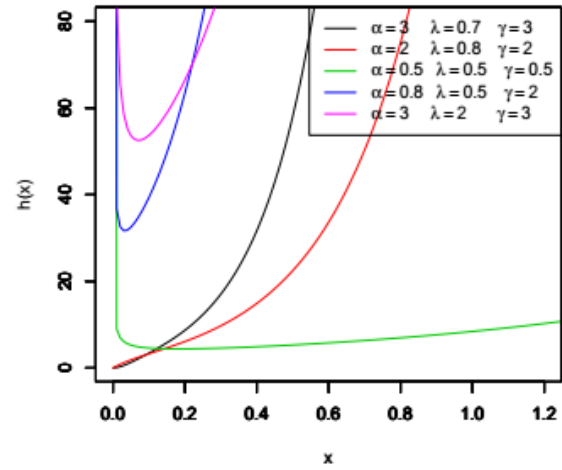


Figure 2: The hazard functions of various PTL-E distributions.

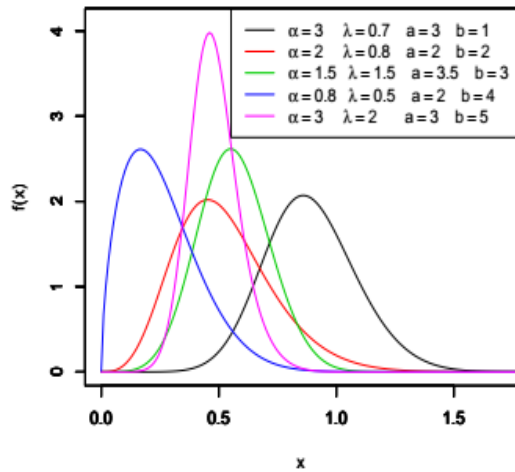


Figure 3: The PDF's of various PTL-W distributions.

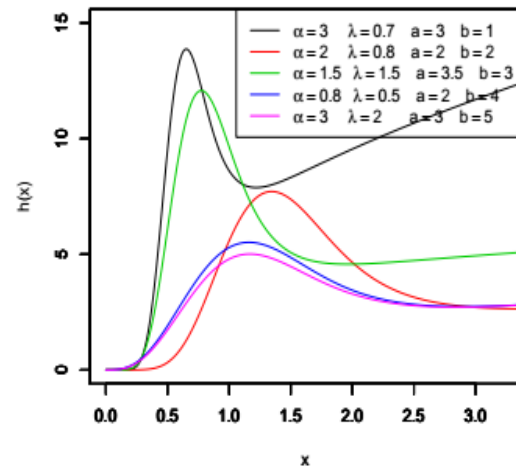


Figure 4: The hazard functions of various PTL-W distributions.

Table 1: Estimated AEs, Biases, and MSEs of the MLEs of parameters of PTL-E distribution based on 500 simulations of with  $n=50, 150$  and  $300$ .

	parameters	50		
		A.E	Bias	MSE
	$\alpha$	3.351	0.759	1.763
	$\lambda$	5.010	0.510	0.351
	$\gamma$	0.735	0.335	0.073
150	$\alpha$	3.559	0.551	0.651
	$\lambda$	5.953	0.553	0.355
	$\gamma$	0.776	0.376	0.103
300	$\alpha$	3.368	0.633	0.539
	$\lambda$	5.996	0.596	0.310
	$\gamma$	0.755	0.355	0.083

Table 2: Estimated parameters for the failure times of 50 components(per 1000h).

Model	Estimates	Standard Error	-logL	AIC	K-S
Weibull	$\alpha=0.429$ $\lambda=0.638$	0.104 0.073	103.0572	210.114	0.175
Modified Weibull	$\alpha=0.043$ $\beta=0.492$ $\gamma=0.619$	0.131 0.181 0.154	102.320	210.640	0.138
<b>PTL-Weibull</b>	$\lambda=2.118$ $\alpha=0.146$ $a=3.902$ $b=1.3 \times 10^{-5}$	0.899 0.024 0.021 $0.13 \times 10^{-5}$	<b>100.818</b>	<b>209.636</b>	<b>0.122</b>
Exponential	$\lambda=0.299$	0.042	110.342	222.685	0.478
Beta-Exponential	$a=0.529$ $b=0.717$ $\lambda=0.281$	0.101 0.308 0.132	102.358	210.716	0.144
Kumaraswamy-Exponential	$a=0.605$ $b=3.274$ $\lambda=0.044$	0.204 1.308 0.012	102.166	210.332	0.131
<b>PTL-Exponential</b>	$\lambda=0.823$ $\alpha=0.606$ $\gamma=0.083$	0.304 0.122 0.029	<b>102.094</b>	<b>210.189</b>	<b>0.128</b>

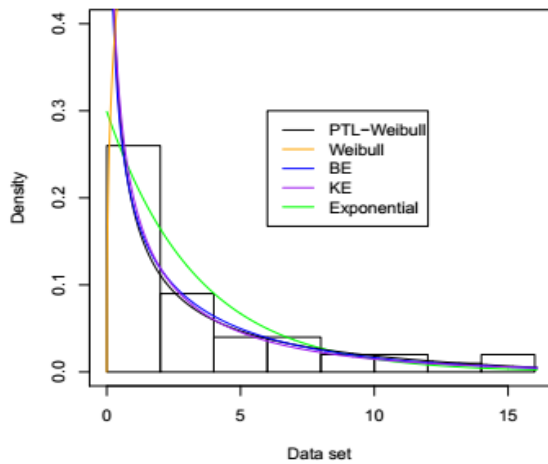


Figure 5: Estimated densities of the models for data set.

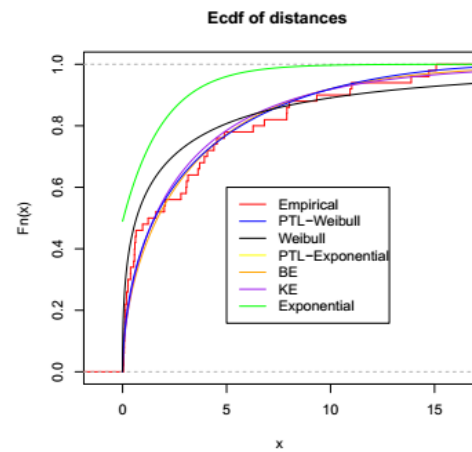


Figure 6: Fitted CDFs plots of the considered distribution for the real data set.

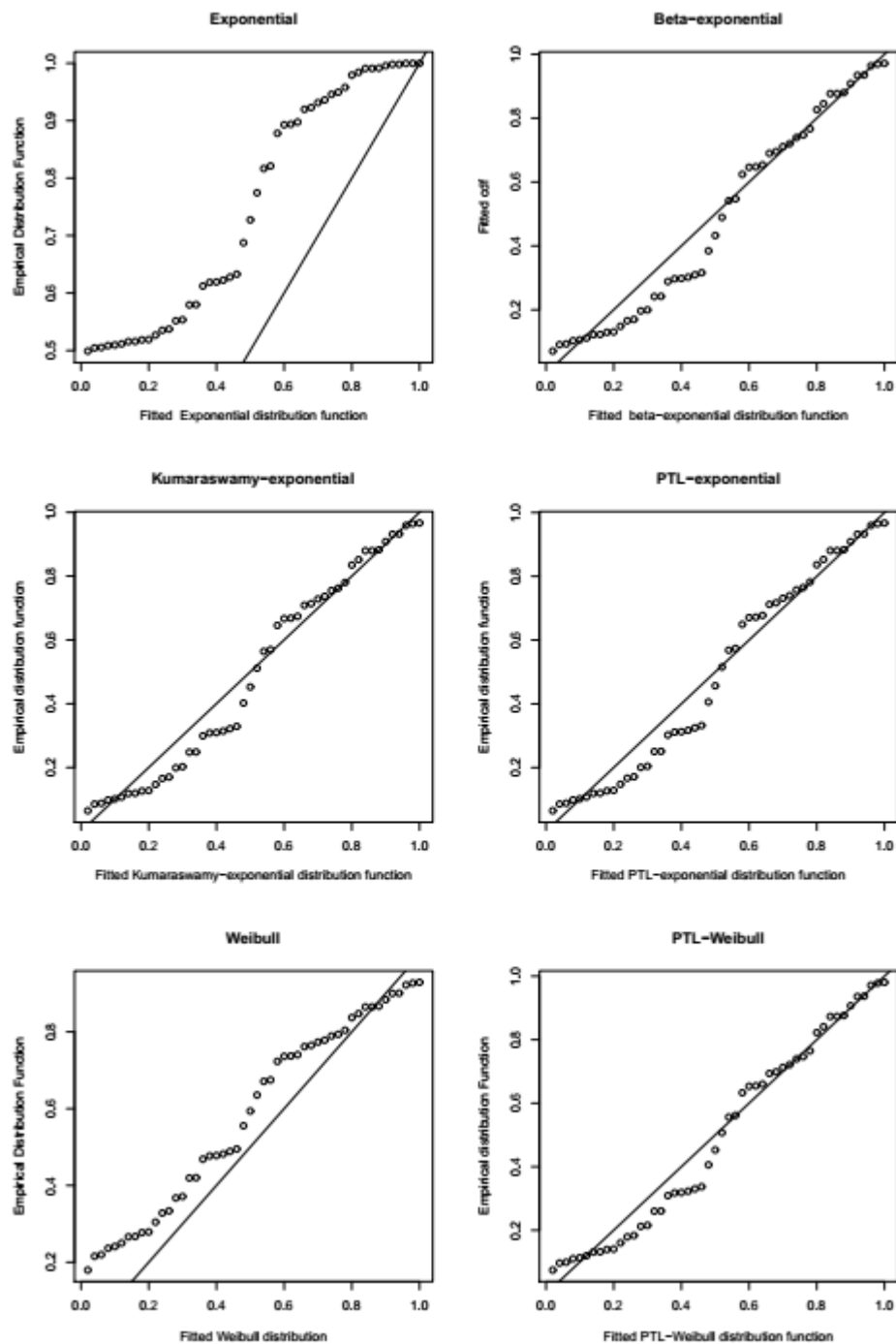


Figure 7: The P-P plots for the real data set