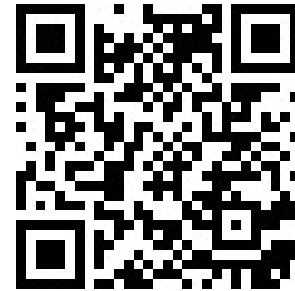


## Transmuted Probability Distributions: A Review

Md. Mahabubur Rahman<sup>1</sup>, Bander Al-Zahrani<sup>2</sup>,  
Saman Hanif Shahbaz<sup>3</sup>, Muhammad Qaiser Shahbaz<sup>4\*</sup>



\*Corresponding author

1. Department of Statistics, King Abdulaziz University, Saudi Arabia  
& Department of Statistics, Islamic University, Kushtia, Bangladesh, mmriu.stat@gmail.com
2. Department of Statistics, King Abdulaziz University, Saudi Arabia, bmalzahrani@kau.edu.sa
3. Department of Statistics, King Abdulaziz University, Saudi Arabia, shmohamad2@kau.edu.sa
4. Department of Statistics, King Abdulaziz University, Saudi Arabia, qshahbaz@gmail.com

### Abstract

Transmutation is the functional composition of the cumulative distribution function (*cdf*) of one distribution with the inverse cumulative distribution function (quantile function) of another. Shaw and Buckley(2007), first apply this concept and introduced quadratic transmuted family of distributions. In this article, we have presented a review about the transmuted families of distributions. We have also listed the transmuted distributions, available in the literature along with some concluding remarks.

**Key Words:** Cubic Transmutation; General Transmutation; Probability Distribution; Quadratic Transmutation; Transmuted Distribution.

**Mathematical Subject Classification:** 60E05, 62E15.

### 1. Background

The probability distributions are being used in many areas of life. The application of a statistical tools depends upon the underlying probability model of the data. As a result huge numbers of probability distributions are being developed by the researchers. However, there still remains large numbers of practical problems that does not follow any of the standard probability distributions. So, developing the new form of the probability distributions is very much common in statistical theory.

Generalizing probability distributions are very common practice in the theory of statistics. In order to generalize probability distributions various methods are proposed in literature which add extra parameter(s) to an existing baseline probability models so that the new model will increase the flexibility of the models to capture the complexity of the data. Several generalized (or G) classes are available in the literature, but our main focus in this paper is to present a review of transmuted-G class of distributions that increase the flexibility of the distribution along with the ability to explore its tail properties and to improve goodness-of-fits.

We first given describe some classical and recent families of distributions in the following.

## 1.1. Classical Pearson and Burr Families

The normal distribution is not suitable for handling all practical problems. So generalization of distributions is very much essential for solving the practical problems that are not well fitted to normal distribution. The first approach for generalization of distributions was due to Karl Pearson and the second approach was due to Burr which are discussed in the following two subsections.

### 1.1.1. The Pearson Family of Distributions

Karl Pearson (1857-1936) developed a family of continuous probability distributions and named it as Pearson distribution. He has listed his four types of distributions; Type-I to Type-IV; in (Pearson, 1895) and has listed normal distribution as Type-V distribution. In an another article, (Pearson, 1901), he has redefined Type-V distribution and has proposed Type-VI distribution. In yet another article, (Pearson, 1916), he has further developed special cases and subtypes VII through XII. More details of the Pearson family of distributions, see Johnson et al.(1994, 1995).

The Pearson probability density function (*pdf*)  $y = f(x)$  is defined to be any valid solution to the differential equation (see Pearson,1895), as follows

$$\frac{f'(x)}{f(x)} + \frac{a + (x - \lambda)}{c_0 + c_1(x - \lambda) + c_2(x - \lambda)^2} = 0, \quad (1)$$

with:

$$\begin{aligned} c_0 &= \frac{4\beta_2 - 3\beta_1}{10\beta_2 - 12\beta_1 - 18}\mu_2, \\ c_1 &= a = \sqrt{\mu_2}\sqrt{\beta_1}\frac{\beta_2 + 3}{10\beta_2 - 12\beta_1 - 18}, \text{ and} \\ c_2 &= \frac{2\beta_2 - 3\beta_1 - 6}{10\beta_2 - 12\beta_1 - 18}. \end{aligned}$$

**The Criterion k:** The Pearson family of distributions is obtained by solving the differential Equation (1), which can also be written as

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{(x - a)}{c_0 + c_1x + c_2x^2} \\ \Rightarrow y &= f(x) = ce^I, \end{aligned}$$

where  $c$  is the constant of integration and  $I = \int \frac{(x-a)}{c_0+c_1x+c_2x^2} dx$ .

**Table 1: Pearson Frequency Curves Type I to Type VII.**

Criterion $k$	Corresponding Frequency Curve
$k < 0$	Type-I
$k = 0$ ( $\beta_1 = 0, \beta_2 < 3$ )	Type-II
$k \rightarrow \pm\infty$	Type-III
$0 < k < 1$	Type-IV
$k = 1$	Type-V
$k > 1$	Type-VI
$k = 0$ ( $\beta_1 = 0, \beta_2 > 3$ )	Type-VII

Hence,  $y = f(x)$  depends on the roots of the equation  $c_0 + c_1x + c_2x^2$ , which can be further expressed as

$$c_0 + c_1x + c_2x^2 = c_2 \left[ \left( x + \frac{c_1}{2c_2} \right)^2 - \frac{4c_0^2}{c_1^2} k(k-1) \right],$$

where  $k = \frac{c_1^2}{4c_2c_0}$  and is used as a criterion for obtaining the form of *pdf*. Basically the Pearson family of distributions is made up of seven distributions (Type I to VII), which are given in Table 1 on the basis of criterion  $k$ .

### 1.1.2. The Burr Family of Distributions

Burr(1942), introduced twelve cumulative frequency functions that could fit the real-life datasets. Most of these distributions are unimodal. The Burr distribution is very similar (and is, in some cases, the same) as many other probability distributions. The *cdf*,  $F(y)$ , of various Burr distributions are given in Table 2 for  $k, r, c \in \mathbb{R}^+$ . More details on Burr distributions can be seen in Johnson et al.(1994, 1995); Tadikamalla and Pandu(1980) among others.

**Table 2: Burr Family of Cumulative Frequency Functions.**

Cumulative Distribution Function	Corresponding Type
$F(y) = y, y \in (0, 1)$	Type-I
$F(y) = (e^{-y} + 1)^{-r}$	Type-II
$F(y) = (y^{-k} + 1)^{-r}, y \in \mathbb{R}^+$	Type-III
$F(y) = \left[ \left( \frac{c-y}{y} \right)^{\frac{1}{c}} + 1 \right]^{-r}, y \in (0, c)$	Type-IV
$F(y) = (ke^{-\tan y} + 1)^{-r}, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	Type-V
$F(y) = (ke^{-c \sinh y} + 1)^{-r}$	Type-VI
$F(y) = 2^{-r}(1 + \tanh y)^r$	Type-VII
$F(y) = \left( \frac{2}{\pi} \arctan e^y \right)^r$	Type-VIII
$F(y) = 1 - \frac{2}{k[(1+e^y)^r - 1] + 2}$	Type-IX
$F(y) = \left( 1 - e^{-y^2} \right)^r, y \in \mathbb{R}^+$	Type-X
$F(y) = \left( y - \frac{1}{2\pi} \sin 2\pi y \right)^r, y \in (0, 1)$	Type-XI
$F(y) = 1 - (1 + y^c)^{-k}, y \in \mathbb{R}^+$	Type-XII

The location and scale parameters  $\mu$  (location) and  $\sigma$  (scale) can be easily introduced in all members of Burr family of distributions, given in Table 2, by using the transformation  $y = \frac{x-\mu}{\sigma}$ . Burr(1942), explored the Type XII distribution in details and the others were studied later.

## 1.2. Some Recent Families

Several families of distributions are available in the literature which extend a probability distributions by adding new parameter(s). Some of these families of distributions are discussed in the following.

### 1.2.1. The Exponentiated Family

Gupta et al.(1998) developed the general form of the exponentiated family of distributions. The *cdf* of this family is

$$F_{\alpha}(x) = [G(x)]^{\alpha}, x \in \mathbb{R},$$

where  $\alpha \in \mathbb{R}^+$ .

### 1.2.2. The Marshall-Olkin Family

Marshall and Olkin(1997) introduced an interesting method to add a new parameter to an existing model. They applied their method to extend exponential and Weibull distributions. The survival function of this family is

$$\bar{F}_{M-O}(x) = \frac{\alpha \bar{G}(x)}{1 - \bar{\alpha} \bar{G}(x)} = \frac{\alpha \bar{G}(x)}{G(x) + \alpha \bar{G}(x)}, \quad x \in \mathbb{R},$$

where  $\alpha \in \mathbb{R}^+$ ,  $\bar{\alpha} = 1 - \alpha$  and  $\bar{G}(x) = 1 - G(x)$  is the survival function of the baseline probability distribution.

### 1.2.3. The Beta-G Family

Eugene et al.(2002) introduced a generalized family of distributions by using logit of the beta random variable. The proposed family is denoted by *Beta* – *G* and has the *cdf*

$$\begin{aligned} F_{B-G}(x) &= I_{G(x)}(\alpha, \beta) = \frac{B_{G(x)}(\alpha, \beta)}{B(\alpha, \beta)} \\ &= \frac{1}{B(\alpha, \beta)} \int_0^{G(x)} t^{\alpha-1} (1-t)^{\beta-1} dt, \quad x \in \mathbb{R}, \end{aligned}$$

where  $\alpha \in \mathbb{R}^+$ ,  $\beta \in \mathbb{R}^+$  and  $B(\alpha, \beta)$  is the Beta function.

### 1.2.4. The Kumaraswamy-G Family

Kumaraswamy-G ( $K_w - G$ ) family of distributions is proposed by Cordeiro and de Castro(2011), by using Kumaraswamy (1980) distribution, and has the *cdf*

$$F_{K_w-G}(x) = 1 - \{1 - G(x)^\alpha\}^\beta, \quad x \in \mathbb{R},$$

where  $\alpha, \beta \in \mathbb{R}^+$  are the shape parameters.

### 1.2.5. The $T - X$ Family

An interesting method for obtaining families of continuous distributions has been introduced by Alzaatreh et al.(2013). In this technique “the transformer” random variable  $X$  is used to transform another “the transformed” random variable  $T$ . The resulting family is known as  $T - X$  family of distributions and is defined below.

Let  $X$  be a random variable with the *pdf*  $g(x)$  and *cdf*  $G(x)$ . Also let  $T$  be a continuous random variable with the *pdf*  $r(t)$ , defined on  $[a, b]$ . The *cdf* of a new family of distributions is defined by

$$F_{T-X}(x) = \int_a^{W[G(x)]} r(t) dt, \quad (2)$$

where  $W(G(x))$  satisfies following conditions

$$\left. \begin{aligned} W(G(x)) &\in [a, b], \\ W(G(x)) &\text{ is differentiable and monotonically non-decreasing,} \\ W(G(x)) &\rightarrow a \text{ as } x \rightarrow -\infty \text{ and } W(G(x)) \rightarrow b \text{ as } x \rightarrow \infty. \end{aligned} \right\}$$

The *cdf*  $F_{T-X}(x)$  in (2) is a composite function of  $(R \cdot W \cdot G)(x)$  and can be written as  $F_{T-X}(x) = R\{W(G(x))\}$ , where  $R(t)$  is the *cdf* of random variable  $T$ .

Several researchers have generated new probability distributions by utilizing recent families of distributions for different choices of  $G(x)$ . Our main interest in this paper is to give a review of work done on transmuted distributions.

The layout plan of this paper follows: The transmuted family of distributions is described in Section 2, along with the expansion of transmuted distributions in Section 3. Section 4 provides the evolution related to cubic and quartic transmuted distributions. In Section 5, we have presented the recent development of general transmuted distributions. Finally, some concluding remarks are given in Section 6.

## 2. Transmuted Family of Distributions

Shaw and Buckley(2007), introduced an interesting method of adding new parameter to an existing distribution for solving the problems related to financial mathematics and named the family as quadratic transmuted family (QT-G, for short) of distributions. The *cdf* of the family has following simple quadratic form

$$F(x) = (1 + \lambda)G(x) - \lambda G(x)^2, \quad x \in \mathbb{R}, \quad (3)$$

where  $\lambda \in [-1, 1]$  and  $G(x)$  is the *cdf* of the baseline distribution.

## 3. Developments in Quadratic Transmuted Distributions

Aryal and Tsokos(2009, 2011) first highlight the technique given in (3) and introduced a couple of transmuted probability distributions that would offer more distributional flexibility in environmental and reliability analysis. The general properties and results for the transmuted family of distributions were described by Bourguignon et al.(2016); Das(2015).

Various researchers have attracted to the quadratic transmuted distributions (3) and they have introduced the new members of this family for various choices of baseline *cdf*  $G(x)$ . Tahir and Cordeiro (2016) have provided a list for quadratic transmuted distributions. At this time, transmuted distributions are very common in the literature. An up-to-date list of popular transmuted-G classes of distributions is given in Table 3 below.

**Table 3: Contributed Work on Quadratic Transmuted Distributions.**

Sl. No.	Distribution	Athor(s) (Year)
1	transmuted extreme value	Aryal and Tsokos (2009)
2	transmuted Weibull	Aryal and Tsokos (2011)
3	transmuted log-logistic	Aryal (2013)
4	transmuted exponentiated exponential	Merovci (2013a)
5	transmuted Fréchet	Mahmoud and Mandouh (2013)
6	transmuted Lomax	Ashour and Eltehiwy (2013c)
7	transmuted Lindley	Merovci (2013b)
8	transmuted quasi-Lindley	Elbatal and Elgarhy (2013)
9	transmuted exponentiated-Lomax	Ashour and Eltehiwy (2013a)
10	transmuted modified inverse Weibull	Elbatal (2013b)
11	transmuted gen. inverted exponential	Elbatal (2013a)
12	transmuted exponent. modif. Weibull	Ashour and Eltehiwy (2013b)
13	transmuted gen. linear exponential	Elbatal et al. (2013)
14	transmuted additive Weibull	Elbatal and Aryal (2013)
15	transmuted modified Weibull	Khan and King (2013)
16	transmuted Pareto	Merovci and Puka (2014)
17	transmuted Maxwell	Iriarte and Astorga (2014)
18	transmuted linear exponential	Tian et al. (2014)
19	transmuted inverse Rayleigh	Ahmad et al. (2014)

20	transmuted generalized Rayleigh	Merovci (2014)
21	transmuted inverse Weibull	Khan et al. (2014)
22	transmuted inverse Weibull	Khan and King (2014a)
23	transmuted gener. inverse Weibull	Merovci et al. (2014)
24	transmuted gener. inverse Weibull	Khan and King (2014b)
25	transmuted exponentiated gamma	Hussian (2014)
26	transmuted exponentiated Fréchet	Elbatal et al. (2014)
27	transmuted Weibull	Ahmad et al. (2015)
28	transmuted generalized gamma	Lucena et al. (2015)
29	transmuted exponential	Owoloko et al. (2015)
30	transmuted Burr type III	Abdul-Moniem (2015)
31	transmuted Gompertz	Abdul-Moniem and Seham (2015)
32	transmuted mod. inverse Rayleigh	Khan and King (2015)
33	transmuted additive Weibull	Mansour et al. (2015)
34	transmuted Marshall-Olkin Fréchet	Afify et al. (2015a)
35	transmuted Weibull-Lomax	Afify et al. (2015b)
36	transmuted generalized Rayleigh	Iriarte and Astorga (2015)
37	transmuted log-logistic	Granzotto and Louzada (2015)
38	transmuted Dagum	Elbatal and Aryal (2015)
39	transmuted exponentiated-Pareto	Luguterah and Nasiru (2015)
40	transmuted exponentiated-Pareto	Fatima and Roohi (2015)
41	transmuted Lindley	Mansour and Mohamed (2015)
42	transmuted Chen lifetime	Khan et al. (2015)
43	transmuted Kumaraswamy	Khan et al. (2016b)
44	transmuted generalized Lindley	Elgarhy et al. (2016)
45	transmuted modified Weibull	Vardhan and Balaswamy (2016)
46	transmuted Weibull-Pareto	Afify et al. (2016)
47	transmuted Dagum	Shahzad and Asghar (2016)
48	transmuted Gompertz	Khan et al. (2016a)
49	transmuted power function	Haq et al. (2016)
50	transmuted geometric	Chakraborty and Bhati (2016)
51	transmuted Birnbaum-Saunders	Bourguignon et al. (2017)
52	transmuted gen. exp. exponential	Khan et al. (2017a)
53	transmuted modified Weibull	Cordeiro et al. (2017)
54	transmuted gener. inverse Weibull	Khan et al. (2017b)
55	transmuted Weibull	Khan et al. (2017c)
56	transmuted exp. add. Weibull	Nofal et al. (2018)
57	transmuted Weibull	Pobočková et al. (2018)
58	transmuted modified Weibull	Khan et al. (2018)
59	transmuted exponential Lomax	Abdullahi and Ieren (2018)
60	transmute half normal	Balaswamy (2018)
61	transmuted gen. inverted expo.	Okorie and Akpanta (2019)
62	transmuted half normal	Abayomi (2019)
63	transmuted Burr type X	Khan et al. (2019)
64	transmuted Ishita	Gharaibeh and Al-Omari(2019)
65	transmuted gen. extreme value	Otiniano et al. (2019)
66	transmuted Logistic	Samuel (2019)

The literature on the transmuted family of distributions is enrich enough and is also rapidly improving. We now describe cubic and quartic transmuted families of distributions in the following.

#### 4. Developments in Cubic and Quartic Transmuted Distributions

The quadratic transmuted distributions, discussed above, capture the complexity of unimodal data but the real-life data can be more complex (multi-modal) and, sometimes, can not be fitted by using above those models. Several researchers have developed different types of cubic transmuted families (CT-G, for short) of distributions. Granzotto et al.(2017) have developed a cubic transmuted family which has following simple form

$$F(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1) G^2(x) + (1 - \lambda_2) G^3(x), \quad x \in \mathbb{R},$$

with  $\lambda_1 \in [0, 1]$  and  $\lambda_2 \in [-1, 1]$ .

Rahman et al.(2018a, 2018c, 2019b) have introduced *three* new cubic transmuted families of distributions. *First* one is defined as

$$F(x) = (1 + \lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x), \quad x \in \mathbb{R}, \quad (4)$$

where  $\lambda_1 \in [-1, 1]$  and  $\lambda_2 \in [-1, 1]$  and  $-2 \leq \lambda_1 + \lambda_2 \leq 1$ . It can be easily observed that the cubic transmuted family of distributions proposed by AL-Kadim and Mohammed(2017) turned out to be a special case of (4) for  $\lambda_2 = -\lambda_1$ .

*Second* family of distributions is given as

$$F(x) = (1 + \lambda_1 + \lambda_2)G(x) - (\lambda_1 + 2\lambda_2)G^2(x) + \lambda_2 G^3(x), \quad x \in \mathbb{R}, \quad (5)$$

where  $\lambda_1 \in [-1, 1]$  and  $\lambda_2 \in [0, 1]$ .

The *third* family of distributions is given as

$$F(x) = (1 - \lambda)G(x) + 3\lambda G^2(x) - 2\lambda G^3(x), \quad x \in \mathbb{R},$$

where  $\lambda \in [-1, 1]$ .

Aslam et al.(2018), introduced another cubic transmuted-G family of distributions and its related properties. Several cubic transmuted distributions are introduced by various researchers and are listed in Table 4.

**Table 4: Contributed Work on Cubic Transmuted Distributions.**

Sl. No.	Distribution	Athor(s) (Year)
1	cubic transmuted Weibull	Granzotto et al. (2017)
2	cubic transmuted log-logistic	Granzotto et al. (2017)
3	cubic transmuted Weibull	AL-Kadim and Mohammed (2017)
4	cubic transmuted exponential	Rahman et al. (2018a)
5	cubic transmuted Pareto	Rahman et al. (2018b)
6	cubic transmuted Pareto	Ansari and Eledum (2018)
7	cubic transmuted exponential	Rahman et al. (2018c)
8	cubic transmuted Fréchet	Celik (2018)
9	cubic transmuted Gumbel	Celik (2018)
10	cubic transmuted Gompertz	Celik (2018)
11	cubic transmuted Weibull	Rahman et al. (2019a)
12	cubic transmuted Burr III-Pareto	Bhatti et al. (2019)
13	cubic transmuted uniform	Rahman et al. (2019b)

The cubic transmuted distributions shows better flexibility to handle more complex (bi-modal) data over quadratic transmuted distributions.

We can easily obtain forth order (quartic) transmuted (FOT-G, for short) families of distributions as extension of (4)

and (5) and has following simple forms

$$F(x) = G(x) + [1 - G(x)] \sum_{i=1}^3 \lambda_i G^i(x), \quad x \in \mathbb{R}, \quad (6)$$

with  $\lambda_i \in [-1, 1]$  for  $i = 1, 2, 3$  and  $-3 \leq \sum_{i=1}^3 \lambda_i \leq 1$ , and

$$F(x) = G(x) + G(x) \sum_{i=1}^3 \lambda_i [1 - G(x)]^i, \quad x \in \mathbb{R}, \quad (7)$$

where  $\lambda_1 \in [-1, 1]$  and  $\lambda_i \in [0, 1]$  for  $i = 2, 3$ .

Using the family (6) and (7) there is huge scope to develop quartic (fourth rank) transmuted distributions. AL-Kadim(2018) has proposed a fourth rank transmuted Weibull distribution which is a special case of the family (6) for  $\lambda_2 = 0$  and  $\lambda_3 = \lambda_1$ .

## 5. Developments in General Transmuted Distributions

Merovci et al.(2016) have introduced and studied general mathematical properties of generalized transmuted family of distributions. Alizadeh et al.(2017) have developed a new generalized transmuted family of distributions and have described it as a linear combination of exponentiated densities in terms of the same baseline distribution. In order to model more complex (multi-modal) data, Rahman et al. (2018a, 2018c) have introduced a couple of general transmuted families (GT-G, for short) of distributions; called  $k$ - transmuted families; which are defined as

$$F(x) = G(x) + [1 - G(x)] \sum_{i=1}^k \lambda_i [G(x)]^i, \quad x \in \mathbb{R}, \quad (8)$$

with  $\lambda_i \in [-1, 1]$  for  $i = 1, 2, \dots, k$  and  $-k \leq \sum_{i=1}^k \lambda_i \leq 1$ , and

$$F(x) = G(x) + G(x) \sum_{i=1}^k \lambda_i [1 - G(x)]^i, \quad x \in \mathbb{R},$$

where  $\lambda_1 \in [-1, 1]$  and  $\lambda_i \in [0, 1]$  for  $i = 2, 3, \dots, k$ .

General transmuted families reduces to the base distribution for  $\lambda_i = 0$  ( $i = 1, 2, \dots, k$ ). AL-Kadim(2018), proposed a generalized family of transmuted distribution which turned out to be a special case of family (8), for  $\lambda_2 = \lambda_4 = \lambda_6 = \dots = -\lambda_1$  and  $\lambda_3 = \lambda_5 = \lambda_7 = \dots = 0$  when  $k$  is even, otherwise  $\lambda_2 = \lambda_4 = \lambda_6 = \dots = 0$  and  $\lambda_3 = \lambda_5 = \lambda_7 = \dots = \lambda_1$  for odd  $k$ .

## 6. Concluding Remarks

Generalization of probability distributions through transmutation was first applied in the area of financial mathematics. After that time, several researchers have successfully applied this technique to model lifetime and survival data. At present, this approach is being applied in the areas of biology, engineering, environmental, medical among others to handle more complex (bi-modal) data.

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