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Extended Reciprocal Rayleigh Distribution: Copula, Properties and Real Data Modeling

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Abstract

A new extension of the reciprocal Rayleigh distribution is introduced. Simple type copula-based construction is presented for deriving and many bivariate and multivariate type distributions of the reciprocal Rayleigh model. The new reciprocal Rayleigh model generalizes another three reciprocal Rayleigh distributions. The performance of the estimation method is assessed using a graphical simulation study.

Key Words: Reciprocal Rayleigh Distribution; Clayton Copula; Morgenstern Family; Moments; Estimation; Odd Log-Logistic Family; Graphical Simulation.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction

A random variable (RV) Z is said to have the reciprocal Rayleigh (RR) distribution if its survival function (SF) is given by

$$S_{\beta}(z) = 1 - G_{\beta}(z) = 1 - exp(-\beta^2 z^{-2}) \,|_{z\geq 0}, \tag{1}$$

where $\beta > 0$ and $G_{\beta}(z)$ refer to the cumulative function (CDF). The density function (PDF) corresponding (1) can be expressed as

$$g_{\beta}(z) = 2\beta^2 z^{-3}\, exp(-\beta^2 z^{-2}) \,|_{z\geq 0} \tag{2}$$

The RR model is a special case from the well-known Reciprocal Weibull (RW) model. Recently, Cordeiro et al. (2016) proposed and studied a new class called the generalized odd log-logistic-G (GOLL-G) with two extra shape parameters $a_1 > 0$ and $a_2 > 0$. For an arbitrary baseline CDF $G_{\underline{\Psi}}(z)$, the SF of the GOLL-G family is given by

$$S_{a_1,a_2,\underline{\Psi}}(z) = 1 - F_{a_1,a_2,\underline{\Psi}}(z) = 1 - \frac{G_{\underline{\Psi}}(z)^{a_1 a_2}}{G_{\underline{\Psi}}(z)^{a_1 a_2} + \left[1 - G_{\underline{\Psi}}(z)^{a_2}\right]^{a_1}}, \tag{3}$$

The PDF corresponding to (3) is given by

$$f_{a_1,a_2,\underline{\Psi}}(z) = \frac{a_1 a_2 g_{\underline{\Psi}}(z) G_{\underline{\Psi}}(z)^{a_1 a_2 - 1}\left[1 - G_{\underline{\Psi}}(z)^{a_2}\right]^{a_1 - 1}}{\left\{G_{\underline{\Psi}}(z)^{a_1 a_2} + \left[1 - G_{\underline{\Psi}}(z)^{a_2}\right]^{a_1}\right\}^2}, \tag{4}$$

where $g_{\underline{\Psi}}(z) = dG_{\underline{\Psi}}(z)/dx.$ For $a_2 = 1$ we get the OLL-G family (Gleaton and Lynch (2006)). For $a_1 = 1$ we get the Proportional reversed hazard rate (ProRHR) family (Gupta and Gupta (2007)). Here, we define a new RW model based on Cordeiro et al. (2016) called GOLL-RR model and then provide some plots of its PDF and hazard rate function (HRF) $\left[h_{a_1,a_2,\beta}(z)\right]$. The SF of the GOLL-RR is given by

$$S_{a_1,a_2,\beta}(z) = 1 - F_{a_1,a_2,\beta}(z) = 1 - \frac{exp(-a_1a_2\beta^2z^{-2})}{exp(-a_1a_2\beta^2z^{-2}) + \{1 - exp(-a_2\beta^2z^{-2})\}^{a_1}}. \quad (5)$$

The PDF corresponding to (5) is given by

$$f_{a_1,a_2,\beta}(z) = \frac{a_1a_22\beta^2z^{-3}\,exp(-a_1a_2\beta^2z^{-2})\,\{1 - exp(-a_2\beta^2z^{-2})\}^{a_1-1}}{(exp(-a_1a_2\beta^2z^{-2}) + \{1 - exp(-a_2\beta^2z^{-2})\}^{a_1})^2}. \quad (6)$$

The HRF for the new model can be get from $f_{a_1,a_2,\beta}(z)/\left[1 - F_{a_1,a_2,\beta}(z)\right]$. Table 1 provides some sub-models of the GOLL-RR model.

Table 1: Sub-models of the GOLL-RR model

a_1	a_2	β	Reduced model	Reduced CDF	Author
1			ProRH-RR	$exp(-a_2\beta^2z^{-2})$	Gusmao et al. (2011)
	1		OLL-RR	$\frac{exp(-a_1\beta^2z^{-2})}{\left\{\begin{matrix} exp(-a_1\beta^2z^{-2}) \\ +[1 - exp(-\beta^2z^{-2})]^{a_1}\end{matrix}\right\}}$	Yousof et al. (2018)
1	1		RR	$exp(-\beta^2z^{-2})$	Trayer (1964)

Fig. 1 give some plots of the GOLL-RR PDF and HRF. From Fig. 1 (left panel) the new GOLL-RR PDF can be a unimodal-PDF with right skewed and symmetric shape. From Fig. 1 (left panel) we note that the new GOLL-RR HRF can be upside-down-HRF with different useful shapes.

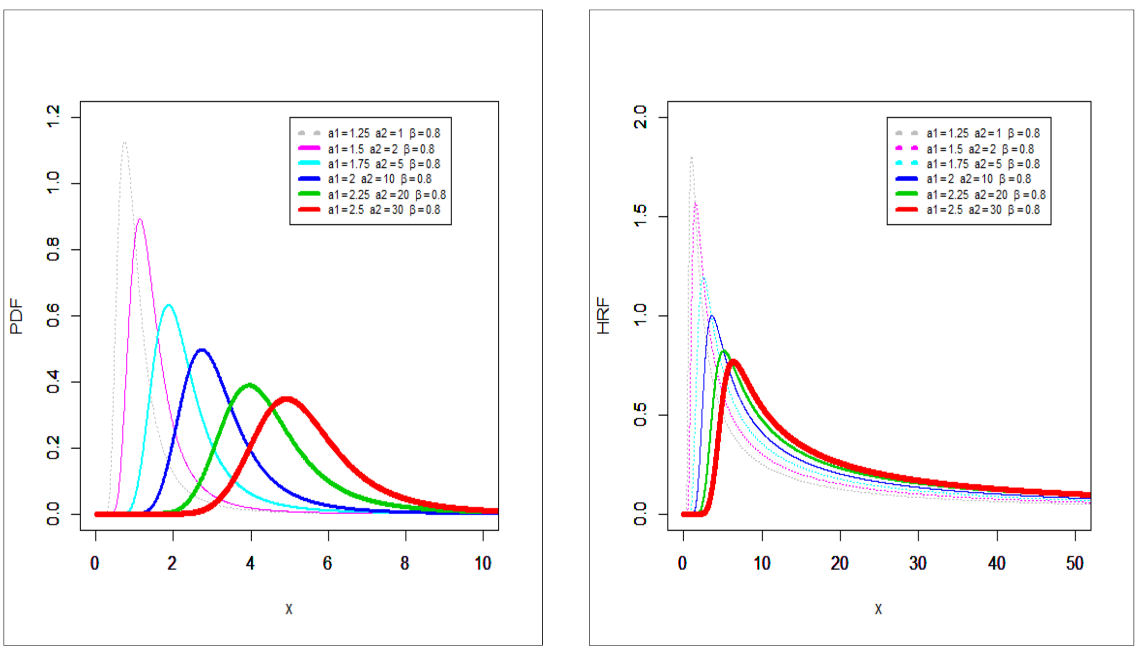

Figure 1: Plots of the GOLL-RR PDF and HRF.

The quantile function (QF) of z (by inverting (5)), say $z_{\mathrm{Q}} = Q(\mathrm{Q}) = F^{-1}(\mathrm{Q})$, as

$$z_Q = \beta / \left(- ln \left\{ \left[\frac{\left(\frac{\mathrm{Q}}{1-\mathrm{Q}}\right)^{\frac{1}{a_1}}}{1+\left(\frac{\mathrm{Q}}{1-\mathrm{Q}}\right)^{\frac{1}{a_1}}} \right]^{\frac{1}{a_2}} \right\} \right)^{\frac{1}{2}}. \quad (7)$$

2. Simple type copula

2.1 Via Morgenstern family

The CDF of the Morgenstern family

$$F_\lambda(z_1, z_2)|_{(|\lambda|\le 1)} = F_{a_1,a_2,\beta_1}(z_1) F_{b_1,b_2,\beta_2}(z_2) \times \left\{1 + \lambda\left[1 - F_{a_1,a_2,\beta_1}(z_1)\right]\left[1 - F_{b_1,b_2,\beta_2}(z_2)\right]\right\},$$

where

$$1 - F_{a_1,a_2,\beta_1}(z_1) = 1 - \frac{exp(-a_1 a_2 \beta_1^2 z_1^{-2})}{exp(-a_1 a_2 \beta_1^2 z_1^{-2}) + \{1 - exp(-a_2 \beta_1^2 z_1^{-2})\}^{a_1}},$$

and

$$1 - F_{b_1,b_2,\beta_2}(z_2) = 1 - \frac{exp(-b_1 b_2 \beta_2^2 z_1^{-2})}{exp(-b_1 b_2 \beta_1^2 z_2^{-2}) + \{1 - exp(-b_2 \beta_1^2 z_2^{-2})\}^{b_1}},$$

then we have a 7-dimension parameter model as

$$F_\lambda(z_1, z_2)|_{(|\lambda|\le 1)} = \frac{exp(-a_1 a_2 \beta_1^2 z_1^{-2})}{exp(-a_1 a_2 \beta_1^2 z_1^{-2}) + \{1 - exp(-a_2 \beta_1^2 z_1^{-2})\}^{a_1}} \times \frac{exp(-b_1 b_2 \beta_2^2 z_1^{-2})}{exp(-b_1 b_2 \beta_1^2 z_2^{-2}) + \{1 - exp(-b_2 \beta_1^2 z_2^{-2})\}^{b_1}} \times \left\{ \begin{array}{c} 1 + \lambda \left[1 - \frac{exp(-a_1 a_2 \beta_1^2 z_1^{-2})}{exp(-a_1 a_2 \beta_1^2 z_1^{-2}) + \{1 - exp(-a_2 \beta_1^2 z_1^{-2})\}^{a_1}}\right] \\ \left[1 - \frac{exp(-b_1 b_2 \beta_2^2 z_1^{-2})}{exp(-b_1 b_2 \beta_1^2 z_2^{-2}) + \{1 - exp(-b_2 \beta_1^2 z_2^{-2})\}^{b_1}}\right] \end{array} \right\}.$$

2.2 Via clayton copula

The bivariate extension

Let

$$H(c_1, c_2)|(\lambda_1 + \lambda_2) \ge 0 = \left[c_1^{-(\lambda_1+\lambda_2)} + c_2^{-(\lambda_1+\lambda_2)} - 1\right]^{-\frac{1}{\lambda_1+\lambda_2}}.$$

Assume that $X \sim$ GOLL-RR (a_1, a_2, β_1) and $Y \sim$ GOLL-RR (b_1, b_2, β_2). Then, setting

$$c_1 = c_1(a_1, a_2, \beta_1) = \frac{exp(-a_1 a_2 \beta_1^2 x^{-2})}{exp(-a_1 a_2 \beta_1^2 x^{-2}) + \{1 - exp(-a_2 \beta_1^2 x^{-2})\}^{a_1}}$$

and

$$c_2 = c_2(b_1, b_2, \beta_2) = \frac{exp(-b_1 b_2 \beta_2^2 y^{-2})}{exp(-b_1 b_2 \beta_1^2 y^{-2}) + \{1 - exp(-b_2 \beta_1^2 y^{-2})\}^{b_1}},$$

the associated CDF bivariate GOLL-RR type distribution will be

$$H(x,y)=\left(\begin{array}{c}\left[\frac{exp(-a_1a_2\beta_1^2x^{-2})}{exp(-a_1a_2\beta_1^2x^{-2})+\{1-exp(-a_2\beta_1^2x^{-2})\}^{a_1}}\right]^{-(\lambda_1+\lambda_2)}\\+\left[\frac{exp(-b_1b_2\beta_2^2y^{-2})}{exp(-b_1b_2\beta_1^2y^{-2})+\{1-exp(-b_2\beta_1^2y^{-2})\}^{b_1}}\right]^{-(\lambda_1+\lambda_2)}\\-1\end{array}\right)^{-\frac{1}{\lambda_1+\lambda_2}}.$$

The Multivariate extension

The m-dimensional type extension can be derived as

$$H(x_1,x_2,\cdots,x_m)=\left[\begin{array}{c}\sum_{i=1}^{m}\left(\frac{exp(-a_ia_i\beta_i^2z_i^{-2})}{\left\{\begin{array}{c}exp(-a_ia_i\beta_i^2z_i^{-2})\\+[1-exp(-a_i\beta_i^2z_i^{-2})]^{a_i}\end{array}\right\}}\right)^{-(\lambda_1+\lambda_2)}\\+1\\-m\end{array}\right]^{-1/(\lambda_1+\lambda_2)}.$$

3. Properties

3.1 Asymptotics

Let

$$\tau=inf\left\{z|_{G(z;\underline{\Psi})>0}\right\},$$

then

$$F_{a_1,a_2,\beta}(z)\sim exp(-a_1a_2\beta^2z^{-2})\ |_{z\to\tau},$$

$$f_{a_1,a_2,\beta}(z)\sim 2a_1a_2\beta^2z^{-3}\,exp(-a_1a_2\beta^2z^{-2})\ |_{z\to\tau}$$
$$h_{a_1,a_2,\beta}(z)\sim 2a_1a_2\beta^2z^{-3}\,exp(-a_1a_2\beta^2z^{-2})\ |_{z\to\tau}.$$
$$1-F_{a_1,a_2,\beta}(z)\sim(a_2\{1-exp(-\beta^2z^{-2})\})^{a_1}|_{z\to\infty},$$

$$f_{a_1,a_2,\beta}(z)\sim 2a_1a_2^{a_1}\beta^2z^{-3}\,exp(-\beta^2z^{-2})\,\{1-exp(-\beta^2z^{-2})\}^{a_1-1}|_{z\to\infty}$$

and

$$h_{a_1,a_2,\beta}(z)\sim\frac{2a_1\beta^2z^{-3}\,exp(-\beta^2z^{-2})}{1-exp(-\beta^2z^{-2})}|_{z\to\infty}.$$

3.2 Representations

Based on Cordeiro et al. (2016), the PDF in (6) can be expressed as

$$f(z)=\sum_{\kappa=0}^{\infty}\ \Upsilon_\kappa\pi_{(1+\kappa)}(z;\beta),\qquad(8)$$

where

$$\Upsilon_\kappa=\frac{a_1a_2}{1+\kappa}\sum_{\zeta_1,\zeta_2=0}^{\infty}\ \sum_{\kappa=0}^{\zeta_3}(-1)^{\zeta_2+\kappa+\zeta_3}\binom{-2}{\zeta_1}$$
$$\times\binom{-a_1(\zeta_1+1)}{\zeta_2}\binom{a_1a_2(\zeta_1+1)+a_2\zeta_2-1}{\zeta_3}\binom{\zeta_3}{\kappa},$$

and $\pi_{(1+\kappa)}(z;\beta)$ is the PDF of the RR model with scale parameter $\beta\sqrt{(1+\kappa)}$ also the CDF of Z becomes

$$F(z)=\sum_{\kappa=0}^{\infty}\ \Upsilon_\kappa\Pi_{(1+\kappa)}(z;\beta),\qquad(9)$$

where $\Pi_{(1+\kappa)}(z;\beta)$ is the CDF of the RR distribution.

3.3 Moments

The r^{th} ordinary moment of Z is given by

$$\mu_r' = E(z^r) = \int_{-\infty}^{\infty} z^r f(z)dz,$$

then we obtain

$$\mu_r' = \sum_{\kappa=0}^{\infty} \Upsilon_\kappa \beta^r (1+\kappa)^{\frac{r}{2}} \Gamma\left(1-\frac{r}{2}\right), \forall\, 2 > r, \tag{10}$$

where

$$\Gamma(1+\omega_1)|_{(\omega_1 \in R^+)} = \omega_1! = \prod_{h=0}^{\omega_1-1}(\omega_1 - h).$$

The r^{th} incomplete moment (IM) can be obtained as

$$\vartheta_r(t) = \int_{-\infty}^{t} z^r f(z)dz = \sum_{\kappa=0}^{\infty} \Upsilon_\kappa \int_{-\infty}^{t} z^r \pi_{(1+\kappa)}(z;\beta)dz,$$

then

$$\vartheta_r(t) = \sum_{\kappa=0}^{\infty} \Upsilon_\kappa \beta^r [(1+\kappa)]^{\frac{r}{2}} \gamma\left(1-\frac{r}{2}, [(1+\kappa)]\left(\frac{\beta}{t}\right)^2\right), \forall\, 2 > r, \tag{11}$$

where $\gamma(\omega_1,\omega_2)$ is the incomplete gamma function

$$\gamma(\omega_1,\omega_2)|_{(\omega_1 \neq 0,-1,-2,\ldots)} = \int_0^{\omega_2} t^{\omega_1 - 1} exp(-t)\, dt$$
$$= \frac{\omega_2^{\omega_1}}{\omega_1}\{1F_1[\omega_1; \omega_1+1; -\omega_2]\}$$
$$= \sum_{\kappa=0}^{\infty} \frac{(-1)^\kappa}{\kappa!\,(\omega_1+\kappa)} \omega_2^{\omega_1+\kappa},$$

and $1F_1[\cdot,\cdot,\cdot]$ is a confluent hypergeometric function (CHF). The first IM given by (11) with $r = 1$ as

$$\vartheta_1(t) = \sum_{\kappa=0}^{\infty} \Upsilon_\kappa\, \beta\sqrt{(1+\kappa)}\,\Gamma\left(1-\frac{1}{2}, (1+\kappa)\left(\frac{\beta}{t}\right)^2\right).$$

The MGF $M_z(t) = E(e^{tz})$ of z can be derived from equation (8) as

$$M_z(t) = \sum_{\kappa=0}^{\infty} \sum_{r=0}^{\infty} (t^r \Upsilon_\kappa / r!)\, \beta^r [(1+\kappa)]^{\frac{r}{2}} \Gamma\left(1-\frac{r}{2}\right), \forall\, 2 > r.$$

The Lorenz (L) and Bonferroni (B) curves are defined by

$$\text{L} = \frac{\vartheta_1(t)}{\mu_1'}$$

and

$$\text{B} = \frac{\vartheta_1(t)}{\mu_1' F(z)} = \frac{\text{L}}{F(z)},$$

respectively. Then, for the new RR we have

$$\text{L} = \frac{\sum_{\kappa=0}^{\infty} \Upsilon_\kappa\, \beta\sqrt{(1+\kappa)}\,\Gamma\left(1-\frac{1}{2}, (1+\kappa)\left(\frac{\beta}{t}\right)^2\right)}{\sum_{\kappa=0}^{\infty} \Upsilon_\kappa\, \beta\sqrt{(1+\kappa)}\,\Gamma\left(1-\frac{1}{2}\right)},$$

and

$$\mathrm{B}=\frac{\sum_{\kappa=0}^{\infty}\Upsilon_{\kappa}\ \beta\sqrt{(1+\kappa)}\Gamma\left(1-\frac{1}{2},(1+\kappa)\left(\frac{\beta}{t}\right)^{2}\right)}{\sum_{\kappa=0}^{\infty}\Upsilon_{\kappa}\ \beta\sqrt{(1+\kappa)}\Gamma\left(1-\frac{1}{2}\right)} \times\{1+[1-exp(-a_2\beta^2z^{-2})]^{a_1}\}.$$

3.4 Residual life and reversed residuals and their moments

The n^{th} moment of the residuals

$$A_{n,z}(t)=E\left[(z-t)^n\ |_{(z>t,n=1,2,\dots)}\right].$$

Then, the n^{th} moment is given by

$$A_{n,z}(t)=\frac{\int_t^{\infty}\ (z-t)^n dF(z)}{1-F(t)}.$$

Therefore,

$$A_{n,z}(t)=\frac{\beta^n}{1-F(t)}\sum_{\kappa=0}^{\infty}\ \zeta_{\kappa}[(1+\kappa)]^{\frac{n}{2}}\Gamma\left(1-\frac{n}{2},[(1+\kappa)]\left(\frac{\beta}{t}\right)^{2}\right),\forall\ 2>n,$$

where

$$\zeta_{\kappa}=\Upsilon_{\kappa}\sum_{r=0}^{n}(-t)^r\binom{n}{r},$$

$$\Gamma(\omega_1,\omega_2)|_{z>0}=\int_{\omega_2}^{\infty}\ t^{\omega_1-1}\,exp(-t)\,dt,$$

and

$$\Gamma(\omega_1,\omega_2)+\gamma(\omega_1,\omega_2)=\Gamma(\omega_1).$$

The n^{th} moment of the reversed residuals

$$B_{n,z}(t)=E\left[(t-z)^n\ |_{(z\le t,t>0\text{ and }n=1,2,\dots)}\right].$$

Then

$$B_{n,z}(t)=\frac{\int_0^{t}\ (t-z)^n dF(z)}{F(t)}.$$

Then, the n^{th} moment becomes

$$B_{n,z}(t)=\frac{\beta^n}{F(t)}\sum_{\kappa=0}^{\infty}\ \eta_{\kappa}[(1+\kappa)]^{\frac{n}{2}}\gamma\left(1-\frac{n}{2},(1+\kappa)\left(\frac{\beta}{t}\right)^{2}\right),\forall\ 2>n,$$

where

$$\eta_{\kappa}=\Upsilon_{\kappa}\sum_{r=0}^{n}(-1)^r\binom{n}{r}t^{n-r}.$$

4. Maximum likelihood estimation (MLE)

The log-likelihood function

$$\ell=\ell(\Theta)=n\,log(2a_1a_2\beta^2)-(2+1)\sum_{i=1}^{n}log(z_i)-a_1a_2\beta^2\sum_{i=1}^{n}\ z_i^{-2}$$
$$+2\sum_{i=1}^{n}\ log\{exp(-a_1a_2\beta^2z_i^{-2})+[1-exp(-a_2\beta^2z_i^{-2})]^{a_1}\}$$
$$+(a_1-1)\sum_{i=1}^{n}\ log[1-exp(-a_2\beta^2z_i^{-2})].$$

The components of the “score vector” is available if needed.

5. Simulation studies

We can perform the simulation experiments to assess of the finite sample behavior of the MLEs based on the following algorithm:

1.Use (7) to generate 1000 samples of size n from the GOLL-RR model

2.Compute the MLEs for the 1000 samples.

3.Compute the SEs of the MLEs for the 1000 samples.

4.The standard errors (SEs) were computed by inverting the observed information matrix.

5.Compute the biases and MSEs for $h = a_1, a_2, \beta$.

We repeated these steps for $n = 50,100, \dots ,200$ with $a_1 = 1$, $a_2 = 1$, $\beta = 1$, so computing biases $(\mathrm{Bias}_h(n))$, mean squared errors $(MSE_h(n))$ for a_1, a_2, β and $n = 50,100, \dots ,200$. Fig.s 2, 3 and 4 give the biases and MSEs for a_1, a_2 and β $\forall$ $n = 50,100, \dots ,200$ for the GOLL-RR model. Fig.s 2, 3 and 4 shows how the biases and MSEs vary with respect to n . The broken line in Fig. 4 corresponds to the biases being 0 . From Fig. 2, 3 and 4, the biases decrease to zero as $n \to \infty$, the MSEs for each parameter decrease to zero as $n \to \infty$.

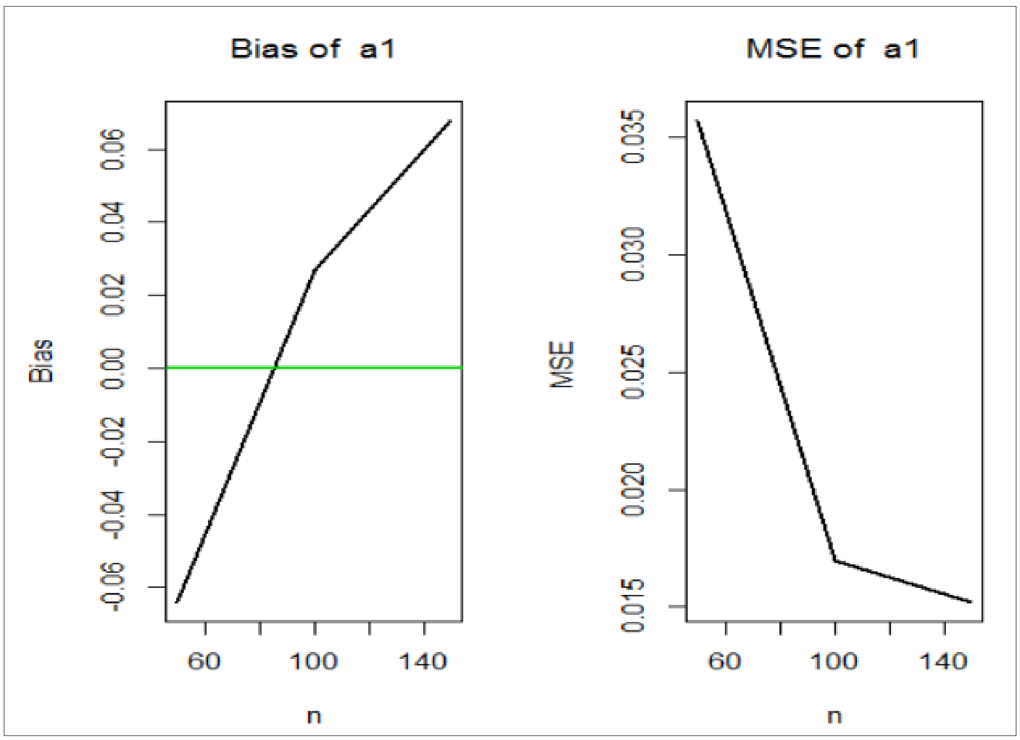

Fig. 2: Biases and MSEs for a_1 for the GOLL-RR model.

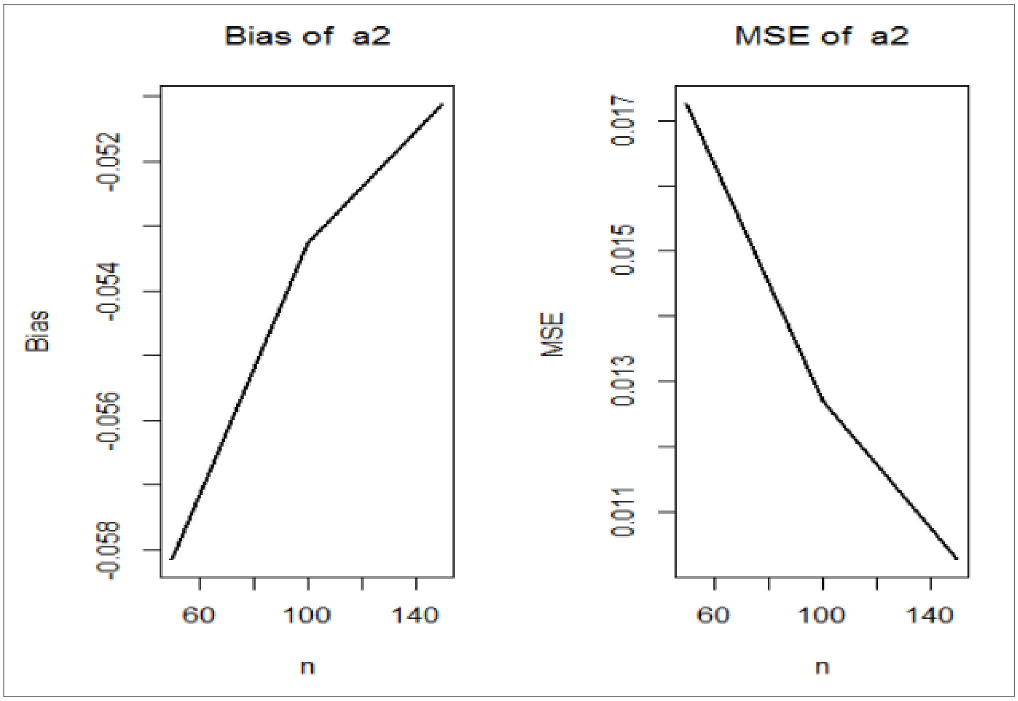

Fig. 3: Biases and MSEs for a_2 for the GOLL-RR model.

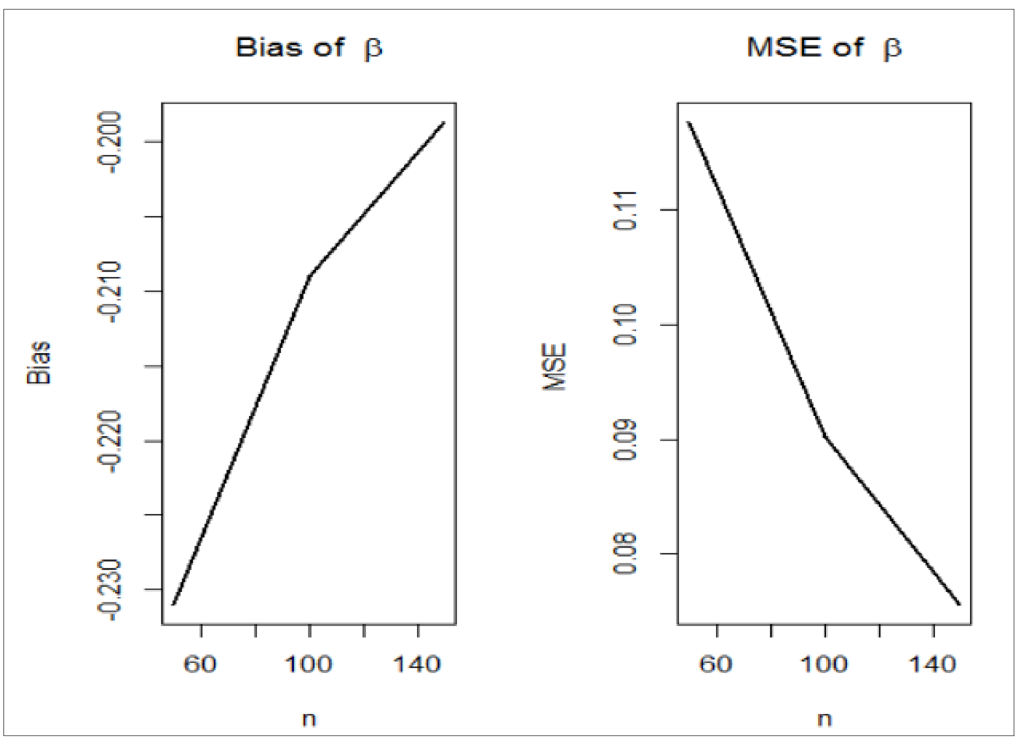

Fig. 4: Biases and MSEs for β for the GOLL-RR model.

6. Real data modeling

We consider the Cramér-Von Mises (C-V-M) and the Anderson-Darling (A-D) and the Kolmogorov-Smirnov (KS) statistic.

We compare the fits of the GOLL-RR distribution with other models such as RW, Kumaraswamy-RW (KFr), exponentiated RW (E-RW), beta-RW (B-RW), transmuted-RW (T-RW), Marshal-Olkin-RW (MO-RW) and McDonald-RW (Mc-RW) distributions given by:

E-RW:

$$f(z) = a_1 b\beta^b z^{-(b+1)}\, exp[\,-(\beta/z)^b]\,\{1 - exp[\,-(\beta/z)^b]\}^{a_1-1};$$

B-RW:

$$f(z) = b\beta^b B^{-1}(a_1, c) z^{-(b+1)}\, exp[\,-a_1(\beta/z)^b]\,\{1 - exp[\,-(\beta/z)^b]\}^{c-1};$$

K-RW:

$$f(z) = a_1 c b\beta^b z^{-(b+1)}\, exp[\,-a_1(\beta/z)^b]\,\{1 - exp[\,-a_1(\beta/z)^b]\}^{c-1};$$

T-RW:

$$f(z) = b\beta^b z^{-(b+1)}\, exp[\,-(\beta/z)^b]\,\{1 + a_1 - 2a_1\, exp[\,-(\beta/z)^b]\};$$

MO-RW:

$$f(z) = a_1 b\beta^b z^{-(b+1)}\, exp[\,-(\beta/z)^b]\,\{a_1 + (1 - a_1)\, exp[\,-(\beta/z)^b]\}^{-2};$$

Mc-RW:

$$f(z) = a_2 b\beta^b z^{-(b+1)} B^{-1}(a_1, c)\, exp[\,-(\beta/z)^b]$$
$$\times \{exp[\,-(\beta/z)^b]\}^{a_1 a_2 - 1}\{1 - exp[\,-a_2(\beta/z)^b]\}^{c-1}$$

OLLE-RW:

$$f(z) = a_2 \beta b a^b z^{-(b+1)}\, exp[-\beta(a/z)^b]$$
$$\times (exp[-\beta(a/z)^b]\,\{1 - exp[-\beta(a/z)^b]\})^{-1+a_2}$$
$$\times (exp[-\beta a_2(a/z)^b] + \{1 - exp[-\beta(a/z)^b]\}^{a_2})^{-2},$$

The parameters of the above densities are all positive real numbers except for the T-RW distribution for which $|a_1| \leq 1$.

The 1st data (see Nichols and Padgett (2006)). Fig. 6 gives the total time test (TTT) plot (see Aarset (1987)) for data set I. It indicates that the empirical HRFs of data sets I is increasing HRF (IHRF). The 2nd data (see Smith and Naylor (1987)). Fig. 6 gives the TTT plot for data set II. It indicates that the empirical HRFs of data sets II is IHRF. The 3rd data set (wingo data). Fig. 7 gives the TTT plot for data set III. It indicates that the empirical HRFs of data sets III is increasing.

Many other useful real data sets can be found in Aryal and Yousof (2017), Merovci et al. (2017), Korkmaz et al. (2017), Hamedani et al. (2017), Alizadeh et al. (2017), Brito et al. (2017), Alizadeh et al. (2018), Korkmaz et al. (2018), Yousof et al. (2018a-d), Hamedani et al. (2018), Cordeiro et al. (2018), Hamedani et al. (2019), Ibrahim (2019), Nascimento et al. (2019), Ibrahim et al. (2019), Goual and Yousof (2019), Korkmaz et al. (2019),Alizadeh et al. (2019), Yousof et al. (2019) and Goual et al. (2020), Ibrahim (2020a,b), Yadav et al. (2020) and Ibrahim et al. (2020).

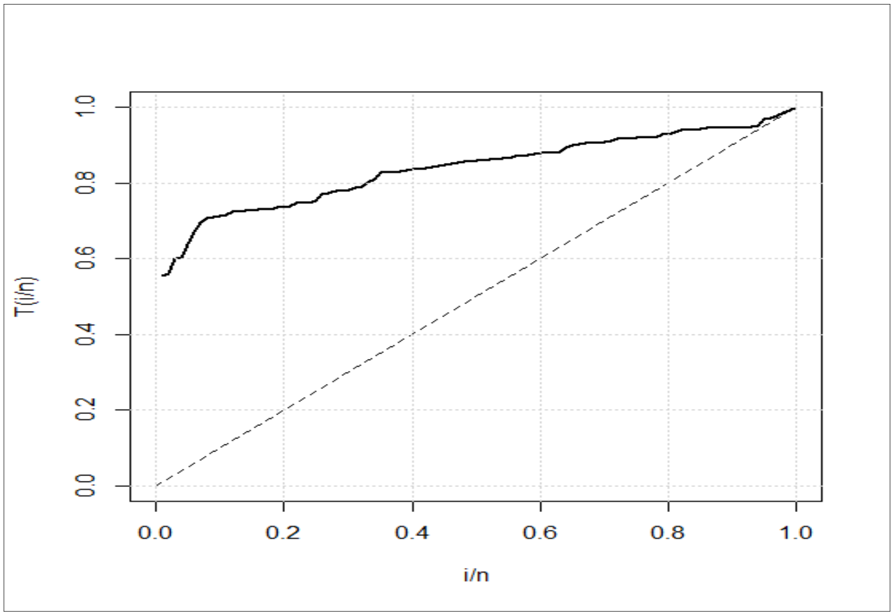

Fig. 5: TTT plot for data set I.

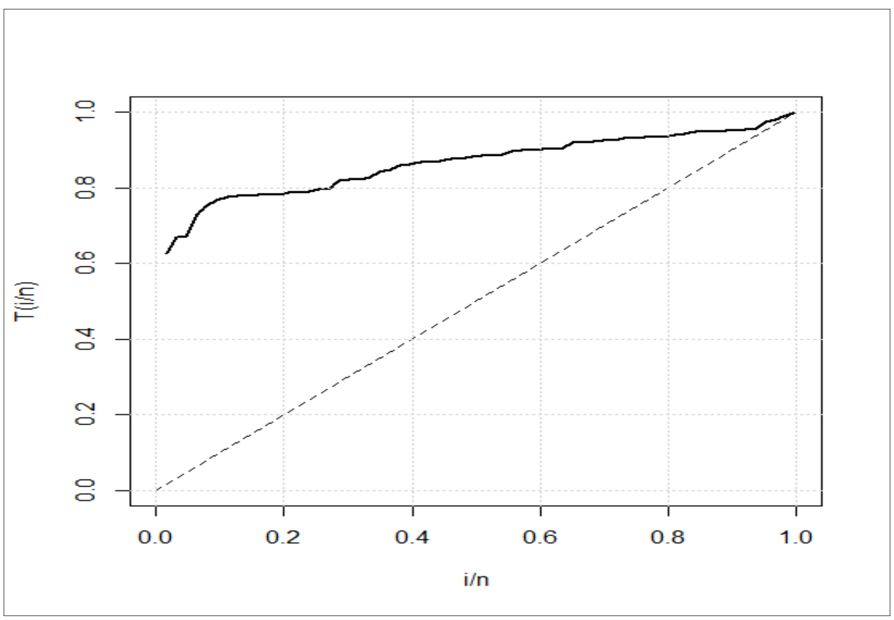

Fig. 6: TTT plot for data set II.

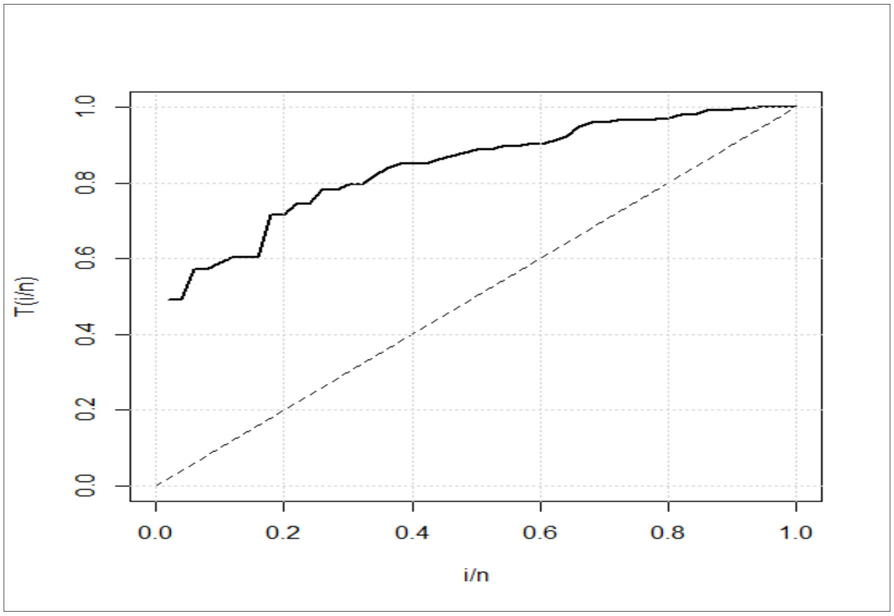

Fig. 7: TTT plot for data set III.

The statistics are presented in Tables 2, 4 and 6 for data sets I-III respectively. The MLEs and corresponding standard errors (SEs) are given in Tables 3, 5 and 7 for data sets I-III respectively. Fig. 5, 6 and 7 gives the estimated density (E-PDF), estimated CDF (E-CDF), P-P plot (P-P) and estimated HRF (E-HRF) for data set I-III respectively. The GOLL-RR distribution in Tables 2, 4 and 6 give the best results.

Table 2

Model	Goodness of fit criteria			
	C-V-M	A-D	K-S	P-value
GOLL-RR	0.0592	0.4845	0.05775	0.8926
OLLE-RW	0.1203	0.9639	0.5561	$2.2\times e^{-16}$
OLLE-RR	0.1553	1.2119	0.6550	$2.2\times e^{-16}$
OLL-RR	0.1553	1.2120	0.6550	$2.2\times e^{-16}$
RW	0.1090	0.7657	0.0874	0.4282
K-RW	0.0812	0.6217	0.0759	0.6118
E-RW	0.1091	0.7658	0.0874	0.4287
BRW	0.0809	0.6207	0.0757	0.6147
T-RW	0.0871	0.6209	0.0782	0.5734
MO-RW	0.0886	0.6142	0.0763	0.5168
Mc-RW	0.1333	1.0608	0.0807	0.5332

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Table 3

Model	Estimates $\widehat{a_1}$	$\widehat{a_2}$	$\hat{c}$	$\hat{\beta}$	$\hat{b}$
GOLL-RR	2.4703	1.6377		1.0039	
	(0.211)	(0.000)		(0.000)	
OLLE-RW	0.1351		3.7216	0.9296	21.319
	(0.011)		(0.0034)	(0.0033)	(0.0034)
OLLE-RR	0.4946		0.067	1.74262	
	(0.0414)		(0.7195)	(9.301)	
OLL-RR	0.49459			0.4524	
	0.04135			0.03869	
RW				1.3968	4.3724
				(0.0336)	(0.3278)
K-RW		0.8489	1.6239	1.6341	3.4208
		(16.083)	(0.6979)	(9.049)	(0.7635)
E-RW		0.9395		1.4169	0.9395
		(3.543)		(2.568)	(0.3278)
B-RW		0.7346	1.5830	1.6684	3.5112
		(1.5290)	(0.7132)	(0.7662)	(0.9683)
T-RW	-0.7166			1.2656	4.7121
	(0.2616)			(0.0579)	(0.3657)
MO-RW		0.0033		6.2296	1.2419
		(0.0009)		(1.0134)	(0.1181)
Mc-RW	0.8503	44.423	19.859	0.0203	46.974
	(0.1353)	(25.100)	(6.706)	(0.0060)	(21.871)

Table 4

Model	Goodness of fit criteria C-V-M	A-D	K-S	P-value
GOLL-RR	0.0525	0.4530	0.0703	0.8926
OLLE-RW	0.1049	0.8325	0.5520	$6.7\times e^{-16}$
OLLE-RR	0.1502	1.1469	0.6795	$6.7\times e^{-16}$
OLL-RR	0.1502	1.1469	0.6795	$6.7\times e^{-16}$
RW	0.0707	0.5332	0.0772	0.8185
K-RW	0.0634	0.4981	0.0715	0.8810
E-RW	0.0707	0.5332	0.0772	0.8187
B-RW	0.0640	0.5008	0.0716	0.8804
T-RW	0.0655	0.4939	0.0735	0.8470
MO-RW	0.0629	0.4902	0.0813	0.7685
Mc-RW	0.1161	0.9193	0.0831	0.7455

Table 5

Model	Estimates $\widehat{a_1}$	$\widehat{a_2}$	$\hat{c}$	$\hat{\beta}$	$\hat{b}$
GOLL-RR	3.032	2.1568		0.86599	
	(0.325)	(9.145)		(1.836)	
OLLE-RW	0.1449		0.00879	1.2997	24.878
	(0.0129)		(0.000)	(0.000)	(0.000)
OLLE-RR	0.5025		0.0716	1.7048	
	(0.0529)		(1.13062)	(13.47)	
OLL-RR	0.50251			0.45599	
	0.05295			0.048652	
RW				1.4108	5.4377
				(0.0344)	(0.5192)
K-RW		0.2855	1.2824	1.9142	4.7731
		(9.1338)	(0.6388)	(12.836)	(1.3134)
E-RW		0.9059		1.4367	5.4379
		(2.764)		(4.324)	(0.5193)
BRW		1.2996	1.2649	1.3945	4.7927
		(4.4378)	(0.6640)	(0.9304)	(1.4641)
T-RW	0.7778			1.5491	4.3139
	(0.2477)			(0.0655)	(0.5849)
MO-RW		0.0023		5.2383	1.4537
		(0.0004)		(0.8209)	(0.1650)
Mc-RW		56.227	14.953	0.0073	29.104
		(30.539)	(4.733)	(0.0013)	(11.304)

Table 6

Model	Goodness of fit criteria C-V-M	A-D	K-S	P-value
GOLL-RR	0.19551	1.3498	0.11008	0.5797
RW	0.3233	2.0301	0.1506	0.2066
E-RW	0.3233	2.0301	0.1506	0.2064
T-RW	0.2823	1.8152	0.1370	0.3045

Table 7

Model	Estimates $\widehat{a_1}$	$\widehat{a_2}$	$\hat{c}$	$\hat{\beta}$	$\hat{b}$
GOLL-RR	1.961	0.111		1.4123	
	(0.234)	(0.000)		(0.000)	
RW				0.4859	3.2078
				(0.023)	(0.3263)
E-RW			0.9047	0.5013	3.2077
			(18.784)	(3.244)	(0.3263)
T-RW	-0.5816			0.4400	3.4974
	(0.2787)			(0.0290)	(0.3527)

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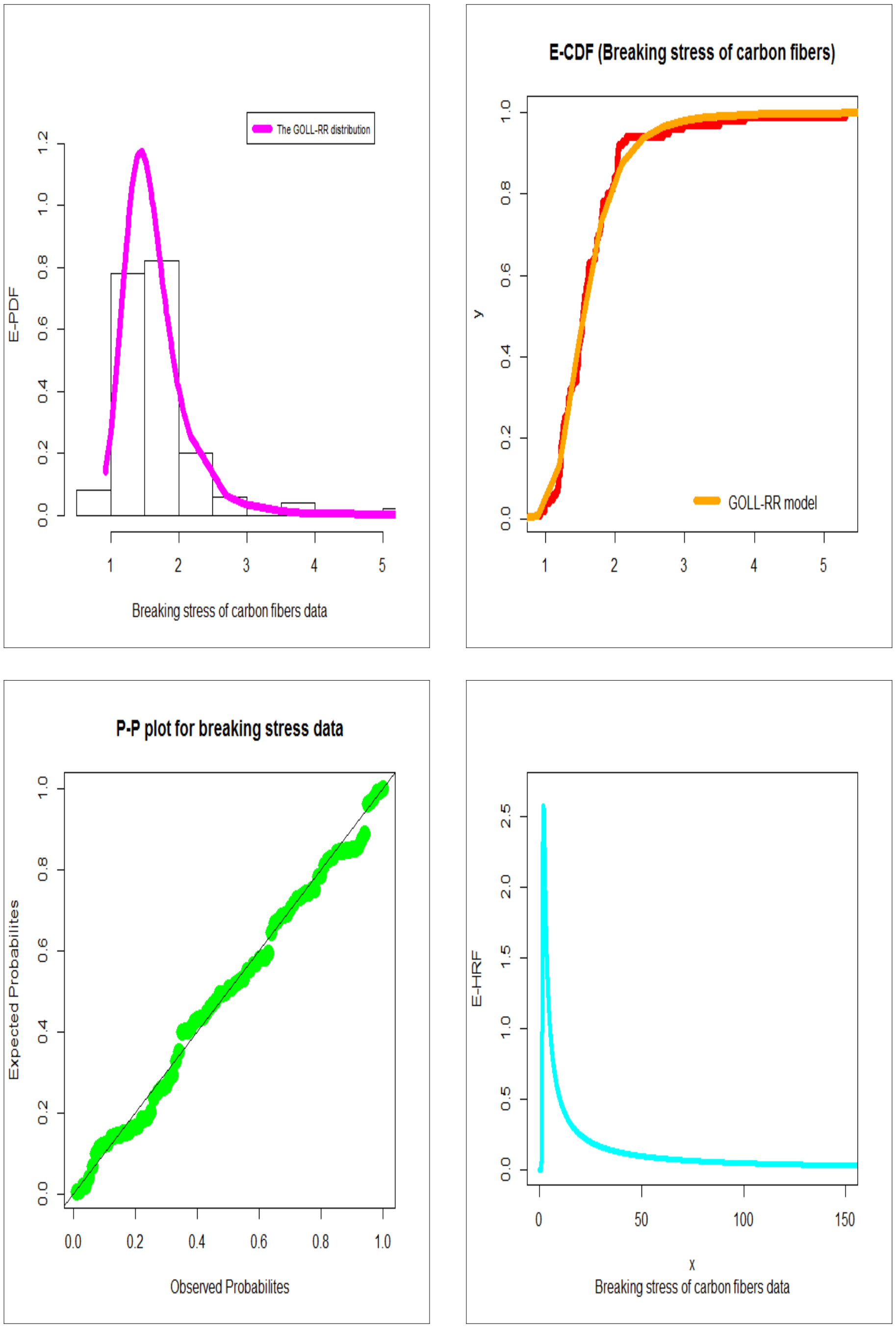

Fig. 5: E-PDF, E-CDF, P-P plot and E-HRF for data set I.

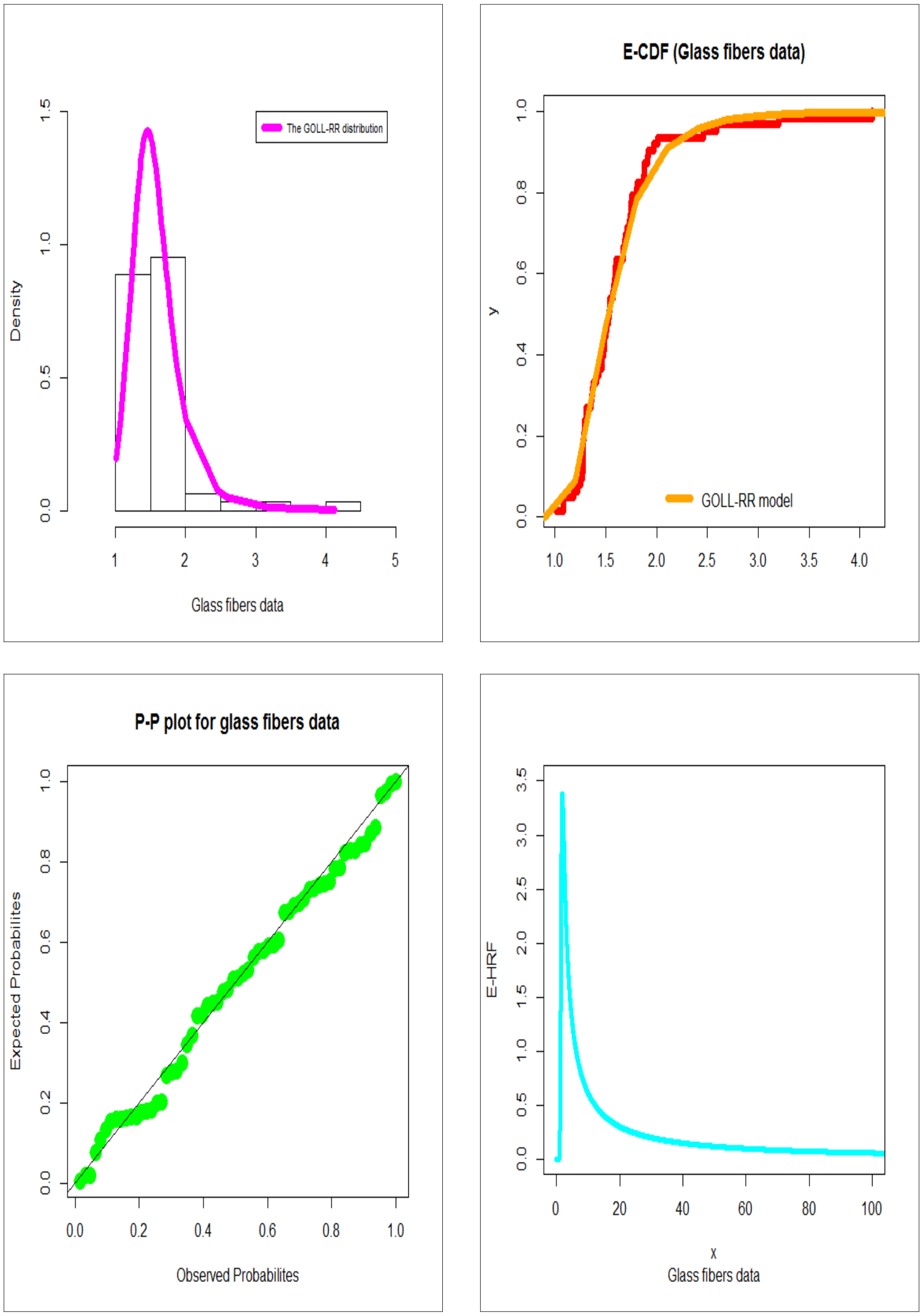

Fig. 6: E-PDF, E-CDF, P-P plot and E-HRF for data set II.

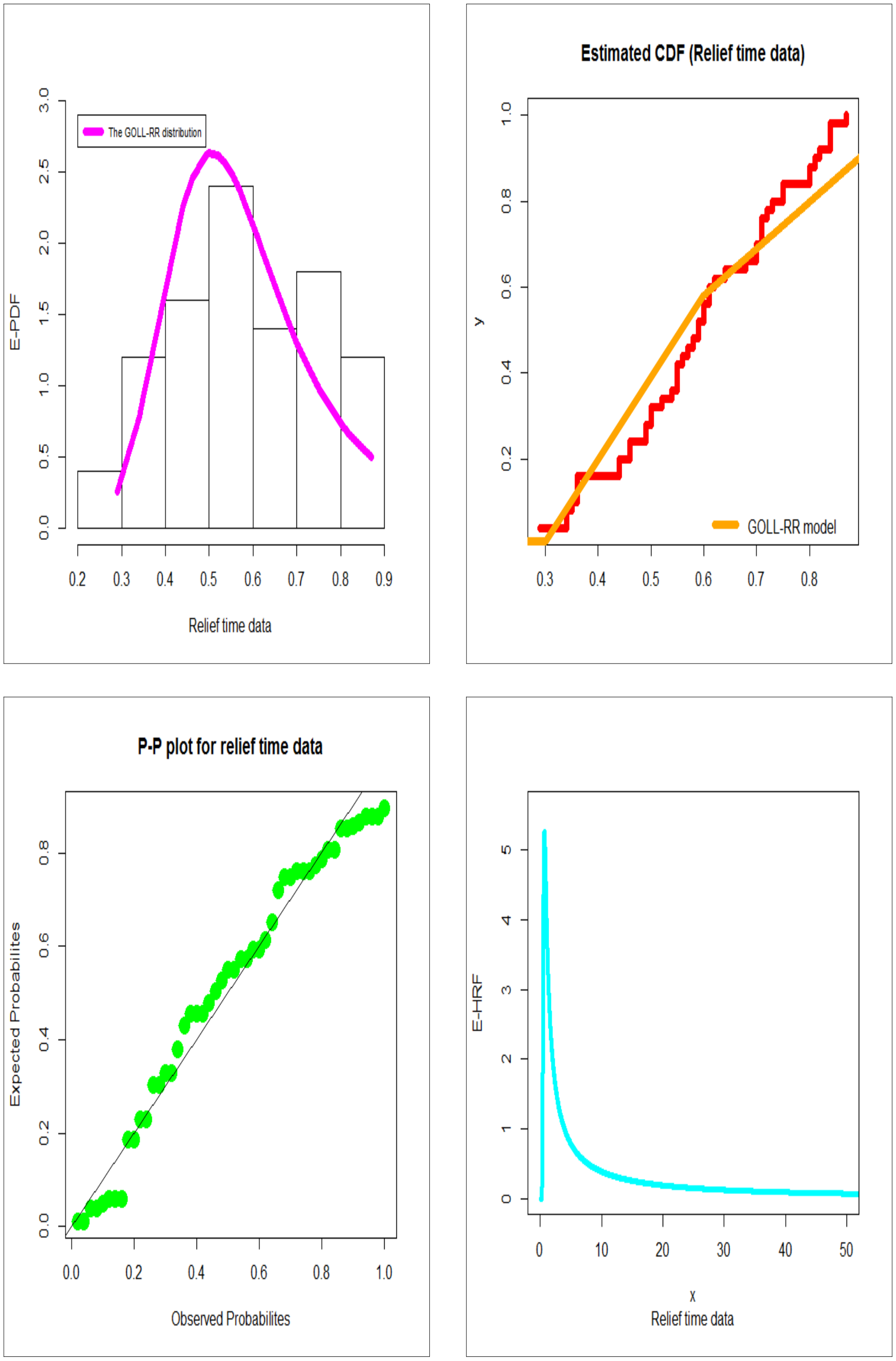

Fig. 7: E-PDF, E-CDF, P-P plot and E-HRF for data set III

7. Concluding remarks

A new extension of the reciprocal Rayleigh distribution is introduced. Simple type copula-based construction is presented for deriving and many multivariate and bivariate type reciprocal Rayleigh models. The new PDF can is expressed as a "double linear mixture" of the reciprocal Rayleigh PDF. The new reciprocal Rayleigh model generalizes another three reciprocal Rayleigh distributions. The performance of the estimation method is assessed using a graphical simulation.

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