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Two-Echelon Supply Chain Model with Demand Dependent On Price, Promotional Effort and Service Level in Crisp and Fuzzy Environments

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Abstract

This article explores a supply chain model consisting of a single manufacturer and two competing retailers in the same market. The manufacturer, as a Stackelberg leader specifies a wholesale price and bears servicing costs of the products. Then, both the retailers advertise the products and sell them to the customers. So, the demand of the products is influenced by selling price, service level and also promotional effort. On the basis of this gaming structure, two mathematical models have been formed - crisp model, where each member of the chain exactly knows all the cost parameters and fuzzy model where those cost parameters are considered as fuzzy numbers. Optimal strategies for the manufacturer and the retailers are determined and some numerical examples have been given. Finally, how little perturbations of parameters affect the profits of the chain members have been determined. The fuzzy model is specially applicable in practical situations where different costs can not be known precisely.

Key Words: Supply chain management; price, service and promotional effort competition; Stackelberg game; retailer advertisement; manufacturer service.

1. Background

Generally consumers buy products on the basis of quality, cost, range of products, availability, after-sale service (12). It is commonly known that more aggressively priced products catch customers' eyes more quickly (7). The manufacturers provide after-sale services to some electronic products like mobile phones, computers etc. Either to prevent their products from becoming obsolete in a few years, by providing software support or by providing free hardware support for the products that are in warranty period. All these types of services are costly for the manufacturer. Again, the manufacturer may choose to provide different amount of service for differently priced products.

Advertisements are also very common tools used by retailers to let consumers know about their products. To advertise, retailers have to invest money. Nowadays, advertisements can be made on newspapers, magazines, television channels, radio, social media etc (6, 12). The main objectives of the advertisements are to provide some useful information about the products and to persuade consumers to purchase the products. There have been a lot of studies for Stackelberg game problems in the supply chain. Han et al. studied a Stackelberg game model with one manufacturer and two competing retailers. In the paper, the manufacturer was the Stackelberg leader and retailers were the followers. The demand function was considered as a function of only price and service level. This paper also explored the equilibrium decisions of the chain members. Choi studied a Stackelberg game model with two competing manufacturers and a common retailer that sold both manufacturers' products. In the study by Choi the retailer was the leader and the manufacturers were the followers. This paper also explored three non-cooperative games of different power structures between the two manufacturers and the retailer. Pal et al. considered a two-echelon supply chain model where there were one manufacturer and one retailer with customer demand dependent on price, promotional effort and product quality. This paper explored the possibility of manufacturer, producing defective items and selling them at a reduced cost to the retailer. Also in the paper, the customers were provided a warranty period in which faulty products were repaired for free. Tsay and Agrawal explored a supply chain model with two competitive retailers and one manufacturer in which both the retailers offered products as well as service to customers. He et al. explored coordination in a two-level supply chain with one manufacturer and two competing retailers under advertising dependent demand. Cárdenas-Barrón and Sana considered a multi-item inventory model with demand dependent on promotional effort. Das and Islam considered a single manufacturer and single retailer system where the demand is dependent on inventory lot-size and production cost.

What we have learnt from the above articles is that no article considered the case where the demand is dependent on three things i.e. price, promotional effort, service level. Also, For real world problems, production costs, wholesale prices may not be exact. So, our paper also considers the case, where it has been assumed the above mentioned cost parameters are fuzzy numbers.

Zadeh first introduced the concept of fuzzy set theory. Then Zimmermann applied the fuzzy set theory concept with some suitable membership functions to solve linear programming problem with several objective functions. Chen introduced function principal which could be used as arithmetic operations between generalized fuzzy numbers. Hsieh and Chen showed that arithmetic operations on fuzzy numbers presented by Chen, not only keep the type of membership function of the fuzzy numbers unchanged after arithmetic operations, but also these operations can bring down the troublesomeness and tediousness of arithmetical calculations. Islam and Roy used the fuzzy numbers in a multi-objective programming problem.

This paper is different from the above papers in the following ways. Firstly, this paper considers demand to be a function of selling price, promotional effort and also the service level. The above papers considered demand to be a function of a combination of any two of the above three at the most. Secondly, the proposed model has been formed in both crisp and fuzzy environments.

The rest of the paper has been organized in the following manner- Assumptions, notations and model formulations are given in section 2. Then some theorems, propositions in crisp environment have been developed in section 3. Then fuzzy model has been formed in section 4. In section 5, some numerical examples have been given. Finally, in section 6 there is a conclusion.

2. Mathematical model

2.1. Assumptions

- Here a two-echelon supply chain model containing one manufacturer and two competing retailers have been considered. The manufacturer is the Stackelberg leader and the two retailers are the followers. The manufacturer can see the decisions made by the retailers, but cannot control their actions.
- The manufacturer produces two products which may or may not differ in production cost and wholesale price. Each retailer buys one type of product before the selling season.
- Both the retailers use advertisements to attract customers.
- The manufacturer bears the production and servicing costs.
- For simplicity , i^{th} retailer buys the i^{th} product.
- All the chain members have the same and complete information while optimizing their profit functions.
- The manufacturer is totally aware of the market demand and produces the exact amount of the products. The retailers also order the exact amount needed to satisfy their respective customers.

2.2. Notations

(a) c_i : production cost for the i^{th} product

(b) f_i: manufacturer's wholesale price for the i^{th} product

(c) g_i: manufacturer's service level for the i^{th} product

(d) e_i : the selling price of the i^{th} retailer

(e) h_i : i^{th} retailer's promotional effort

(f) π_{r_i} : profit of the i^{th} retailer

(g) π_m :profit of the manufacturer

where i =1,2 and i^{th} retailer sells the i^{th} product.

2.3. Model formulations

Here demand functions are assumed to be linear functions of selling price, service level and promotional effort. These linear demand functions ensure equal competing powers in each retailers' hand. The retailer-1 and the retailer-2 face the following demands from the customers:

$$D_1 = d_1 - b_{\mathrm{e}}.e_1 + c_{\mathrm{e}}.e_2 + b_{\mathrm{h}}h_1 - c_{\mathrm{h}}h_2 + b_{\mathrm{g}}.g_1 - c_{\mathrm{g}}.g_2 \quad (1)$$
$$D_2 = d_2 - b_{\mathrm{e}}.e_2 + c_e.e_1 + b_{\mathrm{h}}.h_2 - c_{\mathrm{h}}.h_1 + b_{\mathrm{g}}.g_2 - c_{\mathrm{g}}.g_1 \quad (2)$$

where $d_i, b_{\mathrm{e}}, c_{\mathrm{e}}, b_{\mathrm{g}}, c_{\mathrm{g}}, b_{\mathrm{h}}, c_{\mathrm{h}} > 0$, for i=1,2 .
Here d_i indicates the primary demand for the i^{th} retailer when the selling price, promotional effort and service level of the product is zero. $b_{\mathrm{e}}, b_{\mathrm{g}}, b_{\mathrm{h}}$ give the responsiveness of market demand to i^{th} retailer's own price and promotional effort and service level by the manufacturer for the i^{th} product respectively. $c_{\mathrm{e}}, c_{\mathrm{g}}, c_{\mathrm{h}}$ measure the intensity of competition between the two retailers with regards to pricing, service level and promotional effort. In the paper, we assumed $c_{\mathrm{g}} < b_{\mathrm{g}}, c_{\mathrm{e}} < b_{\mathrm{e}}, c_{\mathrm{h}} < b_{\mathrm{h}}$ ensuring that the response functions are negatively sloped and Nash equilibrium exists. This is reasonable, as sales are relatively more sensitive to service level, selling price and promotional effort of the retailer's own products (6).

The manufacturer pays for the production cost and the servicing cost. The servicing cost is assumed to be $\frac{m.g_i^2}{2}$ (i=1,2), where its quadratic nature suggests diminishing returns on such expenditure. And m is the ultimate servicing cost Han et al.. For similar reason, we assumed the promotional cost for the i^{th} retailer is $\frac{n_i(h_i-1)^2}{2}$ (where $h_i > 1$ and $n_i > 0$, i=1,2)
Now, by Equation 1 and Equation 2, for this model, the profit functions are,

$$\pi_{r_1} = (e_1 - f_1)D_1 - \frac{n_1}{2}.(h_1 - 1)^2 \quad (3)$$
$$\pi_{r_2} = (e_2 - f_2)D_2 - \frac{n_2}{2}.(h_2 - 1)^2 \quad (4)$$
$$\pi_m = (f_1 - c_1)D_1 + (f_2 - c_2)D_2 - \frac{m}{2}.(g_1^2 + g_2^2) \quad (5)$$

So, for the retailer-1, the problem is
maximize π_{r_1}
For the retailer-2, the problem is
maximize π_{r_2}
For the manufacturer, the problem is
maximize π_m
From the above discussion, it is clear that, the i^{th} retailer maximizes his profit with respect to e_i and h_i and the manufacturer maximizes his profit with respect to g_1 and g_2 .

3. Crisp model

Firstly, we discuss this model in crisp environment, where we assume all the cost parameters like wholesale price and production cost are exactly known to every member of the chain.

3.1. Some Results

We have now characterized the profit functions with respect to the respective variables and found out the values of the variables in the respective chain members.

Theorem 3.1. *For given service levels g_1 and g_2 , the profit function $\pi_{r_i}(e_i, h_i)$ is concave with respect to selling price e_i and promotional effort h_i whenever $2.b_e.n_i - b_h^2 > 0$, for i=1,2*

Proof. For i=1,2 and j=3-i, from the i^{th} retailer's profit function (by Equation 3 and Equation 4) we have,

$$\begin{aligned}
\frac{\partial \pi_{r_i}}{\partial e_i} &= (d_i - b_{\mathrm{e}}.e_i + c_{\mathrm{e}}.e_j + b_{\mathrm{h}}.h_i - c_{\mathrm{h}}.h_j + b_{\mathrm{g}}.g_i - c_{\mathrm{g}}.g_j) + (e_i - f_i)(-b_e) \\
\frac{\partial \pi_{r_i}}{\partial h_i} &= (e_i - f_i)(b_h) - n_i(h_i - 1) \\
\frac{\partial^2 \pi_{r_i}}{\partial e_i^2} &= -2b_e \\
\frac{\partial^2 \pi_{r_i}}{\partial h_i^2} &= -n_i \\
\frac{\partial^2 \pi_{r_i}}{\partial e_i h_i} &= b_h = \frac{\partial^2 \pi_{r_i}}{\partial h_i e_i}
\end{aligned}$$

Now, the Hessian matrix, $\mathbb{H}_{\pi_{r_i}} = \begin{pmatrix} \frac{\partial^2 \pi_{r_i}}{\partial e_i^2} & \frac{\partial^2 \pi_{r_i}}{\partial e_i h_i} \\ \frac{\partial^2 \pi_{r_i}}{\partial h_i e_i} & \frac{\partial^2 \pi_{r_i}}{\partial h_i^2} \end{pmatrix} = \begin{pmatrix} -2b_e & b_h \\ b_h & -n_i \end{pmatrix}$

For $\pi_{r_i}(e_i, h_i)$ to be concave, $\mathbb{H}_{\pi_{r_i}}$ has to be negative definite.
$\because$ clearly $-2.b_e < 0$, and $\det \mathbb{H}_{\pi_{r_i}} = 2.b_e.n_i - b_h^2$
$\therefore$ $\mathbb{H}_{\pi_{r_i}}$ is negative definite if $\det \mathbb{H}_{\pi_{r_i}} > 0$
$\therefore 2.b_e.n_i - b_h^2 > 0$
Which proves the theorem. □

Proposition 3.1.1. *For given g_1 and g_2 and $2.b_e.n_i - b_h^2 > 0$, for the i^{th} retailer the globally optimal solution to the selling price and the promotional effort are $e_i' = A_{e_{ij}} - B_{e_{ij}} + C_{e_{ij}}$ and $h_i' = A_{h_{ij}} - B_{h_{ij}} + C_{h_{ij}}$ where $A_{e_{ij}}, B_{e_{ij}}, C_{e_{ij}}, A_{h_{ij}}, B_{h_{ij}}, C_{h_{ij}}$ are functions of g_1 and g_2 only, for i=1,2 and j=3-i.*

Proof. For extreme points we have,

$$\frac{\partial \pi_{r_i}}{\partial e_i} = 0$$

$$\Rightarrow (d_i - b_{\mathrm{e}}.e_i + c_e.e_j + b_{\mathrm{h}}.h_i - c_h.h_j + b_{\mathrm{g}}.g_i - c_{\mathrm{g}}.g_j) + (e_i - f_i)(-b_e) = 0 \tag{6}$$

and

$$\frac{\partial \pi_{r_i}}{\partial h_i} = 0$$

$$(e_i - f_i)(b_h) - n_i(h_i - 1) = 0 \tag{7}$$

solving Equation 6 and Equation 7 simultaneously we get,

$$\begin{aligned} e_i' &= \frac{-b_h^2.f_i + b_h.n_i + d_i.n_i + c_e.e_j.n_i + b_e.f_i.n_i - c_g.g_j.n_i + b_g.g_i.n_i - c_h.h_j.n_i}{2.b_e.n_i - b_h^2} \\ h_i' &= \frac{b_h.d_i + b_h.c_e.e_j - b_e.b_h.f_i - b_h.c_g.g_j + b_g.b_h.g_i - b_h.c_h.h_j + 2.b_e.n_i}{2.b_e.n_i - b_h^2} \end{aligned}$$

putting i=1 and j=3-i=2 we get,

$$e_1' = \frac{-b_h^2.f_1 + b_h.n_1 + d_1.n_1 + c_e.e_2.n_1 + b_e.f_1.n_1 - c_g.g_2.n_1 + b_g.g_1.n_1 - c_h.h_2.n_1}{2.b_e.n_1 - b_h^2} \quad (5)$$

$$h_1' = \frac{b_h.d_1 + b_h.c_e.e_2 - b_e.b_h.f_1 - b_h.c_g.g_2 + b_g.b_h.g_1 - b_h.c_h.h_2 + 2.b_e.n_1}{2.b_e.n_1 - b_h^2} \quad (6)$$

putting i=2 and j=3-i=1 we get,

$$e_2' = \frac{-b_h^2.f_2 + b_h.n_2 + d_2.n_2 + c_e.e_1.n_2 + b_e.f_2.n_2 - c_g.g_1.n_2 + b_g.g_2.n_2 - c_h.h_1.n_2}{2.b_e.n_2 - b_h^2} \quad (7)$$

$$h_2' = \frac{b_h.d_2 + b_h.c_e.e_1 - b_e.b_h.f_2 - b_h.c_g.g_1 + b_g.b_h.g_2 - b_h.c_h.h_1 + 2.b_e.n_2}{2.b_e.n_2 - b_h^2} \quad (8)$$

Now since e_i and h_i are functions of e_j and h_j, we need to solve Equation 5,Equation 6,Equation 7 and Equation 8. Solving we get,

$$e_i' = A_{e_{ij}} - B_{e_{ij}} + C_{e_{ij}} \quad (9)$$

where

$$\begin{aligned} A_{e_{ij}} &= \frac{b_h^4.f_i - b_h^3.n_i + n_i.n_j.(2.b_e^2.f_i + c_e(-c_h + d_j - c_g.g_i + b_g.g_j)}{(b_h^4 + (4b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2.be(n_i + n_j)))} \\ &+ \frac{b_e(-2.c_h + 2d_i + c_e.f_j + 2b_g.g_i - 2c_g.g_j))}{(b_h^4 + (4b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2.be(n_i + n_j)))} \\ B_{e_{ij}} &= \frac{b_h^2(c_h^2 f_i + d_i.n_i + b_e.f_i.n_i + c_e.f_j.n_i + b_g.g_i.n_i - c_g.g_j.n_i + 2b_e.f_i.n_j)}{(b_h^4 + (4b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2.be(n_i + n_j)))} \\ C_{e_{ij}} &= \frac{b_h(c_h^2.n_i + (2b_e + c_e)n_i.n_j + c_h(-d_j.n_i + b_e.f_j.n_i + c_g.g_i.n_i - b_g.g_j.n_i + c_e.f_i.n_j))}{(b_h^4 + (4b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2.be(n_i + n_j)))} \end{aligned}$$

for i=1,2 and j=3-i and

$$h_i' = A_{h_{ij}} - B_{h_{ij}} + C_{h_{ij}} \quad (10)$$

where

$$\begin{aligned} A_{h_{ij}} &= \frac{(-b_h^3.(d_i - b_e.f_i + c_e.f_j + b_g.g_i - c_g g_j) + (4b_e^2 - c_e^2).n_i.n_j}{b_h^4 + (4b_e^2 - c_e^2)n_i.n_j + b_h.c_e c_h(n_i + n_j) - b_h^2(c_h^2 + 2b_e(n_i + n_j)))} \\ B_{h_{ij}} &= \frac{b_h^2(c_h(d_j + c_e.f_i - b_e.f_j - c_g.g_i + b_g.g_j) + 2.b_e.n_i - c_e.n_j)}{(b_h^4 + (4b_e^2 - c_e^2)n_i.n_j + b_h.c_e c_h(n_i + n_j) - b_h^2(c_h^2 + 2b_e(n_i + n_j)))} \\ C_{h_{ij}} &= \frac{b_h(c_e^2.f_i.n_j - 2.b_e(c_h - d_i + b_e.f_i - b_g.g_i + c_g.g_j)n_j}{(b_h^4 + (4b_e^2 - c_e^2)n_i.n_j + b_h.c_e c_h(n_i + n_j) - b_h^2(c_h^2 + 2b_e(n_i + n_j)))} \\ &+ \frac{c_e(c_h.n_i + (d_j + b_e.f_j - c_g.g_i + b_g.g_j)n_j))}{(b_h^4 + (4b_e^2 - c_e^2)n_i.n_j + b_h.c_e c_h(n_i + n_j) - b_h^2(c_h^2 + 2b_e(n_i + n_j)))} \end{aligned}$$

for i=1,2 and j=3-i

$\because 2.b_e.n_i - b_h^2 > 0$,by Theorem 3.1, the above extreme points are globally optimal solutions. Hence the proposition is proved. □

Theorem 3.2. *The manufacturer's profit function $\pi_m(g_1, g_2|e_1', e_2', h_1', h_2')$ is concave with respect to g_1, g_2 .*

Proof. By Equation 5,

$$\pi_m(g_1, g_2|e_1', e_2', h_1', h_2') = (f_1 - c_1)D_1(e_1', h_1') + (f_2 - c_2)D_2(e_2', h_2') - \frac{m}{2}.(g_1^2 + g_2^2)$$

By Equation 9 and Equation 10,we get,

$$\begin{aligned}
\frac{\partial^2 e_1'}{\partial g_1^2} = \frac{\partial^2 e_2'}{\partial g_1^2} = \frac{\partial^2 h_1'}{\partial g_1^2} = \frac{\partial^2 h_2'}{\partial g_1^2} = 0 \\
\frac{\partial^2 e_1'}{\partial g_1 \partial g_2} = \frac{\partial^2 e_2'}{\partial g_1 \partial g_2} = \frac{\partial^2 h_1'}{\partial g_1 \partial g_2} = \frac{\partial^2 h_2'}{\partial g_1 \partial g_2} = 0 \\
\frac{\partial^2 e_1'}{\partial g_2^2} = \frac{\partial^2 e_2'}{\partial g_2^2} = \frac{\partial^2 h_1'}{\partial g_2^2} = \frac{\partial^2 h_2'}{\partial g_2^2} = 0 \\
\frac{\partial^2 e_1'}{\partial g_2 \partial g_1} = \frac{\partial^2 e_2'}{\partial g_2 \partial g_1} = \frac{\partial^2 h_1'}{\partial g_2 \partial g_1} = \frac{\partial^2 h_2'}{\partial g_2 \partial g_1} = 0
\end{aligned}$$

So,

$$\begin{aligned}
\frac{\partial^2 \pi_m(g_1, g_2|e_1', e_2', h_1', h_2')}{\partial g_1^2} &= \frac{\partial^2 \pi_m(g_1, g_2|e_1', e_2', h_1', h_2')}{\partial g_2^2} = -m \\
\frac{\partial^2 \pi_m(g_1, g_2|e_1', e_2', h_1', h_2')}{\partial g_1 \partial g_2} &= \frac{\partial^2 \pi_m(g_1, g_2|e_1', e_2', h_1', h_2')}{\partial g_2 \partial g_1} = 0
\end{aligned}$$

Now, the Hessian matrix

$$\mathbb{H}_{\pi_m} = \begin{pmatrix} \frac{\partial^2 \pi_m(g_1, g_2|e_1', e_2', h_1', h_2')}{\partial g_1^2} & \frac{\partial^2 \pi_m(g_1, g_2|e_1', e_2', h_1', h_2')}{\partial g_1 \partial g_2} \\ \frac{\partial^2 \pi_m(g_1, g_2|e_1', e_2', h_1', h_2')}{\partial g_2 \partial g_1} & \frac{\partial^2 \pi_m(g_1, g_2|e_1', e_2', h_1', h_2')}{\partial g_2^2} \end{pmatrix} = \begin{pmatrix} -m & 0 \\ 0 & -m \end{pmatrix}$$

So, clearly the $\mathbb{H}_{\pi_m}$ is negative definite and hence the profit function is concave. So, the theorem is proved. □

Proposition 3.1.2. *The globally optimal service levels for the manufacturer are given by $g_i' = (A_{g_{ij}} + B_{g_{ij}} + C_{g_{ij}})$ where $A_{g_{ij}}, B_{g_{ij}}, C_{g_{ij}}$ are parameters.(i=1,2 and j=3-i)*

Proof. It is given, $\pi_m(g_1, g_2|e_1', e_2', h_1', h_2') = (f_1 - c_1)D_1(e_1', h_1') + (f_2 - c_2)D_2(e_2', h_2') - \frac{m}{2}.(g_1^2 + g_2^2)$.
Now, by Equation 9 and Equation 10 we get for i=1,2 and j=3-i,

$$\begin{aligned}
\frac{\partial e_i'}{\partial g_i} &= \frac{-b_g.b_h^2.n_i + b_h.c_g.c_h n_i + (2b_e.b_g - c_e.c_g)n_i.n_j}{b_h^4 + (4.b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2.b_e(n_i + n_j))} \\
\frac{\partial e_i'}{\partial g_j} &= \frac{b_h^2.c_g.n_i - b_g.b_h.c_h.n_i + (b_g.c_e - 2b_e.c_g)n_i.n_j}{b_h^4 + (4.b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2b_e(n_i + n_j))} \\
\frac{\partial h_i'}{\partial g_i} &= \frac{-b_g.b_h^3 + b_h^2.c_g.c_h + b_h(2.b_e.b_g.n_j - c_e.c_g.n_j)}{b_h^4 + (4.b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2b_e(n_i + n_j))} \\
\frac{\partial h_i'}{\partial g_j} &= \frac{b_h^3.c_g - b_g.b_h^2.c_h + b_h(b_g.c_e.n_j - 2b_e.c_g.n_j)}{b_h^4 + (4.b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2b_e(n_i + n_j))}
\end{aligned}$$

So, solving

$$\frac{\partial \pi_m(g_1, g_2|e_1', e_2', h_1', h_2')}{\partial g_1} = 0$$

$$\frac{\partial \pi_m(g_1, g_2 | e'_1, e'_2, h'_1, h'_2)}{\partial g_2} = 0$$

we get,

$$g'_i = (A_{g_{ij}} + B_{g_{ij}} + C_{g_{ij}}) \tag{11}$$

where,

$$\begin{aligned}
A_{g_{ij}} &= \frac{c_i.b_e(b_g.b_h^2.n_i - b_h.c_g.c_h.n_i - 2.b_e.b_g.n_i.n_j + c_e.c_g.n_i.n_j)}{m(b_h^4 + (4.b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2b_e(n_i + n_j)))} \\
B_{g_{ij}} &= \frac{c_j.b_e(-b_h^2.c_g.n_j + b_g.b_h.c_h.n_j - b_g.c_e.n_i.n_j + 2.b_e.c_g.n_i.n_j)}{m(b_h^4 + (4.b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2b_e(n_i + n_j)))} \\
C_{g_{ij}} &= \frac{b_e(c_g(b_h c_h f_i n_i + b_h^2 f_j n_j - (c_e f_i + 2b_e f_j)n_i n_j) + b_g(-b_h^2 f_i n_i - b_h c_h f_j n_j + (2b_e f_i + c_e f_j)n_i n_j))}{m(b_h^4 + (4.b_e^2 - c_e^2)n_i.n_j + b_h.c_e.c_h(n_i + n_j) - b_h^2(c_h^2 + 2b_e(n_i + n_j)))}
\end{aligned}$$

By the Theorem 3.2 the values of g'_i's maximize the problem globally and hence the proposition is proved. □

4. Fuzzy Model

For real world problems, the costs associated with a product may not be exactly known to the manufacturer or the retailers beforehand. So, in this section we assume the production cost and wholesale prices are vague in nature. So, we assume that these prices are fuzzy numbers.

4.1. Prerequisite Mathematics

Fuzzy set theory was first introduced by Zadeh as a mathematical technique of representing impreciseness or vagueness in the real world.

Definition 4.1.1. *Fuzzy set: A fuzzy set $\widetilde{A}$ in a universe of discourse X is defined as the ordered pairs $\widetilde{A} = \{(x, M_{\widetilde{A}}(x)) : x \in X\}$ where $M_{\widetilde{A}} : X \to [0,1]$ is a function known as the membership function of the set $\widetilde{A}$. $M_{\widetilde{A}}(x)$ is the degree of membership of $x \in X$ in the fuzzy set $\widetilde{A}$. Higher value of $M_{\widetilde{A}}(x)$ indicates a higher degree of membership in $\widetilde{A}$.*

Definition 4.1.2. *Convex fuzzy set: A fuzzy set $\widetilde{A}$ of the universe of discourse X is convex if and only if $M_{\widetilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geqslant min(M_{\widetilde{A}}(x_1), M_{\widetilde{A}}(x_2))$, $\forall x_1, x_2$ in X and $\lambda \in [0,1]$.*

Definition 4.1.3. *Normal fuzzy set: A fuzzy set $\widetilde{A}$ of the universe of discourse X is called a normal fuzzy set if $\exists$ at least one $x \in X$ such that $M_{\widetilde{A}}(x) = 1$.*

Definition 4.1.4. *Generalized Trapezoidal fuzzy number (GTrFN): Generalized trapezoidal fuzzy number (GTrFN) $\widetilde{A}$, as represented in (2, 3), is a fuzzy subset of $\mathbb{R}$, where $\widetilde{A} = (p, q, r, s; w)$, p,q,r,s $\in \mathbb{R}$ and $w \in (0,1]$ and its membership function satisfies the following properties:*

- *$M_{\widetilde{A}}(x) : \mathbb{R} \to [0,1]$ is a continuous function.*
- *$M_{\widetilde{A}}(x) = 0$, where $x \in (-\infty, p]$*
- *$M_{\widetilde{A}}(x)$ is strictly increasing with constant rate on [p,q]*
- *$M_{\widetilde{A}}(x) = w$, where $x \in [q, r]$*
- *$M_{\widetilde{A}}(x)$ is strictly decreasing with constant rate on [r,s]*
- *$M_{\widetilde{A}}(x) = 0$, where $x \in [s, \infty)$*

Note: $\widetilde{A}$ is a convex fuzzy set. It is normalized for w=1. If w=1 the generalized fuzzy number $\widetilde{A}$ will be called a trapezoidal fuzzy number (TrFN) and denoted by $\widetilde{A} = (p, q, r, s)$.

(i) If p=q and r=s, then $\widetilde{A}$ is called crisp interval [p,s].

(ii) If q=r, then $\widetilde{A}$ is called a generalized triangular fuzzy number (GTFN) and denoted as $\widetilde{A} = (p, q, s; w)$.

(iii) if q=r,w=1 then it is known as a triangular fuzzy number(TFN) as $\widetilde{A} = (p, q, s)$.

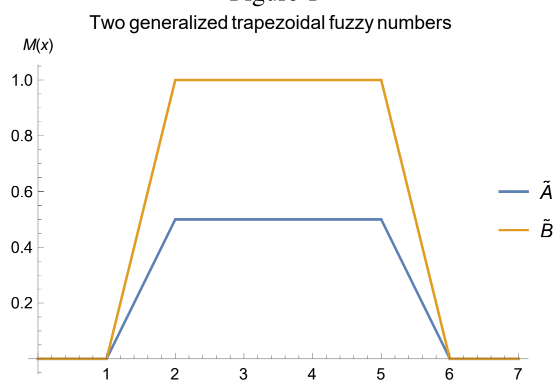
(iv) If p=q=r=s and w=1, then $\widetilde{A}$ is a real number which is p.

Definition 4.1.5. *For a generalized trapezoidal fuzzy number $\widetilde{A} \equiv (p, q, r, s; w)$ the membership function defined as*

$$M^w_{\widetilde{A}}(x) = \begin{cases} M^w_{L\widetilde{A}}(x) = w\left(\frac{x-p}{q-p}\right) & for\ x \in [p, q], \\ w & for\ x \in [q, r], \\ M^w_{R\widetilde{A}}(x) = w\left(\frac{s-x}{s-r}\right) & for\ x \in [r, s], \\ 0 & otherwise. \end{cases}$$

where $p < q < r < s$ and w $\in (0, 1]$.

Figure 1



In the above figure we have shown two different generalized trapezoidal fuzzy numbers $\widetilde{A} = (p, q, r, s; w_1)$ and $\widetilde{B} = (p, q, r, s; w_2)$ for $p = 1, q = 2, r = 5, s = 6, w_1 = .5, w_2 = 1$, which denote two different decision makers' opinions. w_1 and w_2 are the degrees of confidence of the opinions of the decision makers $\widetilde{A}$ and $\widetilde{B}$, respectively, where $w_1 = .8$ and $w_2 = 1$.

Classical fuzzy arithmetic operations are useful for normalized fuzzy numbers, but these operations change the type of membership functions of fuzzy numbers in the process after arithmetical operations. These operations also require troublesome and tedious arithmetical calculations. Thus, Chen proposed the function principle, which could be used as arithmetic operations between generalized fuzzy numbers, as these fuzzy arithmetic operations can deal with the generalized fuzzy numbers. Hsieh and Chen showed that arithmetic operations on fuzzy numbers presented by Chen, not only keep the type of membership function of the fuzzy numbers unchanged after arithmetic operations, but also these operations can bring down the troublesomeness and tediousness of arithmetical calculations. Thus, in this paper, we use arithmetical operators of fuzzy numbers proposed by Chen, to deal with arithmetical operations with generalized fuzzy numbers. These arithmetic operations on generalized fuzzy numbers can deal with both non-normalized and normalized fuzzy numbers, whereas and the traditional operations on fuzzy numbers can only deal with normalized fuzzy numbers.

Let $\lambda \in [0, 1]$ be a pre-assigned parameter, known as the degree of optimism. The total λ -integral value of $\widetilde{A}$ is defined as $I_\lambda(\widetilde{A}) = \lambda I_R(\widetilde{A}) + (1-\lambda)I_L(\widetilde{A})$ where $I_L(\widetilde{A})$ and $I_R(\widetilde{A})$ are left and right integral values of $\widetilde{A}$ respectively and they are defined as

$$I^w_L(\widetilde{A}) = \int_0^1 (M^w_{L\widetilde{A}})^{-1} t dt \tag{12}$$

$$I^w_R(\widetilde{A}) = \int_0^1 (M^w_{R\widetilde{A}})^{-1} t dt \tag{13}$$

Now,

$$(M^w_{L\widetilde{A}})^{-1}t = p + \frac{t}{w}(q-p)$$
$$(M^w_{R\widetilde{A}})^{-1}t = s - \frac{t}{w}(s-r).$$

$\therefore$ the left and right integral values, respectively are

$$I^w_L(\widetilde{A}) = w\left(\frac{p+q}{2}\right)$$
$$I^w_R(\widetilde{A}) = w\left(\frac{r+s}{2}\right)$$

Hence the total λ -integral value of $\widetilde{A}$ is

$$I_\lambda(\widetilde{A}) = \left[\lambda w\left(\frac{r+s}{2}\right) + (1-\lambda)w\left(\frac{p+q}{2}\right)\right] \tag{14}$$

The left integral value gives a pessimistic viewpoint and the right integral gives an optimistic viewpoint of the decision maker. So, higher the value of λ implies higher optimism and lower implies lower optimism. If we take $\lambda = .5$ then the λ-integral gives a moderately optimistic decision-maker's viewpoint and is the same as the defuzzification of the fuzzy number $\widetilde{A}$.

4.2. Fuzzy model with imprecise production costs

In this subsection, we check how the profit function and hence the optimal service level change for the manufacturer if the production cost is imprecisely known to him.
We take the production costs as GTrFN (Definition 4.1.4). So, we denote it by $\widetilde{c}_i(i = 1,2)$ and defined as $\widetilde{c}_i = (c^1_i, c^2_i, c^3_i, c^4_i)$ for i=1,2. Changes in Production cost affect the manufacturer directly. As the service levels change, the selling price and the promotional effort change for the retailers because these are functions of g_1 and g_2 (see Equation 9 and Equation 10). So, the modified service levels are,
For a fixed value of λ , the λ -integral value by Equation 14 is,

$$I^w_\lambda(\widetilde{c}_i) = \left[\lambda w\left(\frac{c^3_i + c^4_i}{2}\right) + (1-\lambda)w\left(\frac{c^1_i + c^2_i}{2}\right)\right] \tag{15}$$

for i=1,2.
$\therefore$ from Equation 5 the new profit function for the manufacturer is

$$\pi_m \quad = \quad (f_1 - \widetilde{c}_1)D_1 + (f_2 - \widetilde{c}_2)D_2 - \frac{m}{2}.(g^2_1 + g^2_2) \tag{16}$$

$\therefore$ from Equation 11 the optimal values for g_i for i=1,2 and j=3-i are

$$g'_i = (A_{g_{ij}} + B_{g_{ij}} + C_{g_{ij}}) \tag{17}$$

where, we can find the values of $A_{g_{ij}}, B_{g_{ij}}, C_{g_{ij}}$ as in Equation 11 by replacing c_i and c_j with $I^w_\lambda(\widetilde{c}_i)$ and $I^w_\lambda(\widetilde{c}_j)$. Now, we can find out the modified service levels and promotional effort for the retailers by replacing the g_i's in Equation 9 and Equation 10 with g'_i's from Equation 17.

4.3. Fuzzy model with imprecise wholesale price

In a similar manner as in the last subsection, here we check how the profit function and hence the optimal selling price for the retailers change if we assume the wholesale price is not precisely known to the retailers.
We take the wholesale prices as GTrFN (Definition 4.1.4). So, we denote it by $\widetilde{f}_i$ (i=1,2) and defined as $\widetilde{f}_i = (f^1_i, f^2_i, f^3_i, f^4_i)$ for i=1,2. Changes in wholesale price affect both the retailers and the manufacturer directly. So,

we have to change the selling price, promotional effort and service level accordingly.
For a fixed value of λ the λ -integral value by eq. Equation 14 is,

$$I_\lambda^w(\widetilde{f}_i) = \left[\lambda w\left(\frac{f_i^3 + f_i^4}{2}\right) + (1-\lambda)w\left(\frac{f_i^1 + f_i^2}{2}\right)\right] \tag{18}$$

for i=1,2.
$\therefore$ from Equation 3 and Equation 4 the new profit functions for the i^{th} retailer respectively are,

$$\pi_{r_1} = (e_1 - \widetilde{f}_1)D_1 - \frac{n_1}{2}.(h_1 - 1)^2 \tag{19}$$

$$\widetilde{\pi}_{r_2} = (e_2 - \widetilde{f}_2)D_2 - \frac{n_2}{2}.(h_2 - 1)^2 \tag{20}$$

From Equation 9 we have the modified optimal values of $\widetilde{e}_i'$ for i=1,2;j=3-i

$$e_i' = A_{e_{ij}} - B_{e_{ij}} + C_{e_{ij}} \tag{21}$$

where, we can find the values of $A_{e_{ij}}, B_{e_{ij}}, C_{e_{ij}}$ as in Equation 9 by replacing f_i and f_j with $I_\lambda^w(\widetilde{f}_i)$ and $I_\lambda^w(\widetilde{f}_j)$ respectively.
And from Equation 10 the modified optimal values of $\widetilde{h}_i'$ for i=1,2;j=3-i are,

$$h_i' = A_{h_{ij}} - B_{h_{ij}} + C_{h_{ij}} \tag{22}$$

where, we can find the values of $A_{h_{ij}}, B_{h_{ij}}, C_{h_{ij}}$ as in Equation 10 by replacing f_i and f_j with $I_\lambda^w(\widetilde{f}_i)$ and $I_\lambda^w(\widetilde{f}_j)$ respectively.
And from Equation 11 we have the modified optimal values of g_i' for i=1,2 and j=3-i as,

$$g_i' = (A_{g_{ij}} + B_{g_{ij}} + C_{g_{ij}}) \tag{23}$$

where, we can find the values of $A_{g_{ij}}, B_{g_{ij}}, C_{g_{ij}}$ as in Equation 11 by replacing f_i and f_j with $I_\lambda^w(\widetilde{f}_i)$ and $I_\lambda^w(\widetilde{f}_j)$.

5. Numerical examples

5.1. Crisp environment

Here, firstly, we give some numerical examples in crisp environment. In table 1 we provide the values of different parameters.

Table 1: Parameters

d_1	d_2	b_e	c_e	b_g	c_g	b_h	c_h	c_1	c_2	f_1	f_2	n_1	n_2	m
30	40	.1	.01	.8	.1	.2	.02	157.5	177.5	190	230	2	6	4
30	40	.15	.01	.8	.1	.2	.02	157.5	177.5	190	230	2	6	4
30	40	.1	.05	.8	.1	.2	.02	157.5	177.5	190	230	2	6	4
50	35	.1	.01	.8	.1	.2	.02	217	177	290	230	7	3	6
50	50	.1	.01	.8	.1	.2	.02	250	197	290	230	5	3	6

For the above table of parameters, we have the following values of the variables and profit functions.

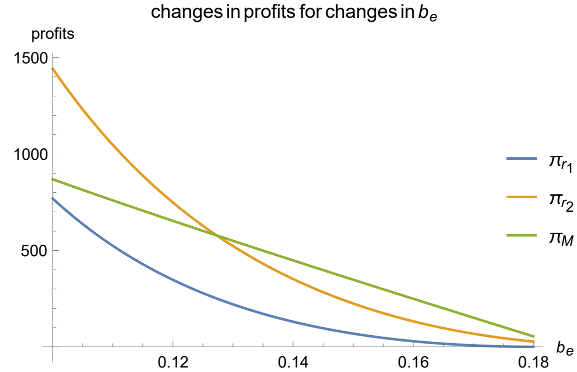
Table 2: Variables and Profits

e_1	e_2	h_1	h_2	g_1	g_2	π_{r_1}	π_{r_2}	π_m
282.40	352.13	10.24	5.07	3.16	5.13	768.51	1441.79	868.80
212.16	269.16	3.22	2.30	2.95	5.03	68.75	224.88	348.35
390.72	440.86	9.14	4.50	5.00	6.81	3626.11	4298.19	1642.16
432.93	329.96	21.07	8.03	4.58	6.13	1984.65	932.55	1472.67
429.89	405.78	6.60	12.72	2.58	2.13	1878.67	2883.83	1105.97

As we can see from table 2, even little perturbations of b_e and c_e result in huge changes in profits. So, we have given some graphs to show how changes in these parameters change profits of the chain members. For $d_1 = 30, d_2 = 40, c_e = .01, b_g = .8, c_g = .1, b_h = .2, c_h = .02, c_1 = 157.5, c_2 = 177.5, f_1 = 190, f_2 = 230, n_1 = 2, n_2 = 6, m = 4$ we have fig. 2.

Figure 2

changes in profits for changes in b_e

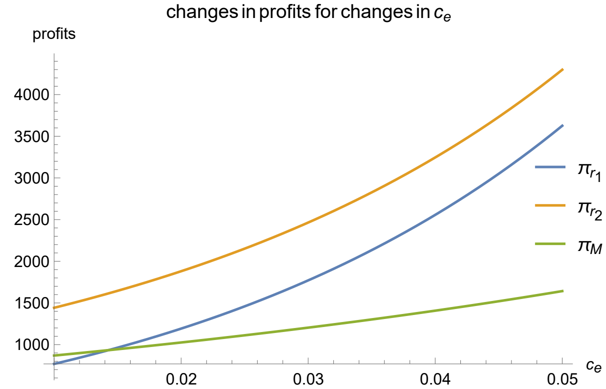


As we can see in fig. 2 as b_e, which is the responsiveness of the market demand of retailer's own price, increases the profits of all the chain members decrease.

For $d_1 = 30, d_2 = 40, b_e = .1, b_g = .8, c_g = .1, b_h = .2, c_h = .02, c_1 = 157.5, c_2 = 177.5, f_1 = 190, f_2 = 230, n_1 = 2, n_2 = 6, m = 4$ we have fig. 3. Here in fig. 3, we can see as c_e ,which is the intensity of competition

Figure 3

changes in profits for changes in c_e



between the retailers, increases the profits of all the chain members increase. We can find out similar graphs for the other parameters.

5.2. Fuzzy environment

Next we give some examples in fuzzy environment. We have taken w=.9 for our calculations. Firstly, we take production costs as fuzzy numbers.

Input data:

$$\begin{pmatrix} c_1^1 & c_1^2 & c_1^3 & c_1^4 \end{pmatrix} = \begin{pmatrix} 150 & 180 & 190 & 230 \end{pmatrix}$$

and

$$\begin{pmatrix} c_2^1 & c_2^2 & c_2^3 & c_2^4 \end{pmatrix} = \begin{pmatrix} 170 & 190 & 240 & 290 \end{pmatrix}$$

Then, using eq. (15) we have following table of parameters.

Corresponding to these parameter in table 3 we have the following table containing values of profit functions and variables.

Table 3

λ	d_1	d_2	b_e	c_e	b_g	c_g	b_h	c_h	c_1	c_2	f_1	f_2	n_1	n_2	m
.3	30	40	.1	.01	.8	.1	.2	.02	160.65	184.95	190	230	2	6	4
.5	30	40	.1	.01	.8	.1	.2	.02	168.75	200.25	190	230	2	6	4
.7	30	40	.1	.01	.8	.1	.2	.02	176.85	215.55	190	230	2	6	4

Table 4

e_1	e_2	h_1	h_2	g_1	g_2	π_{r_1}	π_{r_2}	π_m
281.39	349.17	10.14	4.97	2.87	4.39	751.73	1372.73	749.97
278.50	343.20	9.85	4.77	2.10	2.88	704.97	1238.8	499.35
275.62	337.24	9.56	4.57	1.33	1.38	659.72	1111.74	260.19

So from table 4 it is clear, increasing of λ implies decreasing of the profit of each member.
Next we take wholesale prices as fuzzy numbers. We have taken w=.9 for our calculations. The values of parameters are given in table 5.

Table 5

λ	d_1	d_2	b_e	c_e	b_g	c_g	b_h	c_h	c_1	c_2	f_1	f_2	n_1	n_2	m
.3	30	40	.1	.01	.8	.1	.2	.02	157.5	177.5	193.5	205.65	2	6	4
.5	30	40	.1	.01	.8	.1	.2	.02	157.5	177.5	202.5	213.75	2	6	4
.7	30	40	.1	.01	.8	.1	.2	.02	157.5	177.5	211.5	221.85	2	6	4

Table 6

e_1	e_2	h_1	h_2	g_1	g_2	π_{r_1}	π_{r_2}	π_m
286.75	329.77	10.32	5.14	3.75	2.59	782.6	1489.25	643.52
294.85	336.77	10.23	5.1	4.68	3.35	767.53	1462.86	795.26
302.95	343.76	10.14	5.06	5.61	4.11	752.61	1436.71	937.83

Corresponding to these values we have the following table. In table 6 we can see, increasing of λ implies decreasing of the profit of each retailer but the increasing of manufacturer's.

6. Concluding Remarks

In this paper, a two-echelon supply chain model with a single manufacturer and two competing retailers was considered. The manufacturer was taken to be the Stackelberg leader and the retailers to be followers. Firstly, a crisp model was solved using classical optimization techniques and then a fuzzy model was formed using generalized trapezoidal fuzzy numbers and solved using λ-integral. The fuzzy model is practically more useful for real world problems.

In this article we assumed the demand is deterministic. So, the problem can be expanded by assuming the demand is stochastic by considering the whole demand is a triangular fuzzy number. Also neutrosophic triangular numbers instead of fuzzy triangular numbers can be used to extend the model to include indeterminacy to this model.

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