Pak.j.stat.oper.res. Vol.16 No. 2 2020 pp305-315 DOI: http://dx.doi.org/10.18187/pjsor.v16i2.3087

Pakistan Journal of Statistics and Operation Research

Some New Goodness-of-fit Tests for Rayleigh Distribution

S. M. A. Jahanshahi[1*], A. Habibirad[2], V. Fakoor[3]

* Corresponding Author

1. Department of Statistics, University of Sistan and Baluchetan, Zahedan, Iran, mjahan@math.usb.ac.ir
2. Department of Statistics, Ferdowsi University of Mashhad, Iran, ahabibi@um.ac.ir
3. Department of Statistics, Ferdowsi University of Mashhad, Iran, fakoor@um.ac.ir

Abstract

This article deals with goodness-of-fit test for Rayleigh distribution. Three new tests based on Jeffreys, Lin-Wong and Rényi divergence measures are proposed, and shown to be consistent. Monte Carlo simulations are performed for various alternatives and sample sizes in order to compare the proposed tests with other goodness-of-fit tests for Rayleigh distribution in the literature. Simulation results showed that in comparison with the existing tests, our proposed tests have shown better performances in terms of power and among them Jeffreys test is better than other. Finally, illustrative example for use of some considered more powerful tests are presented and analyzed.

Key Words: Divergence measure; Entropy; Goodness-of-fit; Jeffreys; Lin-Wong; Monte Carlo Simulation; Power; Rayleigh distribution; Rényi.

Mathematical Subject Classification: 94A17, 40A30, 65C05

Introduction

Identifying true distribution of real data is an important part of reliability, life testing and survival analysis. In this paper, the Rayleigh distribution which is a special case of Weibull distribution has a momentous role.

The Rayleigh distribution was originally offered by a physicist, Lord Rayleigh (1880), in connection with an acoustics problem. But more recently, it has been used to model data that are skewed to the right; such as life data which arises in many areas of applications.

As mentioned, Rayleigh distribution has wide applications. Polovko (1968) and Dyer and Whisenand (1973) showed the importance of this distribution in electro-vacuum devices and communication engineering. Siddiqui (1962) discussed the origin and properties of the Rayleigh distribution. One major application of this model is used in analyzing wind speed data. The origin and other aspects of this distribution can be found in Siddiqui (1962), Miller and Sackrowttz (1967). For more details on the Rayleigh distribution the reader is referred to Johnson et al. (1994).

In statistics, goodness-of-fit (GOF) is the nearness of agreement between a set of observations and a hypothetical model that is suggested. Usually, there are three types of GOF families widely used in statistical applications:

i) The GOF measures derived from empirical distribution functions. Kolmogorov-Smirnov statistic, Cramér-Von-Mises statistic and Anderson-Darling statistic belong to this family.

ii) The GOF measures based on the differences between observed and expected frequencies. Likelihood ratio and chi-square GOF statistics belong to this category.

iii) Measures derived from entropy and relative entropy notions. Among them, Kullback-Leibler divergence (KL-divergence), Jeffreys' divergence are the most popular representatives. In fact, these divergences can be studied with their close relations with Shannon entropy, Rényi entropy.

As a matter of fact, divergence measures are generally used to find appropriate distance or difference between two probability distributions. These measures have been applied in several fields like probability distributions, signal processing, pattern recognition, finance, economics, etc.

While GOF tests for exponentiality have a large literature, GOF tests for the Rayleigh distribution (testing Rayleighity) have only recently been considered. The GOF tests for the Rayleigh distribution were proposed in Meintanis and Iliopoulos (2003), Morris and Szynal (2008), Best et al. (2010), Alizadeh et al. (2012), Baratpour and Khodadadi (2012), Safavinejad et al. (2015) and Jahanshahi et al. (2016), Al-Omari and Zamanzade (2016), Zamanzade and Mahdizadeh (2017), Badr (2019) and Ahrari et al. (2019).
The Rayleigh distribution has the following probability density function

$$f(x)=\frac{x}{\theta^2}exp\left(-\frac{x^2}{2\theta^2}\right), x>0, \theta>0,$$

and cumulative distribution function

$$F(x)=1-exp\left(-\frac{x^2}{2\theta^2}\right), x>0.$$

Maximum likelihood (ML) estimator of θ is

$$\hat{\theta}=\left(\sum_{i=1}^{n}\frac{x_i^2}{2n}\right)^{0.5}.$$

Meanwhile, an unbiased ML estimate of θ^2 is

$$\hat{\theta}^2=\sum_{i=1}^{n}\frac{x_i^2}{2n},$$

although $\hat{\theta}$ is a biased estimator.

The article is organized as follows. In section 2, some new tests based on the Jeffreys, Lin-Wong and Rényi divergence measures are developed and some theorems about their properties are presented. In Section 3, using p-value method for checking Rayleighity is described. In Section 4, the GOF tests of Rayleighity in literature are reviewed. In Section 5, the power and type I error rate estimation techniques and alternative distributions have been described. Then, the power of the proposed tests against competitors' tests are compared. In section 6, the performance of considered tests for a real data is evaluated. Finally, in section 7 we appraise the ability of considered tests and choose better tests for each category of alternative distributions.

Proposed Tests for the Rayleigh distribution

One of the important problems in many applications is finding a suitable measure of distance (or divergence or difference or discrimination) between two probability distributions.

A measure of divergence is used as a way to appraise the distance (divergence) between any two populations or functions. Let f_1 and f_2 be two probability density functions which may or may not depend on an unknown parameter of fixed finite dimension. As mentioned, the most well-known measure of divergence is the KL-divergence.
Actually, a divergence is a measure of difference or closeness between two distributions f_1 and f_2. Here, the focus is on continuous distributions with densities $f_1(x)$ and $f_2(x)$, for which the real-valued functional $D(f_1, f_2)$ is defined to be a divergence if

i) $D(f_1, f_2)\geq 0$
ii) $D(f_1, f_2)=0$ iff $f_1(x)=f_2(x)$ almost everywhere.
iii) $D(f_1, f_2)$ is a convex function of both f_1 and f_2 .

Note that $D(f_1, f_2)$ is not a metric distance in general (symmetry and the triangle inequality are not required).

In fact, some measures proposed for determining GOF do not possess all properties of a metric function. Therefore, they are rather called as divergence measures. For example, KL-divergence is nonnegative but is not symmetric. Also, it is easy to see that it does not satisfy the triangular inequality (Garrido, 2009) thus it is not a metric function. Hence, it must be interpreted as a pseudo-metric measure only. So we propose some new GOF tests for the Rayleigh distribution which are based on some divergence measures.

Accordingly, in this section we propose using some popular divergence measures such as Jeffreys, Lin-Wong and Rényi divergence for checking the Rayleigh distribution and we guess using these divergence measures leads to a better performance in terms of power against other proposed tests in the statistical literature.

Jeffreys Test

As mentioned earlier, KL-divergence is not symmetric, so in order to obtain a symmetric measure Jeffreys (1946) proposed a symmetric variant of the KL-divergence which is called the Jeffreys' divergence (or sometimes J-divergence) in the literature. This divergence has several practical uses in statistics such as detecting influential data in regression analysis and model comparisons (see Arellano-Valle et al., 2000). The Jeffreys' divergence is defined as

$$D_J(f_1,f_2) = D_{KL}(f_1,f_2) + D_{KL}(f_2,f_1),$$

where

$$D_{KL}(f_1,f_2) = \int f_1(x)\, log\{f_1(x)/f_2(x)\}\, dx.$$

Clearly, $D_{KL}(f_1,f_2)$ and $D_J(f_1,f_2)$ divergences share most of their properties. As pointed out in Ullah (1996), Jeffreys measure does not satisfy the triangular inequality of metric and hence it is also a pseudo-metric measure.

Let $X_1,\dots,X_n$be nonnegative; independent and identically distributed (iid) random variables from a continuous distribution function F with order statistics $X_{(1)} \leq \dots \leq X_{(n)}$. Let $f_0(x)$ denote a Rayleigh distribution, where θ is the unknown parameter. The H_0 and H_1 hypotheses are as follows

$$H_0:\ f(x) = f_0(x,\theta), \qquad H_1:\ f(x) \neq f_0(x,\theta).$$

To discriminate between two hypotheses H_0 and H_1, we propose using the Jeffreys' divergence measure of two density functions $f(x)$ and $f_0(x)$ as

$$D_J(f,f_0) = D_{KL}(f,f_0) + D_{KL}(f_0,f) = \int\{f(x) - f_0(x)\}\, log\frac{f(x)}{f_0(x)}dx. \qquad (1)$$

Remark 1 The Jeffreys' statistic for testing Rayleighity can be defined as below

$$J_{mn} = \frac{1}{n}\sum_{i=1}^{n}\left\{1 - \frac{X_{(i)}e^{-X_{(i)}^2/2\hat{\theta}^2}/\hat{\theta}^2}{(\frac{n}{2m}(X_{(i+m)} - X_{(i-m)}))^{-1}}\right\} log\left\{\frac{\left[\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right]^{-1}}{X_{(i)}e^{-X_{(i)}^2/2\hat{\theta}^2}/\hat{\theta}^2}\right\}.$$

where $X_{(i)} = X_{(1)}$ for $i < 1$ and $X_{(i)} = X_{(n)}$ for $i > n$ and $\hat{\theta}$ is the ML estimator of θ.

Proof: we can write (1) easily as

$$D_J(f,f_0) = \int\left\{1 - \frac{f_0(x)}{f(x)}\right\} log\left\{\frac{f(x)}{f_0(x)}\right\} f(x)dx.$$

Now, by replacing f_0 the Jeffreys' divergence will be

$$D_J(f,f_0) = \int_0^\infty\left\{1 - \frac{x_{(i)}e^{-x_{(i)}^2/2\theta^2}/\theta^2}{f(x)}\right\} log\left\{\frac{f(x)}{x_{(i)}e^{-x_{(i)}^2/2\theta^2}/\theta^2}\right\} f(x)dx.$$

Under the null hypothesis $D_J(f,f_0) = 0$ and we expect large values of $D_J(f,f_0)$ under H_1. Based on Vasicek's method (1976) and supposing $F(x) = p$ we have

$$D_J(f,f_0) = \int_0^1\left\{1 - \frac{F^{-1}(p)e^{-(F^{-1}(p))^2/2\theta^2}/\theta^2}{\left(\frac{d}{dp}F^{-1}(p)\right)^{-1}}\right\} log\left\{\frac{\left(\frac{d}{dp}F^{-1}(p)\right)^{-1}}{F^{-1}(p)e^{-(F^{-1}(p))^2/2\theta^2}/\theta^2}\right\} dp.$$

Now, the proposed statistic in Remark 1 will be easily derived by using the relation below

$$\begin{aligned} f(x) &= \left(\frac{d}{dp}F^{-1}(p)\right)^{-1}, \\ &\simeq \left(\frac{n}{2m}(X_{(i+m)} - X_{(i-m)})\right)^{-1}. \qquad (2) \end{aligned}$$

Generally, the Null Hypothesis for large values of J_{mn} will be rejected, that is, H_0 is rejected if $J_{mn} \geq J_{mn}^*$ for some critical values J_{mn}^* . Although, we will reject the null hypothesis when the p-value is less than the level of significance.

Theorem 1 The statistic of J_{mn} is scale-free and invariant with respect to scale transformation.

Proof: As θ is a scale parameter of the Rayleigh distribution we consider a scale transformation group as $G = \{g_c: g_c(X) = cX,\ c > 0\}$. Since

$$J_{mn}\big(g(X)\big) = J_{mn}(X)\ \ \forall X, \forall g \in G,$$

therefore, J_{mn} is a scale-free statistic and invariant under G.

Theorem 2 The distribution of J_{mn} does not depend on θ.
Proof: Since the test statistic J_{mn} is invariant under the scale transformations and the parameter space is transitive, the distribution of the proposed test statistic J_{mn} is free of θ.

Based on Theorem 2, the critical values do not depend on θ. Therefore, the critical values of the test statistic can be obtained by simulation for any values of θ such as $\theta = 1$.

Theorem 3 Let F be an unknown continuous distribution with a positive support and F_0 be the Rayleigh distribution with unspecified parameter θ. Then under H_1 the test based on J_{mn} is consistent.
Proof: We use Alizadeh and Arghami (2011) technique to prove the consistency of J_{mn}. For $n, m \to \infty$ and $m/n \to 0$, we have

$$\begin{aligned}\frac{2m}{n} &= F_n\big(X_{(i+m)}\big) - F_n\big(X_{(i-m)}\big) \simeq F\big(X_{(i+m)}\big) - F\big(X_{(i-m)}\big)\\ &\simeq \frac{f(X_{(i+m)}) + f(X_{(i-m)})}{2}\big(X_{(i+m)} - X_{(i-m)}\big)\\ &\simeq f(X_{(i)})\big(X_{(i+m)} - X_{(i-m)}\big),\end{aligned}$$

where F_n is the empirical distribution function. Since $\hat{\theta}$ is a consistent estimator of θ as $n \to \infty$ and based upon strong law of large numbers we have

$$\begin{aligned}J_{mn} &= \frac{1}{n}\sum_{i=1}^{n}\left\{1 - \frac{X_{(i)}e^{-X_{(i)}^2/2\hat{\theta}^2}/\hat{\theta}^2}{\left(\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right)^{-1}}\right\} log\left\{\frac{\left(\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right)^{-1}}{X_{(i)}e^{-X_{(i)}^2/2\hat{\theta}^2}/\hat{\theta}^2}\right\}\\ &\simeq \frac{1}{n}\sum_{i=1}^{n}\left\{1 - \frac{X_{(i)}e^{-X_{(i)}^2/2\theta^2}/\theta^2}{f(X_{(i)})}\right\} log\left\{\frac{f(X_{(i)})}{X_{(i)}e^{-X_{(i)}^2/2\theta^2}/\theta^2}\right\}\\ &= \frac{1}{n}\sum_{i=1}^{n}\left\{1 - \frac{f_0(X_i)}{f(X_i)}\right\} log\left\{\frac{f(X_i)}{f_0(X_i)}\right\} \xrightarrow{a.s.} E\left(\left\{1 - \frac{f_0(X)}{f(X)}\right\} log\left\{\frac{f(X)}{f_0(X)}\right\}\right)\\ &= \int_0^{\infty}\left\{1 - \frac{f_0(x)}{f(x)}\right\} log\left\{\frac{f(x)}{f_0(x)}\right\} dx\\ &= D_J(f, f_0).\end{aligned}$$

Thus, J_{mn} is a consistent test.

Meanwhile, Safavinejad et al. (2015) showed that the ML estimator of θ in the Rayleigh distribution is equivariant. Therefore, the J_{mn} test is a reasonable test for the Rayleigh distribution which has some good properties such as to be scale-free, invariancy and consistency.

Lin-Wong Test
In this subsection, for constructing the test statistic we estimate the Lin-Wong distance similar to Vasicek's (1976) method for estimating the Shannon entropy.
To discriminate between the two hypotheses H_0 and H_1, we propose using the Lin and Wong (1990) divergence measure of two density functions $f(x)$ and $f_0(x)$ as

$$D_{LW}(f, f_0) = \int_0^{\infty} f(x)\, log\, \frac{2f(x)}{f(x) + f_0(x)} dx. \qquad (3)$$

Since Lin-Wong divergence belongs to Csiszer family, we have $D_{LW}(f, f_0) \geq 0$ and the equality holds if and only if $f(x) = f_0(x)$ (See Kapur and Kesavan, 1992). So, it motivates us to use Lin-Wong divergence as a test statistic for Rayleighity.
Remark 2 The Lin-Wong statistic for testing Rayleighity can be defined as below

$$LW_{mn} = -\frac{1}{n}\sum_{i=1}^{n} log\left\{\frac{1}{2} + \frac{X_{(i)}}{2\hat{\theta}^2} e^{-\frac{X_{(i)}^2}{2\hat{\theta}^2}} \frac{n}{2m}\big(X_{(i+m)} - X_{(i-m)}\big)\right\}.$$

where $X_{(i)} = X_{(1)}$ for $i < 1$ and $X_{(i)} = X_{(n)}$ for $i > n$ and $\hat{\theta}$ is the ML estimator of θ.
Proof: Similar to proof of Remark 1 by replacing f_0 in (3) we have

$$D_{LW}(f,f_0)=\int_{-\infty}^{\infty} f(x)\log\frac{2f(x)}{f(x)+\frac{x}{\theta^2}e^{-x^2/2\theta^2}}dx.$$

Based on Vasicek's method (1976) and supposing $F(x)=p$ we have

$$D_{LW}(f,f_0)=\int_0^1 \log\frac{2\left(\frac{d}{dp}F^{-1}(p)\right)^{-1}}{\left(\frac{d}{dp}F^{-1}(p)\right)^{-1}+\frac{F^{-1}(p)}{\theta^2}e^{-\left(F^{-1}(p)\right)^2/2\theta^2}}dp. \qquad (4)$$

Finally, the proposed statistic in Remark 2 will be derived by using (2) in (4).
By using similar method of Theorems 1, 2 and 3 easily we can prove that

- D_{LW} is invariant under parameter transformations.
- LW_{mn} is a scale-free and invariant test with respect to scale transformation.
- Distribution of LW_{mn} does not depend on θ .
- LW_{mn} is a consistent test.

Based upon the mentioned properties, $LW_{mn}$is a reasonable test for the Rayleigh distribution which has some good properties such as to be scale-free, invariancy and consistency.

Rényi Test

In this subsection, for constructing the test statistic we estimated the Rényi divergence similar to Vasicek's (1976) method for estimating the Shannon entropy.

Remark 3 The Rényi statistic for testing Rayleighity can be defined as below

$$R_{mn}=2\,\widehat{\log\theta}+\frac{1}{s-1}\log\left\{\frac{1}{n}\sum_{i=1}^{n}\left\{\left[\frac{n}{2m}X_{(i)}\left(X_{(i+m)}-X_{(i-m)}\right)\right]^{1-s}e^{(s-1)X_{(i)}^2/2\hat{\theta}^2}\right\}\right\}.$$

where $X_{(i)}=X_{(1)}$ for $i<1$ and $X_{(i)}=X_{(n)}$ for $i>n$ and $\hat{\theta}$ is the ML estimator of θ.
To discriminate between the two hypotheses H_0 and H_1, we propose using the Rényi (1961) divergence of two density functions $f(x)$ and $f_0(x)$ as

$$D_s(f,f_0)=\frac{1}{s-1}\log\int_0^{\infty}\left\{\frac{f(x)}{f_0(x)}\right\}^{s-1}f(x)dx. \qquad (5)$$

where $s>0$ and $s\neq 1$.
Proof: Similar proof of Remark 1 by replacing f_0 in (5) the Rényi information will be

$$D_s(f,f_0)=\frac{1}{s-1}\log\int_0^{\infty}\left\{\frac{f(x)}{\frac{x}{\theta^2}e^{-x^2/2\theta^2}}\right\}^{s-1}f(x)dx$$

$$=2\log\theta+\frac{1}{s-1}\log\int_0^{\infty}\{f(x)\}^{s-1}x^{1-s}e^{(s-1)x^2/2\theta^2}f(x)dx.$$

Based on Vasicek's method (1976) and supposing $F(x)=p$ we have

$$D_s(f,f_0)=2\log\theta+\frac{1}{s-1}\log\int_0^1\left\{\frac{d}{dp}F^{-1}(p)\right\}^{s-1}\{F^{-1}(p)\}^{1-s}e^{\frac{(s-1)\{F^{-1}(p)\}^2}{2\theta^2}}dp. \qquad (6)$$

Finally, the proposed statistic in Remark 3 will be derived by using (2) in (6).
Given that, determining optimum value of s is very important, based upon a simulation study and by comparing power values of the Rényi test for various values of s, we conclude that the Rényi test has its maximum power when $s=0.9$. Thus, we use $s=0.9$ in testing Rayleighity. It should be noted that for $s\to 1$ Rényi measure becomes the KL-divergence.
By using similar method of Theorems 1, 2, and 3 we can easily prove that

- D_s is invariant under parameter transformations.
- R_{mn} is a scale-free and invariant test with respect to scale transformation.
- Distribution of R_{mn} does not depend on θ .
- R_{mn} is a consistent test.

Based upon mentioned properties, R_{mn} is a reasonable test for the Rayleigh distribution which has some good properties such as to be scale-free, invariancy and consistency.

The GOF tests of Rayleighity in the literature
Kullback-Leibler Test

Alizadeh et al. (2012) proposed a GOF test based on KL-divergence for checking Rayleighity. Their proposed statistic is

$$KL_{mn} = -H_{mn} + 2\,log(\hat{\theta}) - \frac{1}{n}\sum_{i=1}^{n} log(X_i) + 1,$$

where $\hat{\theta}$ is the ML estimate of θ and H_{mn} is the Vasicek's sample entropy estimator which is

$$H_{mn} = \frac{1}{n}\sum_{i=1}^{n} log\left\{\frac{n}{2m}\left(X_{(i+m)} - X_{(i-m)}\right)\right\},$$

where the window size m is a positive integer smaller than $n/2$, $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$ and $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ are the order statistics based on a random sample of size n.

The statistic KL_{mn} estimates the KL-divergence between the underlying data distribution and the Rayleigh distribution.

The null hypothesis, H_0, is rejected at the significance level α if $KL_{mn} \geq KL_{mn,1-\alpha}$, and $KL_{mn,1-\alpha}$ is $100(1-\alpha)$ percentile of KL_{mn} under H_0. Alizadeh et al. (2012) proved KL_{mn} is a non-negative and consistent test.

Cumulative Residual Entropy-Based Test

Baratpour and Khodadadi (2012) based on cumulative Kullback-Leibler defined another GOF test for Rayleighity which is

$$CK_n = \frac{\sum_{i=1}^{n}\left(1-\frac{i}{n}\right) ln\left(1-\frac{i}{n}\right)\left(X_{(i+1)} - X_{(i)}\right) + \sqrt{\frac{\pi}{2}}\sqrt{\frac{\sum_{i=1}^{n} X_i^3}{3\sum_{i=1}^{n} X_i}}}{\bar{X}},$$

where H_0 is rejected at the significance level α in favor of H_1 if $CK_n \geq CK_{n,1-\alpha}$, where $CK_{n,1-\alpha}$ is $100(1-\alpha)$ percentile of CK_n under H_0. Baratpour and Habibi (2012) proved that CK_n is a non-negative and consistent test.

Empirical Laplace Transform Test

Meintanis and Iliopoulos (2003) proposed a GOF test of Rayleighity based on the empirical Laplace transform which was defined as

$$L = \frac{n}{a} + \frac{\sqrt{2}}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{\hat{Y}_j + \hat{Y}_k + a} + \frac{\hat{Y}_j + \hat{Y}_k}{\left(\hat{Y}_j + \hat{Y}_k + a\right)^2} + \frac{2\left(\hat{Y}_j\hat{Y}_k + 2\right)}{\left(\hat{Y}_j + \hat{Y}_k + a\right)^3} + \frac{6\left(\hat{Y}_j + \hat{Y}_k\right)}{\left(\hat{Y}_j + \hat{Y}_k + a\right)^4}$$
$$+ \frac{24}{\left(\hat{Y}_j + \hat{Y}_k + a\right)^5} - 2\sqrt{2}\sum_{j=1}^{n}\left\{\frac{1}{\left(\hat{Y}_j + a\right)} + \frac{\hat{Y}_j}{\left(\hat{Y}_j + a\right)^2} + \frac{2}{\left(\hat{Y}_j + a\right)^3}\right\},$$

where $a = 2\sqrt{2}$, $\hat{Y}_j = X_j/\hat{\theta}$ and $\hat{\theta}$ denotes the consistent estimator of θ. The null hypothesis, H_0, is rejected for large values of L.

Empirical Likelihood Ratio Test

Safavinejad et al. (2015), based on the empirical likelihood ratio methodology, proposed the following statistic

$$R_n = \frac{\min_{1\leq m<n^{\delta}} \prod_{i=1}^{n}\left\{\frac{2m}{n}\left(X_{(j+m)} - X_{(j-m)}\right)\right\}}{\left(\prod_{i=1}^{n}\frac{X_i}{\hat{\theta}^{2n}}\right) exp\left\{-\sum_{i=1}^{n}\frac{X_i^2}{2\hat{\theta}^2}\right\}},$$

where $\hat{\theta}$ is the ML estimator of θ and $0 < \delta < 1$. The null hypothesis is rejected if $log(R_n) > c_\alpha$, where c_α is a critical value. Meanwhile, Alizadeh et al. (2014) proved the invariancy and asymptotic consistency of their proposed test. The null hypothesis reject for large values of R_n.

Empirical Distribution Function Tests

Empirical distribution is a very important estimator in statistics. Many statistical procedures depend on its performance. Consider a random variable X with the distribution function $F(x)$. A random sample of size n, $X_1, \cdots, X_n$, is given from $F(x)$ and let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics. We assume this distribution to be continuous. The empirical distribution function, $F_n(x)$, is defined as

$$F_n(x) = \sum_{i=1}^{n} I(X_i \leq x)/n$$

where I is the indicator function.
The Kolmogorov-Smirnov D and Kuiper V statistics are defined as

$$D = max(D_n^+, D_n^-),$$

$$V = D_n^+ + D_n^-,$$

where

$$D_n^+ = \max_i\left\{i/n - Z_{(i)}\right\},$$
$$D_n^- = \max_i\left\{Z_{(i)} - (i-1)/n\right\}$$

and $Z = F(x)$; when $F(x)$ is the true distribution of X, the new random variable Z is uniformly distributed between 0 and 1. If a sample contains $X_1, \cdots, X_n$, it gives values $Z_i = F(x_i; \theta),\ i = 1, \cdots, n$. The formulas involve the Z-values arranged in ascending order, $Z_{(1)} < Z_{(2)} < \cdots < Z_{(n)}$, with $\bar{Z} = \sum_i Z_i/n$.
The Cramer-von Mises, Watson, Anderson-Darling, and Finkelstein-Schafers (1971) statistics are respectively as follows

$$W^2 = \sum_{i=1}^{n}\left\{Z_{(i)} - \frac{2i-1}{2n}\right\}^2 + \frac{1}{12n},$$

$$U^2 = W^2 - n(\bar{Z} - 0.5)^2,$$

$$A^2 = -n - \sum_{i=1}^{n}\frac{2i-1}{n}\left\{log(Z_{(i)}) + log(1 - Z_{(n-i+1)})\right\},$$

$$S^* = \sum_{i=1}^{n} max\left\{\left|Z_{(i)} - \frac{i}{n}\right|, \left|Z_{(i)} - \frac{i-1}{n}\right|\right\}.$$

where θ has been substituted by $\hat{\theta}$ which is the ML estimator of θ. The null hypothesis, H_0, is rejected for large values of W^2, U^2, A^2 and S^*.

Power Study

In this section, the power of considered tests are estimated against several alternatives. The method is that of Monte Carlo simulation of the distribution of considered tests under alternative distributions. For each alternative, 10000 samples of size n were generated. Then, the power was calculated by proportion of rejecting null hypothesis for simulated data from alternative distribution in some iterations. In other words, the power can be estimated by

$$\text{power} \simeq \frac{1 + \sum_{j=1}^{B} I\left\{\text{p-value}(T_{obs}^{(j)}) < \alpha | H_1\right\}}{1 + B},$$

where p-value $(T_{obs}^{(j)})$ is the probability value of T_{obs} at j-th iteration and H_1 is the alternative distribution. This form of definition of power ensures that the approximate power is a number strictly between 0 and 1.
The p-value can be easily calculated using Monte Carlo simulation proposed by North et al. (2003) through simulating B samples from the distribution specified in the null hypothesis and computing the test statistic, T_{obs}, for each of these samples. Then the p-value will be

$$\text{p-value} \simeq \frac{1 + \sum_{k=1}^{n} I\{T_k(x_1, \dots, x_n) > T_{obs}\}}{1 + B},$$

where $T_k(x_1, \dots, x_n)$ is the statistic from the simulated Rayleigh data $(x_1, \dots, x_n)$ at k-th iteration and T_{obs} is the observed statistic.
It is important to mention here that as we checked the power values which were calculated by the method mentioned in this paper and the classical method which is based on using critical values to estimate powers, the calculated powers based on these methods are very close together and the difference between them are usually in thousandths decimal which are negligible.
The continuous alternative distributions are classified in the three classes below:

- Monotonic decreasing hazard (Dec. Hazard) rate: Chisquare(1), Weibull(0.5,1),
- Monotonic increasing hazard (Inc. Hazard) rate: Chisquare(3), Beta(3,1),
- Non-Monotonic hazard (Non-Mon. Hazard) rate: Beta(1,0.5), Exponential(2).

The powers of the considered tests are compared to that of some other tests for Rayleighity against the same alternatives which are tabulated in Tables 2 and 3.

Determining Optimum values of window size (m)

The GOF test based on entropy involves choosing the best integer parameter m. Unfortunately, there is no choice criterion of m, and in general it depends on n. Ebrahimi et al. (1992) tabulated the values of m which maximize the power of the test. Meanwhile, similar tables are given by Abbasnejad (2011), Alizadeh Noughabi and Balakrishnan (2014).

Meanwhile, the optimum value of m is a value of n which leads to the smallest value of bias and mean square error (MSE). Therefore, we determined the optimum values of m based on 10000 iterations which their results are presented in Table 1.

To determine optimum values of m for any value of n, we compute bias and MSE of H_{mn} for different values of m, 1 to $n/2$, by

$$Bias = \frac{1}{k}\sum_{i=1}^{k} H_{mn}^{(i)} - H(X)\,, \quad MSE = \frac{1}{k}\sum_{i=1}^{k}\left\{H_{mn}^{(i)} - H(X)\right\}^2,$$

where $H(X) = 1 + ln(\theta/\sqrt{2}) + \gamma/2$ is the entropy of the Rayleigh distribution, γ is the Euler-Mascheroni constant which is already 0.57721 and k is the number of iterations. From Table 1 it is evident that with increasing n, the optimal choice of m also increases.

Table 1: Optimal values of m for various values of n

n	5-12	13-27	28-50	51-57	58-71	72-100	101-120	121-150	>150
m	2	3	4	5	6	7	8	9	10

Table 2: Comparing powers of considered tests for n=10 and α=0.05

	Dec. Hazard		Inc. Hazard		Non-Mon. Hazard	
Tests	Chisq(1)	Wei(0.5,1)	Chisq(3)	Beta(3,1)	Beta(1,0.5)	Exp(2)
L	0.998	0.996	0.420	0.252	0.140	0.412
D	0.890	0.971	0.330	0.281	0.224	0.221
V	0.820	0.956	0.220	0.351	0.348	0.402
W^2	0.903	0.973	0.345	0.362	0.290	0.231
U^2	0.833	0.956	0.259	0.340	0.345	0.426
A^2	0.977	0.995	0.464	0.417	0.354	0.381
S^*	0.894	0.970	0.350	0.357	0.341	0.236
KL_{mn}	0.915	0.979	0.176	0.451	0.566	0.205
CK_n	0.918	0.979	0.405	0.395	0.441	0.449
R_n	0.718	0.911	0.468	0.477	0.557	0.540
LW_{mn}	0.164	0.384	0.005	0.325	0.308	0.221
R_{mn}	0.897	0.960	0.161	0.382	0.434	0.216
J_{mn}	0.972	0.992	0.458	0.396	0.571	0.727

Table 3: Comparing powers of considered tests for n=10 and α=0.05

	Dec. Hazard		Inc. Hazard		Non-Mon. Hazard	
Tests	Chisq(1)	Wei(0.5,1)	Chisq(3)	Beta(3,1)	Beta(1,0.5)	Exp(2)
L	0.999	0.999	0.623	0.469	0.160	0.413
D	0.998	0.999	0.587	0.514	0.417	0.237
V	0.979	0.999	0.490	0.658	0.668	0.701
W^2	0.997	0.999	0.646	0.668	0.563	0.240
U^2	0.999	0.999	0.632	0.609	0.649	0.716
A^2	0.997	0.999	0.707	0.612	0.615	0.631
S^*	0.993	0.999	0.631	0.708	0.615	0.250
KL_{mn}	0.999	0.999	0.450	0.850	0.901	0.205
CK_n	0.999	0.999	0.653	0.341	0.867	0.873
R_n	0.880	0.987	0.709	0.831	0.902	0.877
LW_{mn}	0.416	0.766	0.013	0.807	0.769	0.624
R_{mn}	0.995	0.999	0.340	0.831	0.853	0.772
J_{mn}	0.999	0.999	0.856	0.862	0.928	0.931

According to Tables 2 and 3 we conclude:

In decreasing hazard rate distributions between entropy type tests, the most powerful tests in priority of order are respectively J_{mn}, CK_n, KL_{mn}. In addition, it is evident that the three most powerful tests in overall are respectively L, A^2, J_{mn}.

Among increasing hazard rate distributions choosing the best test is difficult. But it should be noticed that among entropy type tests the most powerful tests in priority of order are R_n, J_{mn}, CK_n. Also, we can offer three most powerful tests in overall as R_n, A^2, J_{mn}.

Although among non-monotone hazard rate distributions choosing the best test is difficult, three most powerful tests in overall and in entropy type tests are respectively J_{mn}, R_n, CK_n.

Another important property of a test is type I error rate which is calculated as proportion of rejected null hypothesis for the Rayleigh distribution. As we checked, we conclude that type I error rates of all considered tests are about the supposed significance level. So, they have acceptable performance in this field.

Application to average wind speed data

In this section, we consider one average wind speed data analysis reported in Best *et al.* (2010). The following data represent 30 average daily wind speeds (in km/h) for the month of November 2007 recorded at Elanora Heights, a northeastern suburb of Sydney, Australia:

2.7, 3.2, 2.1, 4.8, 7.6, 4.7, 4.2, 4.0, 2.9, 2.9, 4.6, 4.8, 4.3, 4.6, 3.7, 2.4, 4.9, 4.0, 7.7, 10.0, 5.2, 2.6, 4.2, 3.6, 2.5, 3.3, 3.1, 3.7, 2.8, 4.0.

The data were analyzed initially by Best et al. and Alizadeh et al. (2012), who fitted the Rayleigh distribution successfully. The probability values of testing Rayleighity for some of more powerful tests in this paper are presented in Table 4. The probability values of all tests are greater than 0.05, so they are not significant at 5 percent significance level. Thus, we cannot reject the null hypothesis which means the wind speed data follows the Rayleigh distribution.

Table 4: P-value of considered tests in evaluating Rayleighity

Tests	L	R_n	J_{mn}	KL_{mn}	CK_n	A^2
p-value	0.834	0.745	0.753	0.737	0.831	0.810

Conclusion

The aim of this paper is to evaluate the performance of the proposed test. So, we considered and compared thirteen different GOF tests for evaluating Rayleighity. In order to evaluate the ability of mentioned tests in identifying the Rayleigh distribution, by a simulation study the powers and type I error of the proposed tests were computed under several alternatives and different sample sizes in three different class of hazard rates. The results were tabulated in Tables 2 and 3 which are the power estimates of considered tests at significance level of 0.05.

Finally, we concluded that J_{mn} test which is a symmetric version of KL test and belongs to entropy based tests performed better than the other tests. Since for real data we do not know type of hazard rate distribution, the results obtained from this study encourage us using J_{mn} test in all three classes.

The future of this research is to work on proposing GOF tests for censored data such as Type II censored data which is very well-known and very applicable to applied researches.

Acknowledgments
We like to thank referee and associate editor for important and constructive comments, which improved the presentation.

References

1. Abbasnejad, M. (2011). Some goodness of fit tests based on Rényi information, Applied Mathematical Sciences, 5**,** 1921-1934.
2. Abbasnejad, M., Arghami, N. R. & Tavakoli, M. (2012). A Goodness of Fit Test for Exponentiality Based on Lin-Wong Information, JIRSS, 11(2), 191-202.
3. Abd-Elfattah, A. M. (2011). Goodness of fit test for the generalized Rayleigh distribution with unknown parameters, Journal of Statistical Computation and Simulation, 81(3), 357-366.
4. Ahrari, V., Baratpour, S., Habibirad, A. & Fakoor, V. (2019). Goodness of fit tests for Rayleigh distribution based on quantiles, Communications in Statistics - Simulation and Computation, DOI: 10.1080/03610918.2019.1651336.
5. Alizadeh, R., Alizadeh, H. & Ebrahimi, A. (2012). An entropy test for the Rayleigh distribution and power comparison, Journal of Statistical Computation and Simulation, 151-158.
6. Alizadeh Noughabi, H. & Arghami, N. R. (2011a). Monte Carlo comparison of five exponentiality tests using different entropy estimates, J. Statist. Comput. Simul., 81, 1579-1592.
7. Alizadeh Noughabi, H. & Arghami, N. R. (2011). Testing exponentiality based on characterizations of the exponential distribution, Journal of Statistical Computation and Simulation, 81, 1641-1651.
8. Alizadeh Noughabi, H. & Balakrishnan, N. (2014). Goodness of Fit Using a New Estimate of Kullback-Leibler Information Based on Type II Censored Data, IEEE Transactions on Reliability, 99.
9. Alizadeh Noughabi, H. (2010). A new estimator of entropy and its application in testing normality, J. Stat. Comput. Simul, 80(10), 1151-1162.
10. Arellano-Valle, R. B., Galea-Rojas, M. & Iglesias, P. (2000). Bayesian sensitivity analysis in elliptical linear regression models, J. Stat. Plan. Infer, 86, 175-199.
11. Al-Omari, A. & Zamanzade, E. (2016). Different goodness of fit tests for Rayleigh distribution in ranked set sampling, Pakistan Journal of Statistics and Operation Research, 12(1), 25-39.
12. Badr, M. M. (2019). Goodness-of-fit tests for the Compound Rayleigh distribution with application to real data, Heliyon 5(8).
13. Baratpour, S. & Khodadadi, F. (2012). A Cumulative Residual Entropy Characterization of the Rayleigh Distribution and Related Goodness-of-Fit Test, J. Statist. Res. Iran, 9, 115-131.
14. Baratpour, S. & Habibi, A. (2012). Testing Goodness-of-Fit for Exponential Distribution Based on Cumulative Residual Entropy, Communications in Statistics - Theory and Methods, 41, 1387-1396.
15. Best, D. J., Rayner, J. C. W. & Thas, O. (2010). Easily applied tests of fit for the Rayleigh distribution, Sankhya B, 72, 254-263.
16. Correa, J. C. (1995). A new estimator of entropy, Comm. Statist. Theory Methods, 24, 2439-2449.
17. Dyer, D. D. & Whisenand, C. W. (1973). Best linear unbiased estimator of the parameter of the Rayleigh distribution, IEEE Transaction on Reliability, 22, 27-34.
18. Dudewicz, E. J. & Van der Meulen, E. C. (1981). Entropy-based tests of uniformity, J. Amer. Statist. Assoc. 76, 967-974.
19. Ebrahimi, N., Pflughoeft, K. & Soofi, E. (1994). Two measures of sample entropy, Statist. Probab. Lett. 20, 225-234.
20. Ebrahimi, N., Habibullah, M. & Soofi, E. S. (1992). Testing exponentiality based on Kullback-Leibler information, J. R. Statist. Soc., 54, 739-748.
21. Finkelstein, J. & Schafer, R. E. (1971). Imported goodness of fit tests, Biometrika, 58, 641-645.
22. Garrido, A. (2009). About some properties of the Kullback-Leibler divergence, Advanced Modeling and Optimization, 11(4).
23. Jahanshahi, S. M. A., Habibirad, A., Fakoor, V. (2016). A Goodness-of-Fit Test for Rayleigh Distribution Based on Hellinger Distance, Annals of Data Science, 3(4), 401-411.
24. Jeffreys, H. (1946). An invariant form for the prior probability in estimation problems, Proc. Roy. Soc. Lon., Ser. A, 186, 453-461.

25. Johnson, N. L., Kotz, S. & Balakrishnan, N. (1994). Continuous univariate distributions, Wiley, New York.
26. Lin, J. & Wong, S. K. M. (1990). A new directed divergence measure and its characterization, Int. J. General Systems, 17, 73-81.
27. Kapur, J. N. & Kesavan, H. K. (1992). Entropy optimization principles with application, Academic Press.
28. Meintanis, S. G. and Iliopoulos, G. (2003). Tests of fit for the Rayleigh distribution based on the empirical Laplace transform, Ann. Inst. Statist. Math., 55(1)**,** 137-151.
29. Morris, K. & Szynal, D. (2008). Some U-statistics in goodness-of-fit tests derived from characterizations via record values, Int. J. of Pure and Appl. Math., 46, 507-582.
30. Miller, K. & Sackrowttz, H. (1967). Relationships Between Biased and Unbiased Rayleigh Distributions, SIAM journal on applied Mathematics, 15, 1490-1495.
31. North B., Curtis, D. & Sham, P. (2003). A Note on the Calculation of Empirical p-values from Monte Carlo Procedures, American Journal of Human Genetics, 71(2), 439-441.
32. Pashardes, S. & Christofides, C. (1995). Statistical analysis of wind speed and direction in Cyprus, Solar Energy, 55(5), 405-414.
33. Polovko, A. M. (1968). Fundamentals of reliability theory, Academic Press, New York.
34. Rayleigh, J. W. S. (1880). On the resultant of a large number of vibrations of some pitch and of arbitrary phase, Philosophical Magazine, 5-th Series, 10, 73-78.
35. Safavinejad, M., Jomhoori, S. & Alizadeh Noughabi, H. (2015). A density-based empirical likelihood ratio goodness-of-fit test for the Rayleigh distribution and power comparison, Journal of Statistical Computation and Simulation, 85, 3322-3334.
36. Shannon, C. E. (1948). A mathematical theory of communication. Bell System Technical Journal, 27, 379-423.
37. Siddiqui, M. M. (1962). Some problems connected with Rayleigh distributions, J. Res. Natl. Bur. Stand. 60D, 167-174.
38. Ullah, A. (1996). Entropy, divergence and distance measures with econometric applications, J. Stat. Plan. Infer., 49, 137-162.
39. Vasicek, O. (1976). A test for normality based on sample entropy, J. R. Statist. Soc., 38, 54-59.
40. Van Es, B. (1992). Estimating functional related to a density by a lass of statistic based on spacing, Scand. J. Statist., 19, 61-72.
41. Zamanzade, E. & Mahdizadeh, M. (2017). Goodness of fit tests for Rayleigh distribution based on Phi-divergence, Revista Colombiana de Estadística 40(2), 279-290.