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Length biased Weighted New Quasi Lindley Distribution: Statistical Properties and Applications

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Abstract

In this Paper, we have introduced a new version of new quasi Lindley distribution known as the length-biased weighted new quasi Lindley distribution (LBWNQLD). The length biased distribution is a special case of weighted distribution. The different mathematical and statistical properties of the newly proposed distribution are derived and discussed in detail. The model parameters of the newly executed distribution are estimated by using the method of maximum likelihood estimation and also the Fisher's information matrix have been discussed. Finally, an application of the proposed distribution is demonstrated with two real life data sets for illustrating its supremacy and applicability.

Key Words: Weighted distribution, New quasi Lindley distribution, Order Statistics, Maximum likelihood estimation, Reliability Analysis, Entropies

1. Introduction

The study of weighted distributions are useful in distribution theory because it provides a new understanding of the existing standard probability distributions and it provides methods for extending existing standard probability distributions for modeling lifetime data due to the introduction of additional parameter in the model which creates flexibility in their nature. Weighted distributions occur in modeling clustered sampling, heterogeneity, and extraneous variation in the dataset. The concept of weighted distributions was firstly introduced by Fisher (1934) to model ascertainment biases which were later formalized by Rao (1965) in a unifying theory for problems where the observations fall in non-experimental, non-replicated and non-random manner. When observations are recorded by an investigator in the nature according to certain stochastic model, the distribution of the recorded observations will not have the original distribution unless every observation is given an equal chance of being recorded. Weighted models were formulated in such situations to record the observations according to some weighted function. The weighted distribution reduces to length biased distribution when the weight function considers only the length of the units. The concept of length biased sampling was first introduced by Cox (1969) and Zelen (1974).Warren (1975) was the first to apply the size biased distributions in connection with sampling wood cells. Patil and Rao (1978) studied weighted distributions and size biased sampling with applications to wildlife populations and human families. Van Deusen (1986) arrived at size biased distribution theory independently and applied it in fitting assumed distributions to data arising from horizontal point sampling .More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, resulting distribution is called size-biased. There are various good sources which provide the detailed description of weighted distributions. Different authors have reviewed and studied the various weighted probability models and illustrated their applications in different fields. Weighted distributions are applied in various research areas related to reliability, biomedicine, ecology and branching processes. Afaq *et al* (2016) have obtained the length biased weighted version of lomax distribution with properties and applications. Reyad et al. (2017) obtained the length biased weighted frechet distribution with properties and estimation. Mudasir and Ahmad (2018) discussed the characterization and estimation of length biased Nakagami distribution. Para and Jan (2018) introduced the Weighted Pareto type II Distribution as a new model for handling medical science data and studied its statistical properties and applications. Rather and Subramanian (2019) obtained the length biased erlang truncated exponential distribution with lifetime data. Rather and Ozel (2020) discussed on the weighted power Lindley distribution with application of real life time data. Hassan, Dar, Peer and Para (2019) obtained the weighted version of Pranav distribution with real life data. Hassan, Wani and Para (2018) discussed on the weighted three parameter quasi Lindley distribution with properties and applications. Ganaie, Rajagopalan and Rather (2019) discussed on the length biased Aradhana distribution with applications. Recently, Ganaie, Rajagopalan and Rather (2020) discussed on the weighted two parameter quasi Shanker distribution with properties and its applications, which shows more reliable and efficient than the classical distribution.

Shanker and Ghebretsadik (2013) introduced a new quasi Lindley distribution is a newly proposed two parametric probability distribution and derive its various mathematical and statistical properties as moments, skewness, kurtosis, failure rate function, mean residual life function and stochastic ordering. It is observed that the expression for failure rate function, mean residual life function and stochastic ordering of the new quasi Lindley distribution shows its flexibility over Lindley distribution, exponential distribution and quasi Lindley distribution of Shanker and Mishra (2013). Also, the new quasi Lindley distribution is a particular case of one parameter Lindley distribution.The parameter estimation is also discussed by using the method of moments and method of maximum likelihood estimation. The goodness of fit of new quasi Lindley distribution has been fitted to number of data sets related to survival times, grouped mortality data and waiting times to test its goodness of fit and it is observed that the new quasi Lindley distribution provides closer fit than those by the Lindley and quasi Lindley distribution.

2. Length Biased Weighted New Quasi Lindley Distribution (LBWNQLD)

The probability density function of new quasi Lindley distribution is given by

$$f(x;\theta,\alpha) = \frac{\theta^2}{\theta^2+\alpha}(\theta+\alpha x)e^{-\theta x};\ \ x>0, \theta>0, \alpha<-\theta^2 \quad (1)$$

and the cumulative distribution function of the new quasi Lindley distribution is given by

$$F(x;\theta,\alpha) = 1-\frac{\theta^2+\alpha+\theta\alpha x}{\theta^2+\alpha}e^{-\theta x}\,; x>0, \theta>0, \alpha<-\theta^2 \quad (2)$$

Suppose X is a non-negative random variable with probability density function $f(x)$.Let *w(x)* be the non-negative weight function, then, the probability density function of the weighted random variable X_w is given by

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))},\ x>0.$$

Where $w(x)$ be a non negative weight function and $E(w(x)) = \int w(x)f(x)dx < \infty.$

We should note that the different choices of the weight function *w(x)* gives different weighted distributions. When $w(x) = x^c$, the result is known as weighted distributions and when $w(x) = x$, the result is known as length biased distribution. In this paper, we have to obtain the length biased version of new quasi Lindley distribution. The length biased weighted new quasi Lindley distribution is obtained by taking c =1 in weights x^c to the weighted new quasi Lindley distribution in order to obtain the length biased weighted new quasi Lindley distribution. Therefore, the probability density function of length biased weighted new quasi Lindley distribution is given by

$$f_l(x;\theta,\alpha) = \frac{xf(x;\theta,\alpha)}{E(x)}, x>0 \quad (3)$$

Where

$$E(x) = \int_0^\infty xf(x;\theta,\alpha)dx$$

$$E(x) = \frac{(\theta^2+2\alpha)}{\theta(\theta^2+\alpha)} \quad (4)$$

Substitute the equations (1) and (4) in equation (3), we will obtain the probability density function of length biased weighted new quasi Lindley distribution

$$f_l(x;\theta,\alpha) = \frac{x\theta^3}{(\theta^2+2\alpha)}(\theta+\alpha x)e^{-\theta x} \tag{5}$$

and the cumulative distribution function of length biased weighted new quasi Lindley distribution is given by

$$F_l(x;\theta,\alpha) = \int_0^x f_l(x\,;\theta,\alpha)dx$$

$$F_l(x;\theta,\alpha) = \int_0^x \frac{x\theta^3}{(\theta^2+2\alpha)}(\theta+\alpha x)e^{-\theta x}\,dx$$

$$F_l(x\,;\theta,\alpha) = \frac{\theta^3}{(\theta^2+2\alpha)}\int_0^x x(\theta+\alpha x)e^{-\theta x}dx \tag{6}$$

After the simplification of equation (6), we obtain the cumulative distribution function of length biased weighted new quasi Lindley distribution

$$F_l(x\,;\theta,\alpha) = \frac{1}{(\theta^2+2\alpha)}\left(\theta^2\gamma(2,\theta x)+\alpha\gamma(3,\theta x)\right) \tag{7}$$

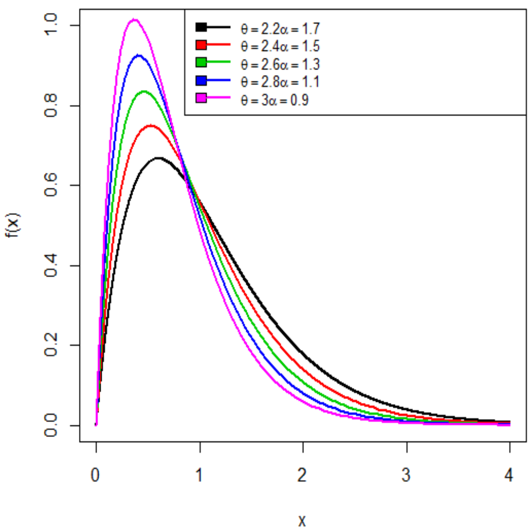

Fig.1:Pdf plot of Length biased weighted new quasi Lindley distribution

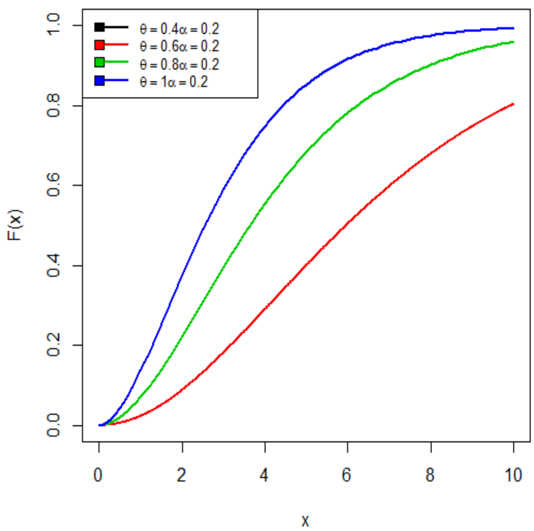

Fig.2 Cdf plot of Length biased weighted new quasi Lindley distribution

3. Reliability Analysis

In this sub section, we obtain the Reliability function, hazard function and Reverse hazard rate function for the proposed length biased weighted new quasi Lindley distribution.

The reliability function or the survival function of length biased weighted new quasi Lindley distribution is given by

$$R(x) = 1 - F_l(x\,;\theta,\alpha)$$

$$\boldsymbol{R(x) = 1 - \frac{1}{(\theta^2+2\alpha)}\left(\theta^2\gamma(2,\theta x)+\alpha\gamma(3,\theta x)\right)}$$

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The hazard function is also known as hazard rate or instantaneous failure rate or force of mortality and is given by

$$h(x) = \frac{f_l(x;\theta,\alpha)}{R(x)}$$

$$h(x) = \frac{x\theta^3}{(\theta^2 + 2\alpha) - (\theta^2\gamma(2,\theta x) + \alpha\gamma(3,\theta x))}(\theta + \alpha x)e^{-\theta x}$$

The reverse hazard function of length biased weighted new quasi Lindley distribution is given by

$$h^r(x) = \frac{f_l(x;\theta,\alpha)}{F_l(x;\theta,\alpha)}$$

$$h^r(x) = \frac{x\theta^3}{(\theta^2\gamma(2,\theta x) + \alpha\gamma(3,\theta x))}(\theta + \alpha x)e^{-\theta x}$$

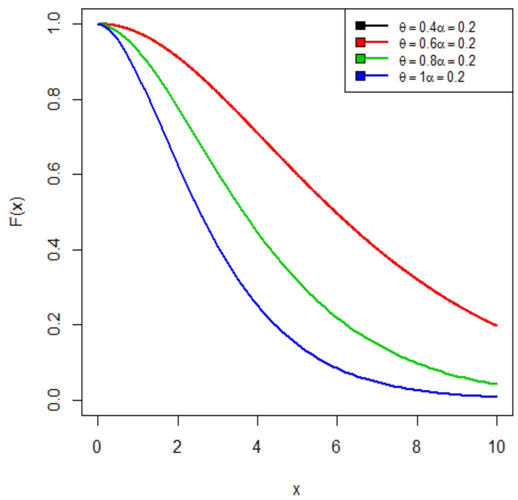

Fig 3 Reliability plot of Length biased weighted new quasi Lindley distributio

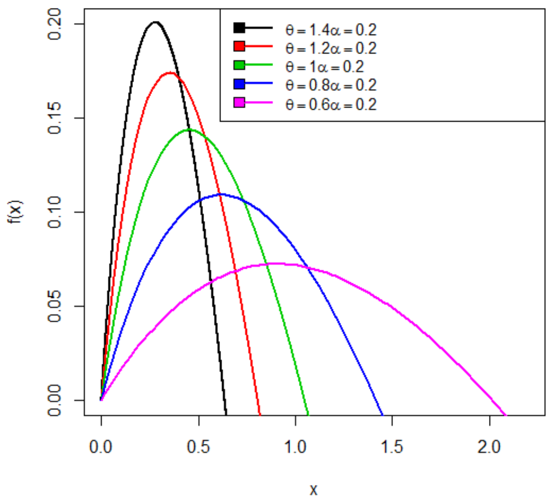

Fig.4 Hazard plot of Length biased weighted new quasi Lindley distribution

4. Statistical properties

In this section we shall discuss the structural properties of length biased weighted new quasi Lindley distribution, especially its moments, harmonic mean, moment generating function and characteristic function.

4.1 Moments

Let X denotes the random variable of length biased weighted new quasi Lindley distribution with parameters θ and α, then the rth order moment $E(X^r)$ of the length biased weighted new quasi Lindley distribution is obtained as

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty x^r \frac{x\theta^3}{(\theta^2 + 2\alpha)}(\theta + \alpha x)e^{-\theta x}dx$$

$$= \frac{\theta^3}{(\theta^2 + 2\alpha)}\int_0^\infty x^{r+1}(\theta + \alpha x)e^{-\theta x}dx$$

$$= \frac{\theta^3}{(\theta^2 + 2\alpha)}\left(\theta\int_0^\infty x^{(r+2)-1}e^{-\theta x}dx + \alpha\int_0^\infty x^{(r+3)-1}e^{-\theta x}dx\right)$$

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After simplification, we obtain

⥂⥂⥂⥂ $E(X^r) = \mu_r{}' = \frac{\theta^2\Gamma(r+2)+\alpha\Gamma(r+3)}{\theta^r(\theta^2+2\alpha)}$ (8)

Substitute r = 1, 2, 3, 4 in equation (8), we obtain the first four moments of length biased weighted new quasi Lindley distribution.

$$E(X) = \mu_1{}' = \frac{2\theta^2 + 6\alpha}{\theta(\theta^2 + 2\alpha)}$$

$$E(X^2) = \mu_2{}' = \frac{6\theta^2 + 24\alpha}{\theta^2(\theta^2 + 2\alpha)}$$

$$E(X^3) = \mu_3{}' = \frac{24\theta^2 + 120\alpha}{\theta^3(\theta^2 + 2\alpha)}$$

$$E(X^4) = \mu_4{}' = \frac{120\theta^2 + 720\alpha}{\theta^4(\theta^2 + 2\alpha)}$$

$$\text{Variance } (\mu_2) = \frac{6\theta^2 + 24\alpha}{\theta^2(\theta^2 + 2\alpha)} - \left(\frac{2\theta^2 + 6\alpha}{\theta(\theta^2 + 2\alpha)}\right)^2$$

$$\text{Standard deviation } \sigma = \sqrt{\left(\frac{6\theta^2+24\alpha}{\theta^2(\theta^2+2\alpha)} - \frac{(2\theta^2+6\alpha)^2}{\theta^2(\theta^2+2\alpha)^2}\right)}$$

$$\text{Coefficient of variation} = \frac{\sigma}{\mu_1{}'} = \sqrt{\left(\frac{6\theta^2 + 24\alpha}{\theta^2(\theta^2 + 2\alpha)} - \frac{(2\theta^2 + 6\alpha)^2}{\theta^2(\theta^2 + 2\alpha)^2}\right)} \times \frac{\theta(\theta^2 + 2\alpha)}{(2\theta^2 + 6\alpha)}$$

4.2 Harmonic mean

The harmonic mean is the reciprocal of the arithmetic mean of the reciprocals. The harmonic mean for the proposed length biased weighted new quasi Lindley distribution is given by

$$H.M = E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_l(x;\theta,\alpha)dx$$

$$= \int_0^\infty \frac{\theta^3}{(\theta^2+2\alpha)}(\theta + \alpha x)e^{-\theta x}dx$$

$$= \frac{\theta^3}{(\theta^2 + 2\alpha)}\left(\theta \int_0^\infty e^{-\theta x}dx + \alpha \int_0^\infty x e^{-\theta x}dx\right)$$

$$= \frac{\theta^3}{(\theta^2 + 2\alpha)}\left(\theta \int_0^\infty x^{1-1}e^{-\theta x}dx + \alpha \int_0^\infty x^{2-1}e^{-\theta x}dx\right)$$

After simplification, we obtain

$$H.M = \frac{\theta^3}{(\theta^2 + 2\alpha)} (\theta\gamma(1, \theta x) + \alpha\gamma(2, \theta x))$$

4.3 Moment generating function

The moment generating function is the expected function of a continuous random variable and the moment generating function for the length biased weighted new quasi Lindley distribution is given by

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f_l(x; \theta, \alpha) dx$$

$$= \int_0^\infty \left(1 + tx + \frac{(tx)^2}{2!} + \ldots\right) f_l(x; \theta, \alpha) dx$$

$$= \int_0^\infty \sum_{j=0}^{\infty} \frac{t^j}{j!} x^j f_l(x ; \theta, \alpha) dx$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \mu_j{}'$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!} \left(\frac{\theta^2 \Gamma(j+2) + \alpha\Gamma(j+3)}{\theta^j (\theta^2 + 2\alpha)}\right)$$

$$\Rightarrow M_X(t) = \frac{1}{(\theta^2 + 2\alpha)} \sum_{j=0}^{\infty} \frac{t^j}{j!\, \theta^j} (\theta^2 \Gamma(j+2) + \alpha\Gamma(j+3))$$

4.4 Characteristic function

The characteristic function is defined as the function of any real valued random variable completely defines the probability distribution of a random variable and the characteristics function exists always even if moment generating function does not exists. The characteristic function of length biased weighted new quasi Lindley distribution is given by

$$\phi_x(t) = M_X(it)$$

$$\Rightarrow M_X(it) = \frac{1}{(\theta^2 + 2\alpha)} \sum_{j=0}^{\infty} \frac{(it)^j}{j!\, \theta^j} (\theta^2 \Gamma(j+2) + \alpha\Gamma(j+3))$$

5. Order Statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ drawn from the continuous population with cumulative distribution function $F_x(x)$ and probability density function $fx(x)$, then the probability density function of rth order statistics $X_{(r)}$ is given by

$f_{x(r)}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x)(F_X(x))^{r-1}(1-F_X(x))^{n-r}$ (9)

For r = 1, 2… n.

Using equations (5) and (7) in equation (9), we obtain the probability density function of *r*th order statistics of length biased weighted new quasi Lindley distribution which is given by

$$f_{x(r)}(x) = \frac{n!}{(r-1)!\,(n-r)!}\left(\frac{x\theta^3}{(\theta^2+2\alpha)}(\theta+\alpha x)e^{-\theta x}\right)\left(\frac{1}{(\theta^2+2\alpha)}(\theta^2\gamma(2,\theta x)+\alpha\gamma(3,\theta x))\right)^{r-1}$$
$$\times\left(1-\frac{1}{(\theta^2+2\alpha)}(\theta^2\gamma(2,\theta x)+\alpha\gamma(3,\theta x))\right)^{n-r}$$

Therefore, the probability density function of higher order statistics $X_{(n)}$ of length biased weighted new quasi Lindley distribution is given by

$$f_{x(n)}(x) = \frac{nx\theta^3}{(\theta^2+2\alpha)}(\theta+\alpha x)e^{-\theta x}\left(\frac{1}{(\theta^2+2\alpha)}(\theta^2\gamma(2,\theta x)+\alpha\gamma(3,\theta x))\right)^{n-1}$$

and the probability density function of the first order statistics $X_{(1)}$ of length biased weighted new quasi Lindley distribution is given by

$$f_{x(1)}(x) = \frac{nx\theta^3}{(\theta^2+2\alpha)}(\theta+\alpha x)e^{-\theta x}\left(1-\frac{1}{(\theta^2+2\alpha)}(\theta^2\gamma(2,\theta x)+\alpha\gamma(3,\theta x))\right)^{n-1}$$

6. Likelihood Ratio Test

Let X_1, X_2, …….X_n be a random sample of size *n* from the new quasi Lindley distribution or length biased weighted new quasi Lindley distribution. We test the hypothesis.

$$H_o: f(x) = f(x\,;\theta,\alpha) \quad \text{against} \quad \mathrm{H}_1: f(x) = f_l(x\,;\theta,\alpha)$$

Thus for testing the hypothesis, whether the random sample of size *n* comes from new quasi Lindley distribution or length biased weighted new quasi Lindley distribution, the following test statistic is used.

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n}\frac{f_l(x\,;\theta,\alpha)}{f(x\,;\theta,\alpha)}$$

$$\Delta = \frac{L_1}{L_o} = \prod_{i=1}^{n}\left(\frac{x_i\theta(\theta^2+\alpha)}{(\theta^2+2\alpha)}\right)$$

$\Delta = \frac{L_1}{L_o} = \left(\frac{\theta(\theta^2+\alpha)}{(\theta^2+2\alpha)}\right)^n \prod_{i=1}^{n} x_i$

We should reject the null hypothesis if

$$\Delta = \left(\frac{\theta(\theta^2+\alpha)}{(\theta^2+2\alpha)}\right)^n \prod_{i=1}^{n} x_i > k$$

Equivalently, we reject the null hypothesis if

$$\Delta^* = \prod_{i=1}^{n} x_i > k\left(\frac{(\theta^2+2\alpha)}{\theta(\theta^2+\alpha)}\right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i > k^*, \text{Where } \; k^* = k\left(\frac{(\theta^2+2\alpha)}{\theta(\theta^2+\alpha)}\right)^n$$

Thus if the sample is large of size *n*, *2log Δ* is distributed as chi-square distribution with one degree of freedom and also p- value is obtained from the chi-square distribution. Also, we reject the null hypothesis, when the probability value is given by

$p(\Delta^* > \beta^*)$, Where $\beta^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance and $\prod_{i=1}^{n} x_i$ is the observed value of the statistic Δ^*

7. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves are used not only in economics to study the distribution of income or wealth or income or poverty, but it is also being used in other fields like reliability, medicine, insurance and demography. The Bonferroni and Lorenz curves are given by

$$B(p) = \frac{1}{p\mu_1'}\int_0^q x f_l(x;\theta,\alpha)dx$$

$$\text{and} \;\; L(p) = pB(p) = \frac{1}{\mu_1'}\int_0^q x f_l(x;\theta,\alpha)dx$$

$$\text{Where} \;\; \mu_1' = E(X) = \frac{(2\theta^2+6\alpha)}{\theta(\theta^2+2\alpha)} \quad \text{and} \;\; q = F^{-1}(p)$$

$$B(p) = \frac{\theta(\theta^2+2\alpha)}{p(2\theta^2+6\alpha))}\int_0^q \frac{\theta^3}{(\theta^2+2\alpha)} x^2(\theta+\alpha x)e^{-\theta x}dx$$

$$B(p) = \frac{\theta(\theta^2+2\alpha)}{p(2\theta^2+6\alpha)}\frac{\theta^3}{(\theta^2+2\alpha)}\int_0^q x^2(\theta+\alpha x)e^{-\theta x}dx$$

$$B(p) = \frac{\theta^4}{p(2\theta^2+6\alpha)}\left(\theta\int_0^q x^{3-1}e^{-\theta x}dx + \alpha\int_0^q x^{4-1}e^{-\theta x}dx\right)$$

After simplification, we obtain

$$B(p) = \frac{\theta^4}{p(2\theta^2+6\alpha)}(\theta\gamma(3,\theta q)+\alpha\gamma(4,\theta q))$$

$$\text{and} \;\; L(p) = pB(p) = \frac{\theta^4}{(2\theta^2+6\alpha)}(\theta\gamma(3,\theta q)+\alpha\gamma(4,\theta q))$$

8. Entropies

The concept of entropies is important in different areas such as probability and statistics, physics, communication theory and economics. Entropy is also called the degree of randomness or disorder in a system.

Entropies quantify the diversity, uncertainty, or randomness of a system. Entropy of a random variable X is a measure of variation of the uncertainty.

8.1 Renyi Entropy

The Renyi entropy is important in ecology and statistics as index of diversity. The entropy is named after Alfred Renyi. The Renyi entropy is important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

$$e(\beta) = \frac{1}{1-\beta} log\left(\int f_l{}^{\beta}(x;\theta,\alpha)\, dx\right)$$

Where, $\beta > 0$ and $\beta \neq 1$

$$e(\beta) = \frac{1}{1-\beta} log \int_0^{\infty} \left(\frac{x\theta^3}{(\theta^2+2\alpha)}(\theta+\alpha x)e^{-\theta x}\right)^{\beta} dx$$

$$e(\beta) = \frac{1}{1-\beta} log\left(\left(\frac{\theta^3}{(\theta^2+2\alpha)}\right)^{\beta} \int_0^{\infty} x^{\beta} e^{-\theta\beta x}(\theta+\alpha x)^{\beta} dx\right) (10)$$

Using Binomial expansion in (10), we obtain

$$e(\beta) = \frac{1}{1-\beta} log\left(\left(\frac{\theta^3}{(\theta^2+2\alpha)}\right)^{\beta} \sum_{j=0}^{\infty} \binom{\beta}{j} \theta^{\beta-j}(\alpha x)^j \int_0^{\infty} x^{\beta} e^{-\theta\beta x} dx\right)$$

$$e(\beta) = \frac{1}{1-\beta} log\left(\left(\frac{\theta^3}{(\theta^2+2\alpha)}\right)^{\beta} \sum_{j=0}^{\infty} \binom{\beta}{j} \theta^{\beta-j}\alpha^j \int_0^{\infty} x^{(\beta+j+1)-1} e^{-\theta\beta x} dx\right)$$

$$e(\beta) = \frac{1}{1-\beta} log\left(\left(\frac{\theta^3}{(\theta^2+2\alpha)}\right)^{\beta} \sum_{j=0}^{\infty} \binom{\beta}{j} \theta^{\beta-j}\alpha^j \frac{\Gamma(\beta+j+1)}{(\theta\beta)^{\beta+j+1}}\right)$$

8.2 Tsallis Entropy

A generalization of Boltzmann-Gibbs (B.G) statistical properties initiated by Tsallis has focused a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy (Tsallis, 1988) for a continuous random variable is defined as follows

$$S_{\lambda} = \frac{1}{\lambda-1}\left(1 - \int_0^{\infty} f_l{}^{\lambda}(x\,;\theta,\alpha)\, dx\right)$$

$$S_{\lambda} = \frac{1}{\lambda-1}\left(1 - \int_0^{\infty} \left(\frac{x\theta^3}{(\theta^2+2\alpha)}(\theta+\alpha x)e^{-\theta x}\right)^{\lambda} dx\right)$$

$$S_{\alpha} = \frac{1}{\lambda-1}\left(1 - \left(\frac{\theta^3}{(\theta^2+2\alpha)}\right)^{\lambda} \int_0^{\infty} x^{\lambda} e^{-\lambda\theta x}(\theta+\alpha x)^{\lambda} dx\right) (11)$$

Using Binomial expansion in equation (11), we get

$$S_{\lambda} = \frac{1}{\lambda-1}\left(1 - \left(\frac{\theta^3}{(\theta^2+2\alpha)}\right)^{\lambda} \sum_{j=0}^{\infty} \binom{\lambda}{j} \theta^{\lambda-j}(\alpha x)^j \int_0^{\infty} x^{\lambda} e^{-\lambda\theta x} dx\right)$$

$$S_\lambda = \frac{1}{\lambda - 1}\left(1 - \left(\frac{\theta^3}{(\theta^2 + 2\alpha)}\right)^\lambda \sum_{j=0}^{\infty}\binom{\lambda}{j}\theta^{\lambda-j}\alpha^j \int_0^\infty x^{(\lambda+j+1)-1}e^{-\lambda\theta x}dx\right)$$

$$S_\lambda = \frac{1}{\lambda - 1}\left(1 - \left(\frac{\theta^3}{(\theta^2 + 2\alpha)}\right)^\lambda \sum_{j=0}^{\infty}\binom{\lambda}{j}\theta^{\lambda-j}\alpha^j \frac{\Gamma(\lambda + j + 1)}{(\lambda\theta)^{\lambda+j+1}}\right)$$

9. Maximum likelihood Estimation and Fisher's Information Matrix

In this section, we will discuss the maximum likelihood estimation for estimating the parameters of length biased weighted new quasi Lindley distribution and also discuss its Fisher's information matrix. Let $X_1, X_2, \ldots,$ X_n be a random sample of size n from the length biased weighted new quasi Lindley distribution, then the likelihood function of length biased weighted new quasi Lindley distribution is given by

$$L(x\,;\theta,\alpha) = \prod_{i=1}^{n} f_l(x\,;\theta,\alpha)$$

$$L(x\,;\theta,\alpha) = \prod_{i=1}^{n}\left(\frac{x_i\theta^3}{(\theta^2 + 2\alpha)}(\theta + \alpha x)e^{-\theta x_i}\right)$$

$$L(x\,;\theta,\alpha) = \frac{\theta^{3n}}{(\theta^2 + 2\alpha)^n}\prod_{i=1}^{n}\left(x_i(\theta + \alpha x_i)e^{-\theta x_i}\right)$$

The log likelihood function is given by

$$\log L\,(x\,;\theta,\alpha) = 3n\log\theta - n\log(\theta^2 + 2\alpha) + \sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}\log(\theta + \alpha x_i) - \theta\sum_{i=1}^{n}x_i \qquad (12)$$

Differentiating the log likelihood equation (12) with respect to θ and α and equating to zero, we obtain the normal equations.

$$\frac{\partial \log L}{\partial\theta} = \frac{3n}{\theta} - n\left(\frac{2\theta}{(\theta^2 + 2\alpha)}\right) + \sum_{i=1}^{n}\left(\frac{1}{(\theta + \alpha x_i)}\right) - \sum_{i=1}^{n}x_i = 0$$

$$\frac{\partial \log L}{\partial\alpha} = -n\left(\frac{2}{(\theta^2+2\alpha)}\right) + \frac{\sum_{i=1}^{n}x_i}{(\theta+\alpha x_i)} = 0$$

Because of the complicated form of the likelihood equations, algebraically it is very difficult to solve the system of non-linear equations. Therefore we use R and wolfram mathematics for estimating the required parameters of the proposed distribution.

To obtain confidence interval we use the asymptotic normality results. we have that if $\hat{\lambda} = (\hat{\theta}, \hat{\alpha})$ denotes the MLE of $\lambda = (\theta, \alpha)$ we can state the results as follows:

$$\sqrt{n}(\hat{\lambda} - \lambda) \rightarrow N_2(0, I^{-1}(\lambda))$$

Where $I(\lambda)$ is the Fisher's Information matrix .i.e.,

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial\theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial\theta\partial\alpha}\right) \\ E\left(\frac{\partial^2 \log L}{\partial\alpha\partial\theta}\right) & E\left(\frac{\partial^2 \log L}{\partial\alpha^2}\right) \end{pmatrix}$$

Where

$$E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) = -\frac{3n}{\theta^2} - n\left(\frac{2(\theta^2+2\alpha)-4\theta^2}{(\theta^2+2\alpha)^2}\right) - \sum_{i=1}^{n}\left(\frac{1}{(\theta+\alpha x_i)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \alpha^2}\right) = n\left(\frac{4}{(\theta^2+2\alpha)^2}\right) - \sum_{i=1}^{n}\left(\frac{E(x_i{}^2)}{(\theta+\alpha x_i)^2}\right)$$

$$E\left(\frac{\partial^2 \log L}{\partial \theta \partial \alpha}\right) = E\left(\frac{\partial^2 \log L}{\partial \alpha \partial \theta}\right) = n\left(\frac{4\theta}{(\theta^2+2\alpha)^2}\right) - \sum_{i=1}^{n}\left(\frac{E(x_i)}{(\theta+\alpha x_i)^2}\right)$$

Since λ being unknown, we estimate $I^{-1}(\lambda)$ by $I^{-1}(\hat{\lambda})$ and this can be used to obtain asymptotic confidence intervals for θ and α

10. Application

In this section, here we analyse and evaluate the two real life data sets for fitting length biased weighted new quasi Lindley distribution and the model has been compared with new quasi Lindley, quasi Lindley, Lindley and exponential distributions. In order to show that the length biased weighted new quasi Lindley distribution is better than the new quasi Lindley, quasi Lindley, Lindley and exponential distributions, the results obtained from the two real life data sets are used. The two real life data sets are given below as:

The first data set denotes the time to failure of turbocharger (*103h*) of one type of engine studied by Xu et al. (2003). The first data set is given as follows

1.6, 8.4, 8.1, 7.9, 3.5, 2, 8.4, 8.3, 4.8, 3.9, 2.6, 8.5, 5.4, 5, 4.5, 3, 6.0, 5.6, 5.1, 4.6, 6.5, 6.1, 5.8, 5.3, 7, 6.5, 6.3, 6, 7.3, 7.1, 6.7, 8.7, 7.7, 7.3, 7.3, 8.8, 8, 7.8, 7.7, 9

The second data set represents 40 patients suffering from blood cancer (leukemia) from one of ministry of Health Hospitals in Saudi Arabia (see Abouammah *et al.*).The ordered lifetimes (in years) is given as follows:

0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.37, 2.532, 2.693, 2.805, 2.91, 2.912, 2.192, 3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 5.381

R software is used to carry out the numerical analysis of two data sets and is also used for estimating the unknown parameters and model comparison criterion values. In order to compare the length biased weighted new quasi Lindley distribution with new quasi Lindley, quasi Lindley, Lindley and exponential distributions, we consider the criterion values like AIC (Akaike information criterion), AICC (corrected Akaike information criterion) and BIC (Bayesian information criterion).The better distribution corresponds to lesser values of AIC, BIC, AICC and -2logL. The formulas for calculation of AIC, AICC and BIC values are

$$AIC = 2k - 2\log L \qquad AICC = AIC + \frac{2k(k+1)}{n-k-1} \quad \text{and} \quad BIC = k\log n - 2\log L$$

Where k is the number of parameters in the statistical model, n is the sample size and -2log$L$ is the maximized value of the log-likelihood function under the considered model.

Table 1: Shows maximum likelihood estimates, corresponding S.errors, criterion values AIC, BIC, AICC and -2logL and comparison of fitted distribution.

Data Set 1						
Distribution	**MLE**	**S.E**	**-2logL**	**AIC**	**BIC**	**AICC**

LBWNQL	$\boldsymbol{\hat{\theta} = 4.7974}$ $\boldsymbol{\hat{\alpha} = 3.6391}$	$\boldsymbol{\hat{\theta} = 4.3797}$ $\boldsymbol{\hat{\alpha} = 4.8440}$	189.011	193.011	196.3888	193.3353
NQL	$\hat{\theta} = 3.1980$ $\hat{\alpha} = 3.3626$	$\hat{\theta} = 3.5756$ $\hat{\alpha} = 3.3560$	201.0361	205.0361	208.4138	205.3604
QL	$\hat{\theta} = 0.3197$ $\hat{\alpha} = 0.0010$	$\hat{\theta} = 0.0311$ $\hat{\alpha} = 0.2123$	201.0589	205.0589	208.4367	205.3832
Lindley	$\hat{\theta} = 0.2844$	$\hat{\theta} = 0.0321$	208.5708	210.5708	212.2597	210.6760
Exponential	$\hat{\theta} = 0.1599$	$\hat{\theta} = 0.0252$	226.6385	228.6385	230.3274	228.7437

Table 2: Shows maximum likelihood estimates, corresponding S.errors, criterion values AIC, BIC, AICC and -2logL and comparison of fitted distribution.

Data Set 2						
Distribution	**MLE**	**S.E**	**2logL**	**AIC**	**BIC**	**AICC**
LBWNQL	$\hat{\theta} = 0.9402$ $\hat{\alpha} = 8.9514$	$\hat{\theta} = 0.1044$ $\hat{\alpha} = 38.4471$	147.461	151.461	154.8388	151.7853
NQL	$\hat{\theta} = 6.3670$ $\hat{\alpha} = 1.6173$	$\hat{\theta} = 7.1183$ $\hat{\alpha} = 1.6778$	152.7528	156.7528	160.1306	157.0771
QL	$\hat{\theta} = 0.6365481$ $\hat{\alpha} = 0.0010000$	$\hat{\theta} = 0.1545506$ $\hat{\alpha} = 0.5536723$	152.766	156.766	160.1437	157.0903
Lindley	$\hat{\theta} = 0.5269$	$\hat{\theta} = 0.0607$	160.5012	162.5012	164.19	162.6064
Exponential	$\hat{\theta} = 0.3183$	$\hat{\theta} = 0.0503$	171.5563	173.5563	175.2452	173.6615

From table 1 and table 2 given above, it has been observed that the length biased weighted new quasi Lindley distribution have the lesser AIC, AICC, BIC and -2logL values as compared to the new quasi Lindley, quasi Lindley, Lindley and exponential distributions. Hence we can conclude that the length biased weighted new quasi Lindley distribution leads to a better fit than the new quasi Lindley, quasi Lindley, Lindley and exponential distributions.

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11. Conclusion

In the present study, we have introduced the length biased weighted new quasi Lindley distribution as a new generalization of new quasi Lindley distribution. The subject distribution is generated by using the length biased technique and taking the two parameter new quasi Lindley distribution as the base distribution. The different mathematical and statistical properties of the newly executed distribution along with the reliability measures are discussed. The parameters of the proposed distribution are obtained by using the method of maximum likelihood estimator and also the Fisher's information matrix has been discussed. Finally, the application of the new distribution has also been illustrated by demonstrating with two real life data sets. The results of the two data sets are used by comparing the length biased weighted new quasi Lindley distribution over new quasi Lindley, quasi Lindley, Lindley and exponential distributions and the results indicate that the length biased weighted new quasi Lindley distribution provides a better fit than the new quasi Lindley, quasi Lindley, Lindley and exponential distributions.

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