The New Exponentiated *T-X* Class of Distributions: Properties, Characterizations and Application

Zubair Ahmad
Department of Statistics, Yazd University, P.O. Box 89175-741, Yazd, Iran
Email: zubair@stu.yazd.ac.ir, z.ferry21@gmail.com

Clement Boateng Ampadu
31 Carrolton Road, Boston, MA 02132-6303, USA
Email: drampadu@hotmail.com

G.G. Hamedani
Department of Mathematical and Statistical Sciences Marquette University Milwaukee, WI 53201-1881, USA
Email: g.hamedani@mu.edu

Farrukh Jamal
Govt. S.A P/G College, Bahawalpur, Pakistan
Email: drfarrukh1982@gmail.com

Muhammad Arslan Nasir
Govt. S.E College, Bahawalpur, Pakistan
Email: arslannasir147@gmail.com

Abstract

In this article, we introduce a new class of distributions called the New Exponentiated *T-X* family of distributions. A special sub-model of the proposed family, called a new exponentiated exponential-Weibull is considered in detail. Some structural properties associated with this new class of distributions are obtained. Certain characterizations of the proposed family are presented. Maximum likelihood estimators of the model parameters are obtained and Monte Carlo simulation study is conducted to evaluate the performances of these estimators. Finally, the importance of the new family is illustrated empirically via a real-life application.

Keywords: Exponentiated family; *T-X* family of distributions; New *T-X* family of distributions; Exponentiated distribution; Characterizations.

1. Introduction

Speaking broadly, statistical distributions are widely used in modeling real phenomena of nature. Among these distributions, the Exponential, Rayleigh and Weibull are some of the important statistical models widely used in many applied areas. However, these distributions have a limited range of capability and thus cannot be applied in all situations. For example, although the exponential distribution is often described as flexible, its hazard function is constant, whereas, the Rayleigh has increasing hazard function only. The limitations of the classical distributions motivated the researchers to introduce new distributions via generalizing the existing ones. Several methods of extending the existing distributions have been introduced which bring more flexibility to these distributions. These methods were pioneered by Marshall and Olkin (1997) and Gupta et al. (1998), who proposed the Marshall-Olkin and exponentiated-G class of distributions, respectively.

In the last couple of years, however, the researchers have shown a deep interest in constructing new family of distributions that brings greater flexibility to the existing models. In this regard, serious attempts have been made that is quite rich and still growing rapidly. Some of the well-known families are Beta-G of Eugene et al. (2002), Kumaraswamy (Kw-G) family by Cordeiro and de Castro (2011), Mc-G of Alexander et al. (2012), Gamma-G Type-1 of Zografos and Balakrishnan (2009), Gamma-G Type-2 of Ristic and Balakrishnan (2012), Gamma-G Type-3 of Torabi and Montazeri (2012), exponentiated generalized family by Cordeiro et al. (2013), Logistic-G of Torabi and Montazeri (2014), The Logistic-*X* of family of Tahir et al. (2016), new Weibull-*X* family of Ahmad et al. (2018), *a*-Zubair-G family of Kyurkchiev et al. (2018), the extended alpha power transformed family of Ahmad et al. (2018) and a new alpha power transformed family of Elbatal et al.(2018), among others.

Let $v(t)$ be the probability density function (pdf) of a random variable, say *T*, where $T \in [m, n]$ for $-\infty \leq m < n < \infty$ and let $W\left[F(x;\xi)\right]$ be a function of the cumulative distribution function (cdf) of a random variable, say *X*, depending on the parameter vector ξ satisfying the conditions given below:

i. $W\left[F(x;\xi)\right] \in [m,n]$,

ii. $W\left[F(x;\xi)\right]$ is differentiable and monotonically increasing, and

iii. $W\left[F(x;\xi)\right] \to m$ as $x \to -\infty$ and $W\left[F(x;\xi)\right] \to n$ as $x \to \infty$.

Recently, Alzaatreh et al. (2013), defined the cdf of the *T-X* family of distributions as

$$G(x) = \int_m^{W[F(x;\xi)]} r(t)\,dt, \qquad x \in \mathbb{R}, \tag{1}$$

certain $W\left[F(x;\xi)\right]$ satisfies the conditions stated above. The pdf corresponding to (1) is

$$g(x) = \left\{\frac{\partial}{\partial x} W\left[F(x;\xi)\right]\right\} r\left\{W\left[F(x;\xi)\right]\right\}, \qquad x \in \mathbb{R}.$$

Using the *T-X* idea, several new classes of distributions have been introduced in the literature. Table 1 provides some $W\left[F(x;\xi)\right]$ functions for some members of the *T-X* family.

Table 1. Some members of the *T-X* family

$W\left[F\left(x;\xi\right)\right]$	Support of T	Members of *T-X* family
$F\left(x;\xi\right)$	$[0,1]$	Beta-G (Eugene et al., 2002), Mc-G (Alexander et al., 2012)
$-\log\left[F\left(x;\xi\right)\right]$	$(0,\infty)$	Gamma-G Type-2 (Risti´c and Balakrishnan, 2012)
$-\log\left[1-F\left(x;\xi\right)\right]$	$(0,\infty)$	Gamma-G Type-1 (Zografos and Balakrishnan, 2009)
$\dfrac{F\left(x;\xi\right)}{1-F\left(x;\xi\right)}$	$(0,\infty)$	Gamma-G Type-3 (Torabi and Montazeri, 2012)
$-\log\left[1-F^{\alpha}\left(x;\xi\right)\right]$	$(0,\infty)$	Exponentiated *T-X* (Alzaghal et al., 2013)
$\log\left\{\dfrac{F\left(x;\xi\right)}{1-F\left(x;\xi\right)}\right\}$	$(-\infty,\infty)$	Logistic-G (Torabi and Montazeri, 2014)
$\log\left[-\log\left\{1-F\left(x;\xi\right)\right\}\right]$	$(-\infty,\infty)$	The Logistic-*X* Family (Tahir et al., 2016)
$\dfrac{\left[-\log\left\{1-F\left(x;\xi\right)\right\}\right]}{1-F\left(x;\xi\right)}$	$(0,\infty)$	New Weibull-*X* Family (Ahmad et al., 2018)
$\dfrac{\left[-\log\left\{1-F\left(x;\xi\right)^{\alpha}\right\}\right]}{1-F\left(x;\xi\right)^{\alpha}}$	$(0,\infty)$	(Proposed)

If T is the exponential random variable with parameter $\lambda>0$, then its cdf is given by

$$R(t;\lambda)=1-e^{-\lambda t},\qquad t>0,\ \lambda>0. \tag{2}$$

The density function corresponding to (2) is

$$r(t;\lambda)=\lambda e^{-\lambda t},\qquad t>0,\ \lambda>0. \tag{3}$$

If $r(t;\lambda)$ follows (3) and setting $W\left[F\left(x;\xi\right)\right]=\dfrac{\left[-\log\left\{1-F\left(x;\xi\right)^{\alpha}\right\}\right]}{1-F\left(x;\xi\right)^{\alpha}}$ in (1), we define the cdf of the NE-Exponential X family by

$$G(x)=1-\exp\left[-\lambda\left\{\frac{-\log\left(1-F\left(x;\xi\right)^{\alpha}\right)}{1-F\left(x;\xi\right)^{\alpha}}\right\}\right],\qquad \alpha,\lambda,\xi>0,\ x\in\mathbb{R}, \tag{4}$$

where, $F\left(x;\xi\right)$ is the cdf of the baseline distribution which depends on the parameter vector ξ. The pdf corresponding to (4) is given by

$$g(x)=\frac{\alpha\lambda f(x;\xi)F(x;\xi)^{\alpha-1}\left[1-\log\left(1-F(x;\xi)^{\alpha}\right)\right]}{\left(1-F(x;\xi)^{\alpha}\right)^{2}}\exp\left[-\lambda\left\{\frac{-\log\left(1-F(x;\xi)^{\alpha}\right)}{1-F(x;\xi)^{\alpha}}\right\}\right],\quad \alpha,\lambda,\xi>0,\ x\in\mathbb{R}. \tag{5}$$

The main goal of this research is to introduce a new family of distributions, called the new exponentiated *T-X*(“NE *T-X* for short) family of which the NE-Exponential *X* family discussed above is a special case. The generic form of the NE *T-X* family is introduced at the beginning of Section 3. We discuss a special sub-model of this family, capable of modeling with monotonic and non-monotonic hazard rates. For the special sub-model of the NE *T-X* family, a real life application is presented. The rest of this paper is structured as follows: In Section 2, a special sub-model of the proposed family is presented. Statistical properties of the proposed family are investigated in Section 3. Section 4 contains some useful characterizations of the proposed class. Section 5, provides estimation of the model parameters using maximum likelihood method. Section 6, provides analysis to a real data set. Simulation results are reported in Section 7. Finally, Section 8 concludes the article.

2. Special Sub-Model

Considering the cdf of the two-parameter Weibull model with shape parameter $\theta>0$ and scale parameter $\gamma>0$, given by $F(x;\xi)=1-e^{-\gamma x^{\theta}},\ x\geq 0,\ \gamma,\theta>0$, and pdf $f(x;\xi)=\gamma\theta x^{\theta-1}e^{-\gamma x^{\theta}}$, where $\xi=(\theta,\gamma)$. Then, the cdf of the new exponentiated exponential Weibull (NEEW) distribution is

$$G(x)=1-\exp\left[-\lambda\left\{\frac{-\log\left(1-\left(1-e^{-\gamma x^{\theta}}\right)^{\alpha}\right)}{1-\left(1-e^{-\gamma x^{\theta}}\right)^{\alpha}}\right\}\right],\qquad x\geq 0,\ \lambda,\alpha,\xi>0. \tag{6}$$

Upon differentiating (6), we have

$$g(x)=\frac{\alpha\lambda\gamma\theta x^{\theta-1}e^{-\gamma x^{\theta}}\left(1-e^{-\gamma x^{\theta}}\right)^{\alpha-1}\left[1-\log\left(1-\left(1-e^{-\gamma x^{\theta}}\right)^{\alpha}\right)\right]}{\left(1-F\left(1-e^{-\gamma x^{\theta}}\right)^{\alpha}\right)^{2}}\exp\left[-\lambda\left\{\frac{-\log\left(1-\left(e^{-\gamma x^{\theta}}\right)^{\alpha}\right)}{1-\left(1-e^{-\gamma x^{\theta}}\right)^{\alpha}}\right\}\right],\quad x>0.$$

For $\lambda=1$, different Plots of the NEEW density and hrf for selected parameter values are presented in Figure 1.

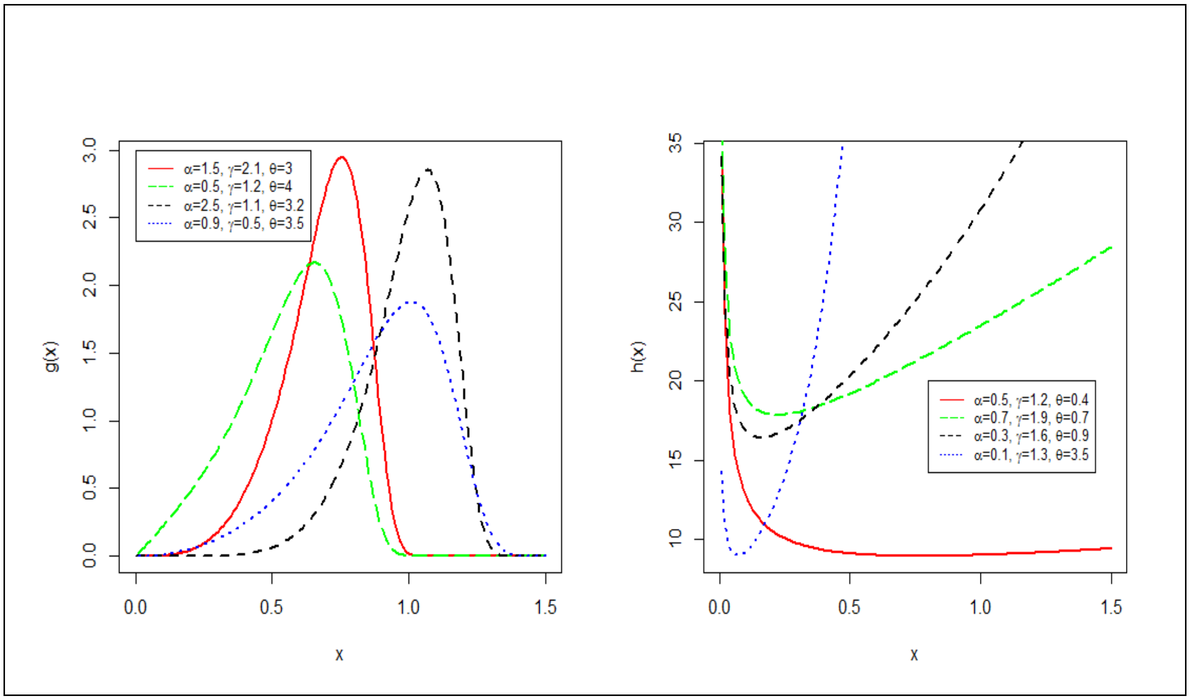

Figure 1: Different plots for the pdf and hrf of NEEW distribution.

3. Mathematical Properties

In this section, we provide some mathematical properties of the proposed class. The generic form of the cdf and pdf of the proposed class are given, respectively, by

$$G_2(x) = R_T\left(\frac{-\log\left(1-F_X^{\alpha}(x)\right)}{1-F_X^{\alpha}(x)}\right), \qquad x \in \mathbb{R}, \tag{7}$$

and

$$g_2(x) = \frac{r_T\left(\frac{-\log\left(1-F_X^{\alpha}(x)\right)}{1-F_X^{\alpha}(x)}\right) f_X^{\alpha}(x)\left(-\log\left(1-F_X^{\alpha}(x)\right)\right)}{\left(1-F_X^{\alpha}(x)\right)^2}, \qquad x \in \mathbb{R}, \tag{8}$$

for some $\alpha > 0$, where the random variable X has cdf $F_X(x)$, pdf $f_X(x)$ and the random variable T has support [0, ∞) and cdf $R_T(t)$.

Theorem 3.1. (Transformation of Random Variables) Let U be uniform on (0,1), the random variable T have quantile Q_T, the random variable X have quantile Q_X, and let $\alpha > 0$, then the random variable

$$Q_X\left\{\left(\frac{Q_T(U) - \text{ProductLog}\left[Q_T(U)\right]}{Q_T(U)}\right)^{\frac{1}{\alpha}}\right\},$$

Belongs to the NE *T-X* class of distributions, where ProductLog[z] gives the principal solution for win $z = we^{w}$.

Proof. Since $Q_X = F_X^{-1}$, $Q_T = R_T^{-1}$,by the transformation technique one can show that the cdf of the random variable in the theorem is given by

$$R_T\left\{\frac{-\log\left(1-F_X^{\alpha}(x)\right)}{1-F_X^{\alpha}(x)}\right\}, \tag{9}$$

for some $\alpha > 0$.

Theorem 3.2. (Quantile Function) The quantile function of the NE *T-X* class of distributions, for $0<p<1$, is given by

$$Q(p) = Q_X\left\{\left(\frac{Q_T(p) - \text{ProductLog}\left[Q_T(p)\right]}{Q_T(p)}\right)^{\frac{1}{\alpha}}\right\},$$

where the random variable T has quantile Q_T, the random variable X has quantile Q_X, $\alpha > 0$, and ProductLog[$\cdot$] is defined as before.

Proof. Since $Q_X = F_X^{-1}$, $Q_T = R_T^{-1}$, it is enough to solve the following equation for *Q(p)*

$$p = R_T\left\{\frac{-\log\left(1-F_X^{\alpha}\left(Q(x)\right)\right)}{1-F_X^{\alpha}\left(Q(x)\right)}\right\}. \tag{10}$$

Corollary 3.1. The quantile function of the new exponentiated Exponential-Weibull family of distributions is given by

$$Q(p,a,\alpha,c,d) = d\left\{-L\log\left\{1-\left(-\frac{a\left(-\frac{Log[1-p]}{a} - \text{Pr}\,oductLog\left(-\frac{Log[1-p]}{a}\right)\right)}{Log[1-p]}\right)^{\frac{1}{\alpha}}\right\}\right\}^{\frac{1}{c}},$$

$0 < p < 1$, and $a,\alpha,c,d > 0$.

Proof. Letting the random variable T follow the exponential distribution with quantile function

$$Q_T(p,a) = -\frac{Log[1-p]}{a},$$

$0 < p < 1$, and $a > 0$, and the random variable X follow the Weibull distribution with quantile function

$$Q_X(p,a,d) = d\left(-Log[1-x]\right)^{\frac{1}{c}}, \tag{11}$$

$0 < p < 1$, and $a,d > 0$, the result follows from Theorem 3.2.

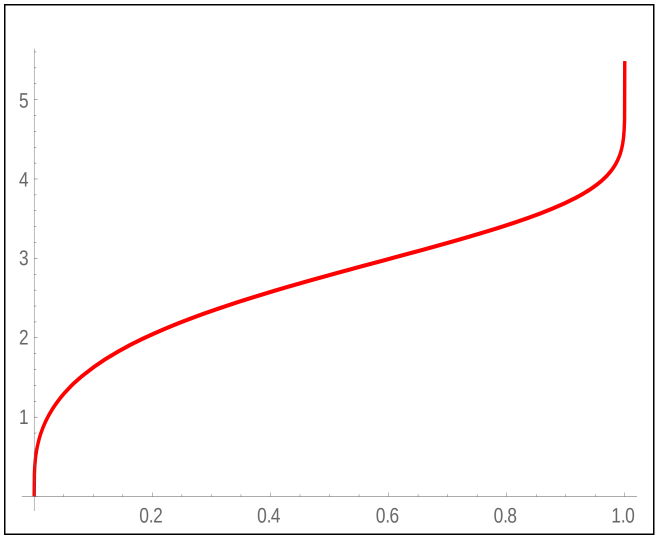

Figure 2: $Q(p,0.1722,4.766,0.943,1)$ for $0<p<1$.

The measure of skewness S defined in (1883) and the measure of kurtosis K defined in (1988) are based on quantile functions and they are defined as

$$S=\frac{Q\left(\frac{6}{8}\right)-2Q\left(\frac{4}{8}\right)+Q\left(\frac{2}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)},$$

and

$$K=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}.$$

Skewness measures the degree of the long tail (towards left or right side). Kurtosis is a measure of the degree of tail heaviness. When the distribution is symmetric, S=0 and when the distribution is right (or left) skewed, S> 0 (or < 0). As K increases, the tail of the distribution becomes heavier.

Theorem 3.4. The measure of skewness associated with the new exponentiated Exponential Weibull family of distributions is given by

$$S=-\frac{AA-2BB+CC}{AA-CC},$$

where, $AA=\left[-\log\left\{1-\left(1-\frac{a\text{ProductLog}\left(A'\right)}{A}\right)^{\frac{1}{\alpha}}\right\}\right]^{\frac{1}{c}}$, $BB=\left[-\log\left\{1-\left(1-\frac{a\text{ProductLog}\left(B'\right)}{B}\right)^{\frac{1}{\alpha}}\right\}\right]^{\frac{1}{c}}$,

$CC=\left[-\log\left\{1-\left(1-\frac{a\text{ProductLog}\left(C'\right)}{C}\right)^{\frac{1}{\alpha}}\right\}\right]^{\frac{1}{c}}$, $A=Log\left[\frac{4}{3}\right]$, $A'=\frac{Log\left[\frac{4}{3}\right]}{a}$, $B=Log[2]$ $B'=\frac{Log[2]}{a}$

$C=Log[4]$ $C'=\frac{Log[4]}{a}$.

Proof. Combine Corollary 3.1 with the definition of skewness given by

$$S=\frac{Q\left(\frac{6}{8}\right)-2Q\left(\frac{4}{8}\right)+Q\left(\frac{2}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}.$$

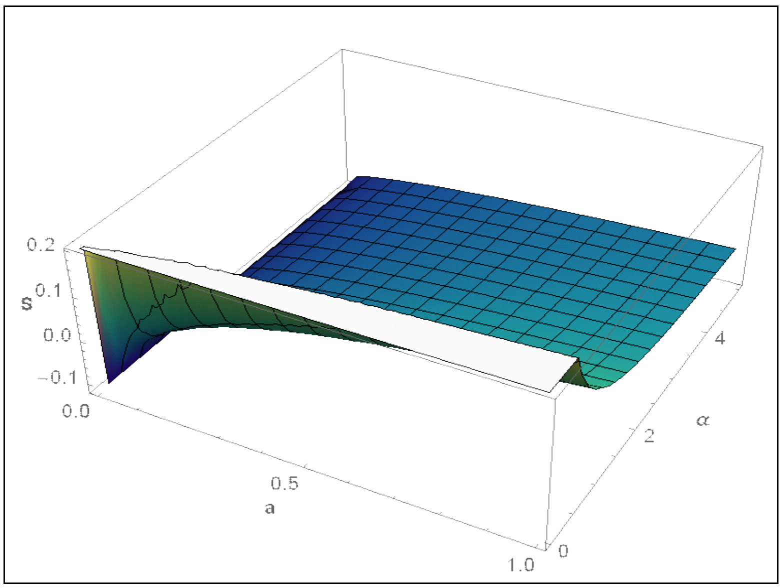

Figure 3: $S(a,\alpha,0.943,1)$ for $0<a<1$ and $0<\alpha<5$.

Theorem 3.5. The measure of kurtosis associated with the new exponentiated Exponential Weibull family of distributions is given by

$$K=\frac{DD-EE+FF-GG}{AA-CC},$$

where, $DD=\left[-\log\left\{1-\left(1-\frac{a\text{ProductLog}\left(D'\right)}{D}\right)^{\frac{1}{\alpha}}\right\}\right]^{\frac{1}{c}}$, $EE=\left[-\log\left\{1-\left(1-\frac{a\text{ProductLog}\left(E'\right)}{E}\right)^{\frac{1}{\alpha}}\right\}\right]^{\frac{1}{c}}$,

$FF=\left[-\log\left\{1-\left(1-\frac{a\text{ProductLog}\left(F'\right)}{F}\right)^{\frac{1}{\alpha}}\right\}\right]^{\frac{1}{c}}$, $GG=\left[-\log\left\{1-\left(1-\frac{a\text{ProductLog}\left(G'\right)}{G}\right)^{\frac{1}{\alpha}}\right\}\right]^{\frac{1}{c}}$,

$D=Log\left[\frac{8}{7}\right]$, $D'=\frac{Log\left[\frac{8}{7}\right]}{a}$, $E=Log\left[\frac{8}{5}\right]$, $E'=\frac{Log\left[\frac{8}{5}\right]}{a}$, $F=Log\left[\frac{8}{3}\right]$, $F'=\frac{Log\left[\frac{8}{3}\right]}{a}$, $F=Log[8]$, $F'=\frac{Log[8]}{a}$.

Proof. Combine Corollary 3.1 with the definition of kurtosis given by

$$K=\frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}.$$

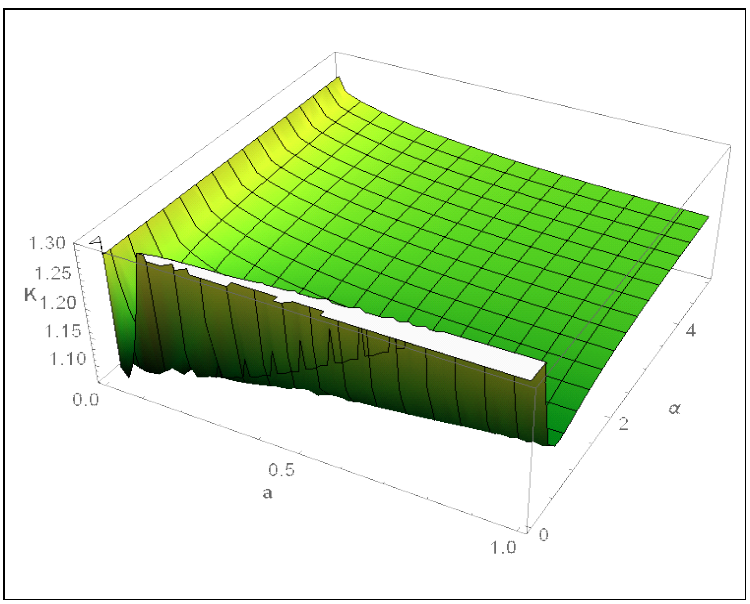

Figure 4: $K(a,\alpha,0.943,1)$ for $0<a<1$ and $0<\alpha<5$.

Theorem 3.6. (Shannon Entropy): Given equation (8), if a random X follows the new exponentiated *T-X* class of distributions, then the Shannon entropy of X is given by

$$S_V=\eta_T-E\left\{\log f_X^{\alpha}\left(Q_X\left(\frac{T-\text{ProductLog}[T]}{T}\right)^{\frac{1}{\alpha}}\right)\right\}-\mu\log\left(1-Log\left(\frac{\text{ProductLog}[T]}{T}\right)\right)+2\mu Log\left(\frac{\text{ProductLog}[T]}{T}\right),$$

where $\alpha>0$, the random variable T has Shannon entropy, η_T, the random variable X has pdf f_X and quantile function Q_X, and given the random variable Q (say), μ_Q is the mean of Q, and ProductLog[·] is defined as before.

Proof. From Theorem 3.1 and the fact that $T=\frac{-\log\left(1-F_X^{\alpha}(x)\right)}{1-F_X^{\alpha}(x)}$ has density *r(t)*, the result follows by noting that

$$-E\left\{\log\ r_T\left(\frac{-\log\left(1-F_X^{\alpha}(x)\right)}{1-F_X^{\alpha}(x)}\right)\right\}=E\left(-\log\ r_T(t)\right)=\eta_T,$$

$$E\left[\log f_X^{\alpha}(X)\right]=E\left[\log f_X^{\alpha}(X)\left\{Q_X\left(\left(\frac{T-\text{ProductLog}[T]}{T}\right)^{\frac{1}{\alpha}}\right)\right\}\right],$$

$$E\left[\log\left(1-Log\left(1-F_X^{\alpha}(X)\right)\right)\right]=E\left[\log\left(1-Log\left(\frac{\text{Pr}oductLog[T]}{T}\right)\right)\right],$$

$$E\left[\log\left(1-Log\left(1-F_X^{\alpha}(X)\right)\right)\right]=\mu\left[\log\left(1-Log\left(\frac{\text{Pr}oductLog[T]}{T}\right)\right)\right],$$

$$E\left[\log\left(1-Log\left(1-F_X^{\alpha}(X)\right)\right)\right]=E\left\{Log\left(\frac{\text{Pr}oductLog[T]}{T}\right)\right\}=\mu Log\left(\frac{\text{Pr}oductLog[T]}{T}\right). \qquad (12)$$

Theorem 3.7. (r^{th} Non-Central Moments) The *rth* non-central moments of the NE *T-X* class of distributions are given by

$$\mu_r^{/}=\sum_{i=0}^{\infty}\sum_{k=0}^{\frac{i}{\alpha}}\sum_{l=k}^{\infty}\frac{\delta_{r,i}(-1)^{k+1}k(-l)^{l-k-1}}{(l-k)!}\binom{\frac{i}{\alpha}}{k}E\left(Q_T(U)^{l-k}\right),$$

where $\delta_{r,i}=(ih_0)^{-1}\sum_{s=1}^{i}\left[s(r+1)-i\right]h_s\delta_{r,i-s}$ with $\delta_{r,0}=h_0^r$ for $i=1,2,...$ (Gradshteyn and Ryzhik, 2012), $\alpha>0$, *U* is uniform on (0, 1), and Q_T is the quantile function of the random variable *T* and *E(·)* is an expectation.

Proof. From Theorem 3.1, the following random variable follows NE *T-X* class of distributions

$$Q_X\left\{\left(\frac{Q_T(U)-\text{ProductLog}\left[Q_T(U)\right]}{Q_T(U)}\right)^{\frac{1}{\alpha}}\right\}.$$

where $Q_X(.)=F_X^{-1}(.)$ is a quantile function. According to Nasiru et al. (2017), we can write

$$Q_X(u)=\sum_{i=0}^{\infty}h_iu^i,$$

where the coefficients are suitably chosen real numbers that depend on the parameters of the *F(x)* distribution. For a power series raised to a positive integer $r\geq 1$, we have

$$\left(Q_X(u)\right)^r=\left(\sum_{i=0}^{\infty}h_iu^i\right)^r=\sum_{i=0}^{\infty}\delta_{r,i}u^i,$$

where $\delta_{r,I}$ are obtained from the recurrence equation as stated in the theorem. Thus we have the following

$$\mu_r^{/}=\sum_{i=0}^{\infty}\delta_{r,i}E\left\{Q_X\left(\frac{Q_T(U)-\text{ProductLog}\left[Q_T(U)\right]}{Q_T(U)}\right)^{\frac{i}{\alpha}}\right\},$$

where $E(\cdot)$ is an expectation.

From the Wikipedia contributors about Binomial theorem (2018) and Lambert W function (2018), we deduce the following

$$\left(\frac{Q_T(U)-\text{ProductLog}\left[Q_T(U)\right]}{Q_T(U)}\right)^{\frac{i}{\alpha}}=\sum_{k=0}^{\frac{i}{\alpha}}\sum_{l=k}^{\infty}\frac{(-1)^{k+1}k(-l)^{l-k-1}}{(l-k)!}\binom{\frac{i}{\alpha}}{k}Q_T(U)^{l-k}.$$

We get the desired result by using the expression immediately above in

$$\mu_r^{/}=\sum_{i=0}^{\infty}\delta_{r,i}E\left\{Q_X\left(\frac{Q_T(U)-\text{ProductLog}\left[Q_T(U)\right]}{Q_T(U)}\right)^{\frac{i}{\alpha}}\right\}. \tag{13}$$

Given a random variable X with pdf $f(x)$, the ordinary moments, for $r\in N$, are given by

$$E\left(X^r\right)=\int_{-\infty}^{\infty}x^r f(x)dx.$$

However, if the random variable X in question has cdf $F(x)$ and quantile function Q_X, then after the substitution $u = F(x)$, the ordinary moments can be expressed as

$$E\left(X^r\right)=\int_0^1\left(Q_X(u)\right)^r du.$$

Thus the following is immediate

Theorem 3.8. The moment generating function of the new exponentiated *T-X* class of distributions are given by

$$M_X(z)=\sum_{r,i=0}^{\infty}\sum_{k=0}^{\frac{i}{\alpha}}\sum_{l=k}^{\infty}\frac{z^r\delta_{r,i}(-1)^{k+1}k(-l)^{l-k-1}}{r!(l-k)!}\binom{\frac{i}{\alpha}}{k}E\left(Q_T(U)^{l-k}\right),$$

where $\delta_{r,i}=(ih_0)^{-1}\sum_{s=1}^{i}\left[s(r+1)-i\right]h_s\delta_{r,i-s}$ with $\delta_{r,0}=h_0^r$ for $i=1,2,\ldots$ [Gradshteyn and Ryzhik [20]), $\alpha > 0$, U is uniform on (0, 1), and Q_T is the quantile function of the random variable T.

Lemma 3.1. Assume U is uniform on (0, 1), then $\left(-\log(1-U)\right)^{l-k}$ admit the following expansion

$$\left(-\log(1-U)\right)^{l-k}=\sum_{p=0}^{\infty}\sum_{q=0}^{p}\frac{(l-k)(-1)^{p+q}\rho_{p,q}}{l-k-q}\binom{p-l+k}{p}\binom{p}{q}U^{l-k+p},$$

where $\rho_{p,q}$ are determined from the recurrence given by equation (3.19) in Almheidat et al. (2015).

Proof. From equation (3.18) in Almheidat et al. (2015), we get

$$\left(-\log\left(1-U\right)\right)^{l-k}=\sum_{p=0}^{\infty}\sum_{q=0}^{p}\frac{\left(l-k\right)\left(-1\right)^{p+q}\rho_{p,q}}{l-k-q}\binom{p-l+k}{p}\binom{p}{q}U^{l-k+p}.$$

(14)

Lemma 3.2. Let $Y=U^{l-k+p}$, where U is uniform on (0, 1), then

$$E\left[Y\right]=\frac{1}{1+l-k+p}$$

Proof. First observe that the pdf of Y is given by

$$f_Y\left(y\right)=\frac{1}{l-k+p}y^{\frac{l-l+k-p}{l-k+p}},$$

for $0 < y < 1$. Thus the result follows by evaluating the following integral

$$\int_0^1 y\, f_Y\left(y\right)dy. \tag{15}$$

Corollary 3.2. The moment generating function of the new exponentiated standard exponential-*X* class of distributions is given by

$$M_X\left(z\right)=\sum_{r,i=0}^{\infty}\sum_{k=0}^{\frac{i}{\alpha}}\sum_{l=k}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\frac{\left(l-k\right)\left(-1\right)^{p+q}\rho_{p,q}z^r\delta_{r,i}\left(-1\right)^{k+1}k\left(-l\right)^{l-k-1}}{r!\left(l-k\right)!\left(l-k-q\right)\left(l+l-k+P\right)}\binom{\frac{i}{\alpha}}{k}\binom{p-l+k}{p}\binom{p}{q},$$

where $\delta_{r,i}=\left(ih_0\right)^{-1}\sum_{s=1}^{i}\left[s\left(r+1\right)-i\right]h_s\delta_{r,i-s}$ with $\delta_{r,0}=h_0^r$ for $i=1,2,...,$ see Gradshteyn and Ryzhik (2014), $\alpha>0$, and ρ, q and p are determined from the recurrence given by equation (3.19) in Almheidat et al. (2015).

Proof. Use Lemma 3.1 and Lemma 3.2 in Theorem 3.10.

4. Characterizations

This section deals with various characterizations of NE *T*-*X* distribution. These characterizations are based on a simple relationship between two truncated moments. It should be mentioned that for these characterizations the cdf may not have a closed form. Due to the nature of the proposed cdf, our characterizations may be the only possible ones. The first characterization result employs a theorem due to Glänzel (1987); see Theorem 4.1 in Appendix A. Note that the result holds also when the interval *H* is not closed. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 4.1. Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let

$$q_1(x)=\frac{\left(1-F(x;\xi)^{\alpha}\right)^{2}\left\{r\left(\frac{-\log\left(1-F(x;\xi)^{\alpha}\right)}{1-F(x;\xi)^{\alpha}}\right)\right\}^{-1}}{1-\log\left(1-F(x;\xi)^{\alpha}\right)} \text{ and } q_2(x)=q_1(x)F(x;\xi)^{\alpha} \text{ for } x\in\mathbb{R}$$

. The random variable X has pdf (5) if and only if the function ξ defined in Theorem 4.1 has the form

$$\xi(x)=\frac{1}{2}\left(1+F(x;\xi)^{\alpha}\right), \qquad x\in\mathbb{R}.$$

Proof. If X has pdf (5), then

$$\left(1-F(x;\xi)\right)E\left(q_1(X)\mid X\geq x\right)=1-F(x;\xi)^{\alpha}, \qquad x\in\mathbb{R},$$

and

$$\left(1-F(x;\xi)\right)E\left(q_2(X)\mid X\geq x\right)=\frac{1}{2}\left\{1-F(x;\xi)^{\alpha}\right\}, \qquad x\in\mathbb{R},$$

and finally

$$\xi(x)q_1(x)-q_2(x)=\frac{1}{2}q_1(x)\left(1-F(x;\xi)^{\alpha}\right)\geq 0, \qquad \text{for} \quad x\in\mathbb{R}.$$

Conversely, if ξ is given as above, then

$$s'(x)=\frac{\xi'(x)q_1(x)}{\xi(x)q_1(x)-q_2(x)}=\frac{\alpha f(x;\xi)F(x;\xi)^{\alpha-1}}{1-F(x;\xi)^{\alpha}}, \qquad x\in\mathbb{R},$$

and hence

$$s(x)=-\log\left(1-F(x;\xi)^{\alpha}\right), \qquad x\in\mathbb{R}.$$

Now, in view of Theorem 4.1, X has density (5).

Corollary 4.1. Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 4.1. Then, X has pdf (5) if and only if there exist functions q_2 and ξ defined in Theorem 4.1 satisfying the differential equation

$$\frac{\xi'(x)q_1(x)}{\xi(x)q_1(x)-q_2(x)}=\frac{\alpha f(x;\xi)F(x;\xi)^{\alpha-1}}{1-F(x;\xi)^{\alpha}}, \qquad x\in\mathbb{R}.$$

Corollary 4.2. The general solution of the differential equation in Corollary 4.1 is

$$\xi(x)=\left(1-F(x;\xi)^{\alpha}\right)^{-1}\left[\int\alpha f(x;\xi)F(x;\xi)^{\alpha-1}\left(q_1(x)\right)^{-1}q_2(x)+D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 4.1 with D = 1/2. However, it should be also noted that there are other triplets (q_1,q_2,ξ) satisfying the conditions of Theorem 4.4.

Theorem 4.2. (Generalized Transmuted New Exponentiated *T-X* Family of Distributions) Let $X:\Omega\rightarrow\mathbb{R}$ be a continuous random variable and let

$$q_2(x) = \left[1+\lambda-2\lambda\left(G_2(x)\right)^{\beta}\right]^{-1},$$

and

$$q_1(x) = q_2(x)G_2(x),$$

for $x \in \mathbb{R}$. The pdf of X is

$$k(x) = \beta g_2(x)\left(G_2(x)\right)^{\beta-1}\left[1+\lambda-2\lambda\left(G_2(x)\right)^{\beta-1}\right],$$

where $1 \le \lambda \le 1, \beta > 0$ and

$$G_2(x) = R_T\left[\frac{-\log\left(1-F_X^{\alpha}(x;\xi)\right)}{1-F_X^{\alpha}(x;\xi)}\right],$$

$$g_2(x) = \frac{r_T\left\{\frac{-\log\left(1-F_X^{\alpha}(x;\xi)\right)}{1-F_X^{\alpha}(x;\xi)}\right\} f_X^{\alpha}(x;\xi)\left(1-\log\left(F_X^{\alpha}(x;\xi)\right)\right)}{\left(1-F_X^{\alpha}(x;\xi)\right)^2},$$

for some α > 0, where the random variable X has cdf $F(x)$ and pdf $f(x)$, the random variable T with support [0, ∞) has pdf $r_T(t)$, and cdf $R_T(t) \Longleftrightarrow$ the function η defined in Theorem 4 [Merovci et al. (2016)] has the form

$$\eta(x) = \frac{\beta}{\beta+1}G_2(x)$$

where β and G_2 are defined as above.

Proof. Let X have pdf

$$k(x) = \beta g_2(x)\left(G_2(x)\right)^{\beta-1}\left[1+\lambda-2\lambda\left(G_2(x)\right)^{\beta-1}\right]$$

then

$$\left(1-K(x;\xi)\right)E\left(q_2(X) \mid X \ge x\right) = \left(G_2(x)\right)^{\beta}, \qquad x \in \mathbb{R},$$

and

$$\left(1-K(x;\xi)\right)E\left(q_1(X) \mid X \ge x\right) = \frac{\beta}{\beta+1}\left(G_2(x)\right)^{\beta+1}, \qquad x \in \mathbb{R},$$

and finally

$$\eta(x)q_2(x) - q_1(x) = \frac{-1}{\beta+1}q_2(x)G_2(x) < 0, \qquad x \in \mathbb{R},$$

Conversely, if η is given as above, then

$$s'(x) = \frac{\eta'(x)q_2(x)}{\eta(x)q_2(x)-q_1(x)} = -\frac{\beta g_2(x;\xi)}{G_2(x;\xi)}, \qquad x \in \mathbb{R},$$

and hence

$$s(x) = -\log\left(G_2(x;\xi)^{\beta}\right), \qquad x \in \mathbb{R}.$$

Now in view of Theorem 4 in Ampadu (2018), X has pdf

$$\beta g_2(x)\left(G_2(x)\right)^{\beta-1}\left[1+\lambda-2\lambda\left(G_2(x)\right)^{\beta-1}\right].$$

5. Estimation

In this subsection, we determine the maximum likelihood estimates of the parameters of the NE *T-X* family. Let $x_1, x_2, \cdots, x_k$ be the observed values from the NE *T-X* distribution with parameters α and ξ. The total log- likelihood function corresponding to (5) is given by

$$\log L(x_i;\alpha,\lambda,\xi) = -2\sum_{i=1}^{k}\log\left(1-F(x_i;\xi)^{\alpha}\right) + k\log\alpha + k\log\lambda + \sum_{i=1}^{k}\log f(x_i;\xi) + (\alpha-1)\sum_{i=1}^{k}\log F(x_i;\xi)$$
$$+\sum_{i=1}^{k}\log\left[1-\log\left(1-F(x_i;\xi)^{\alpha}\right)\right] - \lambda\left\{\frac{-\log\left(1-F(x_i;\xi)^{\alpha}\right)}{1-F(x_i;\xi)^{\alpha}}\right\}, \tag{16}$$

The partial derivates corresponding to (16), are given by

$$\frac{\partial}{\partial\alpha}\log L(x_i;\alpha,\lambda,\xi) = 2\sum_{i=1}^{k}\frac{\left(\log F(x_i;\xi)\right)F(x_i;\xi)^{\alpha}}{\left(1-F(x_i;\xi)^{\alpha}\right)} + \sum_{i=1}^{k}\frac{\left(\log F(x_i;\xi)\right)F(x_i;\xi)^{\alpha}}{\left[1-\log\left(1-F(x_i;\xi)^{\alpha}\right)\right]\left(1-F(x_i;\xi)^{\alpha}\right)}$$
$$+\frac{k}{\alpha} + \sum_{i=1}^{k}\log F(x_i;\xi) - \lambda\left(\log F(x_i;\xi)\right)F(x_i;\xi)^{\alpha}\left\{\frac{1-\log\left(1-F(x_i;\xi)^{\alpha}\right)}{\left(1-F(x_i;\xi)^{\alpha}\right)^{2}}\right\}, \tag{17}$$

$$\frac{\partial}{\partial\lambda}\log L(x_i;\alpha,\lambda,\xi) = \frac{k}{\lambda} - \left\{\frac{-\log\left(1-F(x_i;\xi)^{\alpha}\right)}{1-F(x_i;\xi)^{\alpha}}\right\}, \tag{18}$$

$$\frac{\partial}{\partial\xi}\log L(x_i;\alpha,\lambda,\xi) = -2\sum_{i=1}^{k}\frac{\alpha F(x_i;\xi)^{\alpha-1}\,\partial F(x_i;\xi)/\partial\xi}{\left(1-F(x_i;\xi)^{\alpha}\right)} + +\sum_{i=1}^{k}\frac{\partial f(x_i;\xi)/\partial\xi}{f(x_i;\xi)} + (\alpha-1)\sum_{i=1}^{k}\frac{\partial F(x_i;\xi)/\partial\xi}{F(x_i;\xi)}$$
$$-\lambda\alpha F(x_i;\xi)^{\alpha-1}\,\partial F(x_i;\xi)/\partial\xi\left\{\frac{1-\log\left(1-F(x_i;\xi)^{\alpha}\right)}{1-F(x_i;\xi)^{\alpha}}\right\} \tag{19}$$
$$+\sum_{i=1}^{k}\frac{\alpha F(x_i;\xi)^{\alpha-1}\,\partial F(x_i;\xi)/\partial\xi}{\left[1-\log\left(1-F(x_i;\xi)^{\alpha}\right)\right]\left(1-F(x_i;\xi)^{\alpha}\right)}.$$

Setting $\frac{\partial}{\partial\alpha}\log L(x_i;\alpha,\lambda,\xi)$, $\frac{\partial}{\partial\lambda}\log L(x_i;\alpha,\lambda,\xi)$ and $\frac{\partial}{\partial\xi}\log L(x_i;\alpha,\xi)$ equal to zero and solving numerically these expressions simultaneously yield the maximum likelihood estimators (MLEs) of (α,λ,ξ).

6. Application

In this section, we illustrate the proposed method via analyzing a real data set taken from Saboor and Pogany (2016) representing the breaking strength of carbon fibers (in Gba). For the application section, we keep one parameter constant $(\gamma = 1)$ and reduce the number of parameters to three. The comparison of the proposed distribution is being made with four other well-known extensions of the Weibull distribution. The cumulative functions of the competing models are:

- Beta Weibull (BW) of Famoye et al. (2005)

$$G(x) = I_{1-e^{-\gamma x^{\theta}}}(a,b) \qquad x \geq 0,\ a,b,\alpha,\gamma > 0.$$

- Kumaraswamy Weibull of Cordeiro et al. (2010)

$$G(x) = 1-\left(1-\left(1-e^{-\gamma x^{\theta}}\right)^{a}\right)^{b}, \qquad x \geq 0,\ a,b,\theta,\gamma > 0.$$

- Generalized power Weibull (GPW) of Haghighi and Nikulin (2006)

$$G(x) = 1-\exp\left\{1-\left(1+\gamma x^{\alpha}\right)^{1/\theta}\right\}, \qquad x \geq 0,\ \alpha,\theta,\gamma > 0.$$

- Flexible Weibull extended (FWE) proposed of Ahmad and Hussain (2017)

$$G(x) = 1-\exp\left\{-e^{\alpha x^{2}-\frac{\gamma}{x^{\theta}}}\right\}, \qquad x \geq 0,\ \alpha,\theta,\gamma > 0.$$

The accuracy measures including Anderson–Darling (AD) test statistic, Cramer-von-Misses (CM) test statistic, Kolmogorov–Smirnov (KS) test statistic, Akaike Information Criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian Information Criterion (BIC) and Hannan-Quinn information criterion (HQIC) are being calculated. Based on these measures, it is showed that the proposed model provides greater distributional flexibility. Corresponding to analyzed data set, the maximum likelihood estimates are provided in Table 2, whereas, the analytical measures are provided in Table 3.

Table 2: Estimates of the parameters with standard errors in parentheses for the fitted models.

Dist.	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
NEEW		4.766 (3.0596)	0.943 (0.0762)	0.1722 (0.3131)		
FEW	3.063 (0.4502)	0.826 (0.1753)	0.115 (0.0134)			
GPW	24.77 (11.2077)	6.595 (2.8017)	0.957 (0.1017)			
BW	1.641 (1.3512)		3.430 (0.3785)		0.711 (0.5176)	0.012 (0.0927)
Ku-W	0.0127 (0.0073)		2.917 (0.4676)		1.235 (0.2678)	4.1479 (2.7288)

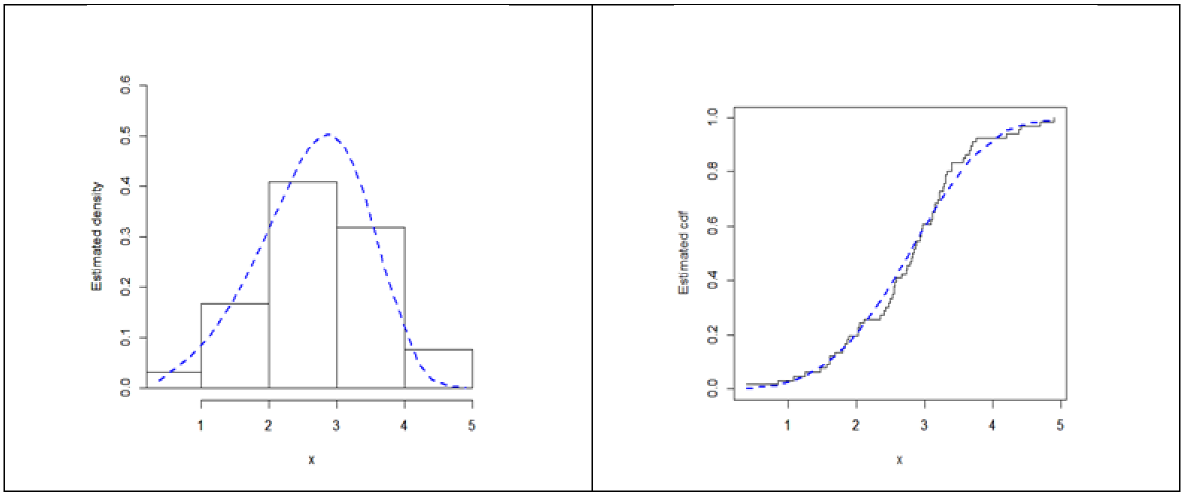

Figure 4: Fitted pdf and cdf of the proposed distribution.

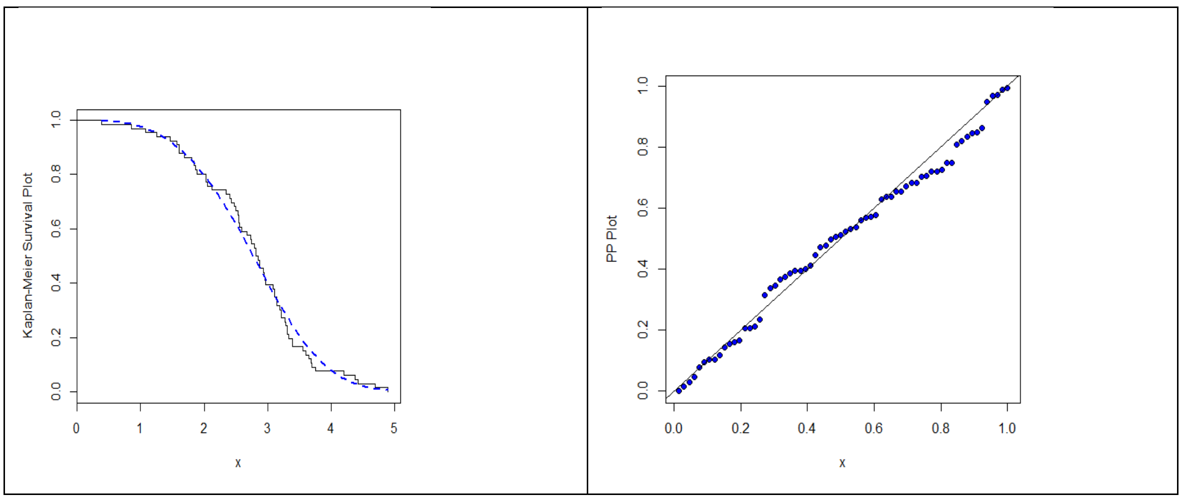

Figure 5: PP-plot and Kaplan Meier survival plot of the proposed model.

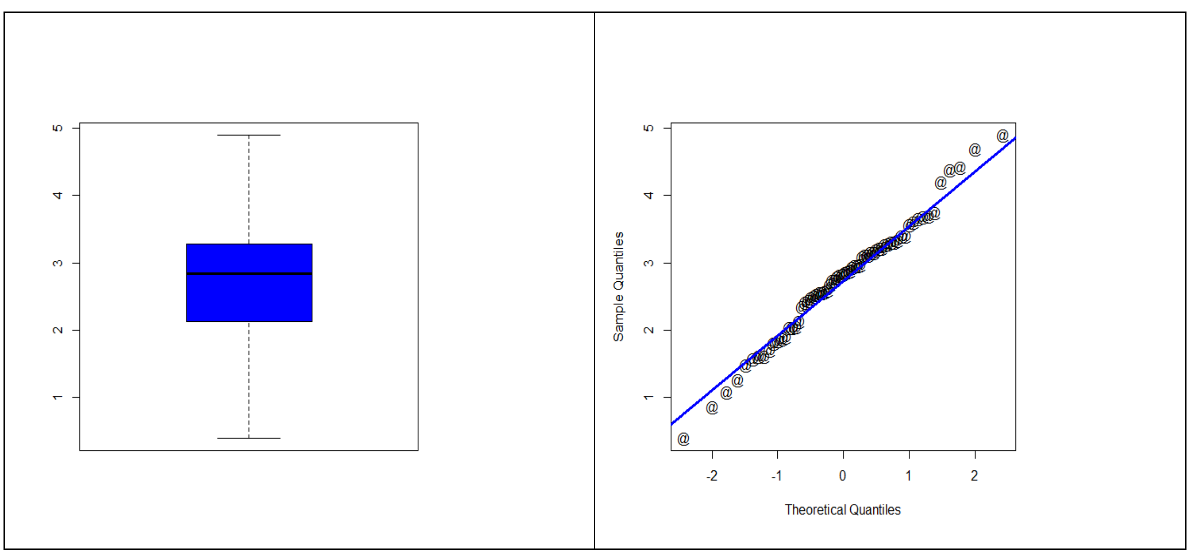

Figure 6: Box plot and Normal *Q-Q* plot of the proposed model.

7. Simulation Study

In this section, we assess the performance of the maximum likelihood estimators in terms of the sample size *n*. A numerical evaluation is carried out to examine the performance of maximum likelihood estimators for NEEW model (as particular case from the family). The evaluation of estimates is performed based on the following quantities for each sample size; the biases and the empirical mean square errors (MSEs) using the *R* software. The numerical steps are listed as follows:

i. A random sample $X_1, X_2, \ldots, X_n$ of sizes; n=30 and 50 are considered, these random samples are generated from the NEEW distribution by using inversion method.

ii. Six sets of the parameters are considered. The MLEs of (Proposed) model are evaluated for each parameter value and for each sample size.

iii. 1000 repetitions are made to calculate the biases and mean square error (MSE) of these estimators.

iv. Formulas used for calculating bias and MSE are given by $Bias(\hat{\alpha}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\alpha}-\alpha)$ and $MSE(\hat{\alpha}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\alpha}-\alpha)^2$, respectively.

v. Step (iv) is also repeated for the other parameters $(\theta, \gamma, \lambda)$.

Empirical results are reported in Table (4). We can detect from these tables that the estimates are quite stable and are close to the true value of the parameters as the sample sizes increase.

Table (4): Simulation Results: MLEs, Biases and MSEs

N	Parameters	Set 1 (α =0.5, θ=0.5, γ=0.9, $\lambda=0.7$)			Set 2 (α =0.6, θ=1, γ=0.5, λ=1)		
		MLEs	Bias	MSE	MLEs	Bias	MSE
30	α	0.4473	0.8270	0.4900	0.5208	0.4592	0.3905
	θ	0.6139	0.9890	1.0334	1.5607	1.3607	2.0295
	γ	1.3805	0.0978	0.8950	0.7140	0.1468	0.6472
	λ	0.9401	0.9903	1.0394	1.3607	1.3089	2.1035
50	α	0.4794	0.7786	0.1078	0.5690	0.3908	0.2701
	θ	0.5401	0.8940	0.9014	1.2906	1.0191	1.4095
	γ	1.0753	0.0690	0.6717	0.5522	0.1109	0.5091
	λ	0.6508	0.8719	0.8107	1.1022	0.9622	1.1901

Continued of **Table (4)**

N	Parameter	Set 3 (α =0.5, θ=0.2, γ=0.5, λ=0.4)			Set 4 (α =0.3, θ=0.5, γ=1, λ=0.5)		
		MLEs	Bias	MSE	MLEs	Bias	MSE
30	α	0.5810	0.9861	0.8096	0.2498	0.6912	0.3149
	θ	0.3150	0.9505	0.9371	0.7128	1.2008	1.4631

	γ	0.5996	0.5120	0.3206	0.8059	0.3161	0.2113
	λ	0.4796	0.7120	0.5206	0.5839	0.2161	0.4013
50	α	0.5309	0.7453	0.6790	0.2791	0.5403	0.1024
	θ	0.2786	0.8091	0.7012	0.5604	0.9024	0.6292
	γ	0.5606	0.4394	0.2493	0.9233	0.1145	0.0164
	λ	0.4590	0.6504	0.4013	0.5507	0.1067	0.1964

Continued of **Table (4)**

N	Parameter	Set 5 (α =0.7, θ=0.5, γ=1.5, λ=1.2)			Set 6 (α =0.5, θ=0.5, γ=2, λ=0.6)		
		MLEs	Bias	MSE	MLEs	Bias	MSE
30	α	0.5823	0.3177	0.6140	0.3835	0.1165	0.4138
	θ	0.6606	0.9132	1.3134	0.7660	1.2260	1.5915
	γ	1.4491	0.8539	0.7216	2.3060	0.1013	0.0609
	λ	1.4361	0.6539	0.9216	0.5201	0.2013	0.0713
50	α	0.6610	0.1049	0.3018	0.4529	0.0474	0.1924
	θ	0.5527	0.6921	0.9249	0.5607	0.5917	0.7860
	γ	1.4690	0.7090	0.6104	1.9173	0.0527	0.0179
	λ	1.4290	0.51700	0.7094	0.5427	0.0594	0.0207

8. Concluding Remarks

We have introduced a new function to extend the existing class of distributions. This effort leads to a new family of lifetime distributions, called the new exponentiated *T-X* family of distributions. General expressions for some of the mathematical properties of the new family are investigated. Maximum likelihood estimates are also obtained. There are certain advantages of using the proposed method like its cdf has a closed form and facilitating data modeling with monotonic and non-monotonic failure rates. A special sub-model of the new family, called the new exponentiated exponential Weibull distribution is considered and a real application is analyzed. In simulation study, the consistency and proficiency of the maximum likelihood estimators of the proposed model are also illustrated. The practical application of the proposed model reveal better fit to real-life data than the other well-known competitors. It is hoped, that the proposed method will attract wider applications in the area.

References

1. Ahmad, Z., & Hussain, Z. (2017). Flexible Weibull extended distribution. MAYFEB Journal of Materials Science. 2, 5–18 (2017).
2. Ahmad, Z., Elgarhy, M., & Hamedani, G. G. (2018). A new Weibull-X family of distributions: properties, characterizations and applications. Journal of Statistical Distributions and Applications, 5(1), 5.

3. Ahmad, Z., Ilyas, M., & Hamedani, G. G. (2018). The Extended Alpha Power Transformed Family of Distributions: Properties and Applications. Journal of Data Science. (Accepted).
4. Alexander, C., Cordeiro, G. M., Ortega, E. M., & Sarabia, J. M. (2012). Generalized beta-generated distributions. Computational Statistics & Data Analysis, 56(6), 1880-1897..
5. Almheidat, M., Famoye, F., & Lee, C. (2015). Some generalized families of Weibull distribution: Properties and applications. International Journal of Statistics and Probability, 4(3), 18.
6. Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. Metron, 71(1), 63-79.
7. Alzaghal, A., Famoye, F., & Lee, C. (2013). Exponentiated $ T $-$ X $ family of distributions with some applications. International Journal of Statistics and Probability, 2(3), 31.
8. Alzaghal, A., Famoye, F., & Lee, C. (2013). Exponentiated T-X family of distributions with some applications. International Journal of Statistics and Probability, 2(3), 31.
9. Ampadu, C. (2018). *Results in Distribution Theory and Its Applications Inspired by Quantile Generated Probability Distributions*. Lulu. com.
10. Cordeiro, G. M., & de Castro, M. (2011). A new family of generalized distributions. Journal of statistical computation and simulation, 81(7), 883-898.
11. Cordeiro, G. M., Ortega, E. M., & da Cunha, D. C. (2013). The exponentiated generalized class of distributions. Journal of Data Science, 11(1), 1-27.
12. Cordeiro, G. M., Ortega, E. M., & Nadarajah, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. Journal of the Franklin Institute, 347(8), 1399-1429.
13. Elbatala, I., Ahmad, Z., Elgarhy, B. M, & Almarashid, A. M. (2018). A New Alpha Power Transformed Familyof Distributions: Properties and Applications to the Weibull Model. Journal of Nonlinear Science andApplications, 12(1).
14. Eugene, N., Lee, C., & Famoye, F. (2002). Beta-normal distribution and its applications. Communications in Statistics-Theory and methods, 31(4), 497-512.
15. Famoye, F., Lee, C., & Olumolade, O. (2005). The beta-Weibull distribution. *Journal of Statistical Theory and Applications*, *4*(2), 121-136.
16. Galton, F. (1883). Inquiries into human faculty and its development. Macmillan and Company.
17. Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. In Mathematical statistics and probability theory (pp. 75-84). Springer, Dordrecht.
18. Glänzel, W. (1990). Some consequences of a characterization theorem based on truncated moments. Statistics, 21(4), 613-618.

19. Gradshteyn, I. S., & Ryzhik, I. M. (2014). Table of integrals, series, and products. Academic press.
20. Gupta, R. C., Gupta, P. L., & Gupta, R. D. (1998). Modeling failure time data by Lehman alternatives. Communications in Statistics-Theory and methods, 27(4), 887-904.
21. Haghighi, F., & Nikulin, M. (2006). A chi-squared test for power generalized Weibull family for the head-and-neck cancer censored data. Journal of Mathematical Sciences, 133(3), 1333-1341.
22. Kyurkchiev, N., Iliev, A., & Rahnev, A. (2018). Comments on a Zubair–g family of cumulative lifetime distributions. Some extensions. Communications in Applied Analysis, 23(1).
23. Marshall, A. W., & Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika, 84(3), 641-652.
24. Merovci, F., Alizadeh, M., & Hamedani, G. G. (2016). Another generalized transmuted family of distributions: properties and applications. *Austrian Journal of Statistics*, *45*(3), 71-93.
25. Moors, J. J. A. (1988). A quantile alternative for kurtosis. The statistician, 25-32.
26. Nasiru, S., Mwita, P. N., & Ngesa, O. (2017).Exponentiated Generalized Transformed-Transformer Family of Distributions. Journal of Statistical and Econometric Methods, 6(4), 1-17.
27. Ristić, M. M., & Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution. Journal of Statistical Computation and Simulation, 82(8), 1191-1206.
28. Saboor, A., & Pogány, T. K. (2016). Marshall–Olkin gamma–Weibull distribution with applications. Communications in Statistics-Theory and Methods, 45(5), 1550-1563.
29. Tahir, M. H., Cordeiro, G. M., Alzaatreh, A., Mansoor, M., & Zubair, M. (2016). The Logistic-X family of distributions and its applications. Communications in Statistics-Theory and Methods, 45(24), 7326-7349.
30. Torabi, H., & Montazeri, N. H. (2012). The gamma-uniform distribution and itsapplications. Kybernetika, 48(1), 16-30.
31. Torabi, H., & Montazeri, N. H. (2014). The logistic-uniform distribution and its applications. Communications in Statistics-Simulation and Computation, 43(10), 2551-2569.
32. Wikipedia contributors. (2018). "Binomial theorem." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia.
33. Wikipedia contributors. (2018). "Lambert W function." Wikipedia, The Free Encyclopedia. Wikipedia, The Free Encyclopedia.
34. Zografos, K., & Balakrishnan, N. (2009). On families of beta-and generalized gamma-generated distributions and associated inference. Statistical Methodology, 6(4), 344-362.

Appendix A.

Theorem 4.1. Let (Ω, *F, P*) be a given probability space and let *H = [d; e]* be an interval for some *d<e* ($d=-\infty$; $e=\infty$ might as well be allowed). Let $X: \Omega \to H$ be a continuous random variable with the distribution function F and let $q_1(x)$ and $q_2(x)$ be two real functions defined on *H* such that

$$E\left(q_2(X) \mid X \geq x\right) = E\left(q_1(X) \mid X \geq x\right)\xi(x), \qquad x \in H,$$

is defined with some real function $\eta(x)$. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(x)$ and F is twice continuously differentiable and strictly monotone function on the set *H*. Finally, assume that the equation $\xi q_1 = q_2$ has no real solution in the interior of *H*. Then F is uniquely determined by the functions q_1, q_2 and ξ particularly

$$F(x) = \int_a^x C\left|\frac{\xi^{/}(u)}{\xi(u)q_1(u)-q_2(u)}\right| \exp(-s(u))\,du,$$

where the function $s(u)$ is a solution of the differential equation $s^{/}(u) = \dfrac{\xi^{/}(u)q_1(u)}{\xi(u)q_1(u)-q_2(u)}$ and *C* is the normalization constant, such that $\int_H dF = 1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see, Glänzel (1990)), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions q_{1n}, q_{2n} and η_n $(n \in \mathbb{N})$ satisfy the conditions of Theorem 4.1 and let $q_{1n} \to q_1$, $q_{2n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F. Under the condition that q_{1n} and q_{2n} are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if ξ_n converges to ξ, where

$$\xi(x) = \frac{E\left(q_2(X) \mid X \geq x\right)}{E\left(q_1(X) \mid X \geq x\right)}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q_1, q_2 and η respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Levy-Smirnov distribution if $\alpha \to \infty$.

A further consequence of the stability property of Theorem 6.1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and, specially, η should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose η as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics. In some cases, one can take $q_1(x) \equiv 1$, which reduces the condition of Theorem 4.1 to $E\left(q_2(X) \mid X \geq x\right) = \xi(x)$, $x \in H$. We, however, believe that employing three functions q_1, q_2 and ξ will enhance the domain of applicability of Theorem 4.1.