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Bayesian and Classical Estimation for the One Parameter Double Lindley Model

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Abstract

The main motivation of this paper is to show how the different frequentist estimators of the new distribution perform for different sample sizes and different parameter values and to raise a guideline in choosing the best estimation method for the new model. The unknown parameters of the new distribution are estimated using the maximum likelihood method, ordinary least squares method, weighted least squares method, Cramer-Von-Mises method and Bayesian method. The obtained estimators are compared using Markov Chain Monte Carlo simulations and we observed that Bayesian estimators are more efficient compared to other the estimators.

Key Words: Different Method of Estimations; Markov Chain Monte Carlo Simulations; Clayton Copula; Bayesian Estimation; Farlie Gumbel Morgenstern Copula; Renyi's entropy Copula.

Mathematical Subject Classification: 62N01; 62N02; 62E10.

1. Introduction

The probability density function (PDF) and the cumulative distribution function (CDF) of Lindley (Li) distribution (see Lindley (1958)) are given by

$$g_{\mathrm{Li}}^{(\lambda)}(x) = \frac{\lambda^2(1+x)}{1+\lambda} exp(-\lambda x)\,|_{x>0,\lambda>0,} \quad (1)$$

and

$$G_{\mathrm{Li}}^{(\lambda)}(x) = 1 - \frac{1+\lambda+\lambda x}{1+\lambda} exp(-\lambda x)|_{x>0,\lambda>0}, \quad (2)$$

respectively. The shape parameter λ is a positive real number and can result in either a unimodal or monotone decreasing (i.e. consistently decreasing) model. The Li model has a thin tail because the it decreases exponentially for large values of x. The Li distribution is one way to describe the lifetime of a process or device. It can be also used in a wide variety of fields like engineering, biology and medicine. Ghitany et al. (2008 a) discussed various properties of the Li model. A discrete version of the Li distribution has been suggested by Deniz and Ojeda (2011) with applications in count data related to insurance. Ghitany et al. (2008 b) obtained size-biased and zero-truncated version of Poisson-Lindley distribution with various properties and applications. Ghitany et. al (2011) state that it's especially useful for modeling in mortality studies.

The goal of this article is to study the One Parameter Double Li (DLi) model, the new model is derived based on the odd Lindley-G (OLi-G) family of distributions firstly introduced by Silva et al. (2017), the PDF and CDF of the OL-G family of distribution are given by

$$f_{\text{OL-G}}^{(\psi)}(x)=\frac{1}{2}\frac{g_{\psi}(x)}{\overline{G}_{\psi}(x)^{3}}exp\left[-O_{\psi}(x)\right], \quad (3)$$

where $O_{\psi}(x)=\frac{G_{\psi}(x)}{\overline{G}_{\psi}(x)}$ and

$$F_{\text{OL-G}}^{(\psi)}(x)=1-\frac{1+\overline{G}(x;\xi)}{2\overline{G}_{\psi}(x)}exp\left[-O_{\psi}(x)\right], \quad (4)$$

respectively. To this end, we use (1), (2) and (3) to obtain the one-parameter DLi CDF and PDF (for $x>0$) as

$$F_{\text{DLi}}^{(\lambda)}(x)=1-\frac{1+exp(-\lambda x)\frac{1+\lambda+\lambda x}{1+\lambda}}{2\,exp(-\lambda x)\frac{1+\lambda+\lambda x}{1+\lambda}}exp\left[-\frac{1-\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)}{\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)}\right], \quad (5)$$

and

$$f_{\text{DLi}}^{(\lambda)}(x)=\frac{\lambda^{2}(1+x)(1+\lambda)^{2}}{2(1+\lambda+\lambda x)^{3}\,exp(-2\lambda x)}exp\left[-\frac{1-\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)}{\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)}\right], \quad (6)$$

respectively. The CDF of X in (5) can be easily expressed as

$$f_{\text{DLi}}^{(\lambda)}(x)=\sum_{i,k=0}^{\infty}\upsilon_{i,k}\;h_{\Delta^{\cdot},\lambda}(x)|_{\Delta^{\cdot}=1+i+k}, \quad (7)$$

where $\upsilon_{i,k}=\frac{(-1)^{k}}{2\Delta^{\cdot}i!}\Gamma(i+k+3)\Gamma^{-1}(k+3)$ and $h_{\Delta^{\cdot},\lambda}(x)$ is PDF of Exp-Li model with positive parameters $\Delta^{\cdot}=1+i+k$ and λ. The CDF of X can be given by integrating (7) as

$$F_{\text{DLi}}^{(\lambda)}(x)=\sum_{i,k=0}^{\infty}\upsilon_{i,k}\;H_{\Delta^{\cdot},\lambda}(x), \quad (8)$$

where $H_{\Delta^{\cdot},\lambda}(x)$ is PDF of Exp-Li model with positive parameters $\Delta^{\cdot}$ and λ. Figure 1 (left panel) displays some plots of the DLi density for different values of λ , these plots show that the new density can be "unimodal" with different flexible shapes. The HRF (right panel) of the DLi distribution can be "increasing" and "**J**-shaped". Many useful "unimodal" real-life data sets can be used in modeling and found in Ibrahim (2019 and 2020a,b), Goual et al. (2019) and Ibrahim et al. (2020).

The corresponding survival function to (4) is given by

$$R_{\text{DLi}}^{(\lambda)}(x)=\frac{1+exp(-\lambda x)\frac{1+\lambda+\lambda x}{1+\lambda}}{2\,exp(-\lambda x)\frac{1+\lambda+\lambda x}{1+\lambda}}exp\left[-\frac{1-\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)}{\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)}\right].$$

The hazard rate functions (HRF) of X becomes

$$h_{\text{DLi}}^{(\lambda)}(x)=\frac{1-\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)}{\left[\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)\right]\left[1+\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)\right]\left[\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)\right]}$$

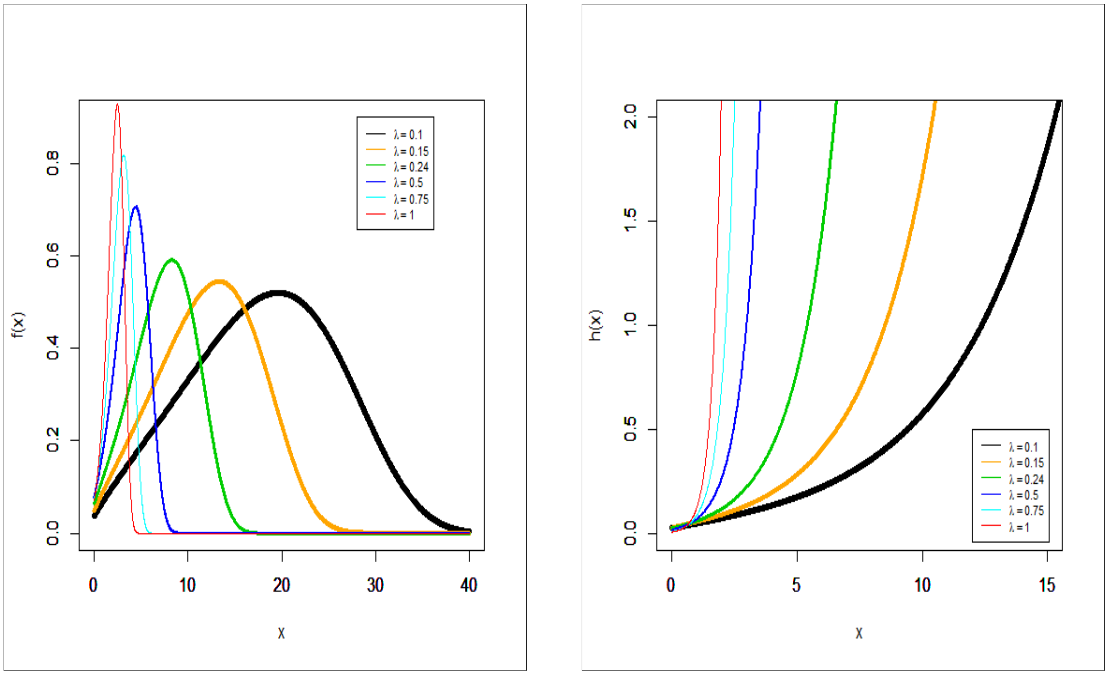

Figure 1: Plots of the PDF and HRF of the DLi model.

The multiplying quantity

$$\frac{1}{\left[\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)\right]\left[1+\frac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x)\right]}$$

works as a corrected factor for the Li HRF. In the literature, certain generalizations on the Li distribution are proposed and studied (see Ghitany et al. (2008a), Ghitany et al. (2008a), Deniz and Ojeda (2011), Nadarajah et al. (2011), Ghitany et al. (2011), Bakouch et al.(2012), Merovci and Sharma (2014), Sharma et al. (2015), Alizadeh et al. (2016) Afify et al. (2016), Ozel et al. (2017), Nofal et al. (2017), Merovci et al. (2017), Alizadeh et al. (2017), Cordeiro et al. (2017), Korkmaz et al. (2017), Alizadeh et al. (2018), Afify et al. (2018a,b), Yousof et al. (2018a,b), Korkmaz et al. (2018a,b), Korkmaz et al. (2019), Ibrahim et al. (2019) and Merovci et al. (2020) among others. The main motivation of the paper is to show how the different frequentist estimators of the DLi distribution perform for different sample sizes and different parameter values and to raise a guideline in choosing the best estimation method for the DLi model. The unknown parameters of the DLi distribution are estimated using the maximum likelihood (ML) method, ordinary least squares (OLS) method, weighted least squares (WLS) method, Cramer-Von-Mises (CVM) method and Bayesian method. The obtained estimators are compared using Markov Chain Monte Carlo (MCMC) simulations and we observed that Bayesian estimators are more efficient compared to other the estimators.

2. Mathematical properties

2.1 Ordinary and incomplete moments

The r^{th} ordinary moment of X is given by $\mu_r' = E(X^r) = x^r$Using (7), we obtain

$$\mu_r' = \sum_{i,k=0}^{\infty} \upsilon_{i,k}\, \varDelta^{\cdot}[\lambda^2/(1+\lambda)]K(\varDelta^{\cdot},\lambda,r,\lambda) \qquad (9)$$

where

$$K(a,b,r,\delta) = \int_0^{\infty} x^r\,(1+x)\left[1-\frac{1+b+bx}{1+b}exp(-bx)\right]^{a-1} exp(-\delta x)\,dx,$$

Which can be expressed as

$$K(a,b,r,\delta)=\sum_{w=0}^{\infty}\sum_{j=0}^{w}\sum_{m=0}^{j+1}\zeta_{w,j,m}^{(r,a)}\,\Gamma(1+r+m).$$

and

$$\zeta_{w,j,m}^{(r,a)}=(-1)^{w}b^{j}(1+b)^{-w}(bw+\delta)^{-(1+r+m)}\binom{a-1}{w}\binom{w}{j}\binom{j+1}{m}$$

The s^{th} incomplete moment of X, say $\phi_s(t)$, is given by $\phi_s(t)=\int_0^t x^s f(x)dx$. Using Equation (7), we obtain

$$\phi_s(t)=\sum_{i,k=0}^{\infty}\upsilon_{i,k}\,\Delta^{\cdot}[\lambda^2/(1+\lambda)]L(\Delta^{\cdot},\lambda,s,\lambda,t) \tag{10}$$

where

$$L(a,b,s,\delta,t)=\int_t^{\infty}x^s\,(1+x)\left[1-\frac{1+b+bx}{1+b}exp(-bx)\right]^{a-1}exp(-\delta x)\,dx$$
$$=\sum_{w=0}^{\infty}\sum_{j=0}^{w}\sum_{m=0}^{j+1}\zeta_{w,j,m}^{(s,a)}\,\Gamma(1+s+m,(bw+\delta)t)$$

and $\Gamma(\nabla,x)=\int_x^{\infty}t^{\nabla-1}\,exp(-t)\,dt$ denotes the complementary incomplete gamma function.

2.2 Order statistics and their moments

The PDF of the i^{th} order statistic, say $X_{i:n}$, can be expressed as

$$f_{i:n}(x)=[B(i,n-i+1)]^{-1}f(x)F(x)^{i-1}[1-F(x)]^{n-i}, \tag{11}$$

where $B(\cdot,\cdot)$ is the beta function. Substituting (5) and (6) in Equation (11), we obtain

$$f_{i\,:\,n}(x)=\sum_{m,p=0}^{\infty}\sum_{j=0}^{k+n-i}\upsilon_{j,m,p}\;\;h_{\tau^{\cdot},\lambda}(x)|_{\tau^{\cdot}=(1+j+m+p)},$$

where

$$\upsilon_{j,m,h}=\sum_{k=0}^{i-1}(-1)^{k+m}\,2^{-(j+1)}[B(i,n-i+1)]^{-1}[m!\,\tau^{\cdot}]^{-1}\binom{\tau^{\cdot}-1}{j+m}\binom{k+n-1}{j}\binom{i-1}{k}.$$

Then, the q^{th} moment of $X_{i:n}$ is given by

$$E\left(X_{i:n}^{q}\right)=\sum_{m,p,h=0}^{\infty}\sum_{j=0}^{k+n-i}\upsilon_{j,m,p}\;\;\tau^{\cdot}[\lambda^2/(1+\lambda)]K(\tau^{\cdot},\lambda,q,\lambda) \tag{12}$$

where

$$K(a,b,q,\delta)=\int_0^{\infty}x^r\,(1+x)\left[1-\frac{1+b+bx}{1+b}exp(-bx)\right]^{a-1}exp(-\delta x)\,dx$$
$$=\sum_{w=0}^{\infty}\sum_{l=0}^{w}\sum_{d=0}^{l+1}\zeta_{w,l,m}^{(q,a)}\,\Gamma(1+q+d).$$

and

$$\zeta_{w,l,m}^{(q,a)}=(-1)^{w}b^{l}(1+b)^{-w}(bw+\delta)^{-(1+q+d)}\binom{a-1}{w}\binom{w}{l}\binom{l+1}{d}$$

Based upon the moments in Equation (12), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable DLi order statistics. They are linear functions of expected order statistics defined by

$$\lambda_r=r^{-1}\sum_{d=0}^{r-1}(-1)^{d}\binom{r-1}{d}E(X_{r-d\,:\,r}),\;\; r\geq 1.$$

2.3 Moment of Residual and Reversed Residual Life

The n^{th} moment of the residual life, say

$$z_n(t) = E[(X-t)^n]|_{X>t, n=1,2,\dots}$$

The n^{th} moment of the residual life of X is given by

$$z_n(t) = [1-F(t)]^{-1}\int_t^{\infty}(x-t)^n\,dF(x).$$

We can write

$$z_n(t) = [1-F(t)]^{-1}\sum_{i,k=0}^{\infty}\sum_{r=0}^{n}\upsilon_{i,k}\,(-t)^{n-r}\;\binom{n}{r}\int_t^{\infty}x^r\,h_{\Delta^\cdot,\lambda}(x)dx$$

Then,

$$z_n(t) = [1-F(t)]^{-1}\sum_{i,k=0}^{\infty}\sum_{r=0}^{n}\upsilon_{i,k}\,(-t)^{n-r}\;\binom{n}{r}\Delta^\cdot\left[\frac{\lambda^2}{(1+\lambda)}\right]L(\Delta^\cdot,\lambda,n,\lambda,t).$$

The mean residual life (MRL), or the life expectation at age t, of X can be obtained by setting $n=1$ in the last equation and it represents the expected additional life length for a unit which is alive at age t. The n^{th} moment of the reversed residual life, say

$$Z_n(t) = E[(t-X)^n]|_{X\le t, t>0 \text{ and } n=1,2,\dots}$$

Then we have

$$Z_n(t) = F(t)^{-1}\int_0^{t}(t-x)^n\,dF(x).$$

Then, the n^{th} moment of the reversed residual life of X becomes

$$Z_n(t) = F(t)^{-1}\sum_{i,k=0}^{\infty}\sum_{r=0}^{n}\upsilon_{i,k}\,(-1)^r\binom{n}{r}t^{n-r}\int_0^{t}\;x^r h_{\Delta^\cdot,\lambda}(x)dx.$$

Then,

$$Z_n(t) = F(t)^{-1}\sum_{i,k=0}^{\infty}\sum_{r=0}^{n}\upsilon_{i,k}\,(-1)^r\binom{n}{r}t^{n-r}\Delta^\cdot\left[\frac{\lambda^2}{(1+\lambda)}\right][K(\Delta^\cdot,\lambda,n,\lambda)-L(\Delta^\cdot,\lambda,n,\lambda,t)].$$

The mean inactivity time (MIT) also called the mean reversed residual life function represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$ and it is defined by

$$Z_n(t) = E[(t-X)^n]|_{X\le t, t>0 \text{ and } n=1}.$$

The MIT of the DLi model can be obtained easily by setting $n=1$ in the above equation.

3. Copula

We derive some new bivariate type DLi (Biv-DLi) model using “Farlie Gumbel Morgenstern” (FGM) Copula (see Morgenstern (1956), Gumbel (1958) and Gumbel (1960)), modified FGM, ” Clayton Copula” and “Renyi's entropy” (Pougaza and Djafari (2011)). The Multivariate DLi (MvDLi) type is also presented. However, future works may be allocated to study these new models. First, we consider the joint CDF of the FGM family, where $C_\nabla(u,\psi) = u\psi(1+\nabla u^*\psi^*)|_{u^*=1-u}$, where the marginal function $u=F_1, \psi=F_2$, $\nabla\in(-1,1)$ is a dependence parameter and for every $u,\psi\in(0,1)$, $C(u,0)=C(0,\psi)=0$ which is "grounded minimum" and $C(u,1)=u$ and $C(1,\psi)=\psi$ which is "grounded maximum", $C(u_1,\psi_1)+C(u_2,\psi_2)-C(u_1,\psi_2)-C(u_2,\psi_1)\ge 0$.

3.1 Biv-DLi type via FGM Copula

A Copula is continuous in u and ψ; actually, it satisfies the “stronger Lipschitz condition”, where

$$|C(u_2,\psi_2)-C(u_1,\psi_1)| \le |u_2-u_1| + |\psi_2-\psi_1|.$$

For $0\le u_1\le u_2\le 1$ and $0\le\psi_1\le\psi_2\le 1$, we have

$$Pr(u_1\le U\le u_2, \psi_1\le W\le\psi_2) = C(u_1,\psi_1)+C(u_2,\psi_2)-C(u_1,\psi_2)-C(u_2,\psi_1)\ge 0.$$

Then, setting $u^* = 1-F_{\psi_1}(x_1)|_{[u^*=(1-u)\in(0,1)]}$ and $\psi^* = 1-F_{\psi_2}(x_2)|_{[\psi^*=(1-\psi)\in(0,1)]}$. We can esaily get the get the joint CDF of the FGM family.The joint PDF can then derived from $c_\nabla(u,\psi) = 1+\nabla u^\cdot\psi^\cdot|_{(u^\cdot=1-2u \text{ and } \psi^\cdot=1-2\psi)}$ or from $f(x_1,x_2) = C(F_1,F_2)f_1f_2$.

3.2 Biv-DLi and MvDLi type via Clayton Copula

The "Clayton Copula" can be considered as $C(\psi_1,\psi_2) = [(1/\psi_1)^{\nabla} + (1/\psi_2)^{\nabla} - 1]^{-\nabla^{-1}}|_{\nabla\in(0,\infty)}$. Setting $\psi_1 = F_{\Psi_1}(t)$ and $\psi_2 = F_{\Psi_2}(x)$. Then, the Biv-DLi type can be derived from $C(\psi_1,\psi_2) = C(F_{\Psi_1}(t), F_{\Psi_2}(x))$. Similarly, the MvDLi **(** m -dimensional extension) from the above can be derived from $C(\psi_i) = [\sum_{i=1}^{m} \psi_i^{-\nabla} + 1 - m]^{-\nabla^{-1}}$.

3.3 Biv-DLi type via Renyi's entropy

Using the theorem of Pougaza and Djafari (2011) where $R(u,\psi) = x_2 u + x_1\psi - x_1 x_2$. Then, the associated Biv-DLi will be $R(u,\psi) = R(F_{\Psi_1}(x_1), F_{\Psi_2}(x_2))$.

2.4 BivDLi type via modified FGM Copula

The modified version of the bivariate FGM copula defined as (Rodriguez-Lallena and Ubeda-Flores (2004)) $C_\Delta(u,\psi) = uv + \Delta\widetilde{\boldsymbol{\phi}(u)}\widetilde{\boldsymbol{\psi}(\psi)}$, where $\widetilde{\boldsymbol{\phi}(u)} = u\overline{\boldsymbol{\phi}(u)}$, and $\widetilde{\boldsymbol{\psi}(\psi)} = \psi\overline{\boldsymbol{\psi}(\psi)}$. Where $\boldsymbol{\phi}(u)$ and $\boldsymbol{\psi}(v)$ are two continuous functions on (0,1) where $\boldsymbol{\phi}(0) = \boldsymbol{\phi}(1) = \boldsymbol{\psi}(0) = \boldsymbol{\psi}(1) = 0$. Let

$$a = inf\left\{\frac{\partial}{\partial u}\widetilde{\boldsymbol{\phi}(u)}: A_1(u)\right\} < 0, b = sup\left\{\frac{\partial}{\partial u}\widetilde{\boldsymbol{\phi}(u)}: A_1(u)\right\} < 0,$$
$$c = inf\left\{\frac{\partial}{\partial \psi}\widetilde{\boldsymbol{\psi}(\psi)}: A_2(\psi)\right\} > 0, d = sup\left\{\frac{\partial}{\partial \psi}\widetilde{\boldsymbol{\psi}(\psi)}: A_2(\psi)\right\} > 0.$$

Then, $min(ab, cd) \geq 1$, where $\frac{\partial}{\partial u}\widetilde{\boldsymbol{\phi}(u)} = \boldsymbol{\phi}(u) + u\frac{\partial}{\partial u}\boldsymbol{\phi}(u)$, $A_1(u) = \left\{u \in (0,1): \frac{\partial}{\partial u}\widetilde{\boldsymbol{\phi}(u)} \text{ exists}\right\}$ and $A_2(\psi) = \left\{\psi \in (0,1): \frac{\partial}{\partial \psi}\widetilde{\boldsymbol{\psi}(\psi)} \text{ exists}\right\}$.

2.3.1 BivDLi-FGM (Type I) model

The BivDLi-FGM **(**Type-**II)** copula can be obtained directly from $C_\Delta(u,v) = u\psi + \Delta\widetilde{\boldsymbol{\phi}(u)}\widetilde{\boldsymbol{\psi}(\psi)}$.

2.3.2 BivDLi-FGM (Type II) model:

Consider the following functional form for both $\boldsymbol{\phi}(u)$ and $\boldsymbol{\psi}(\psi)$ which satisfy all the conditions stated earlier where $\boldsymbol{\phi}(u)|_{(\Delta_1>0)} = u^{\Delta_1}(1-u)^{1-\Delta_1}$ and $\boldsymbol{\psi}(\psi)|_{(\Delta_2>0)} = \psi^{\Delta_2}(1-\psi)^{1-\Delta_2}$. The corresponding bivariate copula (henceforth, BivDLi-FGM **(**Type-**II)** copula) can be derived from

$$C_{\Delta,\Delta_1,\Delta_2}(u,\psi) = u\psi[1 + \Delta u^{\Delta_1}\psi^{\Delta_2}(1-u)^{1-\Delta_1}(1-\psi)^{1-\Delta_2}].$$

2.3.3 BvDLi-FGM (Type III) model:

Consider the following functional form for both $\boldsymbol{\phi}(u)$ and $\boldsymbol{\psi}(\psi)$ which satisfy all the conditions stated earlier where $\boldsymbol{\phi}^{\bullet}(u) = u[log(1+\overline{u})]$ and $\boldsymbol{\psi}^{\bullet}(\psi) = \psi\left[log\left(1+\overline{\psi}\right)\right]$**.** Then, the associated CDF of the BivDLi-FGM (Type-**III**)

$$C_\Delta(u,\psi) = uv + [1 + \Delta\boldsymbol{\phi}^{\bullet}(u)\boldsymbol{\psi}^{\bullet}(\psi)]$$

2.3.4 BvDLi-FGM (Type IV) model:

The BivDLi-FGM (Type-**IV**) model can be derived from $C(u,\psi) = uF^{-1}(\psi) + \psi F^{-1}(u) - F^{-1}(u)F^{-1}(\psi)$where $F^{-1}(u)$ and $F^{-1}(\psi)$ can be easily derived (Silva et al. (2017)).

4. Classical estimation

4.1 Maximum likelihood method

Let $x_1, \cdots, x_n$ be a random sample from the DLi distribution with parameter λ. Then, the log-likelihood function, say $\ell = \ell(\lambda)$, is given by

$$\ell = -n\,log(2) + 2n\,log[\lambda(1+\lambda)] + \sum_{i=0}^{n} log(1+x_i) + 2\lambda\sum_{i=0}^{n} x_i - 3\sum_{i=0}^{n} log\, z_i - \sum_{i=0}^{n}\left(\frac{1}{s_i} - 1\right), \tag{13}$$

where $z_i = 1 + \lambda + \lambda x_i$ $s_i = \frac{z_i}{(1+\lambda)} exp(-\lambda x_i)$. Equation (13) can be maximized either directly by using the **R** (optim function), **SAS** (PROC NLMIXED) or **Ox** program (sub-routine MaxBFGS) or by solving the nonlinear likelihood equations obtained by differentiating (13). Note that ML estimate of the λ cannot be solved analytically so numerical iteration techniques, such as the Newton-Raphson algorithm, are adopted to solve the log-likelihood equation for which (13) is maximized.

4.2 Method of ordinary least square and weighted least square estimation

The theory of OLS and WLS was firstly proposed to estimate the parameters of Beta distribution. It is based on the minimization of the sum of the square of differences of theoretical cumulative distribution function and empirical distribution function. Suppose $F_{\mathrm{DLi}}^{(\lambda)}(x_{i:n})$ denotes the CDF of DLi model and if $x_1 < x_2 < \cdots < x_n$ be the n ordered random sample. The OLS estimators (OLSEs) are obtained upon minimizing

$$OLS(\lambda) = \sum_{i=1}^{n}\left[F_{\mathrm{DLi}}^{(\lambda)}(x_{i\,:\,n}) - i/(n+1)\right]^2. \tag{14}$$

Using (5) and (14), we have

$$OLS(\lambda) = \left(1 - \left\{\begin{matrix} \dfrac{1 + exp(-\lambda x_i)\dfrac{1+\lambda+\lambda x_i}{1+\lambda}}{2\,exp(-\lambda x_i)\dfrac{1+\lambda+\lambda x_i}{1+\lambda}} \\ \times\, exp\left[-\dfrac{1-\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}{\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}\right]\end{matrix}\right\} - i/(n+1)\right)^2$$

The OLSEs are obtained via solving the following nonlinear equation

$$0 = \sum_{i=1}^{n}\left(1 - \left\{\begin{matrix} \dfrac{1 + exp(-\lambda x_i)\dfrac{1+\lambda+\lambda x_i}{1+\lambda}}{2\,exp(-\lambda x_i)\dfrac{1+\lambda+\lambda x_i}{1+\lambda}} \\ \times\, exp\left[-\dfrac{1-\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}{\dfrac{1+\lambda+\lambda x}{1+\lambda}exp(-\lambda x_i)}\right]\end{matrix}\right\} - i/(n+1)\right)\xi_\lambda(x_i),$$

where $\xi_\lambda(x_i)$ the values of 1st derivatives with recpect to (w.r.t.) the parameter λ of the CDF of the DLi distribution. The OLSEs of the parameter λ is obtained by solving the above simultaneous equations by using any numerical approximation technique. The WLSE are obtained by minimizing the given form of equation with respect to the parameters.

$$WLS(\lambda) = \sum_{i=1}^{n} w_i\left[F_{\mathrm{DLi}}^{(\lambda)}(x_{i\,:\,n}) - i/(n+1)\right]^2.$$

The WLSEs of the parameters are obtained by solving the following non-linear equation

$$0 = \sum_{i=1}^{n} w_i\left(1 - \left\{\begin{matrix} \dfrac{1 + exp(-\lambda x_i)\dfrac{1+\lambda+\lambda x_i}{1+\lambda}}{2\,exp(-\lambda x_i)\dfrac{1+\lambda+\lambda x_i}{1+\lambda}} \\ \times\, exp\left[-\dfrac{1-\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}{\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}\right]\end{matrix}\right\} - i/(n+1)\right)\xi_\lambda(x_i),$$

where

$$w_i = [(n+1)^2(n+2)]/[i(n-i+1)].$$

4.3 Method of Cramer-Von-Mises estimation

The CVME of the parameters is based on the theory of minimum distance estimation (MDE). It was first proposed by MacDonald (1971) and justified that the bias of the estimator is smaller than the other MD estimators. So, the CVME of the parameter λ is obtained by minimizing the following expression w.r.t. the parameter λ, then we have

$$C\Phi M(\lambda) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F_{\mathrm{DLi}}^{(\lambda)}(x_{i\,:\,n}) - (2i-1)/2n\right]^2,$$

and

$$CVM(\lambda)=\sum_{i=1}^{n}\left(1-\left\{\begin{matrix}\dfrac{1+exp(-\lambda x_i)\dfrac{1+\lambda+\lambda x_i}{1+\lambda}}{2\,exp(-\lambda x_i)\dfrac{1+\lambda+\lambda x_i}{1+\lambda}}\\ \times exp\left[-\dfrac{1-\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}{\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}\right]\end{matrix}\right\}-(2i-1)/2n\right)^2.$$

The, Cramer-Von-Mises estimators (CVME) of the parameters are obtained by solving the following non-linear equation

$$0=\sum_{i=1}^{n}\left(1-\left\{\begin{matrix}\dfrac{1+exp(-\lambda x_i)\dfrac{1+\lambda+\lambda x_i}{1+\lambda}}{2\,exp(-\lambda x_i)\dfrac{1+\lambda+\lambda x_i}{1+\lambda}}\\ \times exp\left[-\dfrac{1-\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}{\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}\right]\end{matrix}\right\}-(2i-1)/2n\right)\xi_\lambda(x_i).$$

5. Bayesian estimation

In this section, we use Bayesian procedures to construct the estimators of the unknown parameters of DLi distribution. There are many situations where maximum likelihood estimator does not converge especially with higher dimension models. In such cases, the use of Bayesian methods is sought. At first sight, Bayesian methods seem to be very complex as the estimators involve intractable integrals. However, the advanced MCMC techniques make possible to apply Bayesian methods even in higher dimension models. Under Bayesian estimation, we update the likelihoods with prior knowledge explore the posterior probabilities of the parameters. Here we assume the gamma priors (GaP) of the parameter λ of the following forms $\pi_1(\lambda)\sim \mathrm{Ga}(\zeta_1,\zeta_2)$ where Ga (ζ_1,ζ_2) stands for gamma distribution with shape parameter ζ_1 and scale parameter d_i. It is further assumed that the parameters are to be independently distributed. The joint prior distribution is given by

$$\pi(\zeta_1,\zeta_2)=\frac{\zeta_2^{\zeta_1}}{\Gamma(\zeta_1)}\lambda^{\zeta_1-1}\,exp[-(\lambda\zeta_2)].$$

The posterior distribution of the parameters is defined by

$$\pi\big(\lambda|\underline{x}\big)\propto \text{likelihood}(\lambda|\underline{x})\times\pi(\zeta_1,\zeta_2).$$

Under squared error loss, Bayes estimators of parameter λ is the means of their marginal posteriors and defined by

$$\hat{\lambda}_{\text{Bayes}}=\int_{\lambda}\lambda\pi\big(\lambda|\underline{x}\big)d\lambda \tag{15}$$

It is not easy to calculate Bayesian estimates through equations (15) so the numerical approximation techniques are needed. Therefore, we propose the use of MCMC techniques namely Gibbs sampler and Metropolis Hastings (MH) algorithm (see Hastings (1970). Since the conditional posteriors of the parameters cannot be obtained in any standard forms, we, therefore used a hybrid MCMC strategy for drawing samples from the joint posterior of the parameter. To implement the Gibbs algorithm, the full conditional posteriors of λ are given by

$$\pi(\lambda|\underline{x})\propto\lambda^{n+\zeta_1-1}\,exp[-(\lambda\zeta_2)]\prod_{i=1}^{n}\Upsilon_i,$$

where

$$\Upsilon_i=\frac{\dfrac{\lambda^2(1+x_i)}{1+\lambda}exp(-\lambda x_i)}{2\left[\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)\right]^3}exp\left[-\frac{1-\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}{\dfrac{1+\lambda+\lambda x_i}{1+\lambda}exp(-\lambda x_i)}\right]$$

The simulation algorithm, we followed is given by

1) Provide initial value, say $\lambda_{(0)}$ then at i^{th} stage,
2) Using MH algorithm, Generate $\lambda_{(i)}\sim\pi_1\big(\lambda_{(i-1)}|,\underline{x}\big)$,
3) Repeat steps 2 , $M(=10000)$ times to get the samples of size M from the corresponding posteriors of interest.

4) Obtain the Bayesian estimates of λ using the following formula

$$\hat{\lambda}_{\text{Bayes}} = \frac{1}{M - M_0} \sum_{j=1+M_0}^{M} \lambda_j$$

5) where $M_0 (\approx 2000)$ is the burn-in period of the generated Markov chains.

6. Simulation study

A MCMC simulation study is conducted in this section to compare the performance of the different classical estimators of the unknown parameters of the DLi distribution with the Bayesian estimators. This performance of all estimation methods is evaluated regarding their mean squared errors (MSEs). All computations in this section are done by **Mathcad program Version 15.0**. We generate 1000 samples of the DLi distribution, where $n = 20$, 50, 100, 200, 500 and choosing λ =0.6, 1.5, 5 and 0.2. The average values (AVs) of estimates and MSEs of MLEs, LSEs, WLSEs, CVMEs and Bayesian estimators are obtained and reported in Tables 1-5. The Bayesian estimators of the parameters are evaluated with flexible GaP under the SELF by using the MCMC technique. The values of the hyperparameter are assumed known and chosen in such a way that the prior mean is equal to the true value, and prior variance is unity.

Table 1: AVs and the corresponding MSEs (in parentheses) for n=20.

Parameters	Bayesian	MLE	OLS	WLS	CVM
λ=0.6	0.63020	0.60764	0.60795	0.60467	0.61039
	(0.00317)	(0.00319)	(0.00504)	(0.00451)	(0.00495)
λ=1.5	1.55997	1.52858	1.51648	1.50588	1.52447
	(0.01768)	(0.02124)	(0.03562)	(0.04190)	(0.03500)
λ=5	4.62813	5.08080	5.07334	5.04355	5.10549
	(0.23062)	(0.38839)	(0.52048)	(0.46462)	(0.51636)
λ=0.2	0.18563	0.20120	0.20170	0.20082	0.20101
	(0.00041)	(0.03713)	(0.00051)	(0.00045)	(0.00047)

Table 2: AVs and the corresponding MSEs (in parentheses) for n=50.

Parameters	Bayesian	MLE	OLS	WLS	CVM
λ=0.6	0.61534	0.59976	0.60064	0.60046	0.60571
	(0.00119)	(0.00158)	(0.00234)	(0.00179)	(0.00221)
λ=1.5	1.54114	1.50655	1.49974	1.49975	1.51255
	(0.00807)	(0.00883)	(0.01359)	(0.01204)	(0.01506)
λ=5	5.27437	4.97214	5.00496	5.00375	5.05325
	(0.14854)	(0.21833)	(0.19002)	(0.16674)	(0.21326)
λ=0.2	0.20478	0.19833	0.19986	0.19987	0.20136
	(0.00012)	(0.00031)	(0.00020)	(0.00018)	(0.00022)

Table 3: AVs and the corresponding MSEs (in parentheses) for n=100.

Parameters	Bayesian	MLE	OLS	WLS	CVM
λ=0.6	0.58540	0.60051	0.59927	0.59826	0.60189
	(0.00064)	(0.00080)	(0.00150)	(0.00160)	(0.00114)
λ=1.5	1.45971	1.50163	1.49817	1.50018	1.50271
	(0.00449)	(0.00536)	(0.00778)	(0.00669)	(0.00769)
λ=5	4.87430	4.90537	4.99668	4.99982	5.01522
	(0.05599)	(0.19889)	(0.10419)	(0.11327)	(0.10374)
λ=0.2	0.19532	0.19574	0.19972	0.19997	0.20025
	(0.00007)	(0.00032)	(0.00012)	(0.00010)	(0.00012)

Table 4: AVs and the corresponding MSEs (in parentheses) for n=200.

Parameters	Bayesian	MLE	OLS	WLS	CVM
λ=0.6	0.59664	0.60001	0.59876	0.59070	0.59882
	(0.00029)	(0.00057)	(0.00112)	(0.00127)	(0.00075)
λ=1.5	1.49366	1.50023	1.49909	1.50139	1.49743
	(0.00206)	(0.00375)	(0.00504)	(0.00424)	(0.00489)
λ=5	4.96587	4.85550	4.99245	4.98687	4.98664
	(0.02620)	(0.17180)	(0.08821)	(0.07905)	(0.08694)
λ=0.2	0.19904	0.19355	0.19985	0.19968	0.19964
	(0.00003)	(0.00029)	(0.00008)	(0.00007)	(0.00008)

Table 5: AVs and the corresponding MSEs (in parentheses) for n=500.

Parameters	Bayesian	MLE	OLS	WLS	CVM
λ=0.6	0.61410	0.59978	0.59723	0.59935	0.59732
	(0.00023)	(0.00045)	(0.00054)	(0.00047)	(0.00054)
λ=1.5	1.53792	1.49960	1.49936	1.49866	1.49899
	(0.00157)	(0.00289)	(0.00337)	(0.00311)	(0.00335)
λ=5	5.13452	4.80657	4.98519	4.99582	4.99911
	(0.01933)	(0.13499)	(0.06232)	(0.03727)	(0.03933)
λ=0.2	0.20460	0.19111	0.19974	0.19981	0.19985
	(0.00002)	(0.00026)	(0.00006)	(0.00005)	(0.00006)

From Tables 1-5, we observe that all the estimates show the property of consistency, i.e., the MSEs decrease and approaching zero as sample size increase. Also, the MSEs of the Bayesian estimators are less as compared to the rest of other estimators $\forall\ n = 20, 50, 100, 200, 500$. Many useful real life data sets for comparing the new model with other competitive models can be found in Elbiely and Yousof (2018 and 2019a,b), Elsayed and Yousof (2019a,b,c and 2020) and Hamedani et al. (2018a,b and 2019)

7. Conclusions

The main motivation of this paper is to show how the different frequentist estimators of the new distribution perform for different sample sizes and different parameter values and to raise a guideline in choosing the best estimation method for the new model. The unknown parameters of the new distribution are estimated using the maximum likelihood method, ordinary least squares method, weighted least squares method, Cramer-Von-Mises method and Bayesian method. The obtained estimators are compared using Markov Chain Monte Carlo simulations and we observed that Bayesian estimators are more efficient compared to other the estimators. Based on the simulation results, we observe that all the estimates show the property of consistency and the mean squared errors decrease and approaching zero as sample size increase. Also, the mean squared errors of the Bayesian estimators are less as compared to the rest of other estimators for all sample sizes.

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