

## On Erlang-Truncated Exponential Distribution: Theory and Application

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### Abstract

In this article, we introduce a new distribution called the McDonald Erlang-truncated exponential distribution. Various structural properties including explicit expressions for the moments, moment generating function, mean deviation of the new distribution are derived. The estimation of the model parameters is performed by maximum likelihood method. The usefulness of the new distribution is illustrated by two real data sets. The new model is much better than other important competitive models in modeling relief times and survival times data sets.

**Key Words:** Erlang-truncated Exponential distribution, McDonald distribution, moments, Moment generating function, Maximum Likelihood Estimation.

**Mathematical Subject Classification:** 62N02; 62N01; 62E10.

### 1. Introduction and motivation

The Erlang-truncated exponential distribution (ETEx) is widely used in the field of queuing system and stochastic processes. El-Alosey (2007) introduced Erlang-truncated exponential (ETEx) distribution by mixing the Erlang distribution with left truncated one parameter exponential distribution. The ETEx distribution, like the exponential distribution, has a constant failure rate which makes it practically impossible for the model to provide a reasonable parametric fit to data sets with decreasing failure rate, increasing failure rate and non-monotonic failure rate such as the bathtub and unimodal failure rates which are common in reliability studies and other related fields of studies. The cumulative distribution function (CDF) of this distribution is given by

$$G_{\text{ETEx}}^{(\alpha, \theta)}(x) = 1 - e^{-\alpha(1-e^{-\theta})x} \mid_{(x \geq 0)}, \quad (1)$$

where  $\theta > 0$  is the scale parameter and  $\alpha > 0$  is the shape parameter. When  $\theta \rightarrow \infty$ , we have the standard exponential model. The corresponding probability density function (PDF) and hazard rate function (HRF) are given by

$$g_{\text{ETEx}}^{(\alpha, \theta)}(x) = \alpha(1 - e^{-\theta})e^{-\alpha(1-e^{-\theta})x} \mid_{(x \geq 0)}, \quad (2)$$

and

$$h_{\text{ETEx}}^{(\alpha, \theta)}(x) = \alpha(1 - e^{-\theta}) \quad (3)$$

respectively. As illustrated in (3), the HRF of the ETEx distribution is constant and this which makes it inadequate for modelling many complex lifetime data sets that have nonconstant failure rates. So, the main aim of this paper is to extend the ETEx distribution by adding three extra shape parameters to define a new flexible model referred to as the Mc-Donald Erlang-truncated exponential (McETEx) distribution. The role of the three additional parameters is to introduce skewness and to vary tail weights and provide greater flexibility in the shape of the generalized distribution and consequently in modeling observed data. It may be mentioned that although several skewed distribution functions exist on the positive real axis not many skewed distributions are available on the whole real line which are easy to use

for data analysis purpose. The main idea is to introduce three shape parameters, so that the McETEx distribution can be used to model skewed data, a feature which is very common in practice.

The PDF of the Mc-Donald G family (Mc-G) is defined by

$$f_{\text{Mc-G}}^{(a,b,c)}(x; \varphi) = \frac{c}{B(a, b)} g(x; \varphi) G^{ac-1}(x; \varphi) [1 - G^c(x; \varphi)]^{b-1}, \quad (4)$$

where  $a > 0$ ,  $b > 0$  and  $c > 0$  are additional shape parameters. Note that  $g(x; \varphi)$  is the PDF of baseline distribution,  $g(x; \varphi) = dG(x; \varphi)/dx$  and  $\varphi$  is the paramrt vector. The class of distributions (1.4) includes as special sub-models the beta generalized (B-G) family of distributions for  $c = 1$  (see Eugene et al. (2002)) and Kumaraswamy generalized (K-G) family of distributions (see Cordeiro and Castro (2011)) for  $a = 1$ . The corresponding CDF OF (4) is given by

$$F_{\text{Mc-G}}^{(a,b,c)}(x; \varphi) = I_{G^c(x; \varphi)}(a, b) = \frac{1}{B(a, b)} \int_0^{G^c(x; \varphi)} \omega^{a-1} (1 - \omega)^{b-1} d\omega, \quad (5)$$

where

$$I_\zeta(a, b) = \frac{1}{B(a, b)} \int_0^\zeta \omega^{a-1} (1 - \omega)^{b-1} d\omega,$$

denotes the incomplete beta function ratio (Gradshteyn and Ryzhik, (2000)). Equation (5) can also be rewritten as follows

$$F_{\text{Mc-G}}^{(a,b,c)}(x; \varphi) = \frac{G(x; \varphi)^{ac}}{aB(a, b)} {}_2F_1(a, 1 - b; a + 1; G(x; \varphi)^c), \quad (6)$$

where

$${}_2F_1(a, b; c; x) = B(b, c - b)^{-1} \int_0^1 \frac{t^{b-1} (1 - t)^{c-b-1}}{(1 - tx)^a} dt,$$

is the well-known “hypergeometric function” which are well established in the literature (see Gradshteyn and Ryzhik (2000)). The HRF and reverse hazard functions (RHF) of the Mc-G distribution are given by

$$h_{\text{Mc-G}}^{(a,b,c)}(x; \varphi) = \frac{cg(x; \varphi) G^{ac-1}(x; \varphi) [1 - G^c(x; \varphi)]^{b-1}}{B(a, b) [1 - I_{G^c(x; \varphi)}(a, b)]}, \quad (7)$$

and

$$\tau_{\text{Mc-G}}^{(a,b,c)}(x; \varphi) = \frac{cg(x; \varphi) G^{ac-1}(x; \varphi) [1 - G^c(x; \varphi)]^{b-1}}{B(a, b) [I_{G^c(x; \varphi)}(a, b)]},$$

respectively. Recently, many authors used the Mc-G family to expand some existing models such as the McDonald gamma distribution by Marciano et al. (2012), the McDonald Normal distribution by Cordeiro et al. (2012), McDonald half-logistic distribution by Oliveira et al. (2013), McDonald Dagum by Oluyede and Rajasooriya (2013), the McDonald Gompertz distribution by Roozgar et al. (2017).

The rest of the article is organized as follows. In Section 2, we define the McETEx distribution. The expansion for the cumulative and density functions of the (McETEx) distribution and some special cases are proposed in Section 2. Moments, moment generating function and conditional moment. Mean deviation, Lorenz and Bonferroni curves are discussed in Section 3. In Section 4 Residual life and Reversed Failure Rate Function of (McETEx) distribution. Maximum likelihood estimation is performed in Section 5. In Section 6, we provide application to real data set to illustrate the importance of the new distribution.

## 2. The new distribution

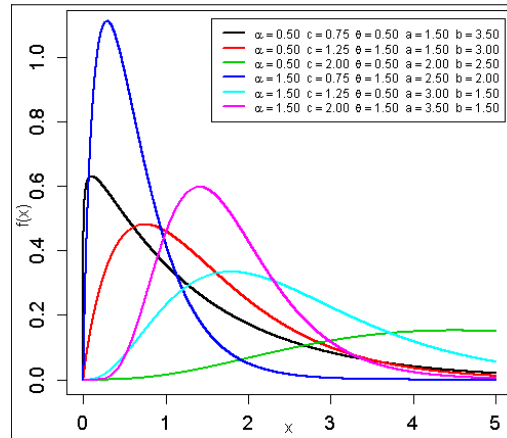
Using (1) and (5) the CDF of (McETEx) can be written as

$$F_{\text{McETEx}}^{(\phi)}(x) = I_{G^c(x; \varphi)}(a, b) = I_{[1 - e^{-\alpha(1 - e^{-\theta})x}]}^c(a, b) |_{(x \geq 0)}, \quad (8)$$

the corresponding PDF is given by

$$f_{\text{McETEx}}^{(\phi)}(x) = \frac{c\alpha(1 - e^{-\theta})}{B(a, b)} e^{-\alpha(1-e^{-\theta})x} \left[1 - e^{-\alpha(1-e^{-\theta})x}\right]^{ac-1} \left\{1 - \left[1 - e^{-\alpha(1-e^{-\theta})x}\right]^c\right\}^{b-1} \mid (x \geq 0). \quad (9)$$

The new distribution will attract wider applications in reliability as well as in other areas of research and it can be used in a variety of problems in modeling survival data (see Section 7). Figure 1 represents some plots of the probability density function of the McETEx distribution for some different parameter values. Table 1 gives some sub models from the McETEx model.



**Figure 1:** Plots of the PDF of McETEx for some different parameter values.

Table 1: Some sub models from the McETEx model.

$a$	$b$	$c$	$\alpha$	$\theta$	Reduced model
$a$	$b$	$c$	$\alpha$	$\rightarrow \infty$	McEx
$a$	$b$	1	$\alpha$	$\theta$	BETEx
$a$	$b$	1	$\alpha$	$\rightarrow \infty$	BEx
1	$b$	$c$	$\alpha$	$\theta$	KETEx
1	$b$	$c$	$\alpha$	$\rightarrow \infty$	KEx

A random variable  $X$  with the PDF (9) is said to have a McETEx distribution and will be denoted by  $X \sim \text{McETEx}(\phi)$  where  $\phi = (a, b, c, \alpha, \theta)$ . The HRF can be defined as

$$h_{\text{McETEx}}(x, \phi) = \frac{c\alpha(1 - e^{-\theta})e^{-\alpha(1-e^{-\theta})x} \left[1 - \left(1 - e^{-\alpha(1-e^{-\theta})x}\right)^c\right]^{b-1}}{B(a, b) \left[1 - I_{(1-e^{-\alpha(1-e^{-\theta})x})^c}(a, b)\right] \left[1 - e^{-\alpha(1-e^{-\theta})x}\right]^{-ac+1}}. \quad (10)$$

Many extension of the ETEx model can be cited such as Okorie et al. (2016) introduced the transmuted Erlang-truncated exponential distribution, Nasiru et al. (2016) investigated the generalized Erlang-truncated exponential distribution and Okorie et al. (2017) proposed and studied the Marshall-Olkin generalized Erlang-truncated exponential distribution, among others. Many other useful extensions can found in Alizadeh et al. (2018), Hamedani et al. (2018), Elbiely and Yousof (2019), Alizadeh et al. (2020), Ibrahim et al. (2020), Karamikabir et al. (2020), Yousof et al. (2018a,b,c, 2019 and 2020), Korkmaz et al. (2018, 2019a,b,c and 2020) and Ibrahim et al. (2020).

### 3. Useful Expansions

In this Section, we present some representations of PDF of McETEx distribution. The mathematical relation given below will be useful in this subsection. The series representation given below will be useful in this subsection. If  $\zeta$  is a positive real non-integer and  $|\tau| < 1$  then

$$(1 - \tau)^\zeta = \sum_{k=0}^{\infty} (-\tau)^k \binom{\zeta}{k}. \quad (11)$$

Substituting from (11) into (9), we get

$$f_{\text{McETEx}}^{(\phi)}(x) = \frac{c\alpha(1-e^{-\theta})}{B(a,b)} \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} e^{-\alpha(1-e^{-\theta})x} \left[1 - e^{-\alpha(1-e^{-\theta})x}\right]^{c(a+i)-1}, \quad (12)$$

again, using the binomial series expansion in (12) we get

$$f_{\text{McETEx}}^{(\phi)}(x) = \sum_{i,j=0}^{\infty} \xi_{i,j} e^{-\vartheta x}, \quad (13)$$

where

$$\xi_{i,j} = c\alpha(1-e^{-\theta}) \frac{(-1)^{i+j}}{B(a,b)} \binom{b-1}{i} \binom{c(a+i)-1}{j},$$

and

$$\vartheta = \alpha(j+1)(1-e^{-\theta}).$$

#### 4. Statistical Properties

##### Moments

The following theorems give the  $r$ th moment ( $\mu_r$ ) and moment generating function  $M_X(t)$  of the distribution McETEx.

**Theorem (4.1):** If  $X$  has the McETEx, then the  $r$ th moment of  $X$  is given by the following

$$\mu'_r = \Gamma(1+r) \sum_{i,j=0}^{\infty} \xi_{i,j}^{(1+r,\vartheta)}, \quad (14)$$

where

$$\xi_{i,j}^{(1+r,\vartheta)} = \frac{1}{\vartheta^{1+r}} \xi_{i,j},$$

Further, the central moments  $\mu_r$  and cumulants  $\kappa_r$  of the McETEx distribution can be determined by

$$\mu_r = \sum_{m=0}^r \binom{r}{m} (-1)^m \mu_1^m \mu'_{r-m},$$

and

$$\kappa_r = \mu'_r - \sum_{m=1}^{r-1} \kappa_m \binom{r-1}{m-1} \mu'_{r-m},$$

respectively, where

$$\kappa_1 = \mu'_1, \kappa_2 = \mu'_2 - \mu_1'^2, \kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3,$$

and

$$\kappa_4 = \mu'_4 - 4\mu'_1\mu'_3 - 3\mu_2'^2 + 12\mu'_2\mu_1'^2 - 6\mu_1'^4.$$

The coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) can be obtained according to the following relation

$$CV = \sqrt{\frac{\mu'_2}{\mu_2}} - 1, CS = \frac{\kappa_3}{\sqrt{\kappa_2^3}},$$

and

$$CK = \frac{\kappa_4}{\kappa_2^2},$$

respectively.

**Theorem (4.2):** If  $X$  has the McETEx ( $\phi$ ), then the the moment generating function (MGF) of  $X$  is given as follows

$$M_X(t) = \Gamma(1+r) \sum_{i,j=0}^{\infty} \frac{\xi_{i,j}}{\vartheta - t}.$$

##### Conditional moments

For lifetime models, it is of interest to know the  $r$ th lower and upper incomplete moments of  $X$  defined by

$$v_s(t) = E(X^s |_{(X < t)}) = \int_0^t x^s f(x) dx,$$

and

$$\eta_s(t) = E(X^s |_{(X > t)}) = \int_t^\infty x^s f(x) dx,$$

respectively, for any real  $s > 0$ . The  $r$ th lower incomplete moment of McETEx distribution is

$$v_s(t) = \int_0^t x^s f(x) dx = \sum_{i,j=0}^{\infty} \xi_{i,j} \int_0^t x^s e^{-\vartheta x} dx = \sum_{i,j=0}^{\infty} \xi_{i,j}^{(1+s, \vartheta t)} \gamma(1+s, \vartheta t),$$

where

$$\xi_{i,j}^{(1+s, \vartheta t)} = \frac{1}{(\vartheta t)^{1+s}} \xi_{i,j},$$

and

$$\gamma(\tau, q) |_{(\tau \neq 0, -1, -2, \dots)} = \int_0^q t^{\tau-1} e^{-t} dt = \frac{q^\tau}{\tau} \{1F_1[\tau; \tau+1; -q]\} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (\tau+k)} q^{\tau+k},$$

is the lower incomplete gamma function. Similarly, the  $r$ th upper incomplete moment of McETEx distribution is

$$\eta_s(t) = \int_t^\infty x^s f(x) dx = \sum_{i,j=0}^{\infty} \xi_{i,j} \int_t^\infty x^s e^{-\vartheta x} dx = \sum_{i,j=0}^{\infty} \xi_{i,j}^{(1+s, \vartheta t)} \Gamma(1+s, \vartheta t),$$

where

$$\Gamma(\tau, q) |_{(q > 0)} = \int_0^q t^{\tau-1} e^{-t} dt \sim \frac{q^{\tau-1}}{e^q} \left[ 1 + \frac{\tau-1}{q} + \frac{(\tau-1)(\tau-2)}{q^2} + \dots \right],$$

is the upper incomplete gamma function,  $1F_1[\cdot, \cdot]$  is a confluent hypergeometric function and

$$\Gamma(\tau, q) + \gamma(\tau, q) = \Gamma(\tau),$$

The MRL has many applications in biomedical sciences, life insurance, maintenance and product quality control, economics and social studies, demography and product technology (see Lai and Xie, 2006). Guess and Proschan (1988) gave an extensive coverage of possible applications of the mean residual life. The MRL (or the life expectancy at age  $t$ ) represents the expected additional life length for a unit, which is alive at age  $t$ . The MRL is given by

$$m_X(t) = E(X |_{(X > t)}) = \frac{\eta_1(t)}{\bar{F}(t)} - t,$$

where  $\eta_1(t)$  is the first incomplete moment of  $X$  and by setting  $s = 1$  in equation (4.4), we get

$$m_X(t) = \sum_{i,j=0}^{\infty} \frac{1}{\bar{F}(t)} \xi_{i,j}^{(2, \vartheta t)} \Gamma(2, \vartheta t) - t.$$

Also, the mean inactivity time (MIT) represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in  $(0; t)$ . The MIT of  $X$  is defined (for  $t > 0$ ) by

$$\tau_X(t) = E(X |_{(X < t)}) = t - \frac{v_1(t)}{F(t)} = t - \frac{1}{F(t)} \xi_{i,j}^{(2, \vartheta t)} \gamma(2, \vartheta t)$$

### Mean deviation, Lorenz and Bonferroni curves

If  $X$  has the McETEx distribution, then we can derive the mean deviations about the mean  $\mu = E(X)$  and the mean deviations about the median  $M$  are defined by

$$\delta_1(x) = \int_0^\infty f(x) |x - \mu| dx = 2[\mu F(\mu) - J(\mu)]$$

and

$$\delta_2(x) = \int_0^\infty f(x) |x - M| dx = \mu - 2J(M)$$

respectively. The measures  $\delta_1(x)$  and  $\delta_2(x)$  can be calculated using the relationships

$$J(d) = \int_0^d f(x) x dx = \sum_{i,j=0}^{\infty} \xi_{i,j}^{(2, \vartheta t)} \gamma(2, \vartheta d).$$

### Bonferroni and Lorenz Curves

The Lorenz curve for a positive random variable  $X$  is defined as

$$L(p) = \frac{1}{\mu} \int_0^q f(x) x dx = \frac{1}{\mu} J(q) = \sum_{i,j=0}^{\infty} \frac{1}{\mu} \xi_{i,j}^{(2,\vartheta t)} \gamma(2, \vartheta d),$$

where  $q = G^{-1}(p)$ . If  $X$  represents annual income,  $L(p)$  is the proportion of total income that accrues to individuals having the 100p% lowest incomes. If all individuals earn the same income, then  $L(p) = p$  for all  $p$ . The area between the line  $L(p) = p$  and the Lorenz curve may be regarded as a measure of inequality of income. Also Bonferroni curve is defined by

$$\begin{aligned} B(p) &= \frac{1}{\mu p} \int_0^q f(x) x dx = \frac{1}{\mu p} J(q) \\ &= \sum_{i,j=0}^{\infty} \frac{1}{\mu} \xi_{i,j}^{(2,\vartheta t)} \gamma(2, \vartheta d), \end{aligned}$$

where the Bonferroni curve has many applications not only in economics to study income and poverty, but also in other fields like reliability, medicine and insurance.

### Residual life and reversed failure rate function

Suppose that a component survives up to time  $t \geq 0$ , the residual life is the period beyond  $t$  until the time of failure and defined by the conditional random variable  $X - t \mid (X > t)$ . Therefore, we obtain the  $r^{th}$  order moment of the residual life via the general formula

$$z_r(t) = E((X - t)^r \mid X > t) = \frac{1}{\bar{F}(t)} \int_t^{\infty} f(x) (x - t)^r dx \mid (r \geq 1).$$

Applying the binomial expansion of  $(x - t)^r$  and substituting  $f_{\text{McETEx}}^{(\phi)}(x)$  given by (2.2) into the above formula gives

$$z_r(t) = \sum_{i,j=0}^{\infty} \frac{\xi_{i,j}}{\bar{F}(t)} \sum_{m=0}^r (-t)^m \binom{r}{m} \int_t^{\infty} e^{-\vartheta x} x^{r-m} dx = \sum_{i,j=0}^{\infty} \sum_{m=0}^r \frac{\xi_{i,j,m}^{(1+r-m,\vartheta t)}}{\bar{F}(t)} \Gamma(1 + r - m, \vartheta t)$$

where

$$\Gamma(s, t) = \int_t^{\infty} e^{-x} x^{s-1} dx$$

and

$$\xi_{i,j,m}^{(1+r-m,\vartheta t)} = \xi_{i,j}^{(1+r-m,\vartheta t)} (-t)^m \binom{r}{m}$$

is the upper incomplete gamma function. The mean residual life (MRL) of the McETEx distribution is given by

$$z(t) = \sum_{i,j=0}^{\infty} \frac{\xi_{i,j}^{(2,\vartheta t)}}{\bar{F}(t)} \Gamma(2, \vartheta t) - t$$

The variance of the residual life of the McETEx distribution can be obtained easily by using  $\mu_2(t)$  and  $\mu(t)$ .

Also, we analogously discuss the reversed residual life and some of its properties. The reversed residual life can be defined as the conditional random variable  $t - X \mid X \leq t$  which denotes the time elapsed from the failure of a component given that its life is less than or equal to  $t$ . This random variable may also be called the inactivity time (or time since failure); for more details you may (see Kundu and Nanda, (2010)). Also, in reliability, the mean reversed residual life and ratio of two consecutive moments of reversed residual life characterize the distribution uniquely. The  $r^{th}$  order moment of the reversed residual life can be obtained by the well-known formula

$$Z_r(t) = E((t - X)^r \mid (X \leq t)) = \frac{1}{F(t)} \int_0^t f(x) (t - x)^r dx, r \geq 1.$$

Applying the binomial expansion of  $(t - x)^r$  and substituting  $f_{\text{McETEx}}^{(\phi)}(x)$  given by (4) into the above formula gives

$$Z_r(t) = \sum_{m=0}^r \binom{r}{m} (-t)^m \int_0^t e^{-\vartheta x} x^{r-m} dx = \sum_{i,j=0}^{\infty} \sum_{m=0}^r \frac{1}{F(t)} \xi_{i,j}^{(1+r-m,\vartheta t)} (-t)^m \binom{r}{m} \gamma(1 + r - m, \vartheta t)$$

where

$$\gamma(s, t) = \int_0^t e^{-x} x^{s-1} dx$$

is the lower incomplete gamma function. Thus, the mean reversed residual life (or mean waiting time) of the McETEx distribution is given by

$$Z(t) = t - \sum_{i,j=0}^{\infty} \frac{1}{\bar{F}(t)} \xi_{i,j}^{(2,\vartheta t)} \gamma(2, \vartheta t).$$

Using  $Z(t)$  and  $Z_2(t)$  one can obtain the variance and the coefficient of variation of the reversed residual life of the McETEx distribution.

## 5. Estimation

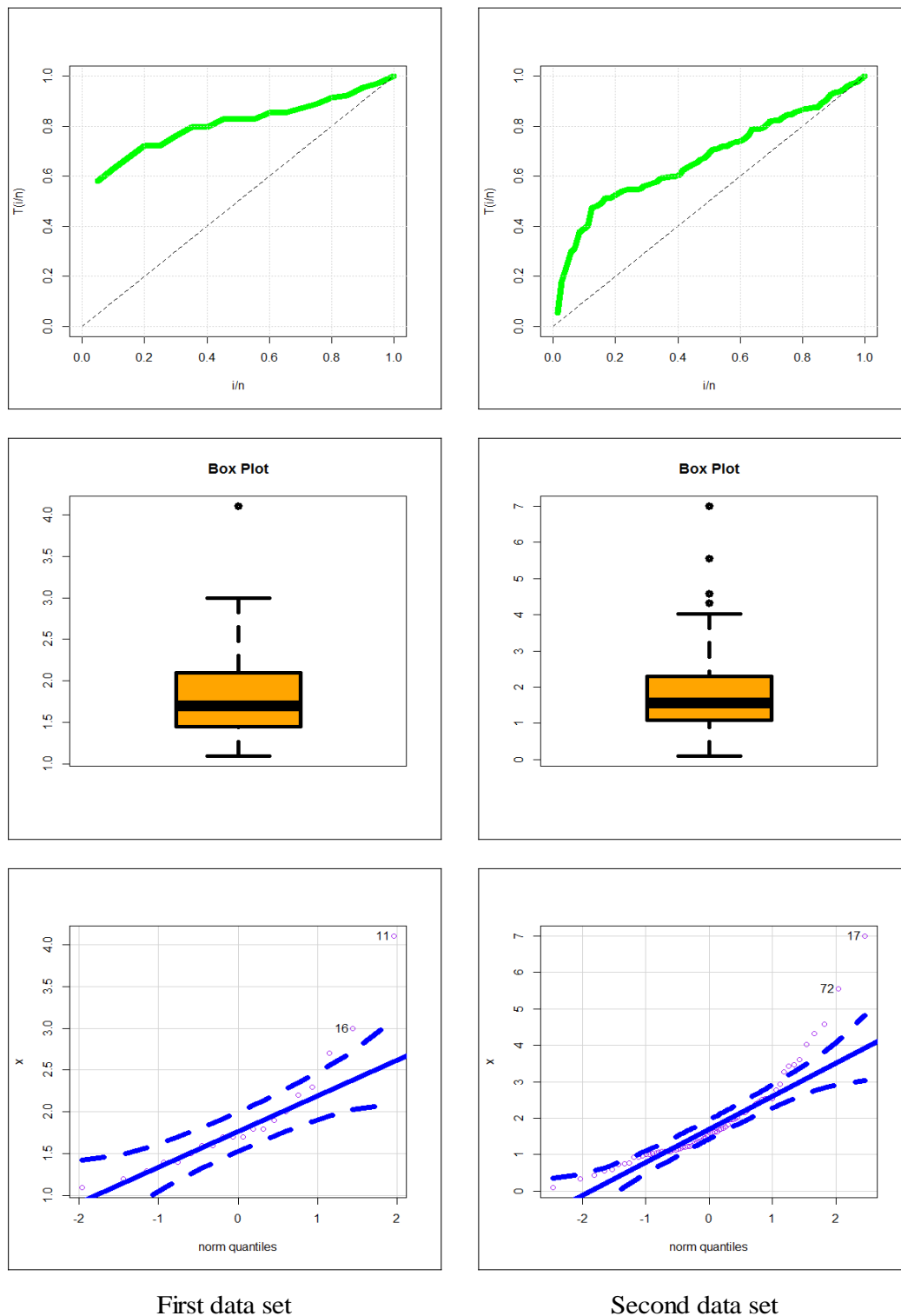
In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the McETEx distribution from complete samples only. Let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from McETEx where  $\mathbf{V} = (\alpha, \theta, a, b, c)^T$  is the parameter vector. The log likelihood function for the vector of parameters  $\mathbf{V}$  can be written easily derived and the components of the score vector as well.

## 6. Application

In this Section, we provide two applications to a real data sets to assess the flexibility of the McETEx model. Figure 2 gives the total time test (TTT) plots (see Aarset (1987)), box plots and Quantile-Quantile (Q-Q) plots for the two real data sets. Based on Figure 2 we note that the HRF of the two data are "increasing", the first data contains an extreme value, the second data contains four extreme values. The kernel density estimation (KDE) is a non-parametric way to estimate the probability density function of a random variable. KDE is a fundamental data smoothing problem where inferences about the population are made based on a finite data sample. Let  $(w_1, w_1, \dots, w_n)$  be a univariate independent and identically distributed sample drawn from some distribution with an unknown density function. Then, kernel density estimator is

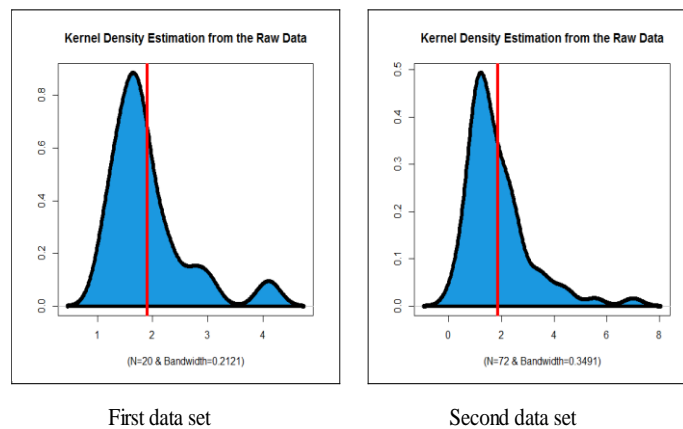
$$\hat{f}_h(w) = \frac{1}{n} \sum_{i=1}^n K_h(w - w_i) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{w - w_i}{h}\right),$$

where  $K$  is the non-negative kernel function, and  $h > 0$  is a smoothing parameter called the bandwidth. Figure 3 gives the KDE plots (see Rosenblatt (1956) and Parzen (1962)). In order to compare the McETEx model with other fitted distributions, we compare the fits of the McETEx distribution with the Exponential ( $\text{Ex}(\theta)$ ), Odd Lindley Exponential (OLiEx), Marshall-Olkin Exponential (MOEx ( $\alpha, \theta$ )), Moment Exponential (MomEx ( $\theta$ )), The Logarithmic Burr-Hatke Exponential (Log BrHEX( $\theta$ )), Generalized Marshall-Olkin Exponential (GMOEx ( $\lambda, \alpha, \theta$ )), Beta Exponential (BEx ( $a, b, \theta$ )), Burr X Ex ( $a, \theta$ ), Marshall-Olkin Kumaraswamy Exponential (MOKEEx ( $\alpha, a, b, \theta$ )), Kumaraswamy Exponential (KEEx ( $a, b, \theta$ )) and Kumaraswamy Marshall-Olkin Exponential (KMOEx ( $\alpha, a, b, \theta$ )) models. The first data set {1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2} called the failure time data: The data represents the lifetime data relating to relief times (in minutes) of patients receiving an analgesic (see Gross and Clark (1975)). The second data set {0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55} called the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960).



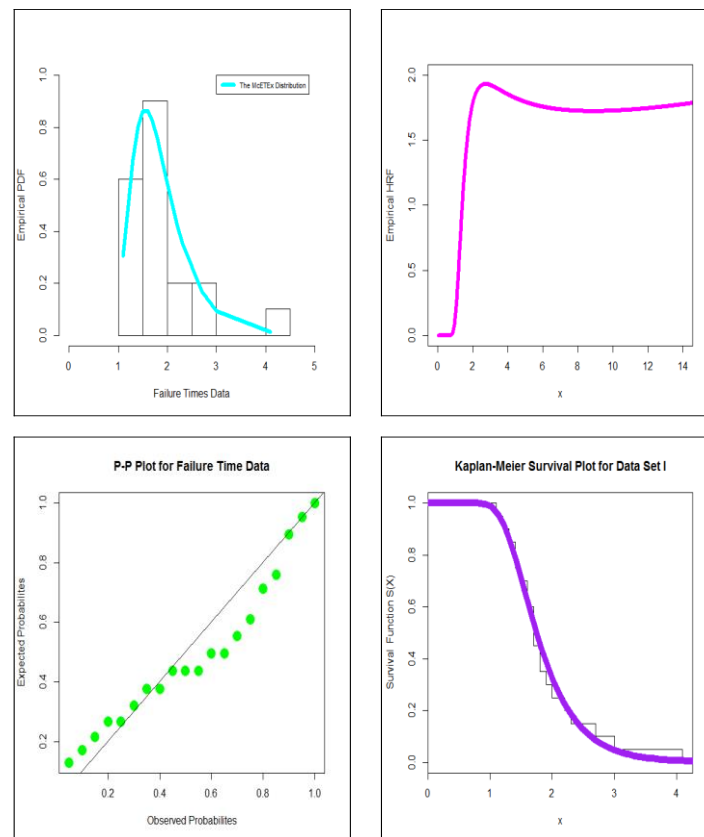
**Figure 2:** TTT plots, box plots and Q-Q plots.



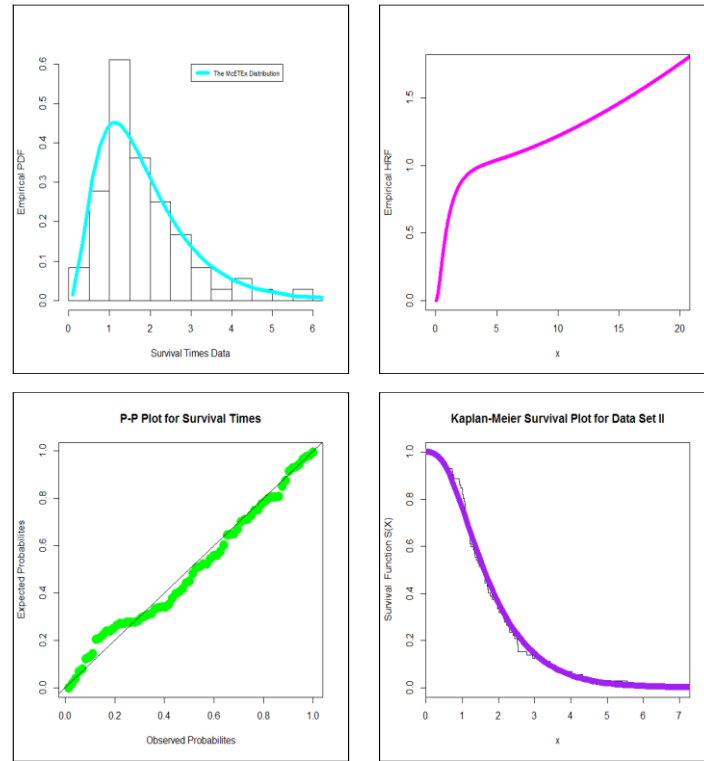


**Figure 3:** Kernel density estimation

We consider the Cramér-Von Mises and the Anderson-Darling ( $A^*, W^*$ ) and the Kolmogorov-Smirnov (KS) statistic. Moreover, we consider some other goodness-of-fit measures including the Akaike Information Criterion, Consistent Akaike Information Criterion, Hannan-Quinn Information Criterion and Bayesian Information Criterion ( $C_1, C_2, C_3$  and  $C_4$ ). Tables 2 and 4 gives the MLEs and SEs values for the two data sets. Table 3 and 5 gives the  $C_1, C_2, C_3, C_4, A^*, W^*, K.S.$  and (p-value) for the two data sets. Figures 4 and 5 gives the estimated PDF (ECDF), EHRF, P-P plot and Kaplan-Meier Survival Plot for the two data sets.



**Figure 4:** EPDF, EHRF, P-P plot and Kaplan-Meier Survival Plot for the 1st data.



**Figure 5:** EPDF, EHRF, P-P plot and Kaplan-Meier Survival Plot for the 2nd data.

**Table 2:** MLEs and SEs values for the relief times data.

Models		Estimates
Ex( $\theta$ )	MLE	0.526
	SE	(0.117)
OLEx( $\theta$ )	MLE	0.6044
	SE	(0.0535)
MEx( $\theta$ )	MLE	0.950
	SE	(0.150)
LBHEx( $\theta$ )	MLE	0.5263
	SE	(0.118)
MOEx( $\alpha, \theta$ )	MLE	54.474, 2.316
	SE	(35.582), (0.374)
GMOEx( $\lambda, \alpha, \theta$ )	MLE	0.519, 89.462, 3.169
	SE	(0.256), (66.278), (0.772)
KwEx( $\alpha, \beta, \theta$ )	MLE	83.756, 0.568, 3.330
	SE	(42.361), (0.326), (1.188)
BEx( $\alpha, \beta, \theta$ )	MLE	81.633, 0.542, 3.514
	SE	(120.41), (0.327), (1.410)
MOKwEx( $\alpha, \beta, \lambda, \theta$ )	MLE	0.133, 33.232, 0.571, 1.669
	SE	(0.332), (57.837), (0.721), (1.814)
KwMOEx( $\alpha, \beta, \lambda, \theta$ )	MLE	8.868, 34.826, 0.299, 4.899
	SE	(9.146), (22.312), (0.239), (3.176)
BrXEx( $a, \theta$ )	MLE	1.1635, 0.3207
	SE	(0.33), (0.03)
McETEx ( $a, b, c, \alpha, \theta$ )	MLE	<b>5.307, 0.866, 6.052, 188.66, 0.012</b>
	SE	<b>(2.201), (0.515), (0.266), (7.458), (0.118)</b>

**Table 3:**  $C_1, C_2, C_3, C_4, A^*, C^*, K.S.$  and (p-value) for the relief times data.

Models	$C_1, C_2, C_3, C_4$	$A^*, C^*$	K.S. (p-value)
<b>McETEx</b>	<b>42.46, 38.96, 46.74, 43.43</b>	<b>0.32, 0.05</b>	<b>0.11(0.96)</b>
BEEx	43.48, 46.45, 44.98, 44.02	0.70, 0.12	0.16(0.80)
Ex	67.67, 68.67, 67.89, 67.87	4.60, 0.96	0.44(0.004)
BrXEx	48.13, 50.51, 48.83, 48.54	1.39, 0.24	0.3(0.171)
OLEx	49.14, 50.13, 49.34, 49.35	1.35, 0.22	0.9(<0.001)
KwEx	41.78, 44.75, 43.28, 42.32	0.45, 0.07	0.14(0.86)
MOKEEx	41.58, 45.54, 44.25, 42.30	0.60, 0.11	0.14(0.87)
LBHEEx	67.67, 68.67, 67.89, 67.87	0.62, 0.11	0.5(<0.001)
MEx	54.32, 55.31, 54.54, 54.50	2.76, 0.53	0.32(0.07)
MOEx	43.51, 45.51, 44.22, 43.90	0.82, 0.14	0.18(0.55)
GMOEx	42.75, 45.74, 44.25, 43.34	0.51, 0.08	0.15(0.78)
KMOEx	42.82, 46.84, 45.55, 43.60	1.08, 0.19	0.15(0.86)

**Table 4:** MLEs and SEs values for the survival times data.

Models	Estimates
Ex( $\theta$ )	MLE 0.540 SE (0.063)
OLEx( $\theta$ )	MLE 0.38145 SE (0.0209)
MEx( $\theta$ )	MLE 0.925 SE (0.077)
LBHEx( $\theta$ )	MLE 0.54 SE (0.064)
MOEx( $\alpha, \theta$ )	MLE 8.778, 1.379 SE (3.555), (0.193)
GMOEx( $\lambda, \alpha, \theta$ )	MLE 0.179, 47.635, 4.465 SE (0.070), (44.901), (1.327)
KwEx( $\alpha, \beta, \theta$ )	MLE 3.304, 1.100, 1.037 SE (1.106), (0.764), (0.614)
BEEx( $\alpha, \beta, \theta$ )	MLE 0.807, 3.461, 1.331 SE (0.696), (1.003), (0.855)
MOKEEx( $\alpha, \beta, \lambda, \theta$ )	MLE 0.008, 2.716, 1.986, 0.099 SE (0.002), 1.316, (0.784), (0.048)
KwMOEx( $\alpha, \beta, \lambda, \theta$ )	MLE 0.373, 3.478, 3.306, 0.299 SE (0.136), (0.861), (0.779), (1.112)
BrXEx( $\alpha, \theta$ )	MLE 0.475, 0.2055 SE (0.06), (0.012)
<b>McETEx (<math>\alpha, \beta, \gamma, \alpha, \theta</math>)</b>	<b>MLE</b> <b>1.88, 3.311, 0.909, 19.415, 3.87</b> <b>SE</b> <b>(0.918), (0.905), (4.801), (2.092), (1.948)</b>

**Table 5:**  $C_1, C_2, C_3, C_4, A^*, C^*, K.S.$  and (p-value) for the survival times data.

Models	$C_1, C_2, C_3, C_4$	$A^*, C^*$	K.S. (p-value)
<b>McETEx</b>	<b>205.81, 212.62, 205.11, 207.88</b>	<b>0.58, 0.09</b>	<b>0.09(0.59)</b>
Ex	234.63, 236.91, 234.68, 235.54	6.53, 1.25	0.27(0.06)
BEx	207.38, 214.22, 207.73, 210.08	0.98, 0.15	0.11(0.34)
BrXEx	235.30, 239.90, 235.50, 237.11	2.91, 0.52	0.22(0.002)
MEx	210.40, 212.68, 210.45, 211.30	1.52, 0.25	0.14(0.13)
LBHEx	234.63, 236.92, 234.73, 235.55	0.71, 0.12	0.28(<0.001)
KMOEx	207.82, 216.94, 208.42, 211.42	0.61, 0.11	0.09(0.53)
OLEx	229.31, 231.43, 229.25, 230.00	1.94, 0.33	0.5(<0.001)
GMOEx	210.54, 217.38, 210.89, 213.24	1.02, 0.16	0.09(0.51)
MOKEx	209.44, 218.56, 210.04, 213.04	0.79, 0.12	0.10(0.44)
MOEx	210.36, 214.92, 210.53, 212.16	1.18, 0.17	0.10(0.43)
KwEx	209.42, 216.24, 209.77, 212.12	0.74, 0.11	0.09(0.50)

From Table 3 and 5 we conclude that the proposed lifetime McETEx model is much better than the Ex, MomEx, MOEx, GMOEx, KEx, BEx, MOKEx and KMOEx models so the new lifetime model is a good alternative to these models in modeling relief times and survival times data sets.

## 7. Conclusions

In this article, we introduced a new distribution called the McDonald Erlang-truncated exponential (McETEx) distribution. Various structural properties including explicit expressions for the moments, moment generating function, mean deviation of the McETEx distribution are derived. The estimation of the model parameters is performed by maximum likelihood method. The usefulness of the McETEx distribution is illustrated by two real data sets. The McETEx model is much better than the Exponential, Odd Lindley Exponential, Marshall-Olkin Exponential, Moment Exponential, The Logarithmic Burr-Hatke Exponential, Generalized Marshall-Olkin Exponential, Beta Exponential, Burr X Exponential, Marshall-Olkin Kumaraswamy Exponential, Kumaraswamy Exponential and Kumaraswamy Marshall-Olkin Exponential models. in modeling relief times and survival times data sets according to Cramér-Von Mises, the Anderson-Darling, Akaike Information Criterion, Consistent Akaike Information Criterion, Hannan-Quinn Information Criterion, Bayesian Information Criterion.

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