Pak.j.stat.oper.res. Vol.16 No.1 2020 pp 789-801 DOI: http://dx.doi.org/10.18187/pjsor.v16i4.2961

Pakistan Journal of Statistics and Operation Research

An Extension of Log-Logistic Distribution for Analyzing Survival Data

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Abstract

In this paper, a new generalization of Log-Logistic distribution using Alpha Power transformation is proposed. The new distribution is named as Alpha Power Log-Logistic distribution. A comprehensive account of some of its statistical properties are derived. The maximum likelihood estimation procedure is used to estimate the parameters. The importance and utility of the proposed model are proved empirically using two real life data sets.

Key Words: Alpha Power Log-Logistic distribution; Reliability Analysis; Entropy; Maximum Likelihood Estimation.

Mathematical Subject Classification: 60E05, 62E15.

1. Introduction

Log-Logistic (LL) distribution also known as Fisk distribution in economics is one of the important continuous probability distribution having massive application in survival analysis. It also finds application in areas like finance and insurance. If logarithm of a random variable X follows Logistic distribution then X follows LL distribution. The properties of LL distribution made it an attractive alternative to distributions which were conventionally used in the analysis of survival data. Kleiber and Kotz (2003) discussed the application of LL distribution in economics, Collet (2003) showed its application in medical field, Ashkar and Mahdi (2006) used LL distribution to analyze stream flow data etc. Some other authors who studied the properties and utility of LL distribution are Singh et al. (1988), Nandram (1989), Diekmann (1992), Bacon (1993), Little et al. (1994) etc.

The cumulative distribution function (cdf) and probability density function (pdf) of LL distribution are given by Eq. (1) and Eq. (2) respectively.

$$F(x) = \frac{x^\theta}{1+x^\theta} \quad ; \quad x,\theta > 0, \tag{1}$$

$$f(x) = \frac{\theta x^{\theta-1}}{(1+x^\theta)^2}. \tag{2}$$

A number of authors extended this distribution to make it more flexible and increase its applicability in diverse fields. Santana et al. (2012) extended LL distribution using Kumaraswamy-G family and named it Kumaraswamy Log-Logistic distribution. Aryal (2013) obtained transmuted Log-Logistic distribution using Quadratic rank Transmutation

Map. Gui (2013) developed a new class of LL distribution using Marshall Olkin transformation. Lemonte (2014) proposed four parameter Beta Log-Logistic distribution and studied its properties.
Lately, a prominent generalization technique Known as Alpha Power (AP) transformation was suggested by Mahadavi and Kundu (2015) which has been exploited by various authors to achieve flexibility. The cdf of AP transformation is given as

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1} & ;\alpha \neq 1, x > 0 \\ F(x) & ;\alpha = 1, x > 0 \end{cases}$$

where $F(x)$ is the cdf of baseline distribution.
The pdf of AP transformation is given as

$$f_{APT}(x) = \begin{cases} \frac{\log(\alpha)\alpha^{F(x)}}{\alpha-1} f(x) & ;\alpha \neq 1 \\ f(x) & ;\alpha = 1 \end{cases}$$

In this article, the LL distribution is generalized by using Alpha Power (AP) transformation and the new model so obtained named as Alpha Power Log-Logistic (APLL) distribution. It is more flexible and exhibits more complex shapes of density and hazard rate functions. Also, the proposed model outclasses some well-established models in terms of two real life data sets. The rest of the article is unfolded as: in Section 2, the pdf and cdf of the proposed model i.e., APLL distribution are defined. Section 3 deals with the reliability measures of the APLL distribution. The expansion of pdf and cdf is discussed in Section 4. Some of the important statistical properties are explored in Section 5. The parameter estimation is discussed in Section 6. The simulation study and applicability of the model is debated in Section 7 and 8 respectively. Finally, some conclusions are provided in Section 9.

2. APLL Distribution

A random variable X is said to follow two parameter APLL distributions with scale parameter $\theta > 0$ and shape parameter $\alpha > 0$ if its cdf takes the following form:

$$F_{APLL}(x) = \begin{cases} \frac{\alpha^{\left(\frac{x^\theta}{1+x^\theta}\right)}-1}{\alpha-1} & \alpha \neq 1 \\ \frac{x^\theta}{1+x^\theta} & \alpha = 1 \end{cases} \quad ; \quad x > 0, \tag{3}$$

The corresponding pdf is given as

$$f_{APLL}(x) = \begin{cases} \frac{log(\alpha)}{\alpha-1} \frac{\theta x^{\theta-1}}{(1+x^\theta)^2} \alpha^{\left(\frac{x^\theta}{1+x^\theta}\right)} & ;\alpha \neq 1 \\ \frac{\theta x^{\theta-1}}{(1+x^\theta)^2} & ;\alpha = 1 \end{cases}. \tag{4}$$

The plots of density function for different parameter combinations are presented in Figure 1.

3. Reliability Analysis

In this section, the reliability measures for APLL distribution are investigated.

3.1. Reliability function

The reliability function denoted by $R_{APLL}(x)$ is the probability that an item does not fail before time say x and for APLL distribution, it is given as

$$R_{APLL}(x) = \begin{cases} \alpha^{\left(\frac{x^\theta}{1+x^\theta}\right)} \left[\frac{\alpha^{\left(\frac{1}{1+x^\theta}\right)}-1}{\alpha-1}\right] & ;\alpha \neq 1 \\ \frac{1}{1+x^\theta} & ;\alpha = 1 \end{cases}$$

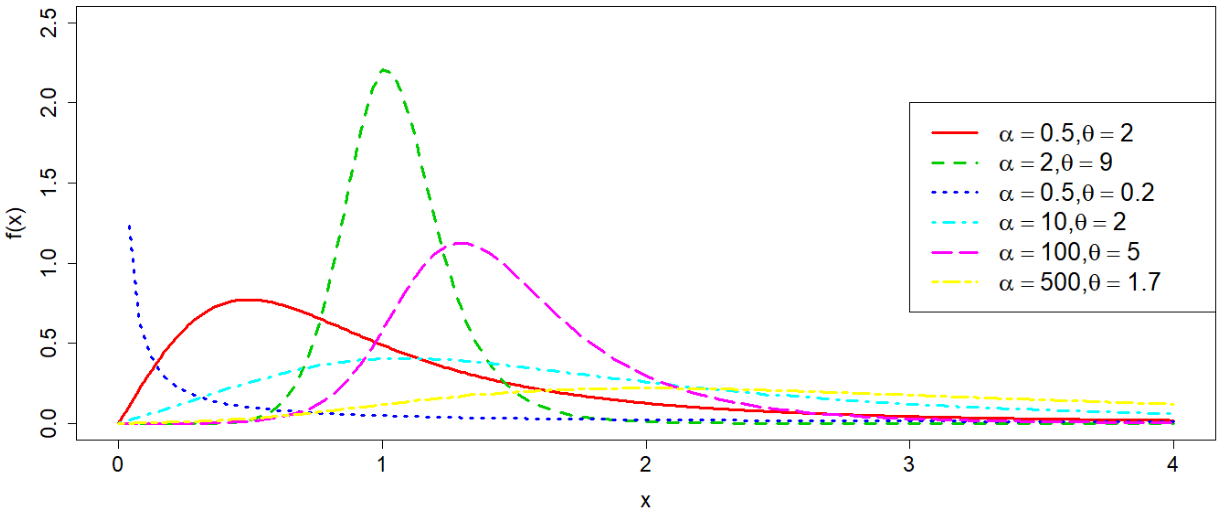

Figure 1: **pdf plots of APLL distribution.**

3.2. Hazard rate function

The Hazard rate Function denoted by $h_{APLL}(t)$ is the probability of instantaneous rate of death and is given as

$$h_{APLL}(x) = \begin{cases} log(\alpha)\frac{\theta x^{\theta-1}}{(1+x^{\theta})^2}\frac{1}{\alpha^{\left(\frac{1}{1+x^{\theta}}\right)}-1} & ;\alpha \neq 1 \\ \frac{\theta x^{\theta-1}}{(1+x^{\theta})} & ;\alpha = 1 \end{cases}$$

3.3. Reverse hazard rate function

The reverse hazard rate function for APLL distribution is denoted by $\phi_{APLL}(x)$ and is given as

$$\phi_{APLL}(x) = \begin{cases} log(\alpha)\frac{\theta x^{\theta-1}}{(1+x^{\theta})^2}\frac{\alpha^{\left(\frac{1}{1+x^{\theta}}\right)}}{\alpha^{\left(\frac{1}{1+x^{\theta}}\right)}-1} & ;\alpha \neq 1 \\ \frac{\theta}{x(1+x^{\theta})} & ;\alpha = 1 \end{cases}$$

The behavior of reliability function and hazard rate function of APLL distribution for different values of the parameters is illustrated in Figure 2.

4. Mixture representation

Using the power series expansion, $b^a = \sum_{j=0}^{\infty} \frac{(logb)^j a^j}{j!}$, the pdf and cdf of APLL distribution can be expressed in terms of an an alternative representation given by Eq. (5) and (6) respectively.

$$f_{APLL}(x) = \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} \frac{\theta x^{\theta(j+1)-1}}{(1+x^{\theta})^{j+2}}, \tag{5}$$

$$F_{APLL}(x) = \frac{1}{\alpha-1}\left[\sum_{j=0}^{\infty} \frac{(log\alpha)^j}{j!}\left(\frac{x^{\theta}}{1+x^{\theta}}\right)^j - 1\right]. \tag{6}$$

Equations (5) and (6) are quite useful in deriving various properties of APLL distribution.

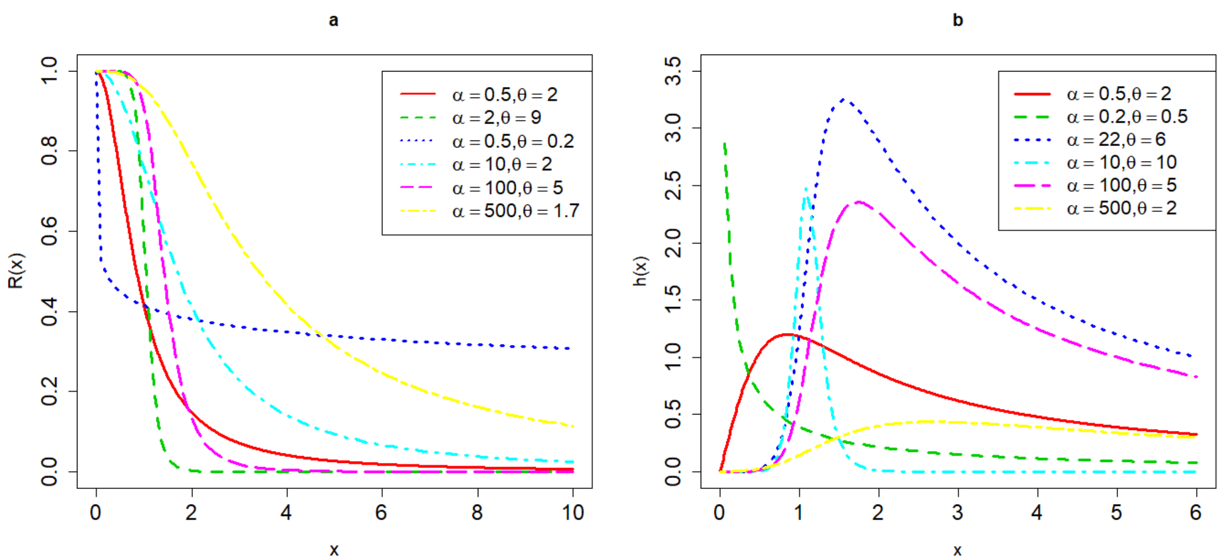

Figure 2: **Plots of APLL distribution reliability Function and hazard rate function for some parameter value.**

5. Statistical properties

In this section, some important statistical properties of APLL distribution are discussed.

5.1. Simulation and quantiles

Upon inverting Eq. (3) , we get

$$x=\left[\frac{log\alpha}{log\{(\alpha-1)U+1\}}-1\right]^{-\frac{1}{\theta}}, \tag{7}$$

Where $U\sim uniform(0,1)$. Using Eq. (7), the APLL distribution can easily be simulated.
Also the p^{th} quantile of APLL distribution is given as

$$x_p=\left[\frac{log\alpha}{log\{(\alpha-1)p+1\}}-1\right]^{-\frac{1}{\theta}}.$$

5.2. Moments

The r^{th} moment about origin of APLL distribution can be obtained as

$$\mu_r^{'}=\int_0^\infty x^r f_{APLL}(x)dx.$$

Upon substituting Eq. (5) in the above given equation, we get

$$\mu_r^{'}=\int_0^\infty x^r\sum_{j=0}^{\infty}\frac{(log\alpha)^{j+1}}{j!(\alpha-1)}\frac{\theta x^{\theta(j+1)-1}}{(1+x^\theta)^{j+2}}dx,$$

$$\mu_r^{'}=\sum_{j=0}^{\infty}\frac{(log\alpha)^{j+1}}{j!(\alpha-1)}\int_0^\infty\frac{\theta x^{r+\theta(j+1)-1}}{(1+x^\theta)^{j+2}}dx$$

Substituting $x^{\theta} = t$ and solving, we get

$$\mu_r^{'} = \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} \int_0^{\infty} \frac{t^{\frac{r}{\theta}+j+1-1}}{(1+t)^{\left(\frac{r}{\theta}+j+1\right)+\left(1-\frac{r}{\theta}\right)}} dt,$$

$$\mu_r^{'} = \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} B\left(\frac{r}{\theta}+j+1, 1-\frac{r}{\theta}\right); r < \theta. \tag{8}$$

Putting r=1 in Eq. (8), we get the mean of APLL distribution

$$\mu_1^{'} = \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} B\left(\frac{1}{\theta}+j+1, 1-\frac{1}{\theta}\right); \theta > 1. \tag{9}$$

The variance of APLL distribution is given as

$$V(X) = \left\{\sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} B\left(\frac{2}{\theta}+j+1, 1-\frac{2}{\theta}\right)\right\} - \left\{\sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} B\left(\frac{1}{\theta}+j+1, 1-\frac{1}{\theta}\right)\right\}^2 ; \theta > 2.$$

The u^{th} incomplete moment about origin is defined by $\psi_u = \int_0^n x^u f_{APLL}(x)dx$ and for APLL distribution, it can be obtained as

$$\psi_u = \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} B_{\left(\frac{n^{\theta}}{1+n^{\theta}}\right)}\left(\frac{u}{\theta}+j+1, 1-\frac{u}{\theta}\right); r < \theta, \tag{10}$$

where $B_x(l,m) = \int_0^x y^{l-1}(1-y)^{m-1}dy$.

5.3. Moment Generating Function

The moment generating function of APLL distribution can be obtained using the relation

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r^{'}, \tag{11}$$

using Eq. (8) in Eq. (11), we get the required expression as given by Eq. (12).

$$M_X(t) = \sum_{j=0}^{\infty}\sum_{r=0}^{\infty} \frac{t^r(log\alpha)^{j+1}}{j!r!(\alpha-1)} B\left(\frac{r}{\theta}+j+1, 1-\frac{r}{\theta}\right); r < \theta. \tag{12}$$

5.4. Mean Deviation about Mean and Median

The mean deviation about mean is defined as

$$D(\mu) = 2\int_0^{\mu} (\mu - x) f_{APLL}(x)dx, \tag{13}$$

using Eq. (5), we get

$$D(\mu) = 2\int_0^{\mu} (\mu - x) \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} \frac{\theta x^{\theta(j+1)-1}}{(1+x^{\theta})^{j+2}} dx,$$

$$D(\mu) = 2\left\{ \mu \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} B_{1-\left(\frac{\mu^{\theta}}{1+\mu^{\theta}}\right)}(j+1,1) \right\} - 2\left\{ \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} B_{1-\left(\frac{\mu^{\theta}}{1+\mu^{\theta}}\right)} \left(\frac{1}{\theta}+j+1, 1-\frac{1}{\theta}\right) \right\}; \theta > 1, \tag{14}$$

Also the expression for Mean deviation about medain for APLL distribution takes the following form:

$$D(M) = \mu - 2\left\{ \sum_{j=0}^{\infty} \frac{(log\alpha)^{j+1}}{j!(\alpha-1)} B_{1-\left(\frac{M^{\theta}}{1+M^{\theta}}\right)} \left(\frac{1}{\theta}+j+1, 1-\frac{1}{\theta}\right) \right\}; \theta > 1.$$

5.5. Mean residual life (MRL) and Mean Waiting Time (MWT)

The MRL is defined as

$$MRL = \frac{E(t) - \psi_1(t)}{1 - F_{APLL}(t)} - t. \tag{15}$$

The MWT is defined as

$$MWT = t - \frac{\psi_1(t)}{F_{APLL}(t)}. \tag{16}$$

Upon substituting Eq. (10) for $u = 1$ and Eq. (5) in Eq. (15) and Eq. (16), we get the required expressions for MRL and MWT for APLL distribution.

5.6. Renyi Entropy

The Renyi Entropy given by Renyi (1961) as a measure of uncertainty is defined as

$$I_{\nu} = \frac{1}{1-\nu} log \int_0^{\infty} \left(f_{APLL}(x)\right)^{\nu} dx; \nu > 0, \nu \neq 1,$$

using Eq. (3), we get

$$\begin{aligned} I_{\nu} =& \frac{1}{1-\nu} log \int_0^{\infty} \left(\frac{log(\alpha)}{\alpha-1} \frac{\theta x^{\theta-1}}{(1+x^{\theta})^2} \alpha^{\left(\frac{x^{\theta}}{1+x^{\theta}}\right)} \right)^{\nu} dx, \\ =& \frac{1}{1-\nu} log \left\{ \frac{(\theta log(\alpha))^{\nu}}{(\alpha-1)^{\nu}} \int_0^{\infty} \frac{x^{\nu(\theta-1)}}{(1+x^{\theta})^{2\nu}} \alpha^{\nu\left(\frac{x^{\theta}}{1+x^{\theta}}\right)} dx \right\}. \end{aligned} \tag{17}$$

Using the expansion $\alpha^s = \sum_{j=0}^{\infty} \frac{(log\alpha)^j s^j}{j!}$ in (17), I_{ν} reduces to

$$I_{\nu} = \frac{1}{1-\nu} log \left\{ \frac{(\theta log(\alpha))^{\nu}}{(\alpha-1)^{\nu}} \int_0^{\infty} \frac{x^{\nu(\theta-1)}}{(1+x^{\theta})^{2\nu}} \sum_{j=0}^{\infty} \frac{(log\alpha)^j \nu^j \left(\frac{x^{\theta}}{1+x^{\theta}}\right)^j}{j!} \right\},$$

$$I_{\nu} = \frac{1}{1-\nu} log \left[\left\{ \sum_{j=0}^{\infty} \frac{\theta^{\nu-1}(log\alpha)^{j+\nu}}{\{j!(\alpha-1)\}^{\nu}} \right\} \nu^j \frac{\Gamma(\nu+j+\frac{1-\nu}{\theta})\Gamma(\nu+\frac{\nu-1}{\theta})}{\Gamma(2\nu+j)} \right]. \tag{18}$$

5.7. L-moments

The L-moments of APLL distribution can be obtained as

$$E(X_{i:n}^r) = \int_0^\infty x^r f_{i:n}(x)dx. \tag{19}$$

We have

$$\begin{aligned}
f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} F_{APLL}^{i-1}(x)\left[1-F_{APLL}(x)\right]^{n-i} f_{APLL}(x) \\
&= \frac{n!}{(i-1)!(n-i)!}\left\{\frac{\alpha^{\left(\frac{x^\theta}{1+x^\theta}\right)}-1}{\alpha-1}\right\}^{i-1}\left\{1-\frac{\alpha^{\left(\frac{x^\theta}{1+x^\theta}\right)}-1}{\alpha-1}\right\}^{n-i} f_{APLL}(x), \\
&= \frac{n!(-1)^{i-1}\alpha^{n-i}}{(i-1)!(n-i)!(\alpha-1)^{n-1}}\left\{1-\alpha^{\left(\frac{x^\theta}{1+x^\theta}\right)}\right\}^{i-1} \\
&\quad \left\{1-\alpha^{\left(\frac{x^\theta}{1+x^\theta}\right)-1}\right\}^{n-i} f_{APLL}(x),
\end{aligned}$$

$$\begin{aligned}
&= \frac{n!(-1)^{i-1}\alpha^{n-i}}{(i-1)!(n-i)!(\alpha-1)^{n-1}}\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\alpha^{j\left(\frac{x^\theta}{1+x^\theta}-1\right)} \\
&\quad \sum_{k=0}^{n-i}\binom{n-i}{k}(-1)^k\alpha^{k\left(\frac{x^\theta}{1+x^\theta}\right)} f(x), \\
&= \sum_{j=0}^{i-1}\sum_{k=0}^{n-i}\frac{n!(-1)^{i+j+k-1}\alpha^{n-i-j}}{(i-1)!(n-i)!(\alpha-1)^{n-1}}\binom{i-1}{j}\binom{n-i}{k} \\
&\quad \alpha^{(j+k+1)\left(\frac{x^\theta}{1+x^\theta}\right)}\frac{log(\alpha)}{\alpha-1}\frac{\theta x^{\theta-1}}{(1+x^\theta)^2}, \\
&= \sum_{j=0}^{i-1}\sum_{k=0}^{n-i}\sum_{l=0}^{\infty}\frac{n!(-1)^{i+j+k-1}\alpha^{n-i-j}}{l!(i-1)!(-cn-i)!(\alpha-1)^n}\binom{i-1}{j}\binom{n-i}{k} \\
&\quad \frac{\theta(log\alpha)^{l+1}(j+k+1)^l x^{l\theta+\theta-1}}{(1+x^\theta)^{l+2}},
\end{aligned}$$

substituting the value of $f_{i:n}(x)$ in Eq. (19) and solving, we get

$$\begin{aligned}
E(X_{i:n}^r) &= \sum_{j=0}^{i-1}\sum_{k=0}^{n-i}\sum_{l=0}^{\infty}\frac{n!(-1)^{i+j+k-1}\alpha^{n-i-j}}{l!(i-1)!(-cn-i)!(\alpha-1)^n}\binom{i-1}{j}\binom{n-i}{k} \\
&\quad (log\alpha)^{l+1}(j+k+1)^l B\left(\frac{r}{\theta}+l+1, 1-\frac{r}{\theta}\right); r<\theta.
\end{aligned} \tag{20}$$

5.8. Stress Strength Reliability

Let $X_1 \sim APLL\ distribution(\alpha_1, \theta)$ and $X_2 \sim APLL\ distribution(\alpha_2, \theta)$ be two independent random variable. Then, the stress strength reliability (SSR) denoted by R for APLL distribution can computed as:
Case 1: $\alpha_1 \neq 1$ and $\alpha_2 \neq 1$, then

$$R = \int_0^\infty \frac{\alpha_1^{\left(\frac{x^\theta}{1+x^\theta}\right)} - 1}{\alpha_1 - 1} \frac{log(\alpha_2)}{\alpha_2 - 1} \frac{\theta x^{\theta-1}}{(1+x^\theta)^2} \alpha_2^{\left(\frac{x^\theta}{1+x^\theta}\right)} dx,$$
$$= \frac{\theta log(\alpha_2)}{(\alpha_2 - 1)(\alpha_1 - 1)} \int_0^\infty \frac{(\alpha_1^{\left(\frac{x^\theta}{1+x^\theta}\right)} - 1) x^{\theta-1} \alpha_2^{\left(\frac{x^\theta}{1+x^\theta}\right)}}{(1+x^\theta)^2} dx,$$
$$= \frac{log(\alpha_2)}{(\alpha_2 - 1)(\alpha_1 - 1)} \left[\frac{\alpha_1\alpha_2 - 1}{log\alpha_1 + log\alpha_2} - \frac{\alpha_2 - 1}{log\alpha_2} \right].$$

Case 2: $\alpha_1 = 1$ and $\alpha_2 \neq 1$, then

$$R = \int_0^\infty \frac{x^\theta}{1+x^\theta} \frac{log(\alpha_2)}{\alpha_2 - 1} \frac{\theta x^{\theta-1}}{(1+x^\theta)^2} \alpha_2^{\left(\frac{x^\theta}{1+x^\theta}\right)} dx,$$

$$= \frac{\theta log(\alpha_2)}{\alpha_2 - 1} \int_0^\infty \frac{(x^{2\theta} - 1)\alpha_2^{\left(\frac{x^\theta}{1+x^\theta}\right)}}{(1+x^\theta)(1+x^\theta)^2} dx,$$
$$= \frac{log(\alpha_2)}{(\alpha_2 - 1)} \left[\frac{\alpha_2 - 1}{log\alpha_2} - \frac{\alpha_2 - log\alpha_2 - 1}{log^2\alpha_2} \right],$$

Case 3: $\alpha_1 \neq 1$ and $\alpha_2 = 1$, then

$$R = \int_0^\infty \frac{\alpha_1^{\left(\frac{x^\theta}{1+x^\theta}\right)} - 1}{\alpha_1 - 1} \frac{\theta x^{\theta-1}}{(1+x^\theta)^2} dx,$$
$$= \frac{\theta}{\alpha_1 - 1} \int_0^\infty \frac{(\alpha_1^{\left(\frac{x^\theta}{1+x^\theta}\right)} - 1) x^{\theta-1}}{(1+x^\theta)^2} dx,$$
$$= \frac{1}{\alpha_1 - 1} \left[\frac{\alpha_1 - 1}{log\alpha_1} - 1 \right].$$

Case 4: $\alpha_1 = 1$ and $\alpha_2 = 1$, then

$$R = \int_0^\infty \frac{x^{\theta_1}}{1+x^{\theta_1}} \frac{\theta_2 x^{\theta_2-1}}{(1+x^{\theta_2})^2} dx,$$

In this case, the model reduces to the SSR of base distribution i.e., LL distribution.

6. Parameter Estimation

Let $\eta = (\alpha, \theta)^T$ be the vector of parameters of APLL distribution. The log-likelihood function denoted by l, computed from a sample of size n drawn from APLL distribution is given as:

$$l = nlog\theta + nlog(log\alpha) - nlog(\alpha - 1) + \sum_{i=0}^{n} \left((\theta - 1)log(x_i) - 2log(1 + x_i^\theta) \right) + \sum_{i=0}^{n} \frac{x_i^\theta}{1 + x_i^\theta} log\alpha. \quad (21)$$

The elements of score matrix $U(\eta) = \frac{\partial l}{\partial \eta} = \left(\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \theta}\right)^T$ are

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha \log(\alpha)} - \frac{n}{\alpha - 1} + \frac{1}{\alpha} \sum_{i=0}^{n} \frac{x_i^\theta}{(1 + x_i^\theta)}, \tag{22}$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=0}^{n} \log x_i - 2 \sum_{i=0}^{n} \frac{x_i^\theta \log x_i}{(1 + x_i^\theta)} + log\alpha \sum_{i=0}^{n} \frac{x_i^\theta \log x_i}{(1 + x_i^\theta)^2}. \tag{23}$$

Upon equating Eqs. (22) and (23) to zero and solving them simultaneously, we obtain the ML estimates of α and θ. Methods such as Newton Raphson can be used to solve such non-linear Equations. For fixed θ in (23) we obtain $\hat{\alpha}(\theta)$ as follows

$$\hat{\alpha}(\theta) = exp \left\{ \frac{\sum_{i=0}^{n} \log x_i + 2 \sum_{i=0}^{n} \frac{x_i^\theta \log x_i}{(1+x_i^\theta)} - \frac{n}{\theta}}{\sum_{i=0}^{n} \frac{x_i^\theta \log x_i}{(1+x_i^\theta)^2}} \right\}, \tag{24}$$

and then we can obtain $\hat{\theta}$ from (22) by solving the following equation:

$$\frac{1}{\hat{\alpha}(\theta) \log(\hat{\alpha}(\theta))} - \frac{1}{\hat{\alpha}(\theta) - 1} + \frac{1}{n\hat{\alpha}(\theta)} \sum_{i=0}^{n} \frac{x_i^\theta}{(1 + x_i^\theta)} = 0. \tag{25}$$

Once $\hat{\theta}$ has been obtained, we can evaluate the value of $\hat{\alpha}$ as $\hat{\alpha} = \hat{\alpha}(\hat{\theta})$.
The second order partial derivatives of APLL distribution for l exist. Thus, the asymptotic sampling distribution of $\hat{\eta}$ is $N_2[0, J(\hat{\eta})^{-1}]$ where $J(\eta)$ is the observed Fisher information matrix given by:

$$J(\eta) = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \tag{26}$$

where

$$V_{11} = \frac{\partial^2 l}{\partial \alpha^2} = -\frac{n(1 + log\alpha)}{(\alpha log\alpha)^2} + \frac{n}{(\alpha - 1)^2} - \frac{1}{\alpha^2} \sum_{i=0}^{n} \frac{x_i^\theta}{(1 + x_i^\theta)},$$

$$V_{22} = \frac{\partial^2 l}{\partial \theta^2} = -\frac{n}{\theta^2} - 2 \sum_{i=0}^{n} \frac{x_i^\theta (\log x_i)^2}{(1 + x_i^\theta)^2} + log\alpha \sum_{i=0}^{n} \frac{x_i^\theta (\log x_i)^2 (1 - x_i^\theta)}{(1 + x_i^\theta)^3},$$

and

$$V_{12} = V_{21} = \frac{\partial^2 l}{\partial \alpha \partial \theta} = \frac{\partial^2 l}{\partial \theta \partial \alpha} = \frac{1}{\alpha} \sum_{i=0}^{n} \frac{x_i^\theta log x_i}{(1 + x_i^\theta)^2}.$$

The $100(1 - \phi)$ confidence interval for α and θ cab be determined as $\hat{\alpha} \pm Z_{\frac{\phi}{2}} \sqrt{\widehat{V_{11}}}$ and $\hat{\theta} \pm Z_{\frac{\phi}{2}} \sqrt{\widehat{V_{22}}}$ where $Z_{\frac{\phi}{2}}$ is the upper $\phi - th$ percentile of standard Normal distribution.

7. Simulation

In this section, we present the simulation study to illustrate the behavior of MLEs for different sample sizes. Samples of sizes (n) $50, 100, 300, 500$ and 1000 were drawn from APLL distribution for the two parameter combination (α, θ) i.e., $(3, 3)$ and $(0.5, 0.8)$. The results are reported in Table 1. It can be clearly seen that the estimates are consistent as the standard deviations of the MLEs show a decreasing trend with increase in the sample size.

Table 1: Simulation Study of ML estimators of APLL distribution.

(α, θ)	n	$\hat{\alpha}$	$\hat{\theta}$
	50	5.355	3.244
		(2.869)	(0.367)
	100	4.674	3.451
		(1.741)	(0.277)
(3,3)	300	3.004	3.384
		(0.623)	(0.160)
	500	3.060	3.146
		(0.493)	(0.115)
	1000	3.035	2.999
		(0.345)	(0.078)
	50	0.743	0.890
		(0.365)	(0.105)
	100	0.590	0.937
		(0.206)	(0.077)
(0.5, 0.8)	300	0.404	0.896
		(0.083)	(0.042)
	500	0.469	0.837
		(0.074)	(0.030)
	1000	0.504	0.800
		(0.0561)	(0.021)

8. Application

To access the flexibility and establish the superiority of the APLL distribution, we compare the fits of APLL distribution with three well- established models for two real life data sets. The three models that are used for comparison are

- Transmuted Log-Logistic (TLL) distribution with pdf

$$f(x) = (1+\alpha)\left(\frac{x^{\theta}}{1+x^{\theta}}\right) - \alpha\left(\frac{x^{\theta}}{1+x^{\theta}}\right)^{2} \quad ; -1 \leqslant \alpha \leqslant 1, \theta > 0,$$

  where α is the transmuted parameter.

- Exponentiated Log-Logistic (ELL) distribution with pdf

$$f(x) = \frac{\alpha\theta x^{\theta(\alpha+1)-2}}{(1+x^{\theta})^{\alpha+1}} \quad ; \alpha, \theta > 0.$$

- LL distribution with pdf

$$f(x) = \frac{\theta x^{\theta-1}}{(1+x^{\theta})^{2}} \quad ; \theta > 0.$$

The first data set consists of the time between failures for 30 repairable items and was also analyzed by Murthy et al. (2004). The Ml estimates and values of comparison criterions for APLL distribution and competitive models are reported in Table 2 and Table 3 respectively.The second data set represents the survival times (in years) after diagnosis of 43 patients with a certain kind of leukemia extracted from Kleiber and Kotz (2003). The Ml estimates and values of comparison criterions for APLL distribution and competitive models are reported in Table 4 and Table 5 respectively. The criterion such as log l, AIC, SIC, AICc and HQIC are used as performance comparing tools. Also, the values of K-S statistic and associated *p*-value is computed.

The results obtained in Table 3 and Table 4 reveal that APLL distribution has the least value of all the comparison criterions, hence APLL distribution can be considered a strong competitor to other distributions compared here for

Table 2: ML Estimates of APLL Distribution and Competitive Models for First Data Set.

Model	$\hat{\alpha}$	$\hat{\theta}$
APLL distribution	2.48	2.220
TLL distribution	-0.417	2.170
ELL distribution	1.239	2.051
LL distribution	-	2.157

Table 3: Comparison of APLL Distribution and Other Competitive Models For First Data Set.

Model	$-\hat{l}$	AIC	SIC	AICc	HQIC	K-S Statistic	p-value
APLL distribution	40	83.85	86	84.30	84	0.075	0.9959
TLL distribution	41	84.95	88	85.39	86	0.299	0.1263
ELL distribution	41	85.66	88	86.29	87	0.108	0.8691
LL distribution	42	85.84	87	85.98	86	0.183	0.2623

Table 4: ML Estimates of APLL Distribution and Competitive Models for Second Data Set.

Model	$\hat{\alpha}$	$\hat{\theta}$
APLL distribution	8.614	1.582
TLL distribution	-0.735	1.431
ELL distribution	1.559	1.302
LL distribution	-	1.374

Table 5: Comparison of APLL Distribution and Other Competitive Models For Second Data Set.

Model	$-\hat{l}$	AIC	SIC	AICc	HQIC	K-S Statistic	p-value
APLL distribution	86	176.60	180	176.90	177	0.105	0.7320
TLL distribution	88	179.05	182	179.35	180	0.243	0.0740
ELL distribution	90	182.99	186	183.29	184	0.187	0.0988
LL distribution	93	188.17	189	188.26	188	0.432	0.0649

fitting data. The relative histogram and fitted APLL distribution for first and second data set are presented in Figure 3(a) and3(b) respectively. Also, to compare the empirical distribution of the data with APLL distribution graphically, the QQ-plot for both the data sets is displayed in Figure 4.

9. Conclusion

In this paper, a new lifetime distribution namely APLL distribution is proposed and studied. The new distribution is more flexible and its hazard rate function exhibits complex shapes. The new distribution is compared with three well-established models using two real life data sets. The results showed that APLL distribution provides better fit than the competitive models. We hope that this distribution attracts wider application in diverse fields.

Acknowledgements

We would like to thank the referees for their comments and suggestions on the manuscript.

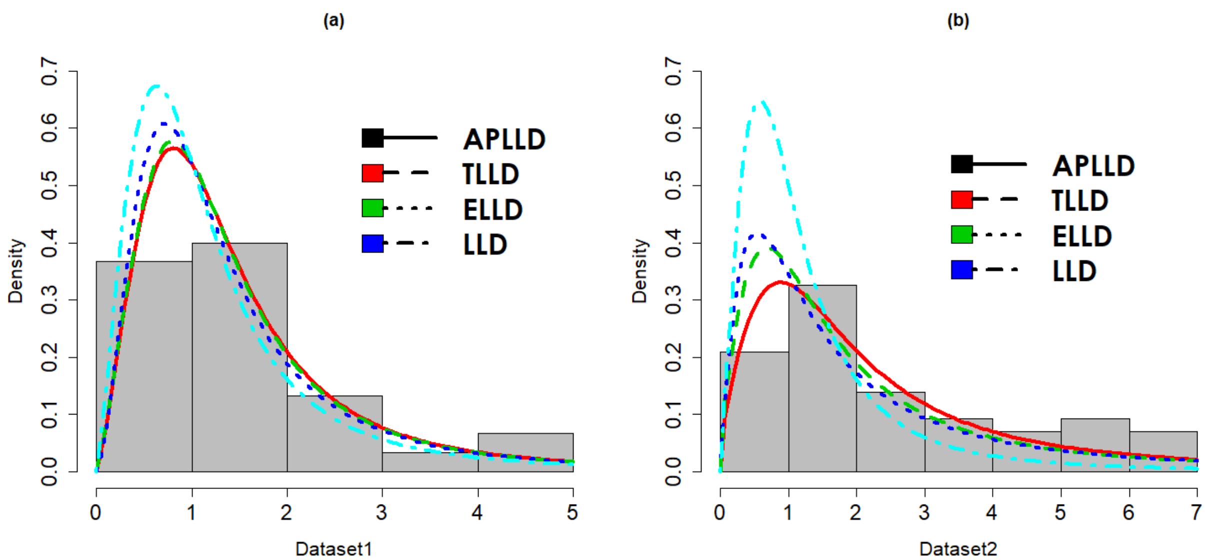

Figure 3: **The Fitted pdf of APLL distribution and other fitted distribution for (a) first data set and (b) second data set respectively.**

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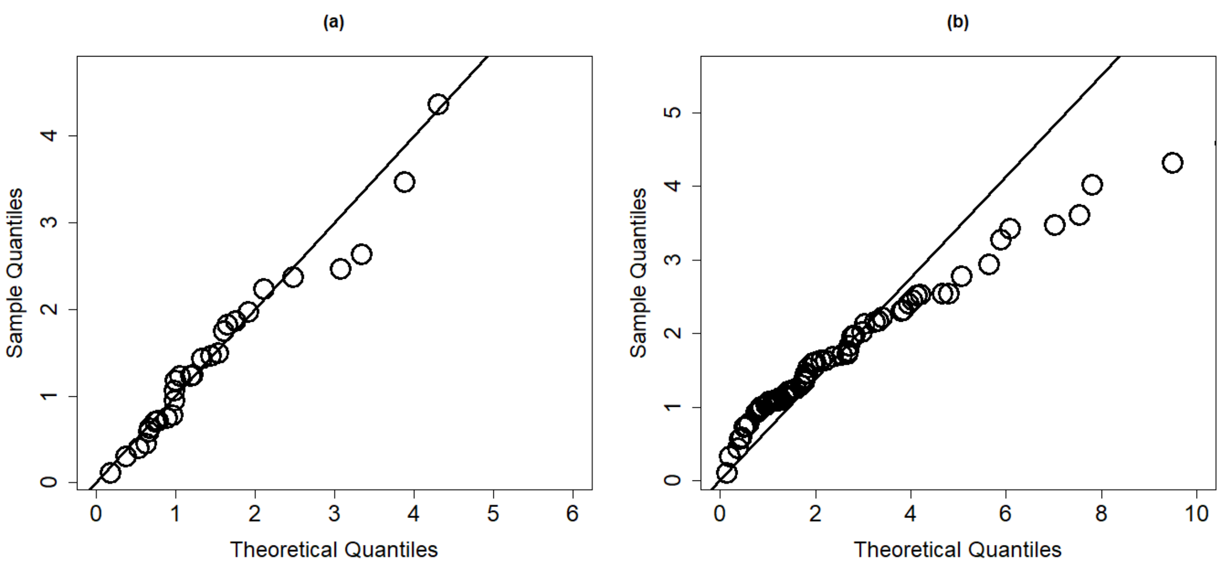

Figure 4: **QQ-Plot for APLL distribution for first and second data set respectively.**