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The Flexible Weibull Extension-Burr XII Distribution: Model, Properties and Applications

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Abstract

This paper is devoted to study a new four- parameter additive model. The newly suggested model is referred to as the flexible Weibull extension-Burr XII distribution. It is derived by considering a serial system with one component following a flexible Weibull extension distribution and another following a Burr XII distribution. The usefulness of the model stems from the flexibility of its failure rate which accommodates bathtub and modified bathtub among other risk patterns. These two patterns have been widely accepted in several fields, especially reliability and engineering fields. In addition, the importance of the new distribution is that it includes new sub-models which are not known in the literature. Some statistical properties of the proposed distribution such as quantile function, the mode, the rth moment, the moment generating function and the order statistics are discussed. Moreover, the method of maximum likelihood is used to estimate the parameters of the model. Also, to evaluate the performance of the estimators, a simulation study is carried out. Finally, the performance of the proposed distribution is compared through a real data set to some well-known distributions including the new modified Weibull, the additive Burr and the additive Weibull distributions. It is shown that the proposed model provides the best fit for the used real data set.

Key Words: Additive models; Flexible Weibull Extension Distribution; Burr XII Distribution; Moments; Maximum Likelihood Estimation.

Mathematical Subject Classification: 62D05

1. Introduction

Usually, in life-testing experiments there are different failure modes affecting the experimental unit in a different way. These failure modes compete simultaneously to cause the failure of this experimental unit. This is known in the statistical literature as competing risks. Also, competing risks occur when the tested item consists of several components connected in series. Each one of these components has a certain distribution with certain parameters and affects the tested item in a different way. Then, the lifetime of this series system is determined by the minimum lifetime of the components and the parameters of the distribution of each component can be estimated by applying a competing risks model. Competing risks data are found in many branches of statistical applications such as reliability engineering, econometrics, demography and biological sciences, etc.

Among the lifetime models, the Weibull distribution is perhaps the most commonly used distributions for lifetime data analysis. It plays a crucial role in reliability theory and life-testing experiments. However, its major weakness is its inability to fit data with non-monotone failure rates. Thus, in the last few years many researchers have proposed various modified forms of the Weibull distribution to achieve non-monotonic shapes. For example, (Mudholkar and Srivastava, (1993)) presented an exponentiated Weibull distribution. (Xie and Lai, (1996)) introduced the additive

Weibull distribution. Another distribution was proposed (by Xie et al., (2002)) called modified Weibull extension distribution. (Lai et al., (2003)) presented a modified Weibull distribution. In addition, (Bebbington et al., (2007)) introduced the flexible Weibull extension distribution. This distribution has a simple and flexible failure function, which can be increasing and modified bathtub shaped. It has gained more attention in the last decade due to the capability of using it in different applications including engineering, reliability, biology, demography and actuarial sciences. On the other hand, in many practical studies, it has been observed that Burr XII distribution can be used quite effectively, in place of other lifetime distributions. Due to its flexibility in modeling many types of data, Burr XII distribution has found important applications in a wide variety of fields. It has been used in many applications such as actuarial science, biology, economics, forestry, life testing and reliability.

In this paper, we propose a new lifetime distribution with four parameters, referred to as the flexible Weibull extension-Burr XII distribution. This new distribution is obtained from the sum of the failure rate functions of the flexible Weibull extension and Burr XII distributions via the use of competing risks to produce a very flexible failure rate function which can be used in many real life situations. Thus, this model can be applied to a component which is mainly affected by two failure modes acting simultaneously on it. One of the modes follows the flexible Weibull distribution and the other follows Burr XII distribution and either one of these two failure modes can cause the component's failure. This new model can also be interpreted as the lifetime of a serial system with two independent components, the lifetime of component 1 follows the flexible Weibull extension distribution and the lifetime of component 2 follows Burr XII distribution and the system's lifetime is the minimum of the lifetimes of the two components. Recently, in the literature, many distributions have been constructed based on the idea of adding the hazard rates of two distributions. For example, (Almalki and Yuan, (2013)) proposed the new modified Weibull distribution. (He et al., (2016)) introduced the additive modified Weibull distribution. In addition, the additive Perks–Weibull distribution which was presented by (Singh, (2016)). More recently, (Tarvirdizade and Ahmadpour, (2019)) proposed the Weibull–Chen distribution. The importance of our proposed model lies in the flexibility of its failure rate to represent several ageing patterns including bathtub and modified bathtub shapes. The usefulness of bathtub-shaped failure rate is well-known in the literature. Several distributions that can model a bathtub pattern are given in (Xie et al., (2003)). On the other hand, components' failure rate may exhibit a more complex failure rate pattern referred to as the modified bathtub shape. In this pattern, the failure rate is initially increasing for a short period perhaps due to defects in the manufacturing process then this is followed by a bathtub shape. (Kuo and Kuo, (1983)) showed the importance of this pattern in reliability and engineering fields based on analyzing different failure data. Thus, our new model allows considerable flexibility in modelling the pre-useful period in the lifetime of a component compared to other existing models which exhibit the bathtub pattern only.

The rest of this paper is organized as follows. The definition of the new distribution, its new sub-models and the failure rate function are presented in Section 2. Section 3 is devoted to studying some of the statistical properties of the new distribution including, quantile function, median, mode, r^{th} moments, the moment generating function (mgf) and the order statistics. Maximum likelihood estimation of the parameters is obtained in Section 4. In Section 5, Monte Carlo simulation results are presented. The importance and flexibility of the new distribution is further emphasized in Section 6 by comparing our distribution to some other existing distributions through a real data set. Finally, Section 7 ends with some conclusions.

2. The Model

2.1. Definition

The failure rate function of a random variable *X* following the flexible Weibull extension-Burr XII (denoted by FWBXII or FWB) lifetime model with four parameters $\Theta = (\alpha, \beta, c, k)$ is derived by the sum of the failure rates of flexible Weibull extension and Burr XII distributions as follows

$$h(x; \Theta) = h_1(x; \alpha, \beta) + h_2(x; k, c), \tag{1}$$

where the failure rate function of the first component (flexible Weibull extension) of (Bebbington et al., (2007)) is given by

$$h_1(x; \alpha, \beta) = \left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}}, \qquad x > 0;\ \alpha, \beta > 0,$$

and the failure rate function of the second component (Burr XII) of (Burr, (1942)) is given by

$$h_2(x; k, c) = ck\, x^{c-1}[1 + x^c]^{-1}, x > 0; c, k > 0.$$

So, the probability density function (pdf) of this distribution is given by

$$f(x; \Theta) = h(x; \Theta) \exp\left(-\int_0^x h(u; \Theta)\, du\right),$$

$$f(x;\Theta) = \left[\left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} + ck\, x^{c-1}[1+x^c]^{-1}\right] exp\left[-e^{\alpha x - \frac{\beta}{x}}\right][1+x^c]^{-k}. \qquad (2)$$

It can be written as

$$f(x;\Theta) = [h_1(x;\alpha,\beta) + h_2(x;c,k)]R_1(x;\alpha,\beta)R_2(x;c,k),$$

where R_1 and R_2 are the reliability functions of flexible Weibull extension and Burr XII distributions, respectively. Evidently, the corresponding cdf $F(x;\Theta)$ and reliability function $R(x;\Theta)$ of this additive model are given by

$$F(x;\Theta) = 1 - exp\left[-e^{\alpha x - \frac{\beta}{x}}\right][1+x^c]^{-k}, \qquad (3)$$

$$R(x;\Theta) = exp\left[-e^{\alpha x - \frac{\beta}{x}}\right][1+x^c]^{-k}. \qquad (4)$$

Figure 1 display plots of the pdf for selected values of the model parameters. The plots suggest that the pdf can be unimodal or bimodal among other shapes for the selected values of the model parameters. In addition, plots of the failure rate function of FWBXII distribution are given in Figure 2. This figure exhibits increasing, bathtub, modified bathtub and bi-bathtub shapes for the selected values of the model parameters. Hence, the FWBXII failure rate function is very flexible and suitable for non-monotonic empirical failure rate behaviors which are more likely to be encountered in practice.

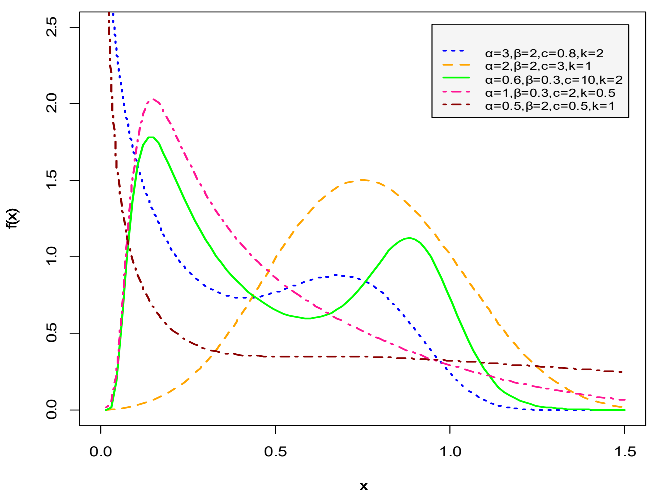

Figure 1: Plots of FWBXII pdf

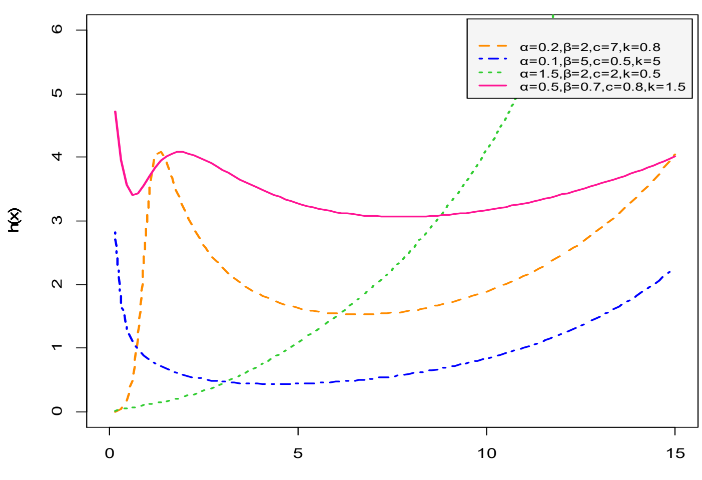

Figure 2: Plots of FWBXII faliure rate function

2.2. Interpretation of the Failure Rate Curves

The shape of a failure rate function can be characterized by studying the limiting behavior of the failure rate function and its derivative. From Equation (1), differentiating the $h(x;\Theta)$ with respect to x gives:

$$\acute{h}(x;\Theta) = \frac{d}{dx} h(x;\Theta) = \acute{h}_1(x;\alpha,\beta) + \acute{h}_2(x;c,k),$$

where

$$\acute{h}_1(x;\alpha,\beta)=\frac{(\alpha x^2+\beta)^2-2\beta x}{x^4}\,e^{\alpha x-\frac{\beta}{x}},$$
$$\acute{h}_2(x;c,k)=\frac{c(c-1)kx^{c-2}}{[1+x^c]}-\frac{c^2kx^{2c-2}}{[1+x^c]^2}.$$

The shapes of the failure rate function of the FWBXII distribution can be explained as follows:

Case (1): For c>1 and $\alpha\beta$ < 27/64:
$\lim_{x\to 0} h(x;\Theta)=0$ and $\lim_{x\to\infty} h(x;\Theta)=\infty$. In this case, $h_1(x;\alpha,\beta)$ is modified bathtub whereas $h_2(x;c,k)$ is unimodal. Let $x_0{}^*$ and $x_0{}^{**}$ define maximum and minimum values of $h(x;\Theta)$ respectively. Initially for $x<x_0{}^*$, both $\acute{h}_1(x;\alpha,\beta)$ and $\acute{h}_2(x;c,k)$ take positive values, so $\acute{h}(x;\Theta)>0$. For $x_0{}^*<x<x_0{}^{**}$, $\acute{h}(x;\Theta)$ will become negative either because both $\acute{h}_1(x;\alpha,\beta)$ and $\acute{h}_2(x;c,k)$ are negative or one of them becomes negative but dominates the other positive term. Eventually, at $x>x_0{}^{**}$, $\acute{h}(x;\Theta)$ will become positive again because the positive term $\acute{h}_1(x;\alpha,\beta)$ will dominate the negative term $\acute{h}_2(x;c,k)$**.** Summarizing, the failure rate of the FWBXII model increases from 0 to a maximum value $x_0{}^*$, on $(0,x_0{}^*)$, decreases on $(x_0{}^*,\ x_0{}^{**})$, and increases again on $(x_0{}^{**},\infty)$, exhibiting a modified bathtub shape.

Case (2): For c>1 and $\alpha\beta$ ≥ 27/64:
$\lim_{x\to 0} h(x;\Theta)=0$ and $\lim_{x\to\infty} h(x;\Theta)=\infty$. In this case, $h_1(x;\alpha,\beta)$ is strictly increasing whereas $h_2(x;c,k)$ is unimodal. So, we have two possibilities either $h(x;\Theta)$ is strictly increasing or $h(x;\Theta)$ is modified bathtub. The first possibility occurs if $h_1(x;\alpha,\beta)$ always dominates $h_2(x;c,k)$. The second possibility happens if $h(x;\Theta)$ has three stages. In the first stage $\acute{h}_1(x;\alpha,\beta)$ and $\acute{h}_2(x;c,k)$ are both positive; in the second stage $\acute{h}_2(x;c,k)$ is negative and dominates the positive term $\acute{h}_1(x;\alpha,\beta)$. Finally in the third stage, the positive term $\acute{h}_1(x;\alpha,\beta)$ dominates the negative term $\acute{h}_2(x;c,k)$.

Case (3): For c ≤ 1 and $\alpha\beta$ ≥ 27/64
$\lim_{x\to 0} h(x;\Theta)=\infty$ and $\lim_{x\to\infty} h(x;\Theta)=\infty$. Here, $\acute{h}_1(x;\alpha,\beta)>0$ and $\acute{h}_2(x;c,k)<0$. So, for $x<x_0$, x_0 is the value of x at which $\acute{h}(x;\Theta)=0$, $\acute{h}_2(x;c,k)$ dominates $\acute{h}_1(x;\alpha,\beta)$ hence $\acute{h}(x;\Theta)<0$. The situation is reversed for $x>x_0$. So, $h(x;\Theta)$ initially decreases and then increases, exhibiting a bathtub shape.

Case (4): For c ≤ 1 and $\alpha\beta$ < 27/64
$\lim_{x\to 0} h(x;\Theta)=\infty$ and $\lim_{x\to\infty} h(x;\Theta)=\infty$. In this case, $\acute{h}(x;\Theta)<0$ for the range $(0,x_1)$ in which $h_2(x;c,k)$dominates $h_1(x;\alpha,\beta)$, then $h_1(x;\alpha,\beta)$ dominates $h_2(x;c,k)$ for the range (x_1,∞). Hence, the failure rate of the FWBXII model decreases to a minimum value x_1 on $(0,\ x_1)$, increases on (x_1,x^*), then decreases to a minimum value x_2 on $(x^*,\ x_2)$, and increases again on (x_0,∞), exhibiting a bi-bathtub shape.

2.3. Sub-models

Three new distributions obtained from the FWBXII distribution as follows

1- Flexible Weibull extension-compound exponential (FWCE) when c=1.
2- Flexible Weibull extension-compound Rayleigh (FWCR) when c=2.
3- Flexible Weibull extension-log logistic (FWLogL) when k=1.

3. Some Statistical Properties

In this Section, some of the statistical properties including, quantile function, the mode, the median, r^{th} moments, the moment generating function (mgf) and the order statistics are presented as follows.

3.1. Mode and Quantile

In this sub-section, the mode(s) and the quantile function of the FWBXII distribution are presented. The mode of a distribution is the value of x corresponding to the maximum value of the probability density function. For a unimodal distribution, x has only one value representing one mode, while for a bimodal distribution x has two values representing two modes. The mode(s) of the FWBXII distribution can be obtained by differentiating the pdf in Equation (2) with respect to x and equating it to zero.

$$\acute{f}(x;\Theta) = 0. \tag{5}$$

Since

$$f(x;\Theta) = h(x;\Theta)e^{-H(x;\Theta)},$$

where $H(x;\Theta)$ is the cumulative hazard rate function which is defined as

$$H(x;\Theta) = \int_0^x h(u;\Theta)\,du.$$

Equation (5) can be written as

$$\left[\acute{h}(x;\Theta) - h^2(x;\Theta)\right]e^{-H(x;\Theta)} = 0.$$

Since

$$R(x;\Theta) = e^{-H(x;\Theta)},$$

then

$$\left[\acute{h}(x;\Theta) - h^2(x;\Theta)\right]R(x;\Theta) = 0. \tag{6}$$

So the mode(s) of the FWBXII distribution is (are) the solution of the following Equation

$$\begin{aligned}
&-2kc\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}\,exp\left[-e^{\alpha x-\frac{\beta}{x}}\right]x^{c-1}(1+x^c)^{-k-1} \\
&\quad+(1+x^c)^{-k}\left[\left(-\frac{2\beta}{x^3}\right)e^{\alpha x-\frac{\beta}{x}}\,exp\left[-e^{\alpha x-\frac{\beta}{x}}\right]+\left(\alpha+\frac{\beta}{x^2}\right)^2 e^{\alpha x-\frac{\beta}{x}}\,exp\left[-e^{\alpha x-\frac{\beta}{x}}\right]\right. \\
&\quad\left.+\left(\alpha+\frac{\beta}{x^2}\right)^2 e^{2\left(\alpha x-\frac{\beta}{x}\right)}\,exp\left[-e^{\alpha x-\frac{\beta}{x}}\right]\right] \\
&\quad+exp\left[-e^{\alpha x-\frac{\beta}{x}}\right]\left[c^2k(-k-1)x^{2c-2}(1+x^c)^{-k-2}+kc(c-1)x^{c-2}(1+x^c)^{-k-1}\right] \\
&\quad=0.
\end{aligned} \tag{7}$$

It is difficult to get an explicit solution of Equation (7), therefore, it can be solved numerically.

Let X be a random variable with distribution function F as defined in Equation (3), and let $q \in (0, 1)$. Then, the quantile x_q of the FWBXII distribution is obtained as a numerical solution of the following nonlinear Equation with respect to x_q:

$$F(x_q;\Theta) = 1 - exp\left[-e^{\alpha x_q-\frac{\beta}{x_q}}\right]\left[1+{x_q}^c\right]^{-k} = q. \tag{8}$$

It is difficult to get an explicit solution of Equation (8), hence, it can be solved numerically. Special quantiles can be obtained using Equation (8). For example, the median which is the 0.5 quantile can be obtained by setting $q = 0.5$ in Equation (8).

Some values of median and mode(s) for various values of the parameters (α, β, c, k) are calculated in Table 1. From Table 1, we note two cases of modes; unimodal and bimodal.

Table 1: The mode(s) and median of the FWBXII distribution

α	β	c	k	Mode(s)		Median
0.6	0.3	0.8	1	0.1161		0.2721
3	2	0.8	0.8	0.7475		0.6285
1	0.3	2	0.5	0.1478		0.3639
2	2	3	1	0.7475		0.7601
0.6	0.3	10	2	0.1412	0.8823	0.4642
0.1	0.5	5	1.5	0.2330	0.7818	0.6680

3.2. The Moments

Moments of a distribution can be used to study many important characteristics. So, in this subsection we will present the moments of the FWBXII distribution. The r^{th} moment of the FWBXII random variable X is given by

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-1)^{i+h}\,(i)^{j+h}\,\alpha^j\,\beta^h}{i!\;\;j!\;\;h!}\left[\frac{r}{c}\;B\left(\frac{r+j-h}{c},k-\frac{r+j-h}{c}\right)\right]. \tag{9}$$

under the condition $0 < \frac{r+j-h}{c} < k$.

Proof:

$$\mu'_r = \int_0^\infty x^r \, dF(x;\Theta).$$

$$\mu'_r = -\int_0^\infty x^r \, dR(x;\Theta).$$

Using integration by parts

$$\mu'_r = \int_0^\infty r\, x^{r-1} exp\left[-e^{\alpha x-\frac{\beta}{x}}\right][1+x^c]^{-k} dx.$$

Using series expansion of $exp\left[-e^{\alpha x-\frac{\beta}{x}}\right]$, we get

$$\mu'_r = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\int_0^\infty r\, x^{r-1}\, e^{i(\alpha x-\frac{\beta}{x})}\,[1+x^c]^{-k}\, dx$$

$$= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\left[\int_0^\infty r\, x^{r-1}\, e^{i(\alpha x)}\, e^{-i(\frac{\beta}{x})}\,[1+x^c]^{-k}\, dx\right].$$

Using series expansion of of $e^{(i)(\alpha x)}$ and $e^{-(i)(\frac{\beta}{x})}$, we get

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-1)^{i+h}\,(i)^{j+h}\,\alpha^j\,\beta^h}{i!\;\; j!\;\; h!}\left[\int_0^\infty r\, x^{r+j-h-1}\,[1+x^c]^{-k}\, dx\right].$$

Let $U = X^c$

Hence

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-1)^{i+h}\,(i)^{j+h}\,\alpha^j\,\beta^h}{i!\;\; j!\;\; h!}\left[r\int_0^\infty u^{\frac{r+j-h-1}{c}}\,[1+u]^{-k}\,\frac{1}{c}u^{\frac{1}{c}-1}\, du\right].$$

Thus;

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\frac{(-1)^{i+h}\,(i)^{j+h}\,\alpha^j\,\beta^h}{i!\;\; j!\;\; h!}\left[\frac{r}{c}\; B\left(\frac{r+j-h}{c}, k-\frac{r+j-h}{c}\right)\right].$$

Some of the most important features and characteristics of a distribution can be studied through moments (e.g., tendency, dispersion, skewness and kurtosis). The variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$\sigma^2 = \mu'_2 - \mu^2,$$

$$CV = \sqrt{\frac{\mu'_2}{\mu^2} - 1},$$

$$CS = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}},$$

and

$$CK = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2}.$$

The r^{th} moment of the FWBXII can be alternatively obtained using numerical integration. The following table displays the first four moments and the corresponding σ^2, CV, CS and CK values for various choices of the parameters (α, β, c, k). It is observed from this table that by choosing different values of the parameters the distribution can cover both cases of positive and negative skewness.

Table 2: Moments of the FWBXII distribution for various choices of the parameters (α, β, c, k)

α	β	c	k	μ'_1	μ'_2	μ'_3	μ'_4	σ^2	CV	CS	CK
0.6	0.3	0.8	1	0.4615	0.4662	0.7037	1.3185	0.2532	1.0904	1.9998	7.4726
3	2	0.8	0.8	0.5669	0.4103	0.3229	0.2674	0.0889	0.5260	-0.3967	2.0876

1	0.3	2	0.5	0.4695	0.3445	0.3340	0.3858	0.1241	0.7502	1.2758	4.4441
2	2	3	1	0.7648	0.6490	0.5958	0.5826	0.0641	0.3310	0.0879	2.7305
0.6	0.3	10	2	0.5177	0.3706	0.3066	0.2725	0.1026	0.6187	0.2594	1.7155
0.1	0.5	5	1.5	0.6769	0.5864	0.5969	0.6944	0.1282	0.5290	0.5751	3.6818

3.3. The Moment Generating Function

If X has FWBXII distribution (Θ) distribution, its moment generating function can be written in the form

$$M_X(t) = \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\frac{t^r}{r!}\frac{(-1)^{i+h}\,(i)^{j+h}\,\alpha^j\,\beta^h}{i!\;\;j!\;\;h!}\left[\frac{r}{c}\;\;B\left(\frac{r+j-h}{c},k-\frac{r+j-h}{c}\right)\right]. \quad (10)$$

under the condition $0<\frac{r+j-h}{c}<$ k.

Proof:

$$M_X(t)=\int_0^{\infty} e^{tx} f(x)dx.$$

Using the expression of e^{tx}

$$e^{tx}=\sum_{r=0}^{\infty}\frac{t^r\,x^r}{r!}.$$

So;

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^{\infty} x^r f(x)dx.$$

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\;\mu'_r. \quad (11)$$

Substituting from (9) into (11); (10) is obtained.

3.4. The Order Statistics

Order statistics play an important role in probability and statistics. In this subsection, we present the distribution of the i^{th} order statistic from the FWBXII distribution. Let $X_{(1)}\le X_{(2)}\le\cdots\le X_{(n)}$ denote the order statistics obtained from a random sample $X_1, X_2, \cdots, X_n$ which is taken from the FWBXII distribution with $f(x;\Theta)$ and $F(x;\Theta)$ given by Equation (2) and Equation (3) respectively. Then the pdf $f_{X_{(i)}}(x)$ of the i^{th} order statistics, say $X_{(i)}$, can be derived as

$$f_{X_{(i)}}(x)=\frac{1}{B(i,n-i+1)}\,[F(x;\Theta)]^{i-1}\,[1-F(x;\Theta)]^{n-i}\,f(x;\Theta). \quad (12)$$

Since $0<F(x)<1$ for $x>0$, by using the binomial expansion theorem for $[1-F(x;\Theta)]^{n-i}$ we obtain

$$[1-F(x;\Theta)]^{n-i}=\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\,[F(x;\Theta)]^j. \quad (13)$$

Substituting $[1-F(x;\Theta)]^{n-i}$ into $f_{X_{(i)}}(x)$ gives

$$f_{X_{(i)}}(x)=\sum_{j=0}^{n-i}\frac{(-1)^j\,n!}{j!\;\;(i-1)!\;\;(n-i-j)!}\,[F(x;\Theta)]^{j+i-1}\;f(x;\Theta). \quad (14)$$

Substituting the pdf and cdf given by (2) and (3) in (14), the pdf $f_{X_{(i)}}(x)$ of the FWBXII distribution is given by

$$f_{X_{(i)}}(x)=\sum_{j=0}^{n-i}\frac{(-1)^j\,n!}{j!\;\;(i-1)!\;\;(n-i-j)!}\left[1-exp\left[-e^{\alpha x-\frac{\beta}{x}}\right][1+x^c]^{-k}\right]^{j+i-1}\;\left[\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}\right.$$
$$\left.+ck\,x^{c-1}[1+x^c]^{-1}\right]exp\left[-e^{\alpha x-\frac{\beta}{x}}\right][1+x^c]^{-k}. \quad (15)$$

4. Maximum Likelihood Estimation

In this section, we discuss the estimation of the 4-dimensional parameter vector $\Theta = (\alpha, \beta, c, k)$ of the FWBXII distribution by using the method of maximum likelihood. Let $X_1, X_2, \dots, X_n$ be a random sample of complete data from the FWBXII distribution defined by (2) and suppose that we are interested in estimating the unknown parameters. The Likelihood function of this sample for $\Theta = (\alpha, \beta, c, k)$ takes the form

$$L = \prod_{i=1}^{n} f(x_i; \Theta). \tag{16}$$

Substituting $f(x_i; \Theta)$ defined by (2) into (16), the above can be written as

$$L = \prod_{i=1}^{n} \left[\left(\alpha + \frac{\beta}{x_i^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}} + ck\, x_i^{c-1}[1 + x_i^c]^{-1}\right] exp\left[-e^{\alpha x_i - \frac{\beta}{x_i}}\right][1 + x_i^c]^{-k}. \tag{17}$$

The corresponding log-likelihood function in this case is then given by

$$\mathcal{L} = \sum_{i=1}^{n} \ln\left[\left(\alpha + \frac{\beta}{x_i^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}} + ck\, x_i^{c-1}[1 + x_i^c]^{-1}\right] - \sum_{i=1}^{n} e^{\alpha x_i - \frac{\beta}{x_i}} - k\sum_{i=1}^{n} \ln(1 + x_i^c). \tag{18}$$

By differentiating the log-likelihood function with respect to the parameters (α, β, c, k) and setting the result to zero, we obtain the following system of nonlinear Equations.

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \sum_{i=1}^{n} \frac{h_\alpha(x_i; \Theta)}{h(x_i; \Theta)} - \sum_{i=1}^{n} x_i e^{\alpha x_i - \frac{\beta}{x_i}} = 0, \tag{19}$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = \sum_{i=1}^{n} \frac{h_\beta(x_i; \Theta)}{h(x_i; \Theta)} + \sum_{i=1}^{n} \frac{1}{x_i} e^{\alpha x_i - \frac{\beta}{x_i}} = 0, \tag{20}$$

$$\frac{\partial \mathcal{L}}{\partial c} = \sum_{i=1}^{n} \frac{h_c(x_i; \Theta)}{h(x_i; \Theta)} - k\sum_{i=1}^{n} \frac{x_i^c \ln(x_i)}{(1 + x_i^c)} = 0, \tag{21}$$

$$\frac{\partial \mathcal{L}}{\partial k} = \sum_{i=1}^{n} \frac{h_k(x_i; \Theta)}{h(x_i; \Theta)} - \sum_{i=1}^{n} \ln(1 + x_i^c) = 0, \tag{22}$$

where

$$h_\alpha(x_i; \Theta) = \frac{\partial h(x_i; \Theta)}{\partial \alpha} = \left(\alpha + \frac{\beta}{x_i^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}}\, x_i + e^{\alpha x_i - \frac{\beta}{x_i}},$$

$$h_\beta(x_i; \Theta) = \frac{\partial h(x_i; \Theta)}{\partial \beta} = \frac{1}{x_i^2} e^{\alpha x_i - \frac{\beta}{x_i}} - \frac{1}{x_i}\left(\alpha + \frac{\beta}{x_i^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}},$$

$$h_c(x_i; \Theta) = \frac{\partial h(x_i; \Theta)}{\partial c} = ck\, x_i^{c-1}[1 + x_i^c]^{-1}\left(\frac{1}{c} + \ln(x_i) - \frac{x_i^c \ln(x_i)}{(1 + x_i^c)}\right),$$

$$h_k(x_i; \Theta) = \frac{\partial h(x_i; \Theta)}{\partial k} = c\, x_i^{c-1}[1 + x_i^c]^{-1},$$

and $h(x_i; \Theta)$ is defined in Equation (1).

It is clear that there is no explicit solution for the above system of nonlinear Equations. Therefore we can use a numerical method such as the Newton-Raphson method to solve these nonlinear Equations and obtain the maximum likelihood estimates of the four parameters, which will be referred to as $\hat{\alpha}, \hat{\beta}, \hat{c}$ and $\hat{k}$. This technique required obtaining the second partial derivatives of the log-likelihood function with respect to the parameters. These derivatives are given in the Appendix.

5. Simulation Study

In this section, a Monte Carlo simulation study is carried out using different sample sizes (n= 30, 50, 100, 200, 400) in order to examine the performance of the FWBXII distribution via the R package. We simulate 1000 samples for the true parameters values (α=1, β=0.3, c=2, k=0.5) and (α=0.05, β=0.02, c=0.5, k=2). Comparison between different estimators is made with respect to their estimated mean square errors (MSEs) and estimated absolute bias (Abs. Bias).

Table (3) displays the average estimates of the four model parameters along with their corresponding MSE and Abs. Bias. From the results of this table we can conclude that the MSE for the estimates of the four parameters always deceases when the sample size increase and in most cases, the Abs. Bias tends to decrease as the sample size increases.

Table (3): Monte Carlo simulation results

n	Parameter	(α=1, β=0.3, c=2, k=0.5)			(α=0.05, β=0.02, c=0.5, k=2)		
		Est.	*MSE*	*Abs. Bias*	*Est.*	*MSE*	*Abs. Bias*
30	α	1.0026	0.5606	0.2083	0.1097	0.4705	0.0713
	β	0.3385	0.6408	0.0676	0.1528	0.4932	0.1348
	c	2.2115	2.9077	0.7326	0.4673	0.1078	0.1603
	k	0.3585	0.5348	0.6014	0.9801	1.2697	1.0466
50	α	0.9576	0.0807	0.1809	0.0585	0.0023	0.0236
	β	0.3090	0.0085	0.0337	0.0580	0.0272	0.0388
	c	2.1311	2.3237	0.7013	0.4525	0.0382	0.1127
	k	0.3890	0.4846	0.5793	1.0244	1.0685	0.9810
100	α	0.9675	0.0547	0.1451	0.0439	0.0002	0.0117
	β	0.3057	0.0063	0.0261	0.0361	0.0013	0.0163
	c	2.0957	1.6520	0.7786	0.4372	0.0082	0.0771
	k	0.3009	0.3383	0.4865	1.0752	0.9142	0.9270
200	α	0.9895	0.0224	0.1062	0.0399	0.0001	0.0106
	β	0.3028	0.0007	0.0179	0.0332	0.0002	0.0132
	c	1.9871	1.0901	0.7333	0.4353	0.0061	0.0690
	k	0.2133	0.2139	0.4168	1.0860	0.8635	0.9153
400	α	0.9932	0.0137	0.0829	0.0388	0.0001	0.0111
	β	0.3019	0.0004	0.0136	0.0328	0.0002	0.0128
	c	1.9140	1.0769	0.7383	0.4337	0.0053	0.0668
	k	0.1569	0.1897	0.3998	1.0815	0.8543	0.9185

6. Data Analysis

In this section, we analyze a real data set to illustrate the applicability and flexibility of the FWBXII distribution for data modeling compared with many known distributions such as additive Weibull (AddW) presented (by Xie and Lai, (1996)), additive Burr XII (AddBXII) presented (by Wang, (2000)), new modified Weibull (NMW) presented (by Almalki and Yuan, (2013)), exponential Flexible Weibull extension (EFW) presented (by El-Desouky et al., (2017)), Flexible Weibull extension (FW) and Burr XII (BXII) distribution. The NMW and EFW pdfs are given by

$$g_{NMW}(x) = \left(\alpha\theta x^{\theta-1} + \beta(\eta + \lambda x)x^{\eta-1}e^{\lambda x}\right)e^{-\alpha x^{\theta}-\beta x^{\eta}e^{\lambda x}}, x > 0;\ \alpha, \theta, \beta, \eta, \lambda > 0.$$

and

$$g_{EFW}(x) = \lambda\left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x-\frac{\beta}{x}} e^{e^{\alpha x-\frac{\beta}{x}}} exp\left[-\lambda\, e^{e^{\alpha x-\frac{\beta}{x}}}\right], x > 0; \alpha, \beta, \lambda > 0,$$

respectively. We have fitted all selected distributions using the method of maximum likelihood for each data set. In order to compare the aforementioned distributions with the proposed distribution, we applied formal goodness-of-fit tests to verify which distribution fits better to the real data set. Here, we calculated the Kolmogorov Smirnov (*K-S*) distance test statistic and its corresponding p-value, the log-likelihood ($\mathcal{L}$), Akaike information criterion (*AIC*), Akaike information criterion with correction (*AICc*) and Bayesian information criterion (*BIC*) values, where

$AIC = 2m - 2\ln(L)$,

$$AICc = AIC + 2\frac{m(m+1)}{n-m-1},$$

$$BIC = m\ln(n) - 2\ln(L),$$

$L = L(\hat{\theta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and m is the number of estimated parameters. In general, the distribution which has the smallest values of these statistics is the better fit for the data.

6.1. Data Set

The data corresponds to the time in months to first failure of any kind of small electric cars used for internal transportation and delivery in a large manufacturing facility that were taken from (Nelson, (1982)). The MLEs of the parameters (standard error in parenthesis), log-likelihood, the *K-S* test statistic with its corresponding p-value, *AIC, AICc* and *BIC* are presented in Table 4. The results of this Table indicate that the FWBXII distribution has the lowest *K-S* value and the highest p-value, which means that the new distribution fits the data better than the other distributions. In addition, our new model has the smallest values of the (*AIC, AICc* and *BIC*) statistics. Thus, we can say that the FWBXII is the best among the other fitted models used here.

More information is provided by a visual comparison of the histogram of the data with the fitted density functions when the distribution is assumed to be AddW, AddBXII, NMW, EFW, FW and BXII in Figure 3(a). In addition, the comparison of the empirical curve of the reliability function by using the Kaplan-Meier method and the fitted reliability functions is displayed in Figure 3(b). From Figure 3, we can conclude that the FWBXII distribution provides a very good fit for these data compared to all other distributions considered here.

Table 4: The MLEs, corresponding standard errors in brackets, K-S, p-values, Log-likelihood, AIC, AICc and BIC of the fitted models

Model	**MLE**		**K-S**	**P-value**	**log L**	**AIC**	**AICc**	**BIC**
	Parameter	**Estimate (S.E.)**						
FWBXII	α	0.0175 (0.0062)	0.1355	0.9781	-67.246	142.493	145.351	146.271
	β	9.4183 (3.8101)						
	c	1.2671 (0.6115)						
	k	0.1737 (0.0957)						
AddW	a	1.778e-5(2.73e-4)	0.7284	0.0000	-108.450	224.903	227.760	228.680
	b	0.2030 (0.0466)						
	c	0.0902 (0.0209)						
	d	0.1038 (0.0236)						
AddBXII	c_1	0.9097 (0.1582)	0.2388	0.5379	-73.203	154.399	157.256	158.176
	k_1	4.238e-6 (0.0045)						
	c_2	1.1920 (0.2707)						
	k_2	0.2861 (0.0763)						
NMW	θ	0.0907 (0.0476)	0.3671	0.1532	-83.027	176.029	180.644	180.751
	α	0.0007 (0.0012)						
	β	0.2033 (0.1210)						
	λ	0.0496 (0.0099)						
	η	0.1025 (0.0633)						
EFW	α	0.0144 (0.0021)	0.2667	0.5379	-77.484	160.969	162.569	163.803
	β	0.9283 (0.3395)						
	λ	0.2829 (0.0798)						
FW	α	0.0213 (0.0040)	0.2666	0.5379	-71.556	147.114	147.864	149.003
	β	1.3689 (0.4233)						
BXII	c	1.7379 (0.5980)	0.2225	0.8081	-71.448	146.896	147.646	148.785
	k	0.2936 (0.1157)						

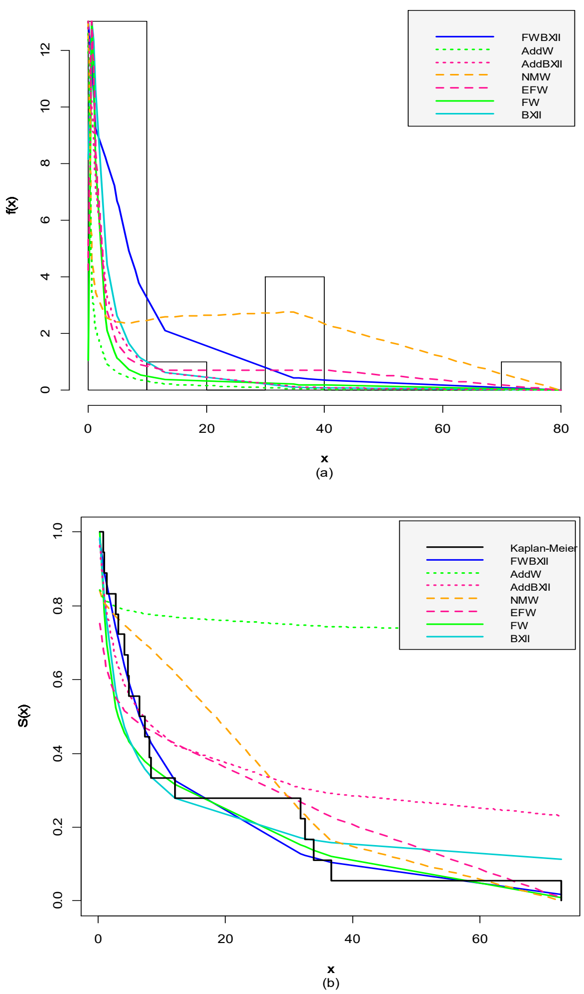

Figure 3: (a) Fitted densities, (b) The Kaplan-Meier estimate and the fitted reliability functions

6.2. Sub-Models of FWBXII

For the sake of simplicity, we will compare the new three distributions, FWCE, FWCR and FWLogL, with the FWBXII distribution to see if any of these three distributions can perform as well as the FWBXII distribution, since it is better to reduce the number of parameters of any distribution. The MLEs of the parameters (standard error in parenthesis), log-likelihood, the *K-S* test statistic with its corresponding p-value, *AIC, AICc* and *BIC* of FWBXII, FWCE, FWCR and FWLogL are presented in Tables 5. It is clear that the FWBXII, FWCE and FWCR distributions fit the data well. All of them have very small K-S values and high p-value. The FWBXII distribution has the larger log-likelihood value. However, the FWCE distribution has the smaller values for AIC, AICc and BIC. Also, Figure 4 show that the FWCE distribution is nearly as good as the FWBXII distribution. In addition, the likelihood ratio test (LRT) is used to test the reduced model $H_0: c = 1$ against the original model $H_a: c \neq 1$. The LRT statistic is 0.220 (p-value=0.6390), with one degrees of freedom. So there is no significant evidence to reject H_0. Therefore, we can conclude that reducing the number of parameters to three by fixing one of them still provides a better fit for the used data set.

Table 5: Results of sub-models of FWBXII

Model	**MLE**		**K-S**	**P-value**	**log L**	**AIC**	**AICc**	**BIC**
	Parameter	**Estimate (S.E.)**						
FWBXII	α	0.0175 (0.0062)	0.1355	0.9781	-67.246	142.493	145.351	146.271
	β	9.4183 (3.8101)						
	c	1.2671 (0.6115)						

	k	0.1737 (0.0957)						
FWCE	α β k	0.0177 (0.0060) 8.7830 (4.1932) 0.1961 (0.0985)	0.1297	0.9781	-67.356	140.713	142.313	143.546
FWCR	α β k	0.0171 (0.0063) 10.4970 (4.0612) 0.1265 (0.1265)	0.1421	0.9781	-67.736	141.472	143.072	144.306
FWLogL	α β c	0.0163 (0.0068) 11.3857 (5.4749) 0.4244 (0.1597)	0.4210	0.1532	-75.455	156.909	158.509	159.743

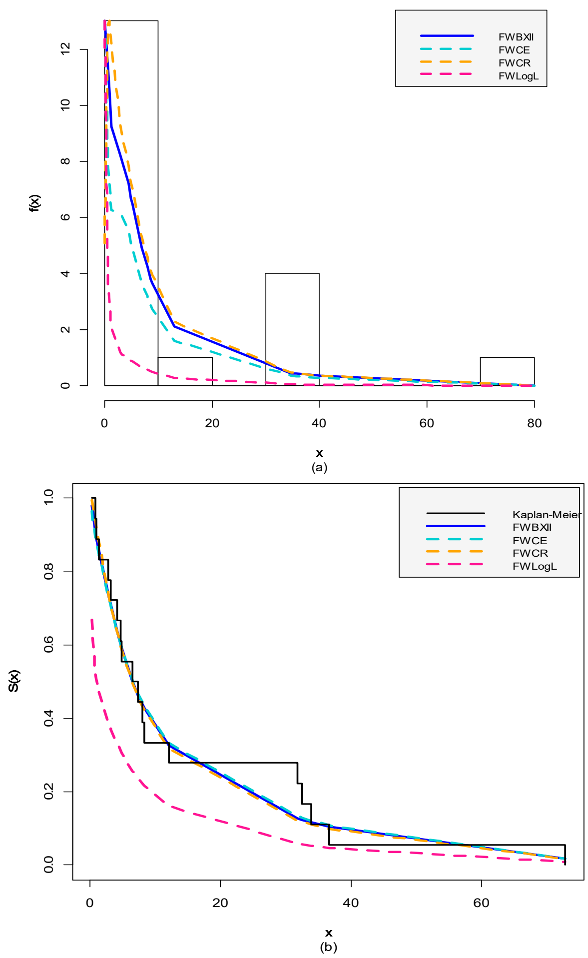

Figure 4: (a) Fitted densities, (b) The Kaplan-Meier estimate and the fitted reliability functions of full model vs. sub-models

7. Conclusion

In this paper, we propose a new distribution called FWBXII distribution. Its definition, the behavior of its failure rate function and some of its statistical properties including the mode, quantile function, moments, moment generating function and order statistics are studied. We use the maximum likelihood method for estimating parameters. Furthermore, the importance of the new distribution is that it includes FWCE, FWCR and FWLogL, which are not known in the literature. In addition, in order to illustrate the usefulness, flexibility and applicability of the distribution, one real data set is analyzed using the new distribution and it is compared with AddW, AddBXII, NMW, EFW, FW and BXII distributions. It is evident from the comparisons that the new distribution is the best distribution for fitting this data set compared to other distributions considered here.

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Appendix
Fisher Information Matrix

The likelihood function of $\Theta = (\alpha, \beta, c, k)$ based on the FWBXII distribution is given by

$$L = \prod_{i=1}^{n} f(x_i; \Theta).$$

By differentiating the log-likelihood function $\mathcal{L}$ with respect to the parameters, we obtain the first order derivatives of $\mathcal{L}$ as given in (19) to (22). Upon differentiating these expressions once again with respect to the parameters, we obtain the partial derivatives of second order as follows:

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} = \sum_{i=1}^{n} \left(\frac{h(x_i;\Theta)\, h_{\alpha\alpha}(x_i;\Theta) - \left(h_\alpha(x_i;\Theta)\right)^2}{(h(x_i;\Theta))^2} - {x_i}^2 e^{\alpha x_i - \frac{\beta}{x_i}} \right),$$

$$\frac{\partial^2 \mathcal{L}}{\partial \beta^2} = \sum_{i=1}^{n} \left(\frac{h(x_i;\Theta)\, h_{\beta\beta}(x_i;\Theta) - \left(h_\beta(x_i;\Theta)\right)^2}{(h(x_i;\Theta))^2} - \frac{1}{{x_i}^2} e^{\alpha x_i - \frac{\beta}{x_i}} \right),$$

$$\frac{\partial^2 \mathcal{L}}{\partial c^2} = \sum_{i=1}^{n} \left(\frac{h(x_i;\Theta)\, h_{cc}(x_i;\Theta) - \left(h_c(x_i;\Theta)\right)^2}{(h(x_i;\Theta))^2} - k\ln(x_i) \frac{{x_i}^c \ln(x_i)\,(1+{x_i}^c) - {x_i}^{2c}\ln(x_i)}{(1+{x_i}^c)^2} \right),$$

$$\frac{\partial^2 \mathcal{L}}{\partial k^2} = -\sum_{i=1}^{n} \left(\frac{\left(h_k(x_i;\Theta)\right)^2}{\left(h(x_i;\Theta)\right)^2} \right),$$

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} \left(\frac{h(x_i;\Theta)\, h_{\alpha\beta}(x_i;\Theta) - h_\alpha(x_i;\Theta) h_\beta(x_i;\Theta)}{(h(x_i;\Theta))^2} + e^{\alpha x_i - \frac{\beta}{x_i}} \right),$$

$$\frac{\partial^2 \mathcal{L}}{\partial\alpha\partial c} = -\sum_{i=1}^{n}\left(\frac{h_\alpha(x_i;\Theta)h_c(x_i;\Theta)}{\left(h(x_i;\Theta)\right)^2}\right),$$
$$\frac{\partial^2 \mathcal{L}}{\partial\alpha\partial k} = -\sum_{i=1}^{n}\left(\frac{h_\alpha(x_i;\Theta)h_k(x_i;\Theta)}{\left(h(x_i;\Theta)\right)^2}\right),$$
$$\frac{\partial^2 \mathcal{L}}{\partial\beta\partial c} = -\sum_{i=1}^{n}\left(\frac{h_\beta(x_i;\Theta)h_c(x_i;\Theta)}{\left(h(x_i;\Theta)\right)^2}\right),$$
$$\frac{\partial^2 \mathcal{L}}{\partial\beta\partial k} = -\sum_{i=1}^{n}\left(\frac{h_\beta(x_i;\Theta)h_k(x_i;\Theta)}{\left(h(x_i;\Theta)\right)^2}\right),$$
$$\frac{\partial^2 \mathcal{L}}{\partial c\partial k} = \sum_{i=1}^{n}\left(\frac{h(x_i;\Theta)\, h_{ck}(x_i;\Theta) - h_c(x_i;\Theta)h_k(x_i;\Theta)}{(h(x_i;\Theta))^2} - \frac{{x_i}^c \ln(x_i)}{(1+{x_i}^c)}\right),$$

where

$$h_{\alpha\alpha}(x_i;\Theta) = \frac{\partial^2 h(x_i;\Theta)}{\partial\alpha^2} = \left(\alpha + \frac{\beta}{{x_i}^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}}\ {x_i}^2 + 2x_i e^{\alpha x_i - \frac{\beta}{x_i}},$$
$$h_{\beta\beta}(x_i;\Theta) = \frac{\partial^2 h(x_i;\Theta)}{\partial\beta^2} = -\frac{2}{{x_i}^3} e^{\alpha x_i - \frac{\beta}{x_i}} + \frac{1}{{x_i}^2}\left(\alpha + \frac{\beta}{{x_i}^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}},$$
$$\begin{aligned} h_{cc}(x_i;\Theta) = \frac{\partial^2 h(x_i;\Theta)}{\partial c^2} \\ &= -ck\, {x_i}^{c-1}[1+{x_i}^c]^{-1}\left(\frac{1}{c^2} + \frac{(1+{x_i}^c){x_i}^c(\ln(x_i))^2 - {x_i}^{2c}(\ln(x_i))^2}{(1+{x_i}^c)^2}\right) \\ &- k\, {x_i}^{c-1}[1+{x_i}^c]^{-1}\left(\frac{1}{c} + \ln(x_i) - \frac{{x_i}^c\ln(x_i)}{(1+{x_i}^c)}\right)(c{x_i}^c[1+{x_i}^c]^{-1}\ln(x_i) - c\ln(x_i) - 1),\end{aligned}$$
$$h_{\alpha\beta}(x_i;\Theta) = \frac{\partial^2 h(x_i;\Theta)}{\partial\alpha\partial\beta} = -\left(\alpha + \frac{\beta}{{x_i}^2}\right) e^{\alpha x_i - \frac{\beta}{x_i}},$$
$$h_{ck}(x_i;\Theta) = \frac{\partial^2 h(x_i;\Theta)}{\partial c\partial k} = c\, {x_i}^{c-1}[1+{x_i}^c]^{-1}\left(\frac{1}{c} + \ln(x_i) - \frac{{x_i}^c\ln(x_i)}{(1+{x_i}^c)}\right).$$